SHARP QUANTITATIVE ESTIMATES OF STRUWE'S DECOMPOSITION

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ABSTRACT. In a seminal work, Struwe proved that if $0 \leq u \in \dot{H}^1(\mathbb{R}^n)$ and $\Gamma(u) := \|\Delta u + u^{\frac{n+2}{n-2}}\|_{H^{-1}} \to 0$ then $dist(u, \mathcal{T}) \to 0$, where $\mathcal{T}$ denotes the manifold of sums of Aubin-Talenti bubbles and $dist(u, \mathcal{T})$ denotes the $\dot{H}^1(\mathbb{R}^n)$-distance of u from $\mathcal{T}$. Ciraolo, Figalli and Maggi obtained the first quantitative version of Struwe's decomposition with one bubble in all dimensions, namely $dist(u, \mathcal{T}) \leq C\Gamma(u)$. For two or more bubbles, Figalli and Glaudo showed a striking dimensional dependent quantitative estimate, namely $dist(u, \mathcal{T}) \leq C\Gamma(u)$ when $3 \leq n \leq 5$ while this is false for $n \geq 6$. In this paper, we show

$$dist(u, \mathcal{T}) \leq C \begin{cases} \Gamma(u) \left|\log \Gamma(u)\right|^{\frac{1}{2}} & \text{if } n = 6, \\ |\Gamma(u)|^{\frac{n+2}{2(n-2)}} & \text{if } n \geq 7. \end{cases}$$

Furthermore, we show that this inequality is sharp.

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1. Introduction

1.1. **Motivation and main results.**

The Sobolev inequality with exponent 2 states that, for any $n \geq 3$ and any $u \in \dot{H}^1(\mathbb{R}^n) := D^{1,2}(\mathbb{R}^n)$, the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm $\|\nabla u\|_{L^2}$, it holds that

$$S\|u\|_{L^{2^*}} \leq \|\nabla u\|_{L^2}, \tag{1.1}$$

where $2^* = \frac{2n}{n-2}$ and $S = S(n)$ is a dimensional constant.

It is well-known that the Euler-Lagrange equation of (1.1), up to scaling, is given by

$$\Delta u + |u|^{p-1}u = 0 \qquad \text{in } \mathbb{R}^n. \tag{1.2}$$

Throughout this paper, we denote $p = \frac{n+2}{n-2}$. By Caffarelli et al. [10] and Gidas et al. [26], it is known that all the positive solutions are *Aubin-Talenti bubbles* [35], which are defined as

$$U[z,\lambda](x) := (n(n-2))^{\frac{n-2}{4}} \left(\frac{\lambda}{1+\lambda^2|x-z|^2}\right)^{\frac{n-2}{2}}. \tag{1.3}$$

Here and after, we shall call z the *center* and λ the *height* of the bubble $U[z,\lambda]$. These bubbles are all the minimizers of the Sobolev inequality, up to scaling. There are many advances in the study of the stability of (1.1). From the perspective of discrepancy in the Sobolev inequality, Bianchi and Egnell [6] gave a quantitative estimate near the minimizers, which is

$$\inf_{z\in\mathbb{R}^n,\lambda>0,\alpha\in\mathbb{R}} \|\nabla(u-\alpha U[z,\lambda])\|_{L^2}^2 \leq C(n)\left(\|\nabla u\|_{L^2}^2 - S^2\|u\|_{L^{2^*}}^2\right). \tag{1.4}$$

A natural and more challenging perspective is through the Euler-Lagrange equation: whether a function u that almost solves (1.2) must be quantitatively close to Aubin-Talenti bubbles. There are many obstacles to addressing this question. First, (1.2) has many sign-changing solutions [18, 20]. Second, even if we restrict to the non-negative functions, u could be a sum of many *weakly interacting* Aubin-Talenti bubbles. In fact, a seminal work of Struwe [34] showed that this is always the case, at least for non-negative functions.

Theorem 1.1 (Struwe [34])**.** *Let $n \geq 3$ and $\nu \geq 1$ be positive integers. Let $(u_k)_{k\in\mathbb{N}} \subseteq \dot{H}^1(\mathbb{R}^n)$ be a sequence of non-negative functions such that $\left(\nu - \frac{1}{2}\right)S^n \leq \int_{\mathbb{R}^n} |\nabla u_k|^2 \leq \left(\nu + \frac{1}{2}\right)S^n$ with $S = S(n)$ as in (1.1), and assume that*

$$\left\|\Delta u_k + u_k^{2^*-1}\right\|_{H^{-1}} \to 0 \qquad \text{as } k \to \infty.$$

Then there exist a sequence $(z_1^{(k)}, \ldots, z_\nu^{(k)})_{k\in\mathbb{N}}$ of ν-tuples of points in $\mathbb{R}^n$ and a sequence $(\lambda_1^{(k)}, \ldots, \lambda_\nu^{(k)})_{k\in\mathbb{N}}$ of ν-tuples of positive real numbers such that

$$\left\|\nabla\left(u_k - \sum_{i=1}^{\nu} U[z_i^{(k)}, \lambda_i^{(k)}]\right)\right\|_{L^2} \to 0 \qquad \text{as } k \to \infty.$$

One can also show that the family of $U[z_i^{(k)}, \lambda_i^{(k)}]$ are *asymptotically orthogonal* in $\dot{H}^1(\mathbb{R}^n)$ as $k \to \infty$ (see Brezis and Coron [9, p. 35]). More precisely, denoting $U_i^{(k)} = U[z_i^{(k)}, \lambda_i^{(k)}]$, by Lemma A.3, we have $\int_{\mathbb{R}^n} \nabla U_i^{(k)} \cdot \nabla U_j^{(k)} = \int_{\mathbb{R}^n} (U_i^{(k)})^p U_j^{(k)} \approx q_{ij}^{(k)} \to 0$ as $k \to \infty$, where $q_{ij}^{(k)} = q(z_i^{(k)}, z_j^{(k)}, \lambda_i^{(k)}, \lambda_j^{(k)})$ is defined as the following.

Definition 1.2 (Interaction of Aubin-Talenti bubbles)**.** Let $U[z_i, \lambda_i]$ and $U[z_j, \lambda_j]$ be two bubbles. Define the interaction of them by

$$q(z_i, z_j, \lambda_i, \lambda_j) = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |z_i - z_j|^2 \right)^{-\frac{n-2}{2}}. \tag{1.5}$$

We shall denote $q_{ij} = q_{ji} = q(z_i, z_j, \lambda_i, \lambda_j)$. Let $\{U_i : 1 \leq i \leq \nu\}$ be a family of Aubin-Talenti bubbles. We say that the family is δ*-interacting* if

$$Q := \max\{q_{ij} : 1 \leq i \neq j \leq \nu\} \leq \delta. \tag{1.6}$$

Despite the difficulty of the non-negativity issue, one can still investigate the problem locally. That is, if u is already near to a sum of weakly interacting Aubin-Talenti bubbles in $\dot{H}^1$-norm, then $\|\Delta u + u|u|^{p-1}\|_{H^{-1}}$ should control the $\dot{H}^1$-distance between u and $\mathcal{T}$. Here $\mathcal{T}$ denotes the manifold of sums of Aubin-Talenti bubbles. Along this direction, Ciraolo et al. [13] obtained the first quantitative estimate $dist(u, \mathcal{T}) \leq C\|\Delta u + u|u|^{p-1}\|_{L^{\frac{2n}{n+2}}}$ for all $n \geq 3$ when $\nu = 1$, i.e., when only one bubble is present. Later, Figalli and Glaudo [22] established the following theorem for any finite number of bubbles.

Theorem 1.3 (Figalli and Glaudo [22])**.** *For any dimension $3 \leq n \leq 5$ and $\nu \in \mathbb{N}$, there exist a small constant $\delta = \delta(n, \nu) > 0$ and a large constant $C = C(n, \nu) > 0$ such that the following statement holds. Let $u \in \dot{H}^1(\mathbb{R}^n)$ be a function such that*

$$\left\| \nabla u - \sum_{i=1}^{\nu} \nabla \tilde{U}_i \right\|_{L^2} \leq \delta,$$

where $\{\tilde{U}_i : 1 \leq i \leq \nu\}$ is a δ-interacting family of Aubin-Talenti bubbles. Then there exist ν Aubin-Talenti bubbles $U_1, U_2, \ldots, U_\nu$ such that

$$\left\| \nabla u - \sum_{i=1}^{\nu} \nabla U_i \right\|_{L^2} \leq C \left\| \Delta u + u|u|^{p-1} \right\|_{H^{-1}}. \tag{1.7}$$

Furthermore, for any $i \neq j$, the interaction between the bubbles can be estimated as

$$\int_{\mathbb{R}^n} U_i^p U_j \leq C \left\| \Delta u + u|u|^{p-1} \right\|_{H^{-1}}.$$

When $n \geq 6$ and $\nu > 1$, Figalli and Glaudo constructed some counterexamples that show that (1.7) is no longer true. They conjectured that one needs to modify the RHS of (1.7) to $\Gamma|\log \Gamma|$ when $n = 6$ and to $|\Gamma|^\gamma$ for some $\gamma < 1$ when $n \geq 7$, where $\Gamma = \left\| \Delta u + u|u|^{p-1} \right\|_{H^{-1}}$. However, the exact value of γ is not known. On the other hand, it is well-known that dimension plays an important role in the analysis of (1.2). The Yamabe problem also has dimension 6 as a threshold, see Aubin [1], Schoen [32]. Prescribing scalar curvature problem has a similar analysis to bubbles, and there the dimension seems to play a more important role. For instance (not intended to be complete), one can see Li [27], Druet [21], Chang and Yang [12], Bahri and Coron [5], Ayed et al. [2], Malchiodi and Mayer [29], and the references therein.

In this paper, we give affirmative answers to both questions of Figalli and Glaudo. Throughout this paper, we define $\zeta_n(x)$ for $x > 0$ and $n \geq 6$ as the following

$$\zeta_n(x) = \begin{cases} x^{\frac{p}{2}} & \text{if } n \geq 7, \\ x|\log x|^{\frac{1}{2}} & \text{if } n = 6. \end{cases} \tag{1.8}$$

It is easy to see that $\zeta_n(x)$ is increasing near zero.

Theorem 1.4. *Suppose $n \geq 6$. There exist a small constant $\delta = \delta(n,\nu) > 0$ and a large constant $C = C(n,\nu) > 0$ such that the following statement holds. Let $u \in \dot{H}^1(\mathbb{R}^n)$ be a function such that*

$$\left\| \nabla u - \sum_{i=1}^{\nu} \nabla \tilde{U}_i \right\|_{L^2} \leq \delta, \tag{1.9}$$

where $\{\tilde{U}_i : 1 \leq i \leq \nu\}$ is a δ-interacting family of Aubin-Talenti bubbles. Then there exist ν Aubin-Talenti bubbles $U_1, U_2, \ldots, U_\nu$ such that

$$\left\| \nabla u - \sum_{i=1}^{\nu} \nabla U_i \right\|_{L^2} \leq C\zeta_n(\Gamma) \tag{1.10}$$

for $\Gamma = \left\| \Delta u + u|u|^{p-1} \right\|_{H^{-1}}$. Furthermore, for any $i \neq j$, the interaction between the bubbles can be estimated as

$$\int_{\mathbb{R}^n} U_i^p U_j \leq C \left\| \Delta u + u|u|^{p-1} \right\|_{H^{-1}}. \tag{1.11}$$

Note that our theorem completely solves the remaining cases in higher dimensions $n \geq 6$. Moreover, we improve the conjecture of [22] when $n = 6$. After finding this intriguing power $\frac{p}{2}$, we went back to check the counterexamples in [22]. Their examples show that there exists $u \in \dot{H}^1(\mathbb{R}^n)$ when $n = 7$ and $\nu = 2$ such that

$$\inf_{\substack{z_1,z_2\in\mathbb{R}^n\\ \lambda_1,\lambda_2>0}} \left\| \nabla u - \sum_{i=1}^{2} \nabla U_i \right\|_{L^2} \geq C\Gamma^{\frac{9}{10}}.$$

Notice the fact that $\frac{9}{10} = \frac{p}{2}$ when $n = 7$ exactly implies that (1.10) is sharp in this case. Indeed, we can prove that our result (1.10) is sharp for all $n \geq 6$.

Theorem 1.5. *For sufficiently large $R > 0$, there exists some ρ such that if $u = U[-Re_1, 1] + U[Re_1, 1] + \rho$ where $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$, then*

$$\inf_{\substack{z_1,z_2\in\mathbb{R}^n\\ \lambda_1,\lambda_2>0}} \left\| \nabla u - \sum_{i=1}^{2} \nabla U[z_i,\lambda_i] \right\|_{L^2} \geq C\zeta_n(\Gamma)$$

for $\Gamma = \left\|\Delta u + u|u|^{p-1}\right\|_{H^{-1}}$.

As a consequence of Theorem 1.4, we obtain the following sharp quantitative estimates of Struwe's decomposition.

Corollary 1.6. *Suppose* $n \geq 6$. *There exists a large constant* $C = C(n, \nu) > 0$ *such that the following statement holds. For any non-negative function* $u \in \dot{H}^1(\mathbb{R}^n)$ *such that*

$$\left(\nu - \frac{1}{2}\right) S^n \leq \int_{\mathbb{R}^n} |\nabla u|^2 \leq \left(\nu + \frac{1}{2}\right) S^n,$$

then there exist ν *Aubin-Talenti bubbles* $U_1, U_2, \ldots, U_\nu$ *such that*

$$\left\|\nabla u - \sum_{i=1}^{\nu} \nabla U_i\right\|_{L^2} \leq C\zeta_n(\Gamma)$$

for $\Gamma = \|\Delta u + u^p\|_{H^{-1}}$. *Furthermore, for any* $i \neq j$, *the interaction between the bubbles can be estimated as*

$$\int_{\mathbb{R}^n} U_i^p U_j \leq C \|\Delta u + u^p\|_{H^{-1}}.$$

Finally, we remark that recently there has been a growing interest in understanding quantitative stability for functional and geometric inequalities, due to important applications to problems in the calculus of variations and PDEs. For extension of (1.4) to Sobolev inequality with general exponents we refer to Figalli and Neumayer [23], Figalli and Zhang [24], and the references therein. Stability results on Sobolev inequality can be used to obtain quantitative convergence rates for fast diffusion equations. We refer to Bonforte and Figalli [7], del Pino and Sáez [15], and the references therein. There is also rich literature on quantitative versions of the isoperimetric inequality and other geometric inequalities analogous to the Sobolev inequality. A nice description of the comparison between Sobolev inequality and isoperimetric inequality can be found in Figalli and Glaudo [22]. We refer to Brasco et al. [8], Cavalletti et al. [11], Delgadino et al. [19], Figalli and Glaudo [22], Fusco et al. [25], Maggi [28], and the references therein.

1.2. **Sketch of the proof.** We briefly explain the ideas of our proof. Throughout this paper, we shall write that $a \lesssim b$ (resp. $a \gtrsim b$) if $a \leq Cb$ (resp. $Ca \geq b$) where C is a constant depending only on the dimension n and the number of bubbles ν. The constant C may change line by line. Also, we say that $a \approx b$ if $a \lesssim b$ and $a \gtrsim b$. The integral $\int$ always means $\int_{\mathbb{R}^n}$ unless specified. We always denote with $o(1)$ any quantity that goes to 0 when δ goes to 0. The common notion $o(Q)$ means $o(Q)/Q$ goes to 0 when Q goes to 0.

Suppose u satisfies (1.9) with a family of δ-interacting bubbles. Consider the following minimization problem

$$dist(u, \mathcal{T}) := \inf_{\substack{z_1, \cdots, z_\nu \in \mathbb{R}^n \\ \lambda_1, \cdots, \lambda_\nu > 0}} \left\|\nabla u - \nabla\left(\sum_{i=1}^{\nu} U[z_i, \lambda_i]\right)\right\|_{L^2}.$$

It is well-known that (for instance, see [4, Appendix A]) if δ is small enough then such an infimum is achieved by *the best approximation*

$$\sigma := \sum_{i=1}^{\nu} U\left[z_i, \lambda_i\right]. \tag{1.12}$$

Let us denote $U_i := U\left[z_i, \lambda_i\right]$. Since the family $\{\tilde{U}_i : 1 \leq i \leq \nu\}$ is δ-interacting, then $\{U_i : 1 \leq i \leq \nu\}$ is δ'-interacting for some δ' that goes to 0 as δ goes to 0.

Let $\rho := u - \sigma$ be the difference between the original function and the best approximation. We call ρ the *error function*. Then ρ satisfies $\|\nabla \rho\|_{L^2} \leq \delta$ and the equation (cf. eq (2.4))

$$\Delta \rho + p\sigma^{p-1}\rho + \sigma^p - \sum_{i=1}^{\nu} U_i^p + N_\sigma(\rho) + f = 0, \tag{1.13}$$

where $N_\sigma(\rho) = (\sigma + \rho)|\sigma + \rho|^{p-1} - \sigma^p - p\sigma^{p-1}\rho$ and $f = -\Delta u - u|u|^{p-1}$. Moreover, ρ also satisfies the following orthogonal conditions

$$\int_{\mathbb{R}^n} \nabla \rho \cdot \nabla Z_i^a = 0 \text{ for any } 1 \leq i \leq \nu;\ 1 \leq a \leq n+1, \tag{1.14}$$

where Z_i^a are the (rescaled) derivatives of $U[z_i, \lambda_i]$ with respect to the a-th component of z_i and λ_i (cf. eq (2.1)).

The linearized operator of (1.13) is $\Delta + p\sigma^{p-1}$, which will have a non-trivial kernel when σ is the sum of a family of weakly interacting bubbles. The non-homogeneous term $\sigma^p - \sum_i U_i^p$ is the main data that encodes the interaction of bubbles. The key idea of this paper is to obtain a precise behavior of the first approximation of ρ.

To illustrate the main idea, we start with the easiest case. Assume $U_i = U[z_i, 1]$ is a family of δ-interacting bubbles with the same height. Since δ is very small, the centers z_i, $i = 1, \cdots, \nu$, are far from each other. Define $R = \min\{\frac{1}{2}|z_i - z_j| : i \neq j\}$ and then $Q \approx R^{2-n}$ from (1.6).

By some standard finite-dimensional reduction method (see for example [17, 37]), given a δ'-interacting family $\{U_i : 1 \leq i \leq \nu\}$, we can find a function ρ_0 (in an appropriate space) and a family of scalars $\left(c_a^i\right)$ such that

$$\begin{cases} \Delta \rho_0 + (\sigma + \rho_0)\left|\sigma + \rho_0\right|^{p-1} - \sum_{j=1}^{\nu} U_j^p = \sum\limits_{i=1}^{\nu} \sum\limits_{a=1}^{n+1} c_a^i U_i^{p-1} Z_i^a, \\ \int \nabla \rho_0 \cdot \nabla Z_i^a = 0, \quad i = 1, \cdots, \nu;\ a = 1, \cdots, n+1. \end{cases} \tag{1.15}$$

We obtained the following point-wise estimate of ρ_0, which is the central part of the paper. Denote $\langle x \rangle = \sqrt{1 + |x|^2}$. When the dimension $n \geq 7$, the point-wise estimate of ρ_0 is

$$|\rho_0(x)| \lesssim \sum_{j=1}^{\nu} \frac{R^{2-n}}{\langle x - z_j \rangle^2} \chi_{\{|x - z_j| \leq R\}} + \frac{R^{-4}}{\langle x - z_j \rangle^{n-4}} \chi_{\{|x - z_j| > R\}}. \tag{1.16}$$

Here χ_Ω is the characteristic function for a set Ω. When the dimension $n = 6$, the point-wise estimate of ρ_0 is

$$|\rho_0(x)| \lesssim \sum_{j=1}^{\nu} \frac{R^{-4}}{\langle x - z_j\rangle^2}\chi_{\{|x-z_j|\le R^2\}} + \frac{R^{-2}}{\langle x - z_j\rangle^3}\chi_{\{|x-z_j|>R^2\}}. \tag{1.17}$$

Notice that ρ_0 is small but decays very slowly in the core of each bubble. Using these point-wise estimates, we multiply (1.15) by ρ_0 and integrate it to get

$$\|\nabla\rho_0\|_{L^2} \lesssim \begin{cases} R^{2-n}R^{\frac{n-6}{2}} \approx Q^{\frac{p}{2}}, & n \ge 7, \\ R^{-4}|\log R|^{\frac12} \approx Q|\log Q|^{\frac12}, & n = 6, \end{cases} = \zeta_n(Q). \tag{1.18}$$

Here the dimension of the space plays an important role in the integration.

Now consider the remaining part of the error function $\rho_1 = \rho - \rho_0$. Then (1.13) and (1.15) imply that ρ_1 satisfies

$$\begin{aligned} \Delta\rho_1 + (\sigma + \rho_0 + \rho_1)\,|\sigma + \rho_0 + \rho_1|^{p-1} - (\sigma + \rho_0)\,|\sigma + \rho_0|^{p-1} \\ + \sum_{i=1}^{\nu}\sum_{a=1}^{n+1} c_a^j U_i^{p-1} Z_i^a + f = 0. \end{aligned} \tag{1.19}$$

Observe that the equation of ρ_1 no longer contains the interaction term $\sigma^p - \sum_{i=1}^{\nu} U_i^p$. Therefore, ρ_1 should be bounded by a higher order term of Q. Indeed, Proposition 6.4 proves that $\|\nabla\rho_1\|_{L^2} \lesssim Q^2 + \|f\|_{H^{-1}}$. Combining with the previous L^2 estimate of $\nabla\rho_0$, we get

$$\|\nabla\rho\|_{L^2} \le \|\nabla\rho_0\|_{L^2} + \|\nabla\rho_1\|_{L^2} \lesssim \|f\|_{H^{-1}} + \zeta_n(Q).$$

On the other hand, we shall multiply (1.13) by some appropriate Z_k^{n+1} and integrate it to arrive (cf. Lemma 2.1)

$$Q \lesssim \|f\|_{H^{-1}} + \left|\int \sigma^{p-1}\rho Z_k^{n+1}\right| + \int |\rho|^p |Z_k^{n+1}|.$$

To establish the above estimates, unlike [22], we did not use cut-off functions. Using the point-wise estimates (1.16) and (1.17) of ρ_0, we can show that the last two terms are higher order terms in Q and then $Q \lesssim \|f\|_{H^{-1}}$. Consequently, $\|\nabla\rho\|_{L^2} \le \zeta_n(\|f\|_{H^{-1}})$. Thus one can establish Theorem 1.4 in this setting.

Things are much more complicated for a general family of bubbles $\{U[z_i, \lambda_i] : 1 \le i \le \nu\}$. We may have *bubbling towers* mixed with *bubbling clusters* (see the definition 3.1). This is one of the major difficulties we have to deal with. The proof of [37] only works for bubbling clusters. To our knowledge, we are the first ones to handle the mixed cases altogether. Also, we remark that there are many papers in the literature concerning the construction of the bubbling cluster or bubbling tower solutions. For bubbling towers, we refer to Del Pino et al. [16], Musso and Pistoia [30], Pistoia and Vétois [31], and the references therein. For bubbling clusters, we refer to Wei and Yan [36, 37], and the references therein. Our strategy is to design a "good" space for the interaction term $\sigma^p - \sum_i U_i^p$ so that (1.15) has a solution ρ_0 with the desired control. Choosing the right norm is a very delicate process. We start with just two bubbles and examine the magnitude of the interaction term

$(U_i+U_j)^p - U_i^p - U_j^p$ on different regions of $\mathbb{R}^n$. Fortunately, we obtain a uniform norm $\|\cdot\|_{**}$ (cf. eq (3.16)) to handle the bubbling tower and bubbling cluster at the same time, which reduces the amount of work significantly. Then $\|\sigma^p - \sum_i U_i^p\|_{**} \le C(n,\nu)$ follows from the estimates of all pairs by a simple inequality.

The existence of ρ_0 satisfying (1.15) is based on some a priori estimates (cf. Lemma 5.1). We use a contradiction argument to establish such estimates and divide $\mathbb{R}^n$ into three regions: core, neck, and exterior region (cf. Proposition 3.4). The core region of a particular bubble is where it dominates all the others. The exterior region is where far from the core of all bubbles. The neck regions are the rest. A standard blow-up argument handles the core regions. The exterior region is excluded by a rough point-wise estimate using Green's representation (cf. Proposition 4.3). The neck region is a new phenomenon we have to deal with. We leverage the fact that neck regions are narrow domains to modify the weight function $W(x)$ (cf. eq (5.14)) to be a super-solution. This is the most crucial and technical part of the proof. After establishing the a priori estimate, we get the existence of ρ_0 from the standard contraction mapping theorem (cf. Proposition 5.4). Consequently $\|\rho_0\|_* \le C(n,\nu)$.

We also construct an example that demonstrates the sharpness of the exponents in (1.10). Suppose $\sigma = U_1 + U_2$ where $U_1 := U[-Re_1, 1]$ and $U_2 = U[Re_1, 1]$. By Proposition 5.4, there exists ρ_0 satisfying (1.15) when $\nu = 2$. Then we let $u = U_1 + U_2 + \rho_0$ and $f = -\Delta u - u|u|^{p-1} = -\sum_{i,a} c_a^i U_i^{p-1} Z_i^a$. Using Green's representation and the point-wise estimates (1.16) and (1.17), we establish that $\|\nabla \rho_0\|_{L^2}$ is (up to some constant) no less than $\zeta_n(\|f\|_{H^{-1}})$. We prove that the $\text{dist}(u,\mathcal{T}) \gtrsim \|\nabla\rho_0\|_{L^2}$ and this finishes the construction.

The organization of the paper is as follows. In the section 2, we prove the main results Theorem 1.4 and Corollary 1.6 assuming several crucial estimates on ρ and $\nabla\rho$. In the section 3, we set up the norms and spaces for the error function. We start with just two bubbles and construct the weight functions V and W. Then we list several integral estimates involving V and W. In the section 4, we prove a rough C^0 bound by Green's representation and establish a bubble tree structure for a family of bubbles with vanishing interaction. Section 5 is the main part of this paper. We use the contradiction argument to prove a priori estimate for ρ_0. The crucial Proposition 5.4 is derived based on that estimate. Section 6 is devoted to proving the L^2 estimate of $\nabla\rho$. With section 5 and section 6, the main results Theorem 1.4 and Corollary 1.6 are justified. In section 7, we construct an example to verify Theorem 1.5. Appendix A mainly consists of various integral estimates between bubbles and their derivatives. Appendix B is devoted to computing the integral required in section 5.

2. Proof of the main theorem

In this section, we will prove Theorem 1.4 and Corollary 1.6 based on some crucial estimates, whose proofs are deferred to the section 6. We first give some basic properties of Aubin-Talenti bubble and its derivatives with respect to parameters z, λ.

For $i = 1, \cdots, \nu$, let us define

$$\begin{aligned} Z_i^a(x) &:= \frac{1}{\lambda_i} \left.\frac{\partial U[z, \lambda_i]}{\partial z^a}\right|_{z=z_i} = (2-n)U[z_i, \lambda_i](x)\frac{\lambda_i(x^a - z^a)}{1+\lambda_i^2|x-z_i|^2}, \\ Z_i^{n+1}(x) &:= \lambda_i \left.\frac{\partial U[z_i, \lambda]}{\partial \lambda}\right|_{\lambda=\lambda_i} = \frac{n-2}{2}U[z_i, \lambda_i](x)\frac{1-\lambda_i^2|x-z_i|^2}{1+\lambda_i^2|x-z_i|^2}, \end{aligned} \tag{2.1}$$

where z^a is the a-th component of z for $a = 1, \cdots, n$.

Since $U_i := U[z, \lambda]$ satisfies $\Delta U + U^p = 0$, by taking derivatives with respect to z^a and λ, we have Z_i^a satisfies

$$\Delta Z_i^a + pU_i^{p-1}Z_i^a = 0, \tag{2.2}$$

for any $i = 1, \cdots, \nu$ and $1 \leq a \leq n+1$. In fact, the kernel of $\Delta + pU_i^{p-1}$ in $\dot{H}^1(\mathbb{R}^n)$ is exactly spanned by $\{Z_i^a : a = 1, \cdots, n+1\}$ (see [6]). We call this property the *non-degeneracy* of Aubin-Talenti bubbles. Using the explicit form of $U[z, \lambda]$ and (2.1), it is easy to verify

$$|Z_i^a| \lesssim U_i, \quad \forall\, i = 1, \cdots, \nu, \forall\, a = 1, \cdots, n+1.$$

It is also well-known that $\|U[z,\lambda]\|_{\dot{H}^1}$, $\|U[z,\lambda]\|_{L^{2^*}}$ are all dimensional constants independent of z and λ. These facts will be utilized repeatedly without being explicitly stated.

Suppose $u = \sigma + \rho$ where $\sigma = \sum_{i=1}^{\nu} U_i$ is the best approximation (see (1.12)). Then

$$\Delta u + u|u|^{p-1} = \Delta\rho + p\sigma^{p-1}\rho + h + N_\sigma(\rho), \tag{2.3}$$

where

$$h = \sigma^p - \sum_{i=1}^{\nu} U_i^p, \quad N_\sigma(\rho) = (\sigma+\rho)|\sigma+\rho|^{p-1} - \sigma^p - p\sigma^{p-1}\rho.$$

Let $f = -\Delta u - u|u|^{p-1}$. Then (2.3) can be reorganized as

$$\Delta\rho + p\sigma^{p-1}\rho + h + N_\sigma(\rho) + f = 0. \tag{2.4}$$

If $n \geq 6$, then $p \in (1, 2]$. We have the following elementary inequality (for instance, see [14, Appendix D])

$$\left|(\sigma+\rho)|\sigma+\rho|^{p-1} - \sigma^p - p\sigma^{p-1}\rho\right| \lesssim |\rho|^p.$$

Thus

$$|N_\sigma(\rho)| \lesssim |\rho|^p.$$

Multiplying (2.4) by Z_k^{n+1} (we specify the choice of k in Lemma 2.3) and integrating over $\mathbb{R}^n$, by the orthogonal condition (1.14), we have

$$\left|\int hZ_k^{n+1}\right| \leq \left|\int fZ_k^{n+1}\right| + \left|\int p\sigma^{p-1}\rho Z_k^{n+1}\right| + C\int |\rho|^p|Z_k^{n+1}|.$$

It follows from (2.2) that $\|\nabla Z_k^{n+1}\|_{L^2}^2 \lesssim \|U_k\|_{L^{2^*}}^{2^*}$ which is a dimensional constant independent of z_k and λ_k. Thus $|\int f Z_k^{n+1}| \le C\|f\|_{H^{-1}}$. Hence

$$\left|\int h Z_k^{n+1}\right| \lesssim \|f\|_{H^{-1}} + \left|\int \sigma^{p-1}\rho Z_k^{n+1}\right| + \int |\rho|^p |Z_k^{n+1}|. \tag{2.5}$$

Lemma 2.1. *Suppose that u satisfies (1.9) with δ small enough. Then*

$$\int h Z_k^{n+1} = \int h\lambda_k \partial_{\lambda_k} U_k = \sum_{i=1, i\neq k}^{\nu} \int U_i^p \lambda_k \partial_{\lambda_k} U_k + o(Q), \tag{2.6}$$

where Q is defined at (1.6).

Proof. The first identity follows from the definition $Z_k^{n+1} = \lambda_k \partial_{\lambda_k} U_k$. Denote $\sigma = \sigma_k + U_k$ where $\sigma_k = \sum_{i=1, i\neq k}^{\nu} U_i$. We make the following decomposition

$$\int h\lambda_k \partial_{\lambda_k} U_k = \int (\sigma^p - \sum_{i=1}^{\nu} U_i^p)\lambda_k \partial_{\lambda_k} U_k = J_1 + J_2 + J_3 + J_4,$$

where

$$\begin{aligned} J_1 &= \int_{\{\nu U_k \ge \sigma_k\}} (pU_k^{p-1}\sigma_k - \sum_{i=1, i\neq k}^{\nu} U_i^p)\lambda_k \partial_{\lambda_k} U_k, \\ J_2 &= \int_{\{\nu U_k \ge \sigma_k\}} (\sigma^p - U_k^p - pU_k^{p-1}\sigma_k)\lambda_k \partial_{\lambda_k} U_k, \\ J_3 &= \int_{\{\sigma_k > \nu U_k\}} (p\sigma_k^{p-1} U_k + \sigma_k^p - \sum_{i=1}^{\nu} U_i^p)\lambda_k \partial_{\lambda_k} U_k, \\ J_4 &= \int_{\{\sigma_k > \nu U_k\}} (\sigma^p - \sigma_k^p - p\sigma_k^{p-1} U_k)\lambda_k \partial_{\lambda_k} U_k. \end{aligned}$$

Notice that $|\lambda_k \partial_{\lambda_k} U_k| \lesssim U_k$. Based on the inequality

$$|(a+b)^p - a^p - pa^{p-1}b| \lesssim a^{p-2}b^2 \quad \textit{if } a \ge b > 0,$$

we have

$$|J_2| \lesssim \int_{\{\nu U_k > \sigma_k\}} U_k^{p-1}\sigma_k^2 \lesssim \int U_k^{p-\epsilon}\sigma_k^{1+\epsilon} \approx Q^{1+\epsilon}. \tag{2.7}$$

Here $\epsilon > 0$ is very small, such that $1+\epsilon < p-\epsilon$, and in the last step we have used Lemma A.3. Similarly $|J_4| \lesssim Q^{1+\epsilon}$. For J_3,

$$\begin{aligned} |J_3| &= \left|\int_{\{\sigma_k \ge \nu U_k\}} (p\sigma_k^{p-1} U_k + \sigma_k^p - \sum_{i=1}^{\nu} U_i^p)\lambda_k \partial_{\lambda_k} U_k\right| \\ &\lesssim \int_{\{\sigma_k \ge \nu U_k\}} p\sigma_k^{p-\epsilon} U_k^{1+\epsilon} + \int \left|\sigma_k^p - \sum_{i=1, i\neq k}^{\nu} U_i^p\right| U_k + \int_{\{\sigma_k \ge \nu U_k\}} U_k^{p+1}. \end{aligned}$$

Using an elementary inequality

$$\left|\sigma_k^p - \sum_{i=1,i\neq k}^{\nu} U_i^p\right| \lesssim \sum_{\substack{1\leq i<j\leq\nu\\ i\neq k, j\neq k}} U_i^{p-1}U_j$$

and the triple integral estimate in Lemma A.4, we have

$$|J_3| \lesssim Q^{1+\epsilon} + \int_{\{\sigma_k\geq\nu U_k\}} U_k^{p+1}.$$

Consider the second term on the RHS. Lemma A.1 implies

$$\int_{\{U_i\geq U_k\}} U_k^{p+1} \leq \int U_k^{p-\epsilon}\inf(U_i^{1+\epsilon}, U_k^{1+\epsilon}) = O(q_{ik}^{\frac{n}{n-2}}|\log q_{ik}|) = o(Q),$$

therefore

$$\int_{\{\sigma_k\geq\nu U_k\}} U_k^{p+1} \leq \sum_{i=1,i\neq k}^{\nu}\int_{\{U_i>U_k\}} U_k^{p+1} = o(Q).$$

Therefore $|J_3| = o(Q)$. Now consider J_1:

$$\begin{aligned}
&J_1 - \sum_{i=1,i\neq k}^{\nu} p\int U_k^{p-1}U_i\lambda_k\partial_{\lambda_k}U_k = J_1 - \int pU_k^{p-1}\sigma_k\lambda_k\partial_{\lambda_k}U_k\\
&= -\int_{\{\nu U_k<\sigma_k\}} pU_k^{p-1}\sigma_k\lambda_k\partial_k U_k - \int_{\{\nu U_k>\sigma_k\}}\sum_{i=1,i\neq k}^{\nu} U_i^p\lambda_k\partial_{\lambda_k}U_k = o(Q).
\end{aligned}$$

We applied the same trick in (2.7) to obtain $o(Q)$. With the above estimates of J_i, $i = 1, 2, 3, 4$, we can get

$$\int h\lambda_k\partial_{\lambda_k}U_k = \sum_{i=1,i\neq k}^{\nu} p\int U_k^{p-1}U_i\lambda_k\partial_{\lambda_k}U_k + o(Q).$$

Simple integration by parts shows that

$$p\int U_k^{p-1}U_i\lambda_k\partial_{\lambda_k}U_k = \int U_i^p\lambda_k\partial_{\lambda_k}U_k.$$

Thus (2.6) holds. □

Now let us go back to (2.5). In Lemma 6.5, we will provide two important estimates

$$\left|\int\sigma^{p-1}\rho Z_k^{n+1}\right| = o(Q) + \|f\|_{H^{-1}}, \qquad \int|\rho|^p|Z_k^{n+1}| = o(Q) + \|f\|_{H^{-1}}. \tag{2.8}$$

Remark 2.2. These two terms have rough bounds easily by Hölder's inequality and Sobolev inequality. Indeed, for instance, when $n\geq 7$, as did in [22, (3.31)],

$$\left|\int\sigma^{p-1}\rho Z_k^{n+1}\right| \lesssim \|\nabla\rho\|_{L^2}Q^{p-1},$$

$$\int|\rho|^p|Z_k^{n+1}| \lesssim \|\nabla\rho\|_{L^2}^p.$$

By Lemma 2.1 and the above two estimates, one can achieve

$$Q \lesssim \|\nabla\rho\|_{L^2} Q^{p-1} + \|\nabla\rho\|_{L^2}^p + \|f\|_{H^{-1}}. \tag{2.9}$$

Multiplying (2.4) by ρ, the approach in [22] would induce $\|\nabla\rho\|_{L^2} \lesssim Q^{p-1} + \|f\|_{H^{-1}}$. Plugging in this fact to (2.9), we obtain $Q \lesssim Q^{2(p-1)} + \|f\|_{H^{-1}}$. One fails to conclude anything when $2(p-1) \leq 1$ (equivalent to $n \geq 10$). This obstacle motivates us to have better control of ρ instead of simply using Hölder's inequality and Sobolev inequality.

Using (2.8), the interaction between the bubbles can be estimated as follows.

Lemma 2.3. *Suppose that u satisfies (1.9) with δ small enough. Then we have*

$$Q \lesssim \|f\|_{H^{-1}}.$$

Proof. We shall prove $q_{ij} \lesssim \|f\|_{H^{-1}}$ for each pair $i < j$ by an iteration argument. Without loss of generality (WLOG), we may assume that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\nu$. Applying Lemma A.2, one has

$$\left| \int U_i^p \lambda_k \partial_{\lambda_k} U_k \right| \lesssim q_{ik}, \quad \forall\, i \neq k, \tag{2.10}$$

$$\int U_i^p \lambda_k \partial_{\lambda_k} U_k \approx -q_{ik}, \quad \forall\, i < k. \tag{2.11}$$

For each $2 \leq l \leq \nu$, let us introduce the following induction hypothesis (P_l).

$$(P_l): \quad \sum_{j=l}^{\nu} \sum_{i=1}^{j-1} q_{ij} \lesssim \|f\|_{H^{-1}} + o(Q).$$

First, we take $k = \nu$ in (2.5) and (2.6). From (2.8) and (2.11) we have

$$\sum_{i=1}^{\nu-1} q_{i\nu} \approx -\sum_{i=1}^{\nu-1} \int U_i^p \lambda_\nu \partial_{\lambda_\nu} U_\nu \lesssim \|f\|_{H^{-1}} + o(Q).$$

It implies that (P_ν) is true. Second, suppose (P_{l+1}) is true. Now we take $k = l$ in (2.5) and (2.6). By (2.8), (2.10), (2.11) and the assumption (P_{l+1}), we get

$$\begin{aligned} \sum_{i=1}^{l-1} q_{il} \approx -\sum_{i=1}^{l-1} \int U_i^p \lambda_l \partial_{\lambda_l} U_l &\lesssim \sum_{i=l+1}^{\nu} \left| \int U_i^p \lambda_l \partial_{\lambda_l} U_l \right| + \|f\|_{H^{-1}} + o(Q) \\ &\lesssim \sum_{i=l+1}^{\nu} q_{il} + \|f\|_{H^{-1}} + o(Q) \lesssim \|f\|_{H^{-1}} + o(Q). \end{aligned}$$

Then (P_l) holds. Inductively, we obtain that (P_2) holds. That is $Q \approx \sum_{i<j} q_{ij} \lesssim \|f\|_{H^{-1}} + o(Q)$. Then $Q \lesssim \|f\|_{H^{-1}}$.

□

Now we can prove the main result Theorem 1.4.

Proof of Theorem 1.4. Write $\rho = \rho_0 + \rho_1$ where ρ_0 solves (1.15). By Proposition 6.1, we have

$$\|\nabla \rho_0\|_{L^2} \lesssim \zeta_n(Q).$$

By Proposition 6.4, we have

$$\|\nabla \rho_1\|_{L^2} \lesssim Q^2 + \|f\|_{H^{-1}}.$$

Since we have shown $Q \lesssim \|f\|_{H^{-1}}$ in the previous Lemma 2.3, then

$$\|\nabla \rho\|_{L^2} \leq \|\nabla \rho_0\|_{L^2} + \|\nabla \rho_1\|_{L^2} \lesssim \zeta_n(\|f\|_{H^{-1}}).$$

Here we have used the fact that $\zeta_n(x)$ is increasing near 0. Therefore (1.10) holds. Finally, (1.11) follows from the fact $\int U_i^p U_j \approx q_{ij} \leq Q$ and Lemma 2.3. $\square$

Proof of Corollary 1.6. The proof is identical to that of Corollary 3.4 in [22]. $\square$

It remains to establish Lemma 6.5, Proposition 6.1 and Proposition 6.4. These depend crucially on a point-wise estimate of ρ_0, which will be done in section 5.

3. Setting up spaces and norms

In this section, we shall introduce two weight functions V and W, which measure the behavior of the interaction between bubbles and the ρ functions defined in (2.4). These are fundamental for obtaining point-wise estimates of ρ.

Let us begin with a rough analysis. Consider the equation (2.4) of ρ. The linearized operator is $\Delta + p\sigma^{p-1}$, $h + N_\sigma(\rho) + f$ is the non-homogeneous term and ρ is the solution. h is the main data that encodes the interaction of bubbles. $N_\sigma(\rho)$ is a higher-order term in ρ and negligible. Therefore, an approximation of ρ can be obtained from studying the linear equation $\Delta\rho_0 + p\sigma^{p-1}\rho_0 = h$. As the interaction of bubbles becomes smaller and smaller, $(\Delta + p\sigma^{p-1})Z_i^a$ will converge to 0. This indicates the linearized operator has a non-trivial approximate kernel when the interaction is small. According to Fredholm's alternative, to solve $\Delta\rho_0 + p\sigma^{p-1}\rho_0 = h$ in a nice space, one needs h to be orthogonal to the approximate kernel. Equivalently, we can project h to the orthogonal space of the approximate kernel. This amounts to solving the equation up to some Lagrange multiplier.

It leads us to consider the following linear equation

$$\begin{cases} \Delta\phi + p\sigma^{p-1}\phi = h + \sum\limits_{i=1}^{\nu}\sum\limits_{a=1}^{n+1} c_a^i U_i^{p-1} Z_i^a, \\ \int U_i^{p-1}\phi Z_i^a = 0, \quad i = 1, \cdots, \nu;\ a = 1, \cdots, n+1, \end{cases} \tag{3.1}$$

where $\sigma = \sum_{i=1}^{\nu} U[z_i, \lambda_i]$ is the sum of a family of δ-interacting bubbles. We always assume δ is very small. We shall use finite-dimensional reduction to prove the solvability of ϕ given a reasonable h in Proposition 5.3. To that end, we need to set up the norms and spaces.

Let $I = \{1, 2, \cdots, \nu\}$. Throughout this paper, we denote $y_i = \lambda_i(x - z_i)$ and

$$R_{ij} := \max_{i \neq j \in I} \left\{ \sqrt{\lambda_i/\lambda_j}, \sqrt{\lambda_j/\lambda_i}, \sqrt{\lambda_i\lambda_j}|z_i - z_j| \right\}. \tag{3.2}$$

Definition 3.1. For any two bubbles U_i, U_j, if $R_{ij} = \sqrt{\lambda_i\lambda_j}|z_i - z_j|$, then we call them a *bubbling cluster*, otherwise call them a *bubbling tower*.

We always denote

$$R := \frac{1}{2}\min_{i\neq j\in I} R_{ij}. \tag{3.3}$$

It holds that $R^{2-n} \approx Q$ (see (1.6)).

Let us define various weight functions designed to measure the behavior of h and ρ. When the dimension $n \geq 7$, for each $i \in I$ and any $t > 1$, the inner (outer) v-weight and w-weight of bubble U_i with radius t are defined by

$$v_i^{\text{in}}(x,t) = \frac{\lambda_i^{\frac{n+2}{2}} t^{2-n}}{\langle y_i\rangle^4}\chi_{\{|y_i|\leq t\}}, \quad v_i^{\text{out}}(x,t) = \frac{\lambda_i^{\frac{n+2}{2}} t^{-4}}{\langle y_i\rangle^{n-2}}\chi_{\{|y_i|>t\}},$$
$$w_i^{\text{in}}(x,t) = \frac{\lambda_i^{\frac{n-2}{2}} t^{2-n}}{\langle y_i\rangle^2}\chi_{\{|y_i|\leq t\}}, \quad w_i^{\text{out}}(x,t) = \frac{\lambda_i^{\frac{n-2}{2}} t^{-4}}{\langle y_i\rangle^{n-4}}\chi_{\{|y_i|>t\}}.$$

When $n = 6$, we also define the following $\hat{v}$-weight and $\hat{w}$-weight:

$$\hat{v}_i^{\text{in}}(x,t) = \frac{\lambda_i^4 t^{-2}}{\langle y_i\rangle^4}\chi_{\{|y_i|\leq t\}}, \quad \hat{v}_i^{\text{out}}(x,t) = \frac{\lambda_i^4 t^{-1}}{\langle y_i\rangle^5}\chi_{\{|y_i|>t\}},$$
$$\hat{w}_i^{\text{in}}(x,t) = \frac{\lambda_i^2 t^{-2}}{\langle y_i\rangle^2}\chi_{\{|y_i|\leq t\}}, \quad \hat{w}_i^{\text{out}}(x,t) = \frac{\lambda_i^2 t^{-1}}{\langle y_i\rangle^3}\chi_{\{|y_i|>t\}}.$$

Now we can define $\|\cdot\|_{**}$ and $\|\cdot\|_*$ norms as

$$\|h\|_{**} = \sup_{x\in\mathbb{R}^n}|h(x)|V^{-1}(x), \quad \|\phi\|_* = \sup_{x\in\mathbb{R}^n}|\phi(x)|W^{-1}(x) \tag{3.4}$$

with the total weights

$$V(x) = \begin{cases}\sum_{i=1}^{\nu}\left(v_i^{\text{in}}(x,R) + v_i^{\text{out}}(x,R)\right), & n\geq 7,\\ \sum_{i=1}^{\nu}\left(\hat{v}_i^{\text{in}}(x,R^2) + \hat{v}_i^{\text{out}}(x,R^2)\right), & n = 6,\end{cases} \tag{3.5}$$

$$W(x) = \begin{cases}\sum_{i=1}^{\nu}\left(w_i^{\text{in}}(x,R) + w_i^{\text{out}}(x,R)\right), & n\geq 7,\\ \sum_{i=1}^{\nu}\left(\hat{w}_i^{\text{in}}(x,R^2) + \hat{w}_i^{\text{out}}(x,R^2)\right), & n = 6.\end{cases} \tag{3.6}$$

For simplicity, we denote $v_i^{\text{in(out)}}(x) = v_i^{\text{in(out)}}(x,R)$ and $w_i^{\text{in(out)}}(x) = w_i^{\text{in(out)}}(x,R)$, while $\hat{v}_i^{\text{in(out)}}(x) = \hat{v}_i^{\text{in(out)}}(x,R^2)$ and $\hat{w}_i^{\text{in(out)}}(x) = w_i^{\text{in(out)}}(x,R^2)$.

Remark 3.2. (1) The ad hoc weight V captures the interaction behavior between bubbles $h = \sigma^p - \sum_{i=1}^{\nu} U_i^p$. See Proposition 3.4. The weight W is designed to solve $\Delta_x W \approx V$, see Lemma 3.6. We have to define a separate norm on $n = 6$, as explained in Remark 3.5. The definitions of V and W depend on $U[z_i,\lambda_i]$, $i = 1, \cdots, \nu$. This is implicitly understood throughout this paper.

(2) For each $i \in I$, let $f_i(x,t) = v_i^{\text{in}}(x,t) + v_i^{\text{out}}(x,t)$. By the explicit form of $v_i^{\text{in(out)}}$, we have

$$f_i(x,t) = (1+o(t^{-1}))\lambda_i^{(n+2)/2} \min\{t^{2-n}\langle y_i\rangle^{-4}, t^{-4}\langle y_i\rangle^{2-n}\}.$$

One can verify that $f_i(x,t)$ is an *approximately non-increasing* function of t. That is, for $1 < t_1 \le t_2$, we have

$$f_i(x,t_2) \lesssim f_i(x,t_1). \tag{3.7}$$

The monotonicity also holds for $w_i^{\text{in}}(x,t)+w_i^{\text{out}}(x,t)$, $\hat{v}_i^{\text{in}}(x,t)+\hat{v}_i^{\text{out}}(x,t)$, and $\hat{w}_i^{\text{in}}(x,t)+\hat{w}_i^{\text{out}}(x,t)$ in dimension 6. This monotonicity of total weight functions will help us to obtain the convenient form of weight functions for any finite number of bubbles in Proposition 3.4.

In this paper, we often need to deal with terms like $U_i^{\gamma_i}U_j^{\gamma_j}$ where $\gamma_i, \gamma_j \ge 0$. Note that $U_i = (n(n-2))^{(n-2)/4}\lambda_i^{(n-2)/2}\langle y_i\rangle^{2-n}$ where $y_i = \lambda_i(x-z_i)$ and $\langle y\rangle = \sqrt{1+|y|^2} \approx 1+|y|$. It is natural to define the following cross-term

$$g_{ij}(x) = \langle y_i\rangle^{-\gamma_i}\langle y_j\rangle^{-\gamma_j}. \tag{3.8}$$

The following lemma gives us an applicable estimate of the cross-term and will be heavily used throughout the rest of this paper. It has the advantage of not having to distinguish between bubbling clusters and towers. To make the estimates more flexible to different scenarios, we introduce parameter τ in the second line of inequality (3.9). In some regions, we set $\tau = 0$ to obtain the fastest decay of $\langle y_1\rangle$, while in other regions, we set $\tau = \gamma_2$ to achieve a more negative power of R_{12}.

Lemma 3.3. *Suppose $\lambda_1 \le \lambda_2$ and $1 \ll R_{12}$ where R_{12} is defined at (3.2). For $\gamma_1 \ge 0, \gamma_2 \ge \tau \ge 0$, it holds that*

$$g_{12}(x) \lesssim \begin{cases} R_{12}^{-\gamma_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{\gamma_1/2}\langle y_2\rangle^{-\gamma_2}, & |y_2| \le \sqrt{\frac{\lambda_2}{\lambda_1}}\frac{R_{12}}{2}, \\ R_{12}^{-\tau}\left(\frac{\lambda_1}{\lambda_2}\right)^{\gamma_2-\tau/2}\langle y_1\rangle^{-\gamma_1-\gamma_2+\tau}, & |y_2| \ge \sqrt{\frac{\lambda_2}{\lambda_1}}\frac{R_{12}}{2}. \end{cases} \tag{3.9}$$

Proof. First, suppose that U_1 and U_2 form a bubbling cluster. That is, $R_{12} = \sqrt{\lambda_1\lambda_2}|z_1-z_2| \ge \sqrt{\lambda_2/\lambda_1}$, and so that $R_{12}\sqrt{\lambda_1/\lambda_2} = \lambda_1|z_1-z_2| \ge 1$.

For $|y_2| \le \sqrt{\lambda_2/\lambda_1}R_{12}/2$, we have

$$|y_1| = \frac{\lambda_1}{\lambda_2}|y_2 - \lambda_2(z_1-z_2)| \ge \frac{\lambda_1}{\lambda_2}\left(\sqrt{\frac{\lambda_2}{\lambda_1}}R_{12} - |y_2|\right) \ge \sqrt{\frac{\lambda_1}{\lambda_2}}\frac{R_{12}}{2}. \tag{3.10}$$

Then

$$g_{12}(x) \lesssim R_{12}^{-\gamma_1}(\lambda_2/\lambda_1)^{\gamma_1/2}\langle y_2\rangle^{-\gamma_2}.$$

For $|y_2| \ge \sqrt{\lambda_2/\lambda_1}R_{12}/2$, equivalently $(\lambda_1/\lambda_2)|y_2| \ge \lambda_1|z_1-z_2|/2 \ge 1/2$, then

$$1+|y_1| \le (\lambda_1/\lambda_2)|y_2| + \lambda_1|z_1-z_2| + 1 \le 5(\lambda_1/\lambda_2)|y_2|.$$

It means that both of $\langle y_2\rangle \gtrsim \sqrt{\lambda_2/\lambda_1}R_{12}$ and $\langle y_2\rangle \gtrsim (\lambda_2/\lambda_1)\langle y_1\rangle$ hold. Then

$$g_{12}(x) \lesssim R_{12}^{-\tau}\left(\lambda_1/\lambda_2\right)^{\gamma_2-\tau/2}\langle y_1\rangle^{-\gamma_1-\gamma_2+\tau}.$$

Second, suppose that U_1 and U_2 form a bubbling tower. That is, $R_{12} = \sqrt{\lambda_2/\lambda_1} > \sqrt{\lambda_1\lambda_2}|z_1 - z_2|$, and so that $\lambda_1|z_1 - z_2| < 1$.

For $|y_2| \le \sqrt{\lambda_2/\lambda_1}R_{12}/2$, we have

$$1 + |y_1| \ge 1 + \lambda_1|z_1 - z_2| - (\lambda_1/\lambda_2)|y_2| \ge 1/2 = \sqrt{\lambda_1/\lambda_2}R_{12}/2. \tag{3.11}$$

Then

$$g_{12}(x) \lesssim \langle y_2\rangle^{-\gamma_2} = R_{12}^{-\gamma_1}\left(\lambda_2/\lambda_1\right)^{\gamma_1/2}\langle y_2\rangle^{-\gamma_2}.$$

For $\sqrt{\lambda_2/\lambda_1}R_{12}/2 \le |y_2|$, equivalently $1/2 \le (\lambda_1/\lambda_2)|y_1|$, it holds that

$$1 + |y_1| \le (\lambda_1/\lambda_2)|y_2| + \lambda_1|z_1 - z_2| + 1 \le 5(\lambda_1/\lambda_2)|y_2|.$$

Consequently,

$$g_{12}(x) \lesssim R_{12}^{-\tau}\left(\lambda_1/\lambda_2\right)^{\gamma_2-\tau/2}\langle y_1\rangle^{-\gamma_1-\gamma_2+\tau}.$$

□

Proposition 3.4. *There exist a small constant $\delta_0 = \delta_0(n,\nu)$ and a constant $C(n,\nu)$ such that if $\delta < \delta_0$, then*

$$\|\sigma^p - \sum_{i=1}^{\nu} U_i^p\|_{**} \le C(n,\nu).$$

Proof. We shall start with the case of just two bubbles and then generalize it to finitely many ones.

• Let $U_1 = U[z_1,\lambda_1]$ and $U_2 = U[z_2,\lambda_2]$. WLOG, we can assume that $\lambda_1 \le \lambda_2$. Since $p \in (1,2]$, we have $h = (U_1 + U_2)^p - U_1^p - U_2^p > 0$.

On the other hand, if $0 < a < b$, then

$$(a+b)^p - a^p - b^p \lesssim b^{p-1}a = a^{p-1}b(a/b)^{2-p} \le a^{p-1}b.$$

If $0 < b \le a$, then

$$(a+b)^p - a^p - b^p \lesssim a^{p-1}b = b^{p-1}a(b/a)^{2-p} \le b^{p-1}a.$$

In conclusion, we have

$$h = (U_1 + U_2)^p - U_1^p - U_2^p \lesssim \min\{U_1^{p-1}U_2, U_2^{p-1}U_1\}.$$

On the set $\{|y_2| \le R_{12}/2\}$, thanks to (3.10) and (3.11), we always have $\langle y_1\rangle \ge \sqrt{\lambda_1/\lambda_2}R_{12}/2$. Hence, it holds that $U_1/U_2 = (\lambda_1/\lambda_2)^{(n-2)/2}\langle y_2\rangle^{n-2}\langle y_1\rangle^{2-n} \le 2^{n-2}$. Thus, applying Lemma 3.3 we get

$$h \lesssim U_2^{p-1}U_1\chi_{\{|y_2|\le R_{12}/2\}} \lesssim \lambda_2^{\frac{n+2}{2}}R_{12}^{2-n}\langle y_2\rangle^{-4}\chi_{\{|y_2|\le R_{12}/2\}}. \tag{3.12}$$

We shall call the region $\{|y_2| \lesssim R_{12}\}$ the core of U_2 (concerning U_1). One can see that h has slow decay in this region. See the illustration of the bubbling tower and cluster in Figure 1 and 2 respectively.

On the set $\{R_{12}/2 \le |y_2| \le \sqrt{\lambda_2/\lambda_1}R_{12}/2\}$, we have $|y_1| \le 2\sqrt{\lambda_1/\lambda_2}R_{12}$. Hence, it holds that $U_2/U_1 \le (\lambda_2/\lambda_1)^{(n-2)/2}\langle y_1\rangle^{n-2}\langle y_2\rangle^{2-n} \le 6^{n-2}$. Thus, applying Lemma 3.3 we get

$$h \lesssim U_1^{p-1}U_2\chi_{\{|y_2|\ge R_{12}/2\}} \lesssim \lambda_2^{\frac{n+2}{2}}R_{12}^{-4}\langle y_2\rangle^{2-n}\chi_{\{|y_2|\ge R_{12}/2\}}. \tag{3.13}$$

We shall call the region $\{R_{12} \lesssim |y_2| \lesssim \sqrt{\lambda_2/\lambda_1}R_{12}\}$ the neck of U_2 (concerning U_1). Within it, h has the same decay as Green's function.

On the set $\{|y_1| \le R_{12}/2\} \cap \{|y_2| \ge \sqrt{\lambda_2/\lambda_1}R_{12}/2\}$, we have $U_2/U_1 = (\lambda_2/\lambda_1)^{(n-2)/2}\langle y_1\rangle^{n-2}\langle y_2\rangle^{2-n} \le 2^{n-2}$. By Lemma 3.3 when $\tau = \gamma_2 = n-2$, we get

$$h \lesssim U_1^{p-1}U_2\chi_{\{|y_1|\le R_{12}/2\}} \lesssim \lambda_1^{\frac{n+2}{2}}R_{12}^{2-n}\langle y_1\rangle^{-4}\chi_{\{|y_1|\le R_{12}/2\}}. \tag{3.14}$$

This region is the core of U_1 (concerning U_2), removing the core and neck of U_2.

In the outer region $\{|y_1| > R_{12}/2\} \cap \{|y_2| > \sqrt{\lambda_2/\lambda_1}R_{12}/2\}$, we have

$$h \lesssim U_1^p + U_2^p \lesssim \frac{\lambda_1^{\frac{n+2}{2}}R_{12}^{-4}}{\langle y_1\rangle^{n-2}}\chi_{\{|y_1|>R_{12}/2\}} + \frac{\lambda_2^{\frac{n+2}{2}}R_{12}^{-4}}{\langle y_2\rangle^{n-2}}\chi_{\{|y_2|>R_{12}/2\}}. \tag{3.15}$$

From (3.12)-(3.15), we conclude

$$h \lesssim \sum_{i=1}^{2}\left[v_i^{\text{in}} + v_i^{\text{out}}\right](x, \frac{R_{12}}{2}) \lesssim \sum_{i=1}^{2}\left[v_i^{\text{in}} + v_i^{\text{out}}\right](x, R). \tag{3.16}$$

Note that, thanks to the monotonicity (3.7), the $\frac{1}{2}R_{12}$ can be replaced by a fixed R in the last inequality in (3.16). It is the key to define the weight V for any finite number of bubbles where the *common* $R = \frac{1}{2}\min_{i\neq j}\{R_{ij}\}$ will be used.

When $n = 6$, i.e., $p = 2$ and $2 - n = -4$, the core, neck, and outer region of U_2 have the same decay. We strengthen (3.15) to have a faster decay in the outer region. That is,

$$h \lesssim \sum_{i=1}^{2}\frac{\lambda_i^4}{\langle y_i\rangle^8} \lesssim \sum_{i=1}^{2}\left(\frac{\lambda_i^4 R_{12}^{-4}}{\langle y_i\rangle^4}\chi_{\{|y_i|>\frac{R_{12}^2}{2}\}} + \frac{\lambda_i^4 R_{12}^{-2}}{\langle y_i\rangle^5}\chi_{\{|y_i|>\frac{R_{12}^2}{2}\}}\right). \tag{3.17}$$

From (3.12)-(3.14) and (3.17), we conclude

$$h \lesssim \sum_{i=1}^{2}\left[\hat{v}_i^{\text{in}} + \hat{v}_i^{\text{out}}\right](x, \frac{R_{12}^2}{2}) \lesssim \sum_{i=1}^{2}\left[\hat{v}_i^{\text{in}} + \hat{v}_i^{\text{out}}\right](x, R^2). \tag{3.18}$$

- For any finite number of bubbles, we shall use the following inequality in Lemma A.6.

$$h = \sigma^p - \sum_{i=1}^{\nu}U_i^p \le \sum_{i<j}|(U_i + U_j)^p - U_i^p - U_j^p|.$$

Each term in the summation on the RHS can be bounded above by the previous estimates of two bubbles in (3.16) and (3.18). Summing them up, one can obtain $h \le C(n,\nu)V(x)$.

□

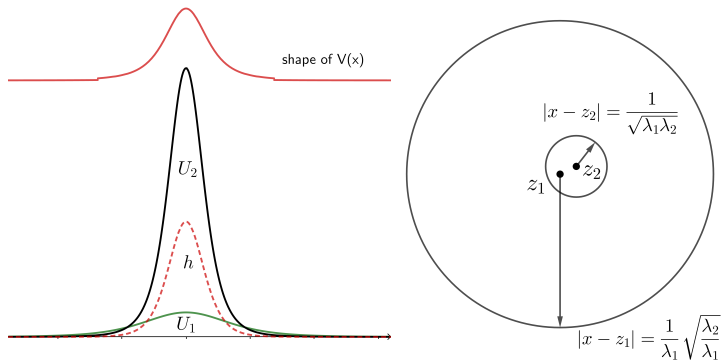

FIGURE 1. U_1 and U_2 form a bubbling tower with $\lambda_1 \ll \lambda_2$. The dotted line denotes the h. The right picture shows that the core region of U_1 (i.e. $\{|y_1| \leq R_{12}\} = \{x : |x - z_1| \leq \sqrt{\lambda_2/\lambda_1}/\lambda_1\}$) contains that of U_2 (i.e. $\{|y_2| \leq R_{12}\} = \{x : |x - z_2| \leq 1/\sqrt{\lambda_1 \lambda_2}\}$).

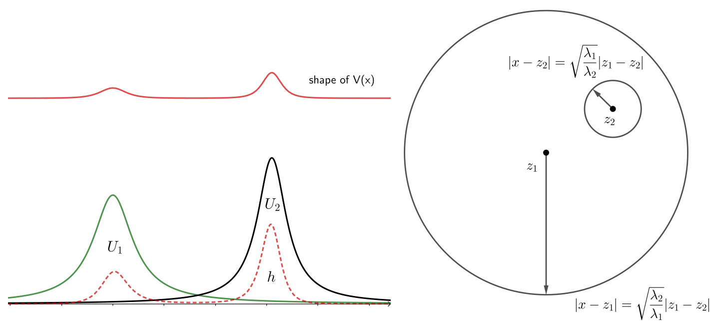

FIGURE 2. U_1 and U_2 form a bubbling cluster. The right picture shows that the core region of U_1 contains that of U_2 like a bubbling tower when $\lambda_1 \ll \lambda_2$. However, if $\lambda_1 \approx \lambda_2$, the core region of them shall be disjoint.

Remark 3.5. To have a simple form of V, we bound h just by $\langle y_i \rangle^{2-n}$ in (3.15). In fact, h decays faster than V at infinity. Such relaxation causes a problem for $n = 6$ when estimating $\int (w_i^{\text{out}})^{p+1}$, because w_i^{out} has the critical decay (in the sense of $\int (w_i^{\text{out}})^{p+1} = \infty$) in the outer region of bubbles. Thus we have to define a separate norm in dimension $n = 6$. Any weight in the outer region that decays faster than w_i^{out} works in dimension $n = 6$. For simplicity, we just choose $\hat{w}_i^{\text{out}}$.

The previous proposition justifies the choice of V. The weight function $W(x)$ is designed to satisfy $\Delta_x W \approx V$. We verify this through Lemma 3.6.

Lemma 3.6. *Suppose that $R \gg 1$. For V and W defined in (3.5) and (3.6) respectively, we have*

$$\int_{\mathbb{R}^n} |\tilde{x} - x|^{2-n} V(x) dx \approx W(\tilde{x}). \tag{3.19}$$

Proof. Let us first consider $n \geq 7$. Denote $y_i = \lambda_i(x - z_i)$ and $\tilde{y}_i = \lambda_i(\tilde{x} - z_i)$. Making a change of variable we get

$$\begin{aligned}\int_{\mathbb{R}^n} |\tilde{x} - x|^{2-n} v_i^{\text{in}}(x) dx &= \int_{\mathbb{R}^n} |\tilde{x} - x|^{2-n} \frac{\lambda_i^{\frac{n+2}{2}} R^{2-n}}{\langle y_i \rangle^4} \chi_{\{|y_i| \leq R\}} dx \\ &= \lambda_i^{\frac{n-2}{2}} R^{2-n} \int_{|y_i| \leq R} |\tilde{y}_i - y_i|^{2-n} \langle y_i \rangle^{-4} dy_i.\end{aligned} \tag{3.20}$$

If $|\tilde{y}_i| \leq \frac{3}{2} R$, on the one hand, we use (A.3) in Lemma A.7 to obtain

$$\int_{|y_i| \leq R} |\tilde{y}_i - y_i|^{2-n} \langle y_i \rangle^{-4} dy_i \leq \int_{\mathbb{R}^n} |\tilde{y}_i - y_i|^{2-n} \langle y_i \rangle^{-4} dy_i \lesssim \langle \tilde{y}_i \rangle^{-2}.$$

On the other hand, on the set $\Omega = \{y_i : |y_i| \leq \langle \tilde{y}_i \rangle / 2\} \subset \{|y_i| \leq R\}$, we have $|\tilde{y}_i - y_i| \leq 3 \langle \tilde{y}_i \rangle / 2$ and

$$\begin{aligned}\int_{|y_i| \leq R} |\tilde{y}_i - y_i|^{2-n} \langle y_i \rangle^{-4} dy_i &\geq \int_{\Omega} |\tilde{y}_i - y_i|^{2-n} \langle y_i \rangle^{-4} dy_i \\ &\gtrsim \langle \tilde{y}_i \rangle^{2-n} \int_{\Omega} \langle y_i \rangle^{-4} dy_i \gtrsim \langle \tilde{y}_i \rangle^{-2}.\end{aligned} \tag{3.21}$$

If $|\tilde{y}_i| \geq \frac{3}{2} R$, then $|\tilde{y}_i - y_i| \approx |\tilde{y}_i|$ on $\{|y_i| \leq R\}$. Consequently,

$$\int_{|y_i| \leq R} |\tilde{y}_i - y_i|^{2-n} \langle y_i \rangle^{-4} dy_i \approx \langle \tilde{y}_i \rangle^{2-n} \int_{|y_i| \leq R} \langle y_i \rangle^{-4} dy_i \approx \langle \tilde{y}_i \rangle^{2-n} R^{n-4}.$$

Inserting the above two inequalities to (3.20), we have

$$\int_{\mathbb{R}^n} |\tilde{x} - x|^{2-n} v_i^{\text{in}}(x) dx \approx \lambda_i^{\frac{n-2}{2}} [R^{2-n} \langle \tilde{y}_i \rangle^{-2} \chi_{\{|\tilde{y}_i| \leq \frac{3}{2} R\}} + R^{-2} \langle \tilde{y}_i \rangle^{2-n} \chi_{\{|\tilde{y}_i| \geq \frac{3}{2} R\}}].$$

Consequently,

$$w_i^{\text{in}}(\tilde{x}) \lesssim \int_{\mathbb{R}^n} |\tilde{x} - x|^{2-n} v_i^{\text{in}}(x) dx \lesssim w_i^{\text{in}}(\tilde{x}) + w_i^{\text{out}}(\tilde{x}).$$

Similarly, we have the following estimate for the integral of v_i^{out},

$$\begin{aligned}&\int_{\mathbb{R}^n} |\tilde{x} - x|^{2-n} v_i^{\text{out}}(x) dx = \lambda_i^{\frac{n-2}{2}} R^{-4} \int_{|y_i| > R} |\tilde{y}_i - y_i|^{2-n} \langle y_i \rangle^{2-n} dy_i \\ &\approx \lambda_i^{\frac{n-2}{2}} R^{-n} \chi_{\{|\tilde{y}_i| \leq \frac{2}{3} R\}} + \lambda_i^{\frac{n-2}{2}} R^{-4} \langle \tilde{y}_i \rangle^{4-n} \chi_{\{|\tilde{y}_i| \geq \frac{2}{3} R\}}.\end{aligned}$$

Consequently,

$$w_i^{\text{out}}(\tilde{x}) \lesssim \int_{\mathbb{R}^n} |\tilde{x} - x|^{2-n} v_i^{\text{out}}(x) dx \lesssim w_i^{\text{in}}(\tilde{x}) + w_i^{\text{out}}(\tilde{x}).$$

The proof of $n \geq 7$ is completed by combining the above two estimates. When $n = 6$, one needs to divide the $\tilde{y}_i$ near R^2 and repeat the above proof. We omit the details. □

We have the following integral estimates for V and W. They are needed in Lemma 5.2 and Proposition 6.1.

Lemma 3.7. *Suppose that $n \geq 6$ and $R \gg 1$. It holds that*

$$\|W\|_{L^{2^*}} \lesssim \begin{cases} R^{-\frac{n+2}{2}}, & n \geq 7, \\ R^{-4}(\log R)^{\frac{1}{3}}, & n = 6, \end{cases} \lesssim \zeta_n(Q), \tag{3.22}$$

$$\|V\|_{L^{(2^*)'}} \lesssim \begin{cases} R^{-\frac{n+2}{2}}, & n \geq 7, \\ R^{-4}(\log R)^{\frac{2}{3}}, & n = 6, \end{cases} \tag{3.23}$$

where $(2^*)' = \frac{2n}{n+2}$ is the Hölder conjugate of 2^*.

Proof. For $n \geq 7$, we have

$$\int (w_i^{\text{in}})^{p+1} = \int_{|y_i| \leq R} \lambda_i^{-n} R^{-2n} \langle y_i \rangle^{-\frac{4n}{n-2}} dx \lesssim R^{-np},$$

$$\int (w_i^{\text{out}})^{p+1} = \int_{|y_i| \geq R} \lambda_i^{-n} R^{-4(p+1)} \langle y_i \rangle^{(4-n)(p+1)} dx \lesssim R^{-np}.$$

Similarly, direct computation yields that, for $n = 6$,

$$\int (\hat{w}_i^{\text{in}})^3 + (\hat{w}_i^{\text{out}})^3 \lesssim R^{-12} \log R.$$

Summing over i, we get

$$\int W^{p+1} \lesssim \begin{cases} R^{-np}, & n \geq 7, \\ R^{-12} \log R, & n = 6. \end{cases}$$

Since $p = (n+2)/(n-2)$ and $R^{2-n} \approx Q$, this implies (3.22).

For $n \geq 7$, we have

$$\int (v_i^{\text{in}})^{\frac{2n}{n+2}} = \int_{|y_i| \leq R} \lambda_i^{-n} R^{-\frac{2n(n-2)}{n+2}} \langle y_i \rangle^{-\frac{8n}{n+2}} dx \lesssim R^{-n},$$

$$\int (v_i^{\text{out}})^{\frac{2n}{n+2}} = \int_{|y_i| \geq R} \lambda_i^{-n} R^{-\frac{8n}{n+2}} \langle y_i \rangle^{-\frac{2n(n-2)}{n+2}} dx \lesssim R^{-n}.$$

Similarly, direct computation yields that, for $n = 6$,

$$\int (\hat{v}_i^{\text{in}})^{\frac{3}{2}} + (\hat{v}_i^{\text{out}})^{\frac{3}{2}} \lesssim R^{-6} \log R.$$

Summing over i, we get

$$\int V^{\frac{2n}{n+2}} \lesssim \begin{cases} R^{-n}, & n \geq 7, \\ R^{-6} \log R, & n = 6. \end{cases}$$

Since $(2^*)' = 2n/(n+2)$, this yields (3.23).

□

4. Analysis of bubbles with weak interaction

This section presents some preliminary analysis leading to the proof of the main point-wise estimate in section 5. In the first subsection, we derive several technical lemmas. Then we prove a rough upper bound of solutions to (3.1) using Green's representation. In the second subsection, we deal with a sequence of ν bubbles with vanishing interaction. We show that there is a bubble tree structure of them. This tree structure facilitates various estimates of bubbles. We expect that it can be used in some other problems.

4.1. **Rough upper bound.** Denote $I = \{1, \cdots, \nu\}$. Suppose $\{U_i = U[z_i, \lambda_i] : i \in I\}$ is a set of ν bubbles such that $Q = \max\{q_{ij} : 1 \le i \ne j \le \nu\} < \delta$. WLOG, assume that they are ordered as

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_\nu.$$

Define $z_{ij} = \lambda_i(z_j - z_i)$. It is easy to see that if $\lambda_i \le \lambda_j$, then $R_{ij}\sqrt{\lambda_i/\lambda_j} \approx \langle z_{ij}\rangle$.

In the following, we frequently have to compare $U_i^{p-1} w_j^{\text{in(out)}}$ with v-weights. It is easy to see that $U_i^{p-1} w_i^{\text{in(out)}} = n(n-2)v_i^{\text{in(out)}}\langle y_i\rangle^{-2}$. For $i \ne j$, we have the following lemma.

Lemma 4.1. *Suppose that $\lambda_i \le \lambda_j$. We have*

$$U_j^{p-1} w_i^{in} \lesssim R_{ij}^{-2} v_j^{in} + R^{-2} v_j^{out} + R^{-2} v_i^{in}, \tag{4.1}$$

$$U_j^{p-1} w_i^{out} \lesssim R_{ij}^{-2} v_j^{in} + R^{-2} v_j^{out} + R^{-2} v_i^{out}, \tag{4.2}$$

$$U_i^{p-1} w_j^{in} \lesssim R_{ij}^{-2} v_j^{in}, \tag{4.3}$$

$$U_i^{p-1} w_j^{out} \lesssim \langle z_{ij}\rangle^{-2}(v_i^{in} + v_i^{out} + v_j^{out}). \tag{4.4}$$

For any $0 < \varepsilon < 1$ and $M > 1$, we also have the following

$$U_i^{p-1} w_j^{out} \lesssim ((\lambda_i/\lambda_j)^2 + \varepsilon^2) v_j^{out}, \quad \text{on } \{x : |y_i - z_{ij}| \le \varepsilon\}, \tag{4.5}$$

$$w_j^{out} \lesssim \langle z_{ij}\rangle^{2n-10}\varepsilon^{4-n}(w_i^{in} + w_i^{out}), \quad \text{on } \{x : |y_i - z_{ij}| > \varepsilon\}, \tag{4.6}$$

$$U_i^{p-1} w_j^{out} \lesssim (|z_{ij}| + M)^{-2} v_j^{out}, \quad \text{on } \{x : |y_i| \ge 2|z_{ij}| + M\}, \tag{4.7}$$

$$U_i^{p-1} w_j^{out} \lesssim \langle z_{ij}\rangle^{\frac{2(n-6)}{n-2}}(\varepsilon^{-\frac{n-4}{2}} v_i^{in} + \varepsilon v_j^{out}), \quad \text{on } \{x : |y_i| \le R\}. \tag{4.8}$$

It follows from (4.4) that $U_i^{p-1} w_j^{\text{out}}$ is smaller compared to the v-weight only when $\langle z_{ij}\rangle$ is large. This is insufficient for late use, especially in the bubbling tower case. However, if $|z_{ij}|$ is small, then the weak interaction will force λ_j/λ_i to be very large, and (4.5) and (4.6) guarantee that $U_i^{p-1} w_j^{\text{out}}$ is still smaller or faster-decaying compared to the v-weight. The last two inequalities (4.7) and (4.8) will be used in a narrow domain, i.e., see $A_i^{(k)}$ in case 3 of the proof of Lemma 5.1, where we only need smallness of the coefficient before v_j^{out}, $j \in S(i)$.

Proof. • To prove (4.1), we divide it into two cases. First, on the set $\{|y_i| \leq R, |y_j| \leq R\}$, using (3.9), we get that $\langle y_i\rangle^{-2}\langle y_j\rangle^{-4} \lesssim R_{ij}^{-2}(\lambda_j/\lambda_i)\langle y_j\rangle^{-4}$. Then

$$U_j^{p-1} w_i^{\text{in}} \approx \frac{\lambda_j^2}{\langle y_j\rangle^4} \frac{\lambda_i^{\frac{n-2}{2}} R^{2-n}}{\langle y_i\rangle^2} \lesssim R_{ij}^{-2} \left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{n-4}{2}} \frac{\lambda_j^{\frac{n+2}{2}} R^{2-n}}{\langle y_j\rangle^4} \leq R_{ij}^{-2} v_j^{\text{in}}.$$

Second, on the set $\{|y_i| \leq R, |y_j| \geq R\}$, denoting $\alpha = \frac{4}{n+2} \leq \frac{1}{2}$, we have

$$\begin{aligned}
U_j^{p-1} w_i^{\text{in}} &\approx R^{\alpha(6-n)} \frac{\langle y_i\rangle^{2-4\alpha}}{\langle y_j\rangle^{4-n\alpha}} \left(v_j^{\text{out}}\langle y_j\rangle^{-2}\right)^\alpha \left(v_i^{\text{in}}\right)^{1-\alpha} \\
&\lesssim R^{2\alpha-2} \left(v_j^{\text{out}}\langle y_j\rangle^{-2}\right)^\alpha \left(v_i^{\text{in}}\right)^{1-\alpha} \\
&\lesssim R^{2\alpha-2}[\varepsilon^{-1}\langle y_j\rangle^{-2} v_j^{\text{out}} + \varepsilon^{\alpha/(1-\alpha)} v_i^{\text{in}}] \lesssim R^{-2} v_j^{\text{out}} + R^{-2} v_i^{\text{in}}.
\end{aligned}$$

We apply Young's inequality in the third step and take $\varepsilon = R^{2\alpha-2}$ in the last step.

• To prove (4.2), we divide it into two cases. First, on the set $\{|y_j| \leq R\}$, using (3.9), we get that $\langle y_i\rangle^{4-n}\langle y_j\rangle^{-4} \lesssim R_{ij}^{4-n} (\lambda_j/\lambda_i)^{\frac{n-4}{2}} \langle y_j\rangle^{-4}$. Then

$$U_j^{p-1} w_i^{\text{out}} \approx \frac{\lambda_j^2}{\langle y_j\rangle^4} \frac{\lambda_i^{\frac{n-2}{2}} R^{-4}}{\langle y_i\rangle^{n-4}} \lesssim R_{ij}^{4-n} \frac{\lambda_i}{\lambda_j} R^{n-6} v_j^{\text{in}} \lesssim R_{ij}^{-2} v_j^{\text{in}}.$$

Second, on the set $\{|y_i| > R, |y_j| \geq R\}$, we have

$$\begin{aligned}
U_j^{p-1} w_i^{\text{out}} &\approx \frac{\lambda_j^2}{\langle y_j\rangle^4} \frac{\lambda_i^{\frac{n-2}{2}} R^{-4}}{\langle y_i\rangle^{n-4}} = \left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{4}{n}} \left(\frac{v_i^{out}}{\langle y_i\rangle^2}\right)^{\frac{n-4}{n}} \left(\frac{v_j^{out}}{\langle y_j\rangle^2}\right)^{\frac{4}{n}} \\
&\lesssim \langle y_i\rangle^{-2} v_i^{\text{out}} + \langle y_j\rangle^{-2} v_j^{\text{out}} \leq R^{-2} v_i^{\text{out}} + R^{-2} v_j^{\text{out}}.
\end{aligned}$$

• To prove (4.3), note that since $\lambda_j \geq \lambda_i$, then $R \leq R_{ij}/2 \leq \sqrt{\lambda_j/\lambda_i} R_{ij}/2$. On the set $\{|y_j| \leq R\}$, using (3.9), we have $\langle y_i\rangle^{-4}\langle y_j\rangle^{-2} \lesssim R_{ij}^{-4}(\lambda_j/\lambda_i)^2\langle y_j\rangle^{-2} \lesssim R_{ij}^{-2}(\lambda_j/\lambda_i)^2\langle y_j\rangle^{-4}$. Then, on the set $\{|y_j| \leq R\}$, we have

$$U_i^{p-1} w_j^{\text{in}} \approx \frac{\lambda_i^2}{\langle y_i\rangle^4} \frac{\lambda_j^{\frac{n-2}{2}} R^{2-n}}{\langle y_j\rangle^2} \lesssim R_{ij}^{-2} \frac{\lambda_j^{\frac{n+2}{2}} R^{2-n}}{\langle y_j\rangle^4} = R_{ij}^{-2} v_j^{\text{in}}.$$

• To prove (4.4), we divide it into three cases. First, on the set $\{R \leq |y_j| \leq \sqrt{\lambda_j/\lambda_i} R_{ij}/2\}$, using (3.9), we can obtain $\langle y_i\rangle^{-4}\langle y_j\rangle^{4-n} \lesssim R_{ij}^{-4}(\lambda_j/\lambda_i)^2\langle y_j\rangle^{4-n} \lesssim R_{ij}^{-2}(\lambda_j/\lambda_i)^3\langle y_j\rangle^{2-n}$. Then

$$U_i^{p-1} w_j^{\text{out}} \approx \frac{\lambda_i^2}{\langle y_i\rangle^4} \frac{\lambda_j^{\frac{n-2}{2}} R^{-4}}{\langle y_j\rangle^{n-4}} \lesssim \frac{\lambda_j}{\lambda_i} R_{ij}^{-2} \frac{\lambda_j^{\frac{n+2}{2}} R^{-4}}{\langle y_j\rangle^{n-2}} \approx \langle z_{ij}\rangle^{-2} v_j^{\text{out}}.$$

We have used the fact that $R_{ij}\sqrt{\lambda_i/\lambda_j} \approx \langle z_{ij}\rangle$ when $\lambda_i \leq \lambda_j$ in the last step.

Second, on the set $\{|y_j| \geq \sqrt{\lambda_j/\lambda_i} R_{ij}/2, |y_i| \leq R\}$, we obtain $\langle y_i\rangle^{-4}\langle y_j\rangle^{4-n} \lesssim R_{ij}^{4-n}(\lambda_i/\lambda_j)^{\frac{n-4}{2}}\langle y_i\rangle^{-4}$. Then

$$U_i^{p-1}w_j^{\text{out}} \approx \frac{\lambda_i^2}{\langle y_i\rangle^4}\frac{\lambda_j^{\frac{n-2}{2}}R^{-4}}{\langle y_j\rangle^{n-4}} \lesssim R^{n-6}R_{ij}^{4-n}\frac{\lambda_j}{\lambda_i}\frac{\lambda_i^{\frac{n+2}{2}}R^{2-n}}{\langle y_i\rangle^4} \lesssim \frac{\lambda_j}{\lambda_i}\frac{v_i^{\text{in}}}{R_{ij}^2} \approx \frac{v_i^{\text{in}}}{\langle z_{ij}\rangle^2}.$$

Third, on the set $\{|y_j| \geq \sqrt{\lambda_j/\lambda_i} R_{ij}/2, |y_i| \geq R\}$,

$$\begin{aligned} U_i^{p-1}w_j^{\text{out}} &\approx \frac{\lambda_i^2}{\langle y_i\rangle^4}\frac{\lambda_j^{\frac{n-2}{2}}R^{-4}}{\langle y_j\rangle^{n-4}} = \langle y_j\rangle^{-2}\left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{8}{n-2}}\left(v_i^{\text{out}}\right)^{\frac{4}{n-2}}\left(v_j^{\text{out}}\right)^{\frac{n-6}{n-2}} \\ &\lesssim \left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{8}{n-2}}[v_i^{\text{out}}+v_j^{\text{out}}]\langle y_j\rangle^{-2} \leq \langle z_{ij}\rangle^{-2}[v_i^{\text{out}}+v_j^{\text{out}}], \end{aligned}$$

where we have used Young's inequality and

$$\left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{8}{n-2}}\langle y_j\rangle^{-2} \lesssim R_{ij}^{-2}\left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{10-n}{n-2}} \lesssim R_{ij}^{-2}\frac{\lambda_j}{\lambda_i} \approx \langle z_{ij}\rangle^{-2}.$$

• Now we prove (4.5). On the set $\{|y_i - z_{ij}| \leq \varepsilon\}$, we have $\langle y_j\rangle^2 = 1 + (\lambda_j/\lambda_i)^2|y_i - z_{ij}|^2 \leq 1 + (\lambda_j/\lambda_i)^2\varepsilon^2$. Consequently,

$$U_i^{p-1}w_j^{\text{out}} \approx \frac{\lambda_i^2}{\langle y_i\rangle^4}\frac{\lambda_j^{\frac{n-2}{2}}R^{-4}}{\langle y_j\rangle^{n-4}} = \frac{\lambda_i^2\langle y_j\rangle^2}{\lambda_j^2\langle y_i\rangle^4}v_j^{\text{out}} \lesssim ((\lambda_i/\lambda_j)^2+\varepsilon^2)v_j^{\text{out}}.$$

• Now we prove (4.6). On the set $\{|y_i - z_{ij}| > \varepsilon\}$, we have $|y_i - z_{ij}| \geq \frac{\varepsilon}{1+|z_{ij}|+\varepsilon}\langle y_i\rangle$, then $\langle y_j\rangle \geq \frac{\lambda_j}{\lambda_i}\frac{\varepsilon}{1+|z_{ij}|+\varepsilon}\langle y_i\rangle \gtrsim \frac{\lambda_j}{\lambda_i}\frac{\varepsilon}{\langle z_{ij}\rangle}\langle y_i\rangle$. Then

$$\begin{aligned} w_j^{\text{out}} = \frac{\lambda_j^{\frac{n-2}{2}}R^{-4}}{\langle y_j\rangle^{n-4}} &\lesssim \frac{\lambda_i^{\frac{n-2}{2}}R^{-4}}{\langle y_i\rangle^{n-4}}\left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{n-6}{2}}\left(\frac{\langle z_{ij}\rangle}{\varepsilon}\right)^{n-4} \\ &\lesssim R_{ij}^{n-6}\left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{n-6}{2}}\left(\frac{\langle z_{ij}\rangle}{\varepsilon}\right)^{n-4}[w_i^{\text{in}}+w_i^{\text{out}}], \end{aligned}$$

where we have used

$$\lambda_i^{\frac{n-2}{2}}R^{-4}\langle y_i\rangle^{4-n} \lesssim R_{ij}^{n-6}w_i^{\text{in}} + w_i^{\text{out}}.$$

If $\lambda_i \leq \lambda_j$, then $R_{ij}\sqrt{\lambda_i/\lambda_j} \approx \langle z_{ij}\rangle$. This completes the proof of (4.6).

• Now we prove (4.7). On the set $\{|y_i| > 2|z_{ij}| + M\}$, we have $\langle y_j\rangle^2 = 1 + (\lambda_j/\lambda_i)^2|y_i - z_{ij}|^2 \leq 2(\lambda_j/\lambda_i)^2|y_i|^2$. Then

$$U_i^{p-1}w_j^{\text{out}} \approx \frac{\lambda_i^2\langle y_j\rangle^2}{\lambda_j^2\langle y_i\rangle^4}v_j^{\text{out}} \lesssim \langle y_i\rangle^{-2}v_j^{\text{out}} \lesssim (|z_{ij}|+M)^{-2}v_j^{\text{out}}.$$

• To prove (4.8), we denote $\alpha = \frac{2}{n-2}$. Using Young's inequality, we get

$$U_i^{p-1}w_j^{\text{out}} \approx \frac{\lambda_i^2}{\langle y_i\rangle^4}\frac{\lambda_j^{\frac{n-2}{2}}R^{-4}}{\langle y_j\rangle^{n-4}} \lesssim (\tfrac{\lambda_i}{\lambda_j})^{\frac{\alpha}{2}(n-6)}R^{\alpha(n-6)}(v_i^{\text{in}})^{\alpha}(v_j^{\text{out}})^{1-\alpha}$$

$$\lesssim \langle z_{ij}\rangle^{\alpha(n-6)}[\varepsilon^{-\frac{1-\alpha}{\alpha}} v_i^{\text{in}} + \varepsilon v_j^{\text{out}}].$$

We have used $R_{ij}\sqrt{\lambda_i/\lambda_j} \approx \langle z_{ij}\rangle$ when $\lambda_i \le \lambda_j$ in the last step.

□

A similar result holds for dimension $n = 6$. Note that we have $\hat{w}_i^{\text{in}}(x, R^2) = w_i^{\text{in}}(x, R) + w_i^{\text{out}}(x, R)\chi_{\{R\le|y_i|\le R^2\}}$ when plugging in $n = 6$. So does $\hat{v}_i^{\text{in}}(x, R^2)$. Therefore, the proof related to w_i^{out} and w_j^{out} in the previous lemma can establish the same estimates about $\hat{w}_i^{\text{in}}$ and $\hat{w}_j^{\text{in}}$ in dimension $n = 6$.

Lemma 4.2. *Suppose that $\lambda_i \le \lambda_j$. When the dimension $n = 6$, we have*

$$U_j\hat{w}_i^{in} \lesssim R^{-2}[\hat{v}_j^{in} + \hat{v}_j^{out} + \hat{v}_i^{in}], \tag{4.9}$$

$$U_j\hat{w}_i^{out} \lesssim R^{-2}[\hat{v}_j^{in} + \hat{v}_j^{out} + \hat{v}_i^{out}], \tag{4.10}$$

$$U_i\hat{w}_j^{in} \lesssim \langle z_{ij}\rangle^{-2}[\hat{v}_i^{in} + \hat{v}_j^{in}] + R^{-2}\langle z_{ij}\rangle^{-1}\hat{v}_i^{out}, \tag{4.11}$$

$$U_i\hat{w}_j^{out} \lesssim \langle z_{ij}\rangle^{-1}\hat{v}_i^{in} + R^{-2}\hat{v}_i^{out} + \langle z_{ij}\rangle^{-2}\hat{v}_j^{out}. \tag{4.12}$$

For any $0 < \varepsilon < 1$ and $M > 1$, we also have the following

$$U_i[\hat{w}_j^{in} + \hat{w}_j^{out}] \lesssim ((\lambda_i/\lambda_j)^2 + \varepsilon^2)[\hat{v}_j^{in} + \hat{v}_j^{out}], \quad \text{on } \{x : |y_i - z_{ij}| \le \varepsilon\}, \tag{4.13}$$

$$[\hat{w}_j^{in} + \hat{w}_j^{out}] \lesssim \langle z_{ij}\rangle^5\varepsilon^{-3}[\hat{w}_i^{in} + \hat{w}_i^{out}], \quad \text{on } \{x : |y_i - z_{ij}| \ge \varepsilon\}, \tag{4.14}$$

$$U_i\hat{w}_j^{in(out)} \lesssim (|z_{ij}| + M)^{-2}\hat{v}_j^{in(out)}, \quad \text{on } \{x : |y_i| \ge 2|z_{ij}| + M\}, \tag{4.15}$$

$$U_i\hat{w}_j^{in} \lesssim \varepsilon^{-1}\hat{v}_i^{in} + \varepsilon\hat{v}_j^{in}, \quad \text{on } \{x : |y_i| \le R^2\}, \tag{4.16}$$

$$U_i\hat{w}_j^{out} \lesssim \langle z_{ij}\rangle^{\frac{4}{5}}[\varepsilon^{-\frac{3}{2}}\hat{v}_i^{in} + \varepsilon\hat{v}_j^{out}], \quad \text{on } \{x : |y_i| \le R^2\}. \tag{4.17}$$

Proof. • To prove (4.9), we divide it into three cases. First, on the set $\{|y_i| \le R^2, |y_j| \le R\}$, one can use the first case in the proof of (4.1) to get $U_i\hat{w}_i^{\text{in}} \lesssim R_{ij}^{-2}\hat{v}_j^{\text{in}}$. Second, on the set $\{|y_i| \le R^2, R \le |y_j| \le R^2\}$, we have

$$U_j\hat{w}_i^{\text{in}} \approx \frac{\lambda_j^2}{\langle y_j\rangle^4}\frac{\lambda_i^2 R^{-4}}{\langle y_i\rangle^2} = \langle y_j\rangle^{-2}(\hat{v}_j^{\text{in}})^{\frac{1}{2}}(\hat{v}_i^{\text{in}})^{\frac{1}{2}} \le R^{-2}[\hat{v}_j^{\text{in}} + \hat{v}_i^{\text{in}}].$$

Third, on the set $\{|y_i| \le R^2, |y_j| \ge R^2\}$, we have

$$U_j\hat{w}_i^{\text{in}} \approx \frac{\lambda_j^2}{\langle y_j\rangle^4}\frac{\lambda_i^2 R^{-4}}{\langle y_i\rangle^2} = \frac{R^{-1}}{\langle y_j\rangle^{3/2}}\left(\hat{v}_j^{\text{out}}\right)^{1/2}\left(\hat{v}_i^{\text{in}}\right)^{1/2} \lesssim R^{-4}[\hat{v}_j^{\text{out}} + \hat{v}_i^{\text{in}}].$$

• To prove (4.10), we divide it into three cases. First, on the set $\{|y_j| \le R, |y_i| \ge R^2\}$, using (3.9), we get that $\langle y_i\rangle^{-3}\langle y_j\rangle^{-4} \lesssim R^{-2}\langle y_i\rangle^{-2}\langle y_j\rangle^{-4} \lesssim R^{-2}R_{ij}^{-2}(\lambda_j/\lambda_i)\langle y_j\rangle^{-4}$. Then on the set $\{|y_j| \le R, |y_i| \ge R^2\}$, we have

$$U_j\hat{w}_i^{\text{out}} \approx \frac{\lambda_j^2}{\langle y_j\rangle^4}\frac{\lambda_i^2 R^{-2}}{\langle y_i\rangle^3} \lesssim R_{ij}^{-2}\frac{\lambda_i}{\lambda_j}\hat{v}_j^{\text{in}} \lesssim R^{-2}\hat{v}_j^{\text{in}}.$$

Second, on the set $\{R \le |y_j| \le R^2, |y_i| > R^2\}$, one has

$$U_j\hat{w}_i^{\text{out}} \approx \frac{\lambda_j^2}{\langle y_j\rangle^4}\frac{\lambda_i^2 R^{-2}}{\langle y_i\rangle^3} = \left(\frac{R^2\hat{v}_j^{\text{in}}}{\langle y_j\rangle^4}\right)^{1/2}\left(\frac{\hat{v}_i^{\text{out}}}{\langle y_i\rangle}\right)^{1/2} \lesssim \frac{\hat{v}_j^{\text{in}}}{R^2} + \frac{\hat{v}_i^{\text{out}}}{R^2}.$$

Third, on the set $\{|y_j| \geq R^2, |y_i| \geq R^2\}$, one has

$$U_j \hat{w}_i^{\text{out}} \approx \frac{\lambda_j^2}{\langle y_j \rangle^4} \frac{\lambda_i^2 R^{-2}}{\langle y_i \rangle^3} = \frac{1}{\langle y_j \rangle} \left(\frac{\hat{v}_j^{\text{out}}}{\langle y_j \rangle} \right)^{1/2} \left(\frac{\hat{v}_i^{\text{out}}}{\langle y_i \rangle} \right)^{1/2} \lesssim \frac{\hat{v}_i^{\text{out}}}{R^4} + \frac{\hat{v}_j^{\text{out}}}{R^4}.$$

• To prove (4.11), we divide it into three cases. First, on the set $\{|y_j| \leq R^2\} \cap \{|y_j| \leq \sqrt{\lambda_j/\lambda_i} R_{ij}/2\}$, similar the first case in the proof of (4.4), we have

$$U_i \hat{w}_j^{\text{in}} \approx \frac{\lambda_i^2}{\langle y_i \rangle^4} \frac{\lambda_j^2 R^{-4}}{\langle y_j \rangle^2} \lesssim \frac{\lambda_j}{\lambda_i} R_{ij}^{-2} \frac{\lambda_j^4 R^{-4}}{\langle y_j \rangle^4} \lesssim \langle z_{ij} \rangle^{-2} \hat{v}_j^{\text{in}}.$$

Second, on the set $\{|y_j| \leq R^2\} \cap \{|y_j| \geq \sqrt{\lambda_j/\lambda_i} R_{ij}/2, |y_i| \leq R^2\}$, similar to the second case in the proof of (4.4), we have

$$U_i \hat{w}_j^{\text{in}} \approx \frac{\lambda_i^2}{\langle y_i \rangle^4} \frac{\lambda_j^2 R^{-4}}{\langle y_j \rangle^2} \lesssim R_{ij}^{-2} \frac{\lambda_j}{\lambda_i} \frac{\lambda_i^2 R^{-4}}{\langle y_i \rangle^4} \lesssim \langle z_{ij} \rangle^{-2} \hat{v}_i^{\text{in}}.$$

Third, on the set $\{|y_j| \geq \sqrt{\lambda_j/\lambda_i} R_{ij}/2, |y_i| \geq R^2\}$, we have

$$\begin{aligned} U_i \hat{w}_j^{\text{in}} &\approx \frac{\lambda_i^2}{\langle y_i \rangle^4} \frac{\lambda_j^2 R^{-4}}{\langle y_j \rangle^2} = \frac{R^{-\frac{3}{2}}}{\langle y_i \rangle^{1/4} \langle y_j \rangle} \frac{\lambda_j}{\lambda_i} (\hat{v}_j^{\text{in}})^{1/4} (\hat{v}_i^{\text{out}})^{3/4} \\ &\lesssim R^{-2} R_{ij}^{-1} \sqrt{\lambda_j/\lambda_i} [\hat{v}_j^{\text{in}} + \hat{v}_i^{\text{out}}] \lesssim R^{-2} \langle z_{ij} \rangle^{-1} [\hat{v}_j^{\text{in}} + \hat{v}_i^{\text{out}}]. \end{aligned}$$

• To prove (4.12), we divide it into three cases. First, on the set $\{R^2 \leq |y_j| \leq \sqrt{\lambda_j/\lambda_i} R_{ij}/2\}$, using (3.9), we can obtain $\langle y_i \rangle^{-4} \langle y_j \rangle^{-3} \lesssim R_{ij}^{-4} (\lambda_j/\lambda_i)^2 \langle y_j \rangle^{-3} \lesssim R_{ij}^{-2} (\lambda_j/\lambda_i)^3 \langle y_j \rangle^{-5}$. Then

$$U_i \hat{w}_j^{\text{out}} \approx \frac{\lambda_i^2}{\langle y_i \rangle^4} \frac{\lambda_j^2 R^{-2}}{\langle y_j \rangle^3} \lesssim R_{ij}^{-2} \frac{\lambda_j}{\lambda_i} \frac{\lambda_j^4 R^{-2}}{\langle y_j \rangle^5} \approx \langle z_{ij} \rangle^{-2} \hat{v}_j^{\text{out}}.$$

Second, on the set $\{|y_j| \geq \sqrt{\lambda_j/\lambda_i} R_{ij}/2, |y_i| \leq R^2\}$, we obtain that $\langle y_i \rangle^{-4} \langle y_j \rangle^{-3} \lesssim R_{ij}^{-3} (\lambda_i/\lambda_j)^{3/2} \langle y_i \rangle^{-4}$. Then

$$U_i \hat{w}_j^{\text{out}} \approx \frac{\lambda_i^2}{\langle y_i \rangle^4} \frac{\lambda_j^2 R^{-2}}{\langle y_j \rangle^3} \lesssim R_{ij}^{-1} \left(\frac{\lambda_j}{\lambda_i} \right)^{1/2} \hat{v}_i^{\text{in}} \approx \langle z_{ij} \rangle^{-1} \hat{v}_i^{\text{in}}.$$

Third, on the set $\{|y_j| \geq \sqrt{\lambda_j/\lambda_i} R_{ij}/2, |y_i| \geq R^2\}$, we have

$$U_i \hat{w}_j^{\text{out}} \approx \frac{\lambda_i^2}{\langle y_i \rangle^4} \frac{\lambda_j^2 R^{-2}}{\langle y_j \rangle^3} \lesssim \frac{1}{\langle y_i \rangle} \left(\frac{\hat{v}_i^{\text{out}}}{\langle y_i \rangle} \right)^{1/2} \left(\frac{\hat{v}_j^{\text{out}}}{\langle y_j \rangle} \right)^{1/2} \lesssim R^{-2} [\hat{v}_j^{\text{out}} + \hat{v}_i^{\text{out}}].$$

• Now we prove (4.13). On the set $\{|y_i - z_{ij}| \leq \varepsilon\}$, we have $\langle y_j \rangle^2 = 1 + (\lambda_j/\lambda_i)^2 |y_i - z_{ij}|^2 \leq 1 + (\lambda_j/\lambda_i)^2 \varepsilon^2$. Consequently,

$$U_i \hat{w}_j^{\text{in}} \approx \frac{\lambda_i^2}{\langle y_i \rangle^4} \frac{\lambda_j^2 R^{-4}}{\langle y_j \rangle^2} = \frac{\lambda_i^2 \langle y_j \rangle^2}{\lambda_j^2 \langle y_i \rangle^4} \hat{v}_j^{\text{in}} \lesssim ((\lambda_i/\lambda_j)^2 + \varepsilon^2) \hat{v}_j^{\text{in}},$$

$$U_i \hat{w}_j^{\text{out}} \approx \frac{\lambda_i^2}{\langle y_i \rangle^4} \frac{\lambda_j^2 R^{-2}}{\langle y_j \rangle^3} = \frac{\lambda_i^2 \langle y_j \rangle^2}{\lambda_j^2 \langle y_i \rangle^4} \hat{v}_j^{\text{out}} \lesssim ((\lambda_i/\lambda_j)^2 + \varepsilon^2) \hat{v}_j^{\text{out}}.$$

• To prove (4.14), we use idea similar to the proof of (4.6). On the set $\{|y_i - z_{ij}| > \varepsilon\}$, we have $\langle y_j\rangle \geq \frac{\lambda_j}{\lambda_i}\frac{\varepsilon}{1+|z_{ij}|+\varepsilon}\langle y_i\rangle \gtrsim \frac{\lambda_j}{\lambda_i}\frac{\varepsilon}{\langle z_{ij}\rangle}\langle y_i\rangle$. Then

$$\hat{w}_j^{\text{in}} = \frac{\lambda_j^2 R^{-4}}{\langle y_j\rangle^2} \lesssim \left(\frac{\langle z_{ij}\rangle}{\varepsilon}\right)^2 \frac{\lambda_i^2 R^{-4}}{\langle y_i\rangle^2}.$$

On the support of $\hat{w}_j^{\text{in}}$, we also have $R^2 \gtrsim \frac{\lambda_j}{\lambda_i}\frac{\varepsilon}{\langle z_{ij}\rangle}\langle y_i\rangle$. Then

$$\lambda_i^2 R^{-4}\langle y_i\rangle^{-2} \lesssim \hat{w}_i^{\text{in}} + R^{-2}\langle y_i\rangle \hat{w}_i^{\text{out}} \lesssim \hat{w}_i^{\text{in}} + \frac{\langle z_{ij}\rangle}{\varepsilon}\hat{w}_i^{\text{out}}.$$

Thus combining the above two inequalities, we get

$$\hat{w}_j^{\text{in}} \lesssim \left(\frac{\langle z_{ij}\rangle}{\varepsilon}\right)^2 \hat{w}_i^{\text{in}} + \left(\frac{\langle z_{ij}\rangle}{\varepsilon}\right)^3 \hat{w}_i^{\text{out}}.$$

For the other one $\hat{w}_j^{\text{out}}$, we have

$$\begin{aligned}\hat{w}_j^{\text{out}} = \frac{\lambda_j^2 R^{-2}}{\langle y_j\rangle^3} &\lesssim \frac{\lambda_i^2 R^{-2}}{\langle y_i\rangle^3}\frac{\lambda_i}{\lambda_j}\left(\frac{\langle z_{ij}\rangle}{\varepsilon}\right)^3 \lesssim \frac{\lambda_i}{\lambda_j}\left(\frac{\langle z_{ij}\rangle}{\varepsilon}\right)^3\left[\frac{R^2}{\langle y_i\rangle}\hat{w}_i^{\text{in}} + \hat{w}_i^{\text{out}}\right] \\ &\lesssim \langle z_{ij}\rangle^3\varepsilon^{-3}[\langle z_{ij}\rangle^2\hat{w}_i^{\text{in}} + \hat{w}_i^{\text{out}}].\end{aligned}$$

• Now we prove (4.15). On the set $\{|y_i| > 2|z_{ij}| + M\}$, we have $\langle y_j\rangle^2 = 1 + (\lambda_j/\lambda_i)^2|y_i - z_{ij}|^2 \leq 2(\lambda_j/\lambda_i)^2|y_i|^2$. Then

$$U_i\hat{w}_j^{\text{in(out)}} \approx \frac{\lambda_i^2\langle y_j\rangle^2}{\lambda_j^2\langle y_i\rangle^4}\hat{v}_j^{\text{in(out)}} \lesssim \langle y_i\rangle^{-2}\hat{v}_j^{\text{in(out)}} \lesssim (|z_{ij}| + M)^{-2}\hat{v}_j^{\text{in(out)}}.$$

• Now we prove (4.16). On the set $\{|y_i| \leq R^2, |y_j| \leq R^2\}$, we have

$$U_i\hat{w}_j^{\text{in}} \approx \frac{\lambda_i^2}{\langle y_i\rangle^4}\frac{\lambda_j^2 R^{-4}}{\langle y_j\rangle^2} \leq (\hat{v}_i^{\text{in}})^{\frac{1}{2}}(\hat{v}_j^{\text{in}})^{\frac{1}{2}} \lesssim \varepsilon^{-1}\hat{v}_i^{\text{in}} + \varepsilon\hat{v}_j^{\text{in}}.$$

• Now we prove (4.17). On the set $\{|y_i| \leq R^2, |y_j| \geq R^2\}$, we have

$$\begin{aligned}U_i\hat{w}_j^{\text{out}} \approx \frac{\lambda_i^2}{\langle y_i\rangle^4}\frac{\lambda_j^2 R^{-2}}{\langle y_j\rangle^3} &\lesssim (\tfrac{\lambda_i}{\lambda_j})^{\frac{2}{5}}R^{\frac{4}{5}}(\hat{v}_i^{\text{in}})^{\frac{2}{5}}\left(\frac{\hat{v}_j^{\text{out}}}{\langle y_j\rangle}\right)^{\frac{3}{5}} \\ &\lesssim \langle z_{ij}\rangle^{\frac{4}{5}}[\varepsilon^{-\frac{3}{2}}\hat{v}_i^{\text{in}} + \varepsilon\hat{v}_j^{\text{out}}].\end{aligned}$$

□

In the following proposition, we derive a rough C^0 estimate of ρ. Note that $\overline{W}$ decays much faster than W. It shows that the behavior of ρ at infinity can be bounded by $\|h\|_{**}W(x)$ up to some constant and small error. It will be used in Lemma 5.1 to exclude the blow-up points going to infinity.

Proposition 4.3. *There exists a constant $C(n,\nu)$ such that, for any h with $\|h\|_{**} < \infty$, if ϕ satisfies*

$$\Delta\phi + p\sigma^{p-1}\phi = h \quad \textit{in} \quad \mathbb{R}^n,$$

then the following inequality holds for any $M > 1$,

$$\frac{|\phi(x)|}{W(x)} \le C(n,\nu)\left(\|h\|_{**} + M^{3n}\|\phi\|_* \frac{\overline{W}(x)}{W(x)} + M^4R^{-2} + M^{-1}\right), \tag{4.18}$$

where $\overline{W}(x) = \sum_{i\neq j}(\bar{w}_i^{in}(x) + \bar{w}_i^{out}(x))$ is defined by

$$\bar{w}_i^{in}(x) = \begin{cases} \lambda_i^{\frac{n-2}{2}} R^{2-n}\langle y_i\rangle^{-4}\chi_{\{|y_i|\le R\}}, & n \ge 7,\\ \lambda_i^2 R^{-4}\langle y_i\rangle^{-3}(1+\log\langle y_i\rangle)\chi_{\{|y_i|\le R^2\}}, & n = 6,\end{cases}$$

$$\bar{w}_i^{out}(x) = \begin{cases} \lambda_i^{\frac{n-2}{2}} R^{-4}\langle y_i\rangle^{2-n}\log\langle y_i\rangle\chi_{\{|y_i|> R\}}, & n \ge 7,\\ \lambda_i^2 R^{-2}\langle y_i\rangle^{-4}\log\langle y_i\rangle\chi_{\{|y_i|> R^2\}}, & n = 6.\end{cases}$$

Proof. By the Green's representation, we have

$$\begin{aligned}\phi(\tilde{x}) &= C(n)\int_{\mathbb{R}^n} |\tilde{x}-x|^{2-n}\left(p\sigma^{p-1}(x)\phi(x) - h(x)\right)dx\\ &=: P_1 + P_2,\end{aligned}$$

where $C(n)$ is a positive dimensional constant. Applying Lemma 3.6, we get

$$|P_2| \le C(n)\|h\|_{**}\int_{\mathbb{R}^n}|\tilde{x}-x|^{2-n}V(x)dx \lesssim \|h\|_{**}W(\tilde{x}).$$

Consider P_1. Let us assume $n \ge 7$. Using $\sigma^{p-1} \lesssim \sum_{i'=1}^{\nu} U_{i'}^{p-1}$, we have

$$\begin{aligned}|P_1| &\lesssim \|\phi\|_* \sum_{i,i'=1}^{\nu}\int_{\mathbb{R}^n}|\tilde{x}-x|^{2-n}U_{i'}^{p-1}(x)(w_i^{\text{in}}(x) + w_i^{\text{out}}(x))dx\\ &=: \|\phi\|_*\sum_{i,i'=1}^{\nu}(A_{ii'}^{\text{in}} + A_{ii'}^{\text{out}})(\tilde{x}).\end{aligned} \tag{4.19}$$

Consider A_{ii}^{in} (that is $i' = i$). We use a similar trick in estimating the last line of (3.20), that is, dividing into two cases according to the relation of $\tilde{y}_i$ and R, and applying Lemma A.7 to get

$$\begin{aligned}A_{ii}^{\text{in}}(\tilde{x}) &= n(n-2)\lambda_i^{\frac{n+2}{2}}R^{2-n}\int_{\mathbb{R}^n}|\tilde{x}-x|^{2-n}\langle y_i\rangle^{-6}\chi_{\{|y_i|\le R\}}dx\\ &\approx \lambda_i^{\frac{n-2}{2}}R^{2-n}\int_{\mathbb{R}^n}|\tilde{y}_i-y_i|^{2-n}\langle y_i\rangle^{-6}\chi_{\{|y_i|\le R\}}dy_i\\ &\lesssim \lambda_i^{\frac{n-2}{2}}R^{2-n}\left[\langle\tilde{y}_i\rangle^{-4}\chi_{\{|\tilde{y}_i|\le 2R\}} + R^{n-6}\langle\tilde{y}_i\rangle^{2-n}\chi_{\{|\tilde{y}_i|\ge 2R\}}\right]\\ &\lesssim \bar{w}_i^{\text{in}}(\tilde{x}) + \bar{w}_i^{\text{out}}(\tilde{x}).\end{aligned}$$

Similarly,

$$\begin{aligned}A_{ii}^{\text{out}}(\tilde{x}) &\approx \lambda_i^{\frac{n-2}{2}}R^{-4}\int_{\mathbb{R}^n}|\tilde{y}_i-y_i|^{2-n}\langle y_i\rangle^{-n}\chi_{\{|y_i|> R\}}dy_i\\ &\lesssim \lambda_i^{\frac{n-2}{2}}\left[R^{-2-n}\chi_{\{|\tilde{y}_i|\le R/2\}} + R^{-4}\langle\tilde{y}_i\rangle^{2-n}\log\langle\tilde{y}_i\rangle\chi_{\{|\tilde{y}_i|\ge R/2\}}\right]\\ &\lesssim \bar{w}_i^{\text{in}}(\tilde{x}) + \bar{w}_i^{\text{out}}(\tilde{x}).\end{aligned}$$

Consider the case $i' \neq i$. Let us first assume $\lambda_i < \lambda_{i'}$. Using (4.1), (4.2) and Lemma 3.6, we have

$$A^{\text{in}}_{ii'} + A^{\text{out}}_{ii'} \lesssim R^{-2} \int_{\mathbb{R}^n} |\tilde{x} - x|^{2-n} V(x) \lesssim R^{-2} W(\tilde{x}).$$

If $\lambda_i \geq \lambda_{i'}$ and $\langle z_{ii'} \rangle > M$, then using (4.3) and (4.4), we have

$$A^{\text{in}}_{ii'} + A^{\text{out}}_{ii'} \lesssim (R^{-2} + M^{-2}) \int_{\mathbb{R}^n} |\tilde{x} - x|^{2-n} V(x) \lesssim (R^{-2} + M^{-2}) W(\tilde{x}).$$

If $\lambda_i \geq \lambda_{i'}$ and $\langle z_{ii'} \rangle \leq M$, we apply (4.3), (4.5) and (4.6) (taking $\varepsilon = M^{-1}$) to get

$$\begin{aligned}
&A^{\text{in}}_{ii'} + A^{\text{out}}_{ii'} \\
&\quad \lesssim ((\lambda_{i'}/\lambda_i)^2 + M^{-2}) W(\tilde{x}) + M^{3n-14} \int_{\mathbb{R}^n} |\tilde{x} - x|^{2-n} U^{p-1}_{i'} (w^{\text{in}}_{i'} + w^{\text{out}}_{i'}) \\
&\quad \lesssim (M^4 R^{-4} + M^{-2}) W(\tilde{x}) + M^{3n} [A_{i'i'}\text{in} + A_{i'i'}\text{out}](\tilde{x}) \\
&\quad \lesssim (M^4 R^{-4} + M^{-2}) W(\tilde{x}) + M^{3n} \overline{W}(\tilde{x}).
\end{aligned}$$

Here we used $R_{i'i}\sqrt{\lambda_{i'}/\lambda_i} \approx \langle z_{i'i} \rangle$ when $\lambda_i \geq \lambda_{i'}$ to get $\lambda_{i'}/\lambda_i \lesssim R^{-2} M^2$.

Consolidating the estimates of P_1 and P_2, we can prove the proposition when the dimension $n \geq 7$.

When the dimension $n = 6$, the proof is similar to that for $n \geq 7$. We point out modifications without giving details. One shall use $\hat{w}^{\text{in}}_i$ and $\hat{w}^{\text{out}}_i$ in (4.19) to define $\hat{A}^{\text{in}}_{ii'}$ and $\hat{A}^{\text{out}}_{ii'}$. Similarly,

$$\begin{aligned}
\hat{A}^{\text{in}}_{ii} &\approx \lambda_i^4 R^{-4} \int_{|y_i| \leq R^2} |\tilde{x} - x|^{-4} \langle y_i \rangle^{-6} dx \\
&\lesssim \lambda_i^2 R^{-4} \left[\langle \tilde{y}_i \rangle^{-4} (1 + \log\langle \tilde{y}_i \rangle) \chi_{\{|y_i| \leq R^2\}} + (\log R) \langle \tilde{y}_i \rangle^{-4} \chi_{\{|\tilde{y}_i| \geq R^2\}} \right], \\
\hat{A}^{\text{out}}_{ii} &\approx \lambda_i^4 R^{-2} \int_{|y_i| \geq R^2} |\tilde{x} - x|^{-4} \langle y_i \rangle^{-6} dx \\
&\lesssim \lambda_i^2 R^{-2} \left[R^{-8} \chi_{\{|y_i| \leq R^2\}} + \langle \tilde{y}_i \rangle^{-4} (1 + \log\langle \tilde{y}_i \rangle) \chi_{\{|\tilde{y}_i| \geq R^2\}} \right].
\end{aligned}$$

Consider $i' \neq i$. If $\lambda_i < \lambda_{i'}$, then (4.9) and (4.10) imply $\hat{A}^{\text{in}}_{ii'} + \hat{A}^{\text{out}}_{ii'} \lesssim R^{-2} W$. If $\lambda_i \geq \lambda_{i'}$ and $\langle z_{i'i} \rangle > M$, then (4.11) and (4.12) imply that $\hat{A}^{\text{in}}_{ii'} + \hat{A}^{\text{out}}_{ii'} \lesssim (R^{-2} + M^{-1}) W$. If $\lambda_i \geq \lambda_{i'}$ and $\langle z_{i'i} \rangle \leq M$, then (4.13) and (4.14) imply that $\hat{A}^{\text{in}}_{ii'} + \hat{A}^{\text{out}}_{ii'} \lesssim (M^4 R^{-4} + M^{-2}) W + M^8 \overline{W}$.

□

4.2. **Configuration of bubbles tree.**

Denote $I = \{1, \ldots, \nu\}$. Suppose that $\left\{ U^{(k)}_i := U[z^{(k)}_i, \lambda^{(k)}_i] : i \in I \right\}_{k=1}^{\infty}$ is a sequence of ν bubbles with the interaction $Q^{(k)} = \max\{q^{(k)}_{ij} : \forall\, i, j \in I, i \neq j\} \to 0$ as $k \to \infty$, or equivalently,

$$R^{(k)} = \frac{1}{2} \min\{R^{(k)}_{ij} : \forall\, i, j \in I, i \neq j\} \to \infty \quad \text{as} \quad k \to \infty. \tag{4.20}$$

By reordering them and taking subsequences (for finitely many times), we can always assume

$$\lambda_1^{(k)} \le \cdots \le \lambda_\nu^{(k)}, \tag{4.21}$$

$$\text{either } \lim_{k\to\infty} z_{ij}^{(k)} \text{ exists or } \lim_{k\to\infty} |z_{ij}^{(k)}| = \infty \tag{4.22}$$

where $z_{ij}^{(k)} := \lambda_i^{(k)}(z_j^{(k)} - z_i^{(k)})$ for $j \in I \setminus \{i\}$.

There is a geometric interpretation of $z_{ij}^{(k)}$. In the rescaled z_i-centered coordinates $y_i = \lambda_i(x - z_i)$, we see that $U_i(x) = \lambda_i^{(n-2)/2} U(y_i)$, where $U(y) = U[0,1](y)$. Here we omit the superscript (k) to ease the notation. That is, U_i is the $\lambda_i^{(n-2)/2}$ multiple of the standard bubble. Under y_i-coordinates, the other bubbles become new ones with

$$U_j(x) = \left(\frac{(n(n-2))^{1/2}\lambda_j}{1 + (\lambda_j/\lambda_i)^2 |y_i - z_{ij}|^2} \right)^{\frac{n-2}{2}} = \lambda_i^{\frac{n-2}{2}} U[z_{ij}, \lambda_j/\lambda_i](y_i).$$

Then z_{ij} is the new center of U_j, $j \in I \setminus \{i\}$. Under y_i-coordinates and omitting $\lambda_i^{(n-2)/2}$ factor, we obtain a new set of ν bubbles $\{U[0,1], U[z_{ij}, \lambda_j/\lambda_i] : j \in I \setminus \{i\}\}$. It is easy to check that $R_{i'j'}$ remains unchanged in this rescaling for all $i', j' \in I$.

We define a partial order $\prec$ on $I = \{1, \cdots, \nu\}$ as

$$\begin{aligned} i \prec j \quad &\Longleftrightarrow \quad i < j \text{ and } \lim_{k\to\infty} z_{ij}^{(k)} \text{ exists}, \\ i \preceq j \quad &\Longleftrightarrow \quad i \prec j \text{ or } i = j. \end{aligned} \tag{4.23}$$

Lemma 4.4. *$\prec$ is a strict partial order.*

Proof. We can see that it is irreflexive and asymmetry. We only need to check the transitivity. Suppose that $i \prec j$ and $j \prec l$. It follows from the definition that $i < j < l$ and $z_{ij}^{(k)} = \lambda_i^{(k)}(z_j^{(k)} - z_i^{(k)})$ and $z_{jl}^{(k)} = \lambda_j^{(k)}(z_l^{(k)} - z_j^{(k)})$ are both uniformly bounded as $k \to \infty$. Then, using interpolation and $\lambda_i^{(k)} \le \lambda_j^{(k)}$, we get $|z_{il}^{(k)}|$ is also uniformly bounded. That is

$$|z_{il}^{(k)}| \le \lambda_i^{(k)} |z_j^{(k)} - z_i^{(k)}| + \lambda_i^{(k)} |z_l^{(k)} - z_j^{(k)}| \le |z_{ij}^{(k)}| + |z_{jl}^{(k)}|.$$

Then by assumption (4.22), we know $\lim_{k\to\infty} z_{il}^{(k)}$ exists. Thus $i \prec l$. $\square$

Lemma 4.5. *Suppose that $\{U_i^{(k)} : i = 1, \cdots, \nu\}$ is a sequence of ν bubbles satisfying* (4.20), (4.21) *and* (4.22). *Then*

$$C^* := 1 + \max_{i,j\in I, k\ge 0} \{|z_{ij}^{(k)}| : i \prec j\} < \infty. \tag{4.24}$$

Moreover,

$$i \prec j \quad \Longleftrightarrow \quad \sqrt{\lambda_j^{(k)}/\lambda_i^{(k)}} \le R_{ij}^{(k)} \le C^* \sqrt{\lambda_j^{(k)}/\lambda_i^{(k)}}, \quad \forall\, k \ge 0, \tag{4.25}$$

$$i \not\prec j \text{ and } j \not\prec i \quad \Leftrightarrow \quad \sqrt{\lambda_j^{(k)}/\lambda_i^{(k)}} + \sqrt{\lambda_i^{(k)}/\lambda_j^{(k)}} = o(1)R_{ij}^{(k)}. \tag{4.26}$$

Here $o(1)$ denotes some quantity that goes to 0 as $k \to \infty$. We will call i and j incomparable in the case (4.26).

Proof. It is easy to see that C^* is finite by the definition of $\prec$ relation (4.23). Suppose that $i \prec j$. Then $\lambda_i^{(k)} \le \lambda_j^{(k)}$ and thus

$$\begin{aligned} R_{ij}^{(k)} &= \max\{\sqrt{\lambda_j^{(k)}/\lambda_i^{(k)}}, \sqrt{\lambda_i^{(k)}\lambda_j^{(k)}}|z_i^{(k)} - z_j^{(k)}|\} \\ &= \sqrt{\lambda_j^{(k)}/\lambda_i^{(k)}}\max\{1, |z_{ij}^{(k)}|\} \le C^*\sqrt{\lambda_j^{(k)}/\lambda_i^{(k)}}. \end{aligned}$$

Conversely, by $R_{ij}^{(k)} \to \infty$ as $k \to \infty$, we have $\lambda_j^{(k)} > \lambda_i^{(k)}$ when k is large enough. Thus $i < j$. The above inequality shows that $|z_{ij}^{(k)}|$ is uniformly bounded. Thus by (4.22) and (4.23), we have $i \prec j$.

If $i \not\prec j$ and $j \not\prec i$, WLOG, assume $\lambda_i^{(k)} \le \lambda_j^{(k)}$, then $z_{ij}^{(k)}$ must be unbounded, thus

$$\sqrt{\lambda_j^{(k)}/\lambda_i^{(k)}} \le o(1)R_{ij}^{(k)}.$$

□

Recall that a *tree* is a partially ordered set, say $(\mathrm{T}, \prec)$, such that for any $t \in \mathrm{T}$, the set $\{s \in \mathrm{T} : s \prec t\}$ is well-ordered by the relation $\prec$. The following lemma shows that I can be decomposed into several trees.

Lemma 4.6. *For any sequences of $\{U_i^{(k)} : i = 1, \cdots, \nu\}$ satisfying (4.20), (4.21) and (4.22), there exists α^* (depends on the sequences) such that $I = \{1, \cdots, \nu\}$ can be partitioned into T_α, $\alpha = 1, \cdots, \alpha^*$, where each T_α is a tree.*

Proof. Fixing any i_0, let us prove the set $\{s \in I : s \prec i_0\}$ is a well-ordered set. That is, if $s \prec i_0$, $t \prec i_0$, and $s \neq t$, then either $s \prec t$ or $t \prec s$. In fact, by the assumption and the definition of $\prec$ relation in (4.23), we obtain

$$\lim_{k\to\infty} z_{si_0}^{(k)} \text{ and } \lim_{k\to\infty} z_{ti_0}^{(k)} \text{ both exist.}$$

WLOG, assume $s > t$, then $\lambda_s^{(k)} \ge \lambda_t^{(k)}$. Since $z_{ts}^{(k)} = \lambda_t^{(k)}(z_s^{(k)} - z_t^{(k)})$, then

$$|z_{ts}^{(k)}| \le \lambda_t^{(k)}|z_{i_0}^{(k)} - z_t^{(k)}| + \lambda_t^{(k)}|z_s^{(k)} - z_{i_0}^{(k)}| \le |z_{ti_0}^{(k)}| + |z_{si_0}^{(k)}| < \infty$$

as $k \to \infty$. This implies that $t \prec s$.

Suppose that the partially ordered set $(I, \prec)$ has α^* minimal elements, say they $r_1, \cdots, r_{\alpha^*}$. Define $\mathrm{T}_\alpha = \{i \in I : r_\alpha \preceq i\}$. The above proof shows that each T_α is a tree with root r_α.

Moreover, these T_α are mutually disjoint sets. For if $i \in \mathrm{T}_\alpha \cap \mathrm{T}_{\alpha'}$, $\alpha \neq \alpha'$, then the above proof shows that either $r_\alpha \prec r_{\alpha'}$ or $r_{\alpha'} \prec r_\alpha$, which contradicts the fact that r_α and $r_{\alpha'}$ are minimal elements. Thus $\{\mathrm{T}_\alpha : \alpha = 1, \cdots, \alpha^*\}$ is a partition of I. This completes the proof.

□

For each $i \in I$, define the set $S(i) = \{j \in I : i \prec j\}$ (which means the successor of i), that is

$$S(i) = \{j \in I : i < j, \lim_{k\to\infty} z_{ij}^{(k)} \textit{ exists}\}. \tag{4.27}$$

One can have a clear picture of the sequence in each $y_i^{(k)} = \lambda_i^{(k)}(x - z_i^{(k)})$ coordinates.

On the one hand, for any $j \in S(i)$, it is clear that $U_j^{(k)}$ are bubbles "higher"(i.e. $\lambda_j^{(k)} \geq \lambda_i^{(k)}$) than $U_i^{(k)}$ and becomes higher and higher than it and eventually "singular" at $\lim_{k\to\infty} z_{ij}^{(k)}$ as $k \to \infty$, since the interaction of them must become smaller and smaller as $k \to \infty$. Moreover, if $|z_{ij}^{(k)}| < 1$, then $U_i^{(k)}$ and $U_j^{(k)}$ form a bubbling tower, otherwise they form a bubbling cluster. In both cases, one must have $\lambda_j^{(k)}/\lambda_i^{(k)} \to \infty$ for such j, because the interaction of all bubbles is vanishing as $k \to \infty$.

On the other hand, for any $j \notin S(i)$, either $U_j^{(k)}$ is "lower" than $U_i^{(k)}$ or $U_j^{(k)}$ escapes to infinity in $y_i^{(k)}$ coordinates. These bubbles are benign in the limiting process. More precisely,

Lemma 4.7. *Fix any $M > 0$. If $j \notin S(i)$, then as $k \to \infty$,*

$$\begin{aligned} U_j^{(k)}(x) &= o(1)U_i^{(k)}(x), \\ w_j^{in}(x) + w_j^{out}(x) &= o(1)w_i^{in}, \\ v_j^{in}(x) + v_j^{out}(x) &= o(1)v_i^{in}, \end{aligned} \tag{4.28}$$

uniformly on $\{x : |y_i^{(k)}| \leq M\}$. Here $o(1)$ denotes some quantity that goes to 0 as $k \to \infty$. Consequently,

$$\begin{aligned} \sum_{j=1}^{\nu} U_j^{(k)} &= \sum_{j\in S(i)} U_j^{(k)} + (1 + o(1))U_i^{(k)}, \\ W(x) &= \sum_{j\in S(i)} \left(w_j^{in}(x) + w_j^{out}(x)\right) + (1 + o(1))w_i^{in}(x), \\ V(x) &= \sum_{j\in S(i)} \left(v_j^{in}(x) + v_j^{out}(x)\right) + (1 + o(1))v_i^{in}(x), \end{aligned} \tag{4.29}$$

uniformly on $\{x : |y_i^{(k)}| \leq M\}$ as $k \to \infty$.

The above statements also hold replacing $w^{in(out)}$ and $v^{in(out)}$ by $\hat{w}^{in(out)}$ and $\hat{v}^{in(out)}$ respectively in dimension $n = 6$.

Proof. In fact, $j \notin S(i)$ means either $j \prec i$ or j and i are incomparable. We shall prove two cases respectively. We will omit the superscript (k) for various notations, like $\lambda_i^{(k)}$, $y_i^{(k)}$, $U_i^{(k)}$, $R^{(k)}$, and $R_{ij}^{(k)}$.

• In the case $j \prec i$, one must have $\lambda_i/\lambda_j \to \infty$ as $k \to \infty$. Recall that $U_i(x) \approx \lambda_i^{\frac{n-2}{2}}\langle y_i\rangle^{2-n}$ and $U_j(x) \approx \lambda_j^{(n-2)/2}\langle y_j\rangle^{2-n}$. Thus $U_j(x) \lesssim \lambda_j^{\frac{n-2}{2}} =$

$o(1)\lambda_i^{\frac{n-2}{2}} \lesssim o(1)\langle M\rangle^{n-2}U_i(x)$ on the set $\{y_i \le M\}$ as $k \to \infty$. The support of w_j^{out} does not intersect $\{|y_i| \le M\}$, thus we only need to consider w_j^{in}. We have

$$\frac{w_j^{\text{in}}}{w_i^{\text{in}}} = \left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{n-2}{2}} \frac{\langle y_i\rangle^2}{\langle y_j\rangle^2} \lesssim M^2 \left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{n-2}{2}} \to 0. \tag{4.30}$$

The proof for v-weights is completely analogous and we omit it.

• If i and j are incomparable, clearly we have $R_{ij} = \sqrt{\lambda_j/\lambda_i}|z_{ij}|$ and $|z_{ij}| \to \infty$ as $k \to \infty$. Recall that $y_j = \lambda_j/\lambda_i(y_i - z_{ij})$, then $|y_j| \ge \frac{1}{2}\lambda_j/\lambda_i|z_{ij}| = \frac{1}{2}\sqrt{\lambda_j/\lambda_i}R_{ij}$ on the set $\{|y_i| \le M\}$ when k is large enough. Then

$$\frac{U_j(x)}{U_i(x)} \lesssim \left(\frac{\lambda_j}{\lambda_i}\frac{1+M^2}{(R_{ij})^2\lambda_j/\lambda_i}\right)^{\frac{n-2}{2}} \lesssim \left(\frac{1+M^2}{(R_{ij})^2}\right)^{\frac{n-2}{2}} \to 0.$$

If $\lambda_j/\lambda_i > 1$, then the support of $w_j^{\text{in}}(x)$ does not intersect $\{x : |y_i| \le M\}$. If $\lambda_j/\lambda_i \le 1$, then

$$\frac{w_j^{\text{in}}}{w_i^{\text{in}}} = \left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{n-2}{2}} \frac{\langle y_i\rangle^2}{\langle y_j\rangle^2} \lesssim M^2 \left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{n-6}{2}} \langle z_{ij}\rangle \to 0.$$

For w_j^{out}, by (4.26) we have

$$\frac{w_j^{\text{out}}}{w_i^{\text{in}}} = \left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{n-2}{2}} R^{n-6}\frac{\langle y_i\rangle^2}{\langle y_j\rangle^{n-4}} \lesssim M^2\frac{\lambda_j}{\lambda_i}R_{ij}^{-2} \to 0,$$

This finishes the proof of the statement about w-weights in (4.29). The proof for v-weights is completely analogous and we omit it.

By the same argument, it is easy to see that all the above assertions hold for $n = 6$ after some minor modifications.

□

5. Point-wise estimate for the main part of error function

This section is the central part of this paper. We shall establish the C^0 estimates for the main part of ρ, i.e., ρ_0, see Proposition 5.4, which comes from the interaction between bubbles. The crucial part is to obtain a priori estimate in Lemma 5.1.

5.1. **A priori estimate.** Suppose that $\{U_i^{(k)} : i = 1, \cdots, \nu\}$ is a sequence of ν bubbles satisfying (4.20), (4.21), and (4.22). Recall the definition of $S(i) = \{j \in I : i \prec j\}$, see (4.27). Recall that $y_i^{(k)} = \lambda_i^{(k)}(x - z_i^{(k)})$, $z_{ij}^{(k)} = \lambda_i^{(k)}(z_j^{(k)} - z_i^{(k)})$ and the definition of C^*, see (4.24). Let us define

$$\begin{aligned} \Omega^{(k)} &:= \bigcup_{i\in I}\left\{x : |y_i^{(k)}| \le L\right\}, \\ \Omega_i^{(k)} &:= \left\{x : |y_i^{(k)}| \le L, |y_i^{(k)} - z_{ij}^{(k)}| \ge \epsilon_1, \forall\, j \in S(i)\right\} \end{aligned} \tag{5.1}$$

with a large constant $L = L(n, \nu, C^*)$ and a small constant $\epsilon_1 = \epsilon_1(n, \nu, C^*)$ to be determined later. The domain $\Omega_i^{(k)}$ is where $U_i^{(k)}$ dominates over other bubbles, and there is no "singular bubble" in $\Omega_i^{(k)}$. When k is large enough, one can see that

(1) For all $j \in S(i)$, $\{|y_i^{(k)} - z_{ij}^{(k)}| \leq \epsilon_1\} \subset\subset \{|y_i^{(k)}| \leq L\}$. Moreover, $\{|y_j^{(k)}| \leq L\} \subset\subset \{|y_i^{(k)} - z_{ij}^{(k)}| \leq \frac{1}{2}\epsilon_1\}$.

(2) If i and j are incomparable (that is, $i \not< j$ and $j \not< i$), then $\Omega_i^{(k)}$ and $\Omega_j^{(k)}$ are disjoint.

See Figure 3 for an illustration of $\Omega_i^{(k)}$ in a simple case.

Lemma 5.1. *There exist positive constants δ_0 and C, independent of δ, such that for all $\delta \leqslant \delta_0$, if $\{U_i\}_{1\leq i\leq\nu}$ is a δ-interacting bubble family and ϕ solves the equation*

$$\begin{cases} \Delta\phi + p\sigma^{p-1}\phi = h, \\ \int U_i^{p-1}\phi Z_i^a = 0, \quad i = 1, \cdots, \nu;\ a = 1, \cdots, n+1, \end{cases} \tag{5.2}$$

for some h with $\|h\|_{**} < \infty$. Then

$$\|\phi\|_* \leq C\|h\|_{**}. \tag{5.3}$$

Proof. We use contradiction arguments to prove (5.3). Suppose that there exist a sequence of bubbles $\left\{U_i^{(k)} = U[z_i^{(k)}, \lambda_i^{(k)}] : i \in I\right\}_{k=1}^{\infty}$ with interaction no more than $1/k$, and a sequence of functions $h_k(x)$ and $\phi_k(x)$ satisfying (5.2) such that $\|\phi_k\|_* \geq k\|h_k\|_{**}$. Replacing $\phi_k(x)$ by $\phi_k(x)/\|\phi_k\|_*$ and $h_k(x)$ by $h_k(x)/\|\phi_k\|_*$, we can assume $\|\phi_k\|_* = 1$ and $\|h_k\|_{**} \leq 1/k \to 0$ as $k \to \infty$.

Going to a subsequence if necessary, we assume that $\{U_i^{(k)} : i \in I\}_{k=1}^{\infty}$ satisfies (4.20), (4.21), and (4.22). Thus Lemma 4.7 holds for such sequence. We can associate a sequence of weight functions $V(x)$ and $W(x)$ to this sequence. We shall prove that for this sequence there holds $\phi_k(x) < W(x)$ on $\mathbb{R}^n$ when k is large enough. This contradicts the fact $\|\phi_k\|_* = 1$ for any k. We first assume $n \geq 7$. We divide $\mathbb{R}^n$ into the following three regions: core, neck, and exterior.

Case 1: Exterior region $\mathbb{R}^n \setminus \Omega^{(k)}$. Applying Proposition 4.3, there exists $C(n, \nu)$ such that for any $M \geq 1$ the following holds:

$$\frac{|\phi_k(x)|}{W(x)} \leq C(n, \nu)\left(\|h_k\|_{**} + M^{3n}\frac{\overline{W}(x)}{W(x)} + M^4R^{-2} + M^{-1}\right).$$

Now choose $M = M(n, \nu)$ sufficiently large such that $C(n, \nu)M^{-1} < (100\nu)^{-1}$. Note that $\bar{w}_i^{\text{in}}$ and $\bar{w}_i^{\text{out}}$ decay faster than w_i^{in} and w_i^{out}. More precisely, on $\mathbb{R}^n\setminus\Omega^{(k)}$,

$$\bar{w}_i^{\text{in}} \leq 2L^{-2}w_i^{\text{in}}, \quad \bar{w}_i^{\text{out}} \leq 2L^{-2}(\log L)\, w_i^{\text{out}}.$$

Taking $L = L(n, \nu, C^*)$ sufficiently large such that

$$2C(n, \nu)M^{3n}L^{-2}(1 + \log L) \leq (100\nu)^{-1}, \tag{5.4}$$

we shall have $C(n,\nu)M^{3n}\overline{W}(x) < \frac{1}{100\nu}W(x)$ for $x \notin \Omega^{(k)}$. Since we assume $\|h_k\|_{**} \to 0$, then as $k \to \infty$ we have

$$\frac{|\phi_k(x)|}{W(x)} \le o(1) + \frac{1}{50\nu}, \quad \text{ on } \mathbb{R}^n \setminus \Omega^{(k)}.$$

Case 2: Core region $\cup_{i\in I}\Omega_i^{(k)}$. We shall prove $|\phi_k|(x) = o(1)W(x)$ as $k \to \infty$ on this region.

Suppose not. Then there exist $\epsilon_* > 0$ and a sequence $x_k \in \cup_{i\in I}\Omega_i^{(k)}$ such that $\phi_k(x_k) > \epsilon_* W(x_k)$. Going to a subsequence if necessary, we assume that $x_k \in \Omega_{i_0}^{(k)}$ for some fixed $i_0 \in I$ when k is large enough. As explained before, $\Omega_{i_0}^{(k)}$ is the domain where $U_{i_0}^{(k)}$ has domination. One can use a blow-up argument to reach a contradiction. Define

$$\begin{cases} \tilde{\phi}_k(y) := W^{-1}(x_k)\phi_k(y/\lambda_{i_0}^{(k)} + z_{i_0}^{(k)}) \quad \textit{with } y = \lambda_{i_0}^{(k)}(x - z_{i_0}^{(k)}), \\ \tilde{h}_k(y) := (\lambda_{i_0}^{(k)})^{-2}W^{-1}(x_k)h_k(y/\lambda_{i_0}^{(k)} + z_{i_0}^{(k)}), \\ \tilde{\sigma}_k(y) := U[0,1](y) + \sum_{j\in I\setminus\{i_0\}} U[z_{i_0 j}^{(k)}, \lambda_j^{(k)}/\lambda_{i_0}^{(k)}](y). \end{cases} \tag{5.5}$$

Then $\tilde{\phi}_k$ satisfies

$$\begin{cases} \Delta\tilde{\phi}_k(y) + p\tilde{\sigma}_k^{p-1}(y)\tilde{\phi}_k(y) = \tilde{h}_k(y) & \textit{in } \mathbb{R}^n, \\ \int U^{p-1}Z^a\tilde{\phi}_k dy = 0, & 1 \le a \le n+1. \end{cases}$$

Here $U = U[0,1](y)$ and $Z^a = Z^a(y) = Z_i^a(y)$ defined in (2.1) for $z = 0$ and $\lambda_i = 1$. Denote $\bar{z}_j = \lim_{k\to\infty} z_{i_0 j}^{(k)}$ and define

$$\mathcal{K}_l := \{y : |y| \le l, \, |y - \bar{z}_j| \ge 1/l, \, \forall\, j \in S(i_0)\}. \tag{5.6}$$

Suppose $l \ge 2\max\{L, \epsilon_1^{-1}\}$, it is easy to see that

$$\lambda_{i_0}^{(k)}(\Omega_{i_0}^{(k)} - z_{i_0}^{(k)}) = \{y : y = \lambda_{i_0}^{(k)}(x - z_{i_0}^{(k)}), x \in \Omega_{i_0}^{(k)}\} \subset\subset \mathcal{K}_l$$

when k is large enough.

Claim 1. *In each $\mathcal{K}_l$, it holds that, as $k \to \infty$,*

$$\tilde{\sigma}_k(y) \to \; U[0,1](y), \quad |\tilde{h}_k|(y) \to \; 0,$$

uniformly $y \in \mathcal{K}_l$. Moreover, we have

$$|\tilde{\phi}_k|(y) \lesssim \sum_{j\in S(i_0)} \left(\frac{L}{|y - \bar{z}_j|}\right)^{n-4} + L^2, \quad \forall\, y \in \mathcal{K}_l. \tag{5.7}$$

We postpone the proof of Claim 1 to the end of this subsection and finish the blow-up argument in Case 2. By the standard elliptic regularity theorem, the Claim 1 shows a subsequence of $\tilde{\phi}_k$ uniformly converges in each $\mathcal{K}_l$. Furthermore, by the diagonal argument, let $l \to \infty$, we have a subsequence of $\tilde{\phi}_k$ converges to $\tilde{\phi}$ locally uniformly on $\mathbb{R}^n \setminus \{\bar{z}_j : j \in S(i_0)\}$ which satisfies that

$$\begin{cases} \Delta\tilde{\phi} + pU^{p-1}\tilde{\phi} = 0, & \textit{in } \mathbb{R}^n \setminus \{\bar{z}_j : j \in S(i_0)\}, \\ |\tilde{\phi}|(y) \lesssim \sum_{j\in S(i_0)} \left(\frac{L}{|y-\bar{z}_j|}\right)^{n-4} + L^2, & \textit{in } \mathbb{R}^n \setminus \{\bar{z}_j : j \in S(i_0)\}, \\ \int U^{p-1}Z^a\tilde{\phi}dy = 0, & 1 \le a \le n+1. \end{cases}$$

Notice that each singular $\bar{z}_j$ is removable since the singularity near $\bar{z}_j$ is strictly "less" than that of Green's function. Therefore $\tilde{\phi}$ satisfies the equation on the whole $\mathbb{R}^n$. By the orthogonality condition and non-degeneracy of Aubin-Talenti bubbles, we get $\tilde{\phi} \equiv 0$. However, since $|\xi_k| := |\lambda_{i_0}^{(k)}(x_k - z_{i_0}^{(k)})| \le L$ and $|\xi_k - z_{i_0 j}^{(k)}| \ge \epsilon_1$, going to a subsequence if necessary, then $\lim_{k\to\infty} \xi_k = \xi_\infty \notin \{\bar{z}_j : j \in S(i_0)\}$ and consequently $\tilde{\phi}(\xi_\infty) \ge \epsilon_* > 0$. This is a contradiction.

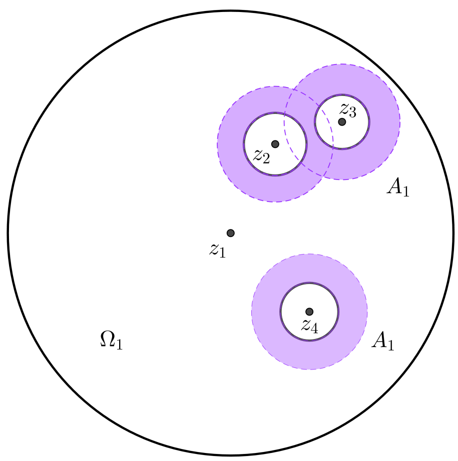

FIGURE 3. Illustration for the blow-up regions of a simple bubble configuration. The solid circles denote $\{y_i = L\}$ for $i = 1, \cdots, 4$. The dashed circles mean $\{|y_1 - z_{1j}| = \epsilon_1\}$. The shaded regions constitute A_1.

Case 3: Neck region $\Omega^{(k)} \setminus \left(\cup_{i\in I}\Omega_i^{(k)}\right)$. In this case, we can not use a blow-up argument directly, because we do not know which bubble will dominate others. Fortunately, this set is a *narrow* domain (see the following $A_i^{(k)}$ in y_i-coordinates), we will construct some barrier functions and use the maximum principle to prove $|\phi_k(x)| \le W(x)/2$ as $k \to \infty$.

Note that $\Omega^{(k)} \setminus \left(\cup_{i\in I}\Omega_i^{(k)}\right) = \cup_{i\in I} A_i^{(k)}$ where $A_i^{(k)}$ is defined by

$$A_i^{(k)} = \cup_{j\in S(i)}\{x : |y_i^{(k)} - z_{ij}^{(k)}| \le \epsilon_1\} \setminus \cup_{j\in S(i)}\{x : |y_j^{(k)}| < L\}. \tag{5.8}$$

One can interpret $A_i^{(k)}$ as the union of the neck regions of $U_j^{(k)}$ with $j \in S(i)$. See Figure 3 for $A_i^{(k)}$ in a simple case. We have the following observations.

(1) $x \in \partial A_i^{(k)}$ implies that either x satisfies $|y_i^{(k)} - z_{ij}^{(k)}| = \epsilon_1$ or $|y_j^{(k)}| = L$ for some $j \in S(i)$. In the first case, $x \in \partial \Omega_i^{(k)}$, and in the second case, $x \in \partial \Omega_j^{(k)}$.

(2) $A_i^{(k)}$ are disjoint from $A_j^{(k)}$ for any $j \neq i$.

To construct a barrier function, we need to modify the weight functions W and V in two steps. First, we need to smooth functions W and V to derive some differential inequality because they are piece-wise defined. We introduce the following function

$$F(a, b) = \frac{a+b}{2} - \sqrt{\left(\frac{a-b}{2}\right)^2 + \frac{1}{4}ab}. \tag{5.9}$$

It can be considered as a smooth $1/4$-approximation of $\min\{a, b\}$. It satisfies that

$$\min\{a, b\}/2 \leq F(a, b) \leq \min\{a, b\}. \tag{5.10}$$

Moreover, it is symmetric, 1-homogeneous, and concave (see [33]).

Second, on $A_i^{(k)}$, w_i^{in} is not a good candidate for barrier because it does not satisfy $\Delta_x w_i^{\text{in}} + pU_i^{p-1} w_i^{\text{in}} \leq -v_i^{\text{in}}$. We need to replace the weight w_i^{in} by $\tilde{w}_i^{\text{in}}$ which is defined by

$$\tilde{w}_i^{\text{in}}(x) = \sum_{j \in S(i)} \lambda_i^{\frac{n-2}{2}} R^{2-n} \left(1 + |z_{ij}^{(k)}|^2 + \epsilon_1^{-2} |y_i^{(k)} - z_{ij}^{(k)}|^2\right)^{-1}. \tag{5.11}$$

Claim 2. *On $A_i^{(k)}$, it holds that*

$$\frac{1}{3} w_i^{in} \leq \tilde{w}_i^{in} \leq 3\nu w_i^{in}, \tag{5.12}$$

$$\Delta_x \tilde{w}_i^{in}(x) \leq -\frac{n-4}{8\epsilon_1^2} v_i^{in}(x). \tag{5.13}$$

Note that $\tilde{w}_i^{\text{in}}$ is comparable to w_i^{in} but has negatively large second-order derivatives. This is exactly where we need to use the narrow property of $A_i^{(k)}$.

Now let us work on a particular $A_{i_0}^{(k)}$. Define the barrier function $\tilde{W}$ by

$$\tilde{W}(x) = \sum_{j \in I \setminus \{i_0\}} \lambda_j^{\frac{n-2}{2}} F\left(\frac{R^{2-n}}{\langle y_j \rangle^2}, \frac{R^{-4}}{\langle y_j \rangle^{n-4}}\right) + \tilde{w}_{i_0}^{\text{in}}. \tag{5.14}$$

Using $w_j^{\text{in}} + w_j^{\text{out}} = (1 + o(R^{-1}))\lambda_j^{\frac{n-2}{2}} \min\{R^{4-n} \langle y_j \rangle^{-2}, R^{-4} \langle y_j \rangle^{2-n}\}$ and the above claim, we have $\tilde{W}(x) \approx W(x)$. Moreover, combining with (4.29), (5.10) and (5.12), we have

$$\frac{1}{4} W(x) \leq \tilde{W}(x) \leq 3\nu W(x) \quad \text{on } A_{i_0}^{(k)}. \tag{5.15}$$

More importantly, we have the following estimate of $\Delta_x \tilde{W}(x)$.

Claim 3. *On $A_{i_0}^{(k)}$, we have*

$$\Delta_x \tilde{W}(x) \leq -(n-4)(1+o(1)) \sum_{j\in S(i_0)} (v_j^{in}(x) + v_j^{out}(x)) - \frac{n-4}{8\epsilon_1^2} v_{i_0}^{in}(x).$$

We also postpone the proofs of Claim 2 and Claim 3 to the end of this subsection.

We will show that $\tilde{W}(x)$ is a super-solution to our problem in the region $A_{i_0}^{(k)}$. That is, for $\sigma_k = \sum_{i=1}^{\nu} U_i^{(k)}$,

$$\Delta_x \tilde{W} + p\sigma_k^{p-1}\tilde{W} \leq -V, \quad on\ A_{i_0}^{(k)}. \tag{5.16}$$

In the region $A_{i_0}^{(k)} \subset \{|y_{i_0}^{(k)}| \leq L\}$, by (4.28), $\sum_{j\in I\setminus S(i_0)} U_j = (1+o(1))U_{i_0}$ for k large. Since $p \in (1,2]$, we have $(\sum_{i=1}^{\nu} a_i)^{p-1} \leq \sum_{i=1}^{\nu} a_i^{p-1}$ for any $a_i \geq 0$. Thus, for k large,

$$\sigma_k^{p-1} \leq \sum_{i=1}^{\nu} U_i^{p-1} \leq \sum_{j\in S(i_0)} U_j^{p-1} + (1+o(1))U_{i_0}^{p-1}.$$

Consequently, thanks to (4.29) and (5.15), we have

$$\begin{aligned}\frac{1}{3\nu}\sigma_k^{p-1}\tilde{W} \leq & \sum_{i,j\in S(i_0)} U_i^{p-1}\left(w_j^{\text{in}} + w_j^{\text{out}}\right) + \sum_{j\in S(i_0)} U_j^{p-1}\tilde{w}_{i_0}^{\text{in}} \\ & + (1+o(1))\left(U_{i_0}^{p-1}\sum_{j\in S(i_0)}\left(w_j^{\text{in}} + w_j^{\text{out}}\right) + U_{i_0}^{p-1}\tilde{w}_{i_0}^{\text{in}}\right).\end{aligned} \tag{5.17}$$

For the first term on the RHS of (5.17), if $\lambda_i \geq \lambda_j$, then we apply (4.1) and (4.2) in Lemma 4.1. If $\lambda_i < \lambda_j$, then we apply (4.3) and (4.7), since $|y_i| \geq L > 2C^*$ for $i \in S(i_0)$ and $x \in A_{i_0}^{(k)}$. Combining these two results, we have

$$U_i^{p-1}\left(w_j^{\text{in}} + w_j^{\text{out}}\right) \lesssim (L^{-2} + o(1))[v_j^{\text{in}} + v_j^{\text{out}} + v_i^{\text{in}} + v_i^{\text{out}}].$$

For the second term on the RHS of (5.17), using (4.1), for $j \in S(i_0)$, we have

$$U_j^{p-1}\tilde{w}_{i_0}^{\text{in}} \approx U_j^{p-1} w_{i_0}^{\text{in}} = o(1)[v_j^{\text{in}} + v_j^{\text{out}} + v_{i_0}^{\text{in}}].$$

For the third term, using (4.3) and (4.8), for $j \in S(i_0)$, we have

$$U_{i_0}^{p-1}\left(w_j^{\text{in}} + w_j^{\text{out}}\right) \lesssim o(1)v_j^{\text{in}} + (C^*)^{\frac{2(n-6)}{n-2}}[\varepsilon^{-\frac{n-4}{2}} v_{i_0}^{\text{in}} + \varepsilon v_j^{\text{out}}]$$

for any $\varepsilon \in (0,1)$. For the fourth term, using (5.12), we have

$$U_{i_0}^{p-1}\tilde{w}_{i_0}^{\text{in}} \approx \lambda_{i_0}^2 \langle y_i\rangle^{-4}\tilde{w}_{i_0}^{\text{in}} \lesssim \lambda_{i_0}^2\langle y_i\rangle^{-4} w_{i_0}^{\text{in}} \lesssim v_{i_0}^{\text{in}}.$$

Plugging in the above four inequalities into (5.17), we have

$$\sigma_k^{p-1}\tilde{W} \lesssim ((C^*)^{\frac{2(n-6)}{n-2}}\varepsilon + \frac{1}{L^2} + o(1)) \sum_{j\in S(i_0)} (v_j^{\text{in}} + v_j^{\text{out}}) + \left(\frac{(C^*)^2}{\varepsilon^{\frac{n-4}{2}}} + o(1)\right) v_{i_0}^{\text{in}}.$$

Combining this with Claim 3, we have

$$\Delta_x \tilde{W} + p\sigma_k^{p-1}\tilde{W}$$

$$\leq [-(n-4)+C((C^*)^{\frac{2(n-6)}{n-2}}\varepsilon+L^{-2})+o(1)]\sum_{j\in S(i_0)}(v_j^{\text{in}}+v_j^{\text{out}})$$
$$+\left(C(C^*)^2\varepsilon^{-\frac{n-4}{2}}+o(1)-\frac{n-4}{8\epsilon_1^2}\right)v_{i_0}^{\text{in}}$$

for some $C = C(n,\nu)$. Then we choose $L = L(n,\nu,C^*)$ large enough, $\varepsilon = \varepsilon(n,\nu,C^*)$ small, and k large enough such that

$$[-(n-4)+C((C^*)^{\frac{2(n-6)}{n-2}}\varepsilon+L^{-2})+o(1)]\leq -1. \tag{5.18}$$

Then we choose $\epsilon_1 = \epsilon_1(n,\nu,C^*)$ small such that

$$\left(C(C^*)^2\varepsilon^{-\frac{n-4}{2}}+o(1)-\frac{n-4}{8\epsilon_1^2}\right)\leq -1. \tag{5.19}$$

This finishes the proof of (5.16). In the following, we will use the maximum principle to prove $|\phi_k(x)|\leq\frac{1}{6\nu}\tilde{W}(x)$ on $A_{i_0}^{(k)}$.

Denote $f_\pm(x)=\frac{1}{6\nu}\tilde{W}\pm\phi_k(x)$. Recall that $\Delta_x\phi_k+p\sigma_k^{p-1}=h_k$. Then, for k large enough, we have

$$\Delta_x f_\pm+p\sigma_k^{p-1}f_\pm\leq-\frac{1}{6\nu}V\pm\|h_k\|_{**}V\leq 0 \quad \textit{on } A_{i_0}^{(k)}. \tag{5.20}$$

By the observation after (5.8), we see that $\partial A_{i_0}^{(k)}\subset\partial\Omega_{i_0}^{(k)}\cup\left(\cup_{j\in S(i_0)}\partial\Omega_j^{(k)}\right)$. It follows from the conclusion of Case 2 and (5.15) that $f_\pm(x)\geq 0$ on $\partial A_{i_0}^{(k)}$ for k large. Let us show $f_\pm(x)\geq 0$ in $A_{i_0}^{(k)}$ by the maximum principle.

Consider $g(x)=f_\pm(x)/\tilde{W}(x)$. Then

$$\Delta_x g(x)=-2\nabla g(x)\cdot\frac{\nabla\tilde{W}(x)}{\tilde{W}(x)}+\frac{1}{\tilde{W}(x)}\left(\Delta_x f_\pm-\frac{\Delta_x\tilde{W}(x)}{\tilde{W}(x)}f_\pm(x)\right).$$

Suppose $g(x)$ takes its minimum in $A_{i_0}^{(k)}$ at x_0 and $g(x_0)<0$. Then $\nabla g(x_0)=0$ and $\Delta_x g(x_0)\geq 0$. Then using $f_\pm(x_0)<0$ and $-\Delta\tilde{W}\geq p\sigma_k^{p-1}\tilde{W}+V$ and (5.20), we have

$$\Delta_x f_\pm(x_0)-\frac{\Delta_x\tilde{W}}{\tilde{W}}f_\pm(x_0)\leq\Delta_x f_\pm(x_0)+\frac{p\sigma_k^{p-1}\tilde{W}+V}{\tilde{W}}f_\pm(x_0)<0,$$

which is a contradiction. This shows that $g(x)\geq 0$ for $x\in A_{i_0}^{(k)}$. Thanks to (5.15), we obtain

$$|\phi_k(x)|\leq\frac{1}{6\nu}\tilde{W}(x)\leq\frac{1}{2}W(x) \quad \textit{on } A_{i_0}^{(k)},$$

when k is large enough. This finishes the proof of Case 3.

Combining all three cases above, we always have $|\phi_k(x)|\leq W(x)/2$ for k large. It is a contradiction because $\|\phi_k\|_*=1$. Thus we prove (5.3).

To complete the whole proof, it suffices to prove the three claims we have used.

Proof of Claim 1. By (4.28) in Lemma 4.7,

$$\tilde{\sigma}_k(y) = (1+o(1))U[0,1](y) + \sum_{j\in S(i_0)} U[z_{i_0 j}, \lambda_j/\lambda_{i_0}](y), \quad y \in \mathcal{K}_l.$$

We shall show the sum of the right-hand side is also $o(1)U[0,1](y)$ on $\mathcal{K}_l$. Invoking the definition of $\mathcal{K}_l$ in (5.6), we see that $|y - z_{i_0 j}| \geq |y_i - \bar{z}_j|/2 \geq 1/2l$ when k is large enough. On the other hand, $j \in S(i_0)$ implies that $\lambda_j/\lambda_{i_0} \to \infty$ as $k \to \infty$. Thus,

$$\frac{U[z_{i_0 j}, \lambda_j/\lambda_{i_0}]}{U[0,1]} = \frac{(\lambda_j/\lambda_{i_0})^{\frac{n-2}{2}}(1+|y|^2)^{\frac{n-2}{2}}}{(1+(\lambda_j/\lambda_{i_0})^2|y-\bar{z}_j|^2)^{\frac{n-2}{2}}} \leq \left(\frac{\lambda_j}{\lambda_{i_0}}\right)^{\frac{2-n}{2}} (2l)^{2n-4} \to 0$$

as $k \to \infty$. Hence

$$\tilde{\sigma}_k(y) = (1+o(1))U[0,1](y), \quad y \in \mathcal{K}_l.$$

By Lemma 4.7, for k large enough, we have

$$\begin{aligned} \frac{|\phi_k(x)|}{W(x_k)} &\lesssim \frac{w_{i_0}^{\text{in}}(x) + \sum_{j\in S(i_0)}(w_j^{\text{in}} + w_j^{\text{out}})(x)}{w_{i_0}^{\text{in}}(x_k) + \sum_{j\in S(i_0)}(w_j^{\text{in}} + w_j^{\text{out}})(x_k)}, \\ \frac{|h_k(x)|}{\lambda_{i_0}^2 W(x_k)} &\lesssim \frac{v_{i_0}^{\text{in}}(x) + \sum_{j\in S(i_0)}(v_j^{\text{in}} + v_j^{\text{out}})(x)}{\lambda_{i_0}^2\left[w_{i_0}^{\text{in}}(x_k) + \sum_{j\in S(i_0)}(w_j^{\text{in}} + w_j^{\text{out}})(x_k)\right]}\|h_k\|_{**}, \end{aligned}$$

when $x = y/\lambda_{i_0} + z_{i_0}$ and $y \in \mathcal{K}_l$. To prove the rest of the claim, we need to use a simple inequality

$$\frac{\sum_i a_i}{\sum_i b_i} \leq \max_i\{\frac{a_i}{b_i}\}, \tag{5.21}$$

which holds for any positive a_i, b_i. It suffices to establish the following estimates for each ratio on $\mathcal{K}_l$.

• For the inner weight functions of U_{i_0}, let $y = \lambda_{i_0}(x - z_{i_0})$ and $\xi_k = \lambda_{i_0}(x_k - z_{i_0})$, we have

$$\begin{aligned} \frac{w_{i_0}^{\text{in}}(x)}{w_{i_0}^{\text{in}}(x_k)} &= \frac{1+|\xi_k|^2}{1+|y|^2} \leq 1 + L^2, \\ \frac{v_{i_0}^{\text{in}}(x)}{\lambda_{i_0}^2 w_{i_0}^{\text{in}}(x_k)} &= \frac{1+|\xi_k|^2}{(1+|y|^2)^2} \leq 1 + L^2. \end{aligned}$$

• If $i_0 \prec j$, then $\lambda_{i_0}/\lambda_j \to 0$ as $k \to \infty$. In this case, $\mathcal{K}_l$ is contained in the support of w_j^{out} when k is large enough. Using $|z_{i_0 j}| \leq C^*$ and $|\xi_k| \leq L$, we have

$$\begin{aligned} \frac{w_j^{\text{out}}(x)}{w_j^{\text{out}}(x_k)} &= \left(\frac{(\lambda_{i_0}/\lambda_j)^2 + |\xi_k - z_{i_0 j}|^2}{(\lambda_{i_0}/\lambda_j)^2 + |y - z_{i_0 j}|^2}\right)^{\frac{n-4}{2}} \lesssim \left(\frac{L}{|y-\bar{z}_j|}\right)^{n-4}, \\ \frac{v_j^{\text{out}}(x)}{\lambda_{i_0}^2 w_j^{\text{out}}(x_k)} &= \frac{((\lambda_{i_0}/\lambda_j)^2 + |\xi_k - z_{i_0 j}|^2)^{\frac{n-4}{2}}}{((\lambda_{i_0}/\lambda_j)^2 + |y - z_{i_0 j}|^2)^{\frac{n-2}{2}}} \lesssim l^{n-2}L^{n-4}. \end{aligned}$$

Combining the above two cases and using (5.21), we have

$$|\tilde{\phi}_k(y)| \lesssim L^2 + \left(\frac{L}{|y-\bar{z}_j|}\right)^{n-4},$$
$$|\tilde{h}_k(y)| \lesssim \|h_k\|_{**}\left(L^2 + l^{n-2}L^{n-4}\right) \to 0 \quad \text{as } k \to \infty.$$

From this, the assertion follows.

□

Proof of Claim 2. • To prove (5.12), using (5.11), we obtain that

$$\frac{\tilde{w}_i^{\text{in}}}{w_i^{\text{in}}} = \sum_{j\in S(i)} \frac{1+|y_i|^2}{1+|z_{ij}|^2+\epsilon_1^{-2}|y_i-z_{ij}|^2}. \tag{5.22}$$

Here we have omitted the superscript (k) for y and z.

Pick any $j \in S(i)$. On the set $\{|y_i - z_{ij}| \le \epsilon_1\}$, it holds that

$$\begin{aligned} \frac{1+|y_i|^2}{1+|z_{ij}|^2+\epsilon_1^{-2}|y_i-z_{ij}|^2} &\le \frac{1+2\epsilon_1^2+2|z_{ij}|^2}{1+|z_{ij}|^2} \le 2,\\ \frac{1+|y_i|^2}{1+|z_{ij}|^2+\epsilon_1^{-2}|y_i-z_{ij}|^2} &\ge \frac{1-2\epsilon_1^2+2|z_{ij}|^2}{2+|z_{ij}|^2} \ge \frac{1}{3}. \end{aligned} \tag{5.23}$$

Let us consider any other $l \in S(i)$. There are the following two cases:

Case 1: $|z_{ij} - z_{il}| \le 2\epsilon_1$. In this case, for $|y_i - z_{ij}| \le \epsilon_1$ we have

$$\frac{1+|z_{ij}|^2+\epsilon_1^{-2}|y_i-z_{ij}|^2}{1+|z_{il}|^2+\epsilon_1^{-2}|y_i-z_{il}|^2} \le \frac{2+8\epsilon_1^2+2|z_{il}|^2}{1+|z_{il}|^2} \le 3. \tag{5.24}$$

Case 2: $|z_{ij}-z_{il}| \ge 2\epsilon_1$. Note $|y_i-z_{il}| \ge |z_{ij}-z_{il}|-|y_i-z_{ij}| \ge |z_{ij}-z_{il}|/2$. Then $|z_{il}|^2+\epsilon_1^{-2}|y_i-z_{il}|^2 \ge |z_{il}|^2+|z_{ij}-z_{il}|^2 \ge |z_{ij}|^2/2$. We have

$$\frac{1+|z_{ij}|^2+\epsilon_1^{-2}|y_i-z_{ij}|^2}{1+|z_{il}|^2+\epsilon_1^{-2}|y_i-z_{il}|^2} \le \frac{2+|z_{ij}|^2}{1+|z_{ij}|^2/2} \le 2. \tag{5.25}$$

Plugging in (5.23)-(5.25) to (5.22), we have

$$\frac{1}{3} \le \frac{\tilde{w}_i^{\text{in}}}{w_i^{\text{in}}} \le 2 + \sum_{l\in S(i), l\neq j} 3 \le 3\nu, \quad \text{on } \{|y_i - z_{ij}| \le \epsilon_1\}.$$

Since $j \in S(i)$ is arbitrary, then the above inequalities hold on $A_i^{(k)}$.

• Towards proving (5.13), we denote $f_j(x) = (1+|z_{ij}|^2+\varepsilon_1^{-2}|y_i-z_{ij}|^2)^{-1}$ temporarily. Recall that $y_i = \lambda_i(x-z_i)$. By direct calculation, we have

$$\Delta_x f_j = \lambda_i^2[-2(n-4)\epsilon_1^{-2}f_j^2 - 8\epsilon_1^{-2}(1+|z_{ij}|^2)f_j^3].$$

Since $\tilde{w}_i^{\text{in}}(x) = \sum_{l\in S(i)} \lambda_l^{\frac{n-2}{2}} R^{2-n} f_l$, then on the set $\{|y_i - z_{ij}| \le \epsilon_1\}$, we have

$$\Delta_x \tilde{w}_i^{\text{in}}(x) \le \lambda_i^{\frac{n-2}{2}} R^{2-n}\Delta_x f_j \le -2(n-4)\lambda_i^{\frac{n+2}{2}} R^{2-n} f_j^2 \le -\frac{n-4}{8\epsilon_1^2}\tilde{v}_i^{\text{in}}.$$

We use (5.23) to get $f_j \ge \frac{1}{3}\langle y_i\rangle^{-2}$ in the last step.

By the arbitrariness of j, we can prove Claim 2 on $A_i^{(k)}$. □

Proof of Claim 3. It is easy to verify that F is a smooth function except $a = b = 0$. Also, F is homogeneous and increasing on a, b with $0 \le \frac{\partial F}{\partial a} \le 1$ and $0 \le \frac{\partial F}{\partial b} \le 1$. Moreover, F is concave on a, b (see [33]). Denote $a_j = R^{2-n}/\langle y_j\rangle^2$ and $b_j = R^{-4}/\langle y_j\rangle^{n-4}$. Therefore,

$$\Delta_x \tilde{W} \le \sum \lambda_j^{\frac{n-2}{2}} \left(\frac{\partial F}{\partial a}(a_j, b_j)\Delta_x a_j + \frac{\partial F}{\partial b}(a_j, b_j)\Delta_x b_j \right) + \Delta_x \tilde{w}_{i_0}^{\text{in}}(x).$$

We have

$$\begin{aligned} \Delta_x a_j &= \Delta_x \frac{R^{2-n}}{\langle y_j\rangle^2} = -2(n-4)\frac{\lambda_j^2 R^{2-n}}{\langle y_j\rangle^4} - \frac{8\lambda_j^2 R^{2-n}}{\langle y_j\rangle^6} \le -2(n-4)\frac{\lambda_j^2 a_j}{\langle y_j\rangle^2}, \\ \Delta_x b_j &= -2(n-4)\frac{\lambda_j^2 R^{-4}}{\langle y_j\rangle^{n-2}} - (n-2)(n-4)\frac{\lambda_j^2 R^{-4}}{\langle y_j\rangle^n} \le -2(n-4)\frac{\lambda_j^2 b_j}{\langle y_j\rangle^2}. \end{aligned}$$

Since F is homogeneous degree 1, we get $a\frac{\partial F}{\partial a} + b\frac{\partial F}{\partial b} = F(a, b)$. Moreover, it also implies $\frac{\partial F}{\partial a}, \frac{\partial F}{\partial b}$ are homogeneous 0, that is $\frac{\partial F}{\partial a}(a_j, b_j) = \frac{\partial F}{\partial a}(a_j/\langle y_j\rangle^2, b_j/\langle y_j\rangle^2)$. Applying the above facts to $\Delta_x \tilde{W}$, we get

$$\begin{aligned} \Delta_x \tilde{W}(x) &\le -2(n-4) \sum_{j\in S(i_0)} \lambda_j^{\frac{n+2}{2}} F\left(\frac{a_j}{\langle y_j\rangle^2}, \frac{b_j}{\langle y_j\rangle^2} \right) + \Delta_x \tilde{w}_{i_0}^{\text{in}}(x) \\ &\le -(n-4) \sum_{j\in S(i_0)} \lambda_j^{\frac{n+2}{2}} \min\{R^{2-n}\langle y_j\rangle^{-4}, R^{-4}\langle y_j\rangle^{n-4}\} - \frac{n-4}{8\epsilon_1^2} v_{i_0}^{\text{in}} \\ &\le -(n-4)(1+o(1)) \sum_{j\in S(i_0)} [v_j^{\text{in}} + v_j^{\text{out}}] - \frac{n-4}{8\epsilon_1^2} v_{i_0}^{\text{in}}. \end{aligned}$$

We have used (5.10) and (5.13) in the second step. The proof of Claim 3 is complete. □

The proof of Lemma 5.1 is complete when the dimension $n \ge 7$.

Replacing $w_i^{\text{in(out)}}$ by $\hat{w}_i^{\text{in(out)}}$ and $v_i^{\text{in(out)}}$ by $\hat{v}_i^{\text{in(out)}}$, one can prove Lemma 5.1 when the dimension $n = 6$ by following the above one verbatim. We point out some necessary modifications.

When $n = 6$, on $\mathbb{R}^n \setminus \Omega^{(k)}$, we have

$$\bar{w}_i^{\text{in}} \le 2L^{-1}(\log L)\hat{w}_i^{\text{in}}, \quad \bar{w}_i^{\text{out}} \le 2L^{-1}(\log L)\hat{w}_i^{\text{out}}.$$

Therefore, Case 1 can be established by choosing L large enough.

For Case 2, in the statement of claim 1, $\tilde{\phi}_k$ should be modified to

$$|\tilde{\phi}_k|(y) \lesssim \sum_{j\in S(i_0)} \left(\frac{L}{|y - \bar{z}_j|} \right)^3 + L^2, \quad \forall\, y \in \mathcal{K}_l. \tag{5.26}$$

Using this upper bound, the other parts of Case 2 still hold in $n = 6$.

For Case 3, the barrier function is

$$\tilde{W}(x) = \sum_{j\in I\setminus\{i_0\}} \lambda_j^2 F\left(\frac{R^{-4}}{\langle y_j\rangle^2}, \frac{R^{-2}}{\langle y_j\rangle^3}\right) + \tilde{w}_{i_0}^{\text{in}}, \tag{5.27}$$

where $\tilde{w}_{i_0}^{\text{in}}$ is again (5.11). Since $\Delta_x\langle y_j\rangle^{-3} \le -3\lambda_i^2\langle y_j\rangle^{-5}$, Claim 3 must be modified to

$$\Delta_x\tilde{W}(x) \le -\frac{3}{2}(1+o(1))\sum_{j\in S(i_0)}(\hat{v}_j^{\text{in}}+\hat{v}_j^{\text{out}}) - \frac{1}{4\epsilon_1^2}v_{i_0}^{\text{in}}(x).$$

To estimate $\sigma_k^{p-1}\tilde{W}$, one should use Lemma 4.2. The other parts of the proof still hold. $\square$

5.2. **Existence and point-wise estimate.** In this subsection, we shall use the a priori estimate we have derived to prove the existence of ρ_0.

First, we estimate the coefficients c_b^j in (3.1). See the definition of $\zeta_n(x)$ in (1.8).

Lemma 5.2. *Suppose σ is the sum of a family of δ-interacting bubbles. If ϕ, h and c_b^j satisfy* (3.1)*, then*

$$|c_b^j| \lesssim Q\|h\|_{**} + \zeta_n(Q)^2\|\phi\|_*, \quad 1\le j\le\nu,\ 1\le b\le n+1.$$

Proof. Multiplying (3.1) by Z_j^b and integrating we get

$$\int p\sigma^{p-1}\phi Z_j^b = \int hZ_j^b + \sum_{i,a}\int c_a^i U_i^{p-1}Z_i^a Z_j^b, \tag{5.28}$$

for any $1\le j\le\nu$, $1\le b\le n+1$. Here we used the orthogonal condition in (3.1).

By the Lemma A.5, for $a,b\le n+1$, there exist some constants $\gamma^b>0$ such that

$$\sum_{i,a}\int c_a^i U_i^{p-1}Z_i^a Z_j^b = c_b^j\gamma^b + \sum_{i\ne j}\sum_{a=1}^{n+1} c_a^i O(q_{ij}).$$

Plugging in the above estimates to (5.28), we see that $\{c_b^j\}$ satisfies the linear system

$$c_b^j\gamma^b + \sum_{i\ne j}\sum_{a=1}^{n+1} c_a^i O(q_{ij}) = \int p\sigma^{p-1}\phi Z_j^b - \int hZ_j^b.$$

Denote $\vec{c}^{\,j} := (c_1^j,\cdots,c_{n+1}^j)\in\mathbb{R}^{n+1}$ for $j=1,\cdots,\nu$. We concatenate these vectors to $\vec{c} = (\vec{c}^{\,1},\cdots,\vec{c}^{\,\nu})\in\mathbb{R}^{\nu(n+1)}$ and think of the above equations as a linear system on $\vec{c}$. Since $q_{ij}\le Q\le\delta$, the coefficient matrix is diagonally dominant and hence solvable. It remains to estimate the terms on the right-hand side.

For each j and b, by the orthogonal condition in (3.1) and $|Z_j^b|\lesssim U_j$, we have

$$\left|\int p\sigma^{p-1}\phi Z_j^b\right| = \left|\int p\left(\sigma^{p-1}-U_j^{p-1}\right)Z_j^b\phi\right| \lesssim \|\phi\|_*\int\left(\sigma^{p-1}-U_j^{p-1}\right)U_jW. \tag{5.29}$$

Thanks to the fact that $(\sigma^{p-1} - U_i^{p-1})U_i \geq 0$ for each i, we have

$$(\sigma^{p-1} - U_j^{p-1})U_j \leq \sum_i (\sigma^{p-1} - U_i^{p-1})U_i = \sigma^p - \sum_i U_i^p.$$

By Proposition 3.4, we have $|\sigma^p - \sum_i U_i^p| \lesssim V$. Then by Lemma 3.7 and Hölder's inequality, (5.29) can be bounded by

$$\left|\int p\sigma^{p-1}\phi Z_j^b\right| \lesssim \|\phi\|_* \int VW \leq \|V\|_{L^{(2^*)'}} \|W\|_{L^{2^*}} \|\phi\|_* \lesssim \zeta_n(Q)^2 \|\phi\|_*.$$

By Lemma B.1, we also have

$$\left|\int hZ_j^b\right| \lesssim \|h\|_{**} \int VU_j dx \lesssim R^{2-n}\|h\|_{**} \approx Q\|h\|_{**}.$$

With the above two inequalities, the Lemma 5.2 is proved.

□

From Lemma 5.1 and Lemma 5.2, using a standard argument as in the proof of Proposition 4.1 in [17], we can prove the following result.

Proposition 5.3. *There exist positive constants δ_0 and C, independent of δ, such that for all $\delta \leqslant \delta_0$ and all h with $\|h\|_{**} < \infty$, problem (3.1) has a unique solution $\phi \equiv \mathcal{L}(h)$. Besides,*

$$\|\mathcal{L}(h)\|_* \leqslant C\|h\|_{**}, \quad \left|c_a^i\right| \leqslant C\delta\|h\|_{**}.$$

Proof. Let $\{U_i : 1 \leq i \leq \nu\}$ be a family of bubbles with δ-interaction, i.e., $Q \leq \delta$. Let us consider the space

$$H := \{\phi \in \dot{H}^1(\mathbb{R}^n) : \int \phi Z_i^a U_i^{p-1} = 0, 1 \leq i \leq \nu, 1 \leq a \leq n+1\}$$

endowed with the inner product $\langle \phi, \psi \rangle = \int \nabla\phi \cdot \nabla\psi$. Problem (3.1) expressed in a weak form is equivalent to that of finding a $\phi \in H$ such that

$$\langle \phi, \psi \rangle = \int \left(p\sigma^{p-1}\phi - h\right)\psi, \ \forall \psi \in H. \tag{5.30}$$

With the aid of Riesz's representation theorem, we can rewrite this equation in the operational form

$$\phi = T(\phi) + \tilde{h} \tag{5.31}$$

with certain $\tilde{h}$ which depends linearly in h and where T is a compact operator in H. Fredholm's alternative guarantees the unique solvability of this problem for any h provided that the homogeneous equation

$$\phi = T(\phi) \tag{5.32}$$

has only the zero solution in H. Observe that this equation (5.32) is equivalent to

$$\begin{cases} \Delta\phi + p\sigma^{p-1}\phi = \sum_{i=1}^{\nu} \sum_{a=1}^{n+1} c_a^i U_i^{p-1} Z_i^a, & \text{in } \mathbb{R}^n, \\ \int U_i^{p-1}\phi Z_i^a = 0, \quad i = 1, \cdots, \nu; a = 1, \cdots, n+1, \end{cases} \tag{5.33}$$

for certain constants c_a^i. Assume it has a nontrivial solution $\phi = \phi_0$, which with no loss of generality may be taken so that $\|\phi_0\|_* = 1$. But with the aid of Lemma 5.1 and Lemma 5.2, we have

$$\|\phi_0\|_* \leq C\zeta_n(Q)^2\|\phi_0\|_* \leq \frac{1}{2}\|\phi_0\|_*.$$

This is certainly a contradiction that proves this equation only has the zero solution in H. We conclude then that for each h, problem (3.1) admits a unique solution. Denote this solution as $\phi = \mathcal{L}(h)$. Note that $|U_i^{p-1}Z_i^a| \lesssim U_i^p \lesssim R^{n-2}(v_i^{\text{in}} + v_i^{\text{out}})$ or $R^4(\hat{v}_i^{\text{in}} + \hat{v}_i^{\text{out}})$ in dimension $n = 6$. Thus $\|U_i^{p-1}Z_i^a\|_{**} \lesssim R^{n-2} \approx Q^{-1}$. It follows from Lemma 5.1 and Lemma 5.2 that

$$\begin{aligned}\|\mathcal{L}(h)\|_* &\leq C\|h\|_{**} + C\left[Q\|h\|_{**} + \zeta_n(Q)^2\|\mathcal{L}(h)\|_*\right]Q^{-1}\\ &\leq C\|h\|_{**} + o(Q)\|\mathcal{L}(h)\|_*.\end{aligned}$$

Thus $\|\mathcal{L}(h)\|_* \leq C\|h\|_{**}$ when δ is small enough. Consequently, $|c_a^i| \lesssim Q\|h\|_{**} + o(Q)\|\mathcal{L}(h)\|_* \leq C\delta\|h\|_{**}$. □

With the aid of the above linear theory, we can solve the following nonlinear equation of ρ_0,

$$\begin{cases}\Delta\rho_0 + (\sigma+\rho_0)|\sigma+\rho_0|^{p-1} - \sum\limits_{i=1}^{\nu} U_i^p = \sum\limits_{i,a} c_a^i U_i^{p-1}Z_i^a \quad \text{in } \mathbb{R}^n,\\ \int U_i^{p-1}Z_i^a\rho_0 = 0, \quad i = 1,\cdots,\nu; a = 1,\cdots,n+1.\end{cases} \tag{5.34}$$

Proposition 5.4. *Suppose that δ is small enough. There exist a solution ρ_0 and a family of scalars (c_a^i) which solve (5.34). Moreover,*

$$|\rho_0|(x) \leq CW(x). \tag{5.35}$$

Proof. Let us consider the following equation

$$\begin{cases}\Delta\phi + (\sigma+\phi)|\sigma+\phi|^{p-1} - \sum_{i=1}^{\nu} U_i^p = \sum_{i,a} c_a^i U_i^{p-1}Z_i^a \quad \text{in } \mathbb{R}^n,\\ \int U_i^{p-1}Z_i^a\phi = 0, \quad i = 1,\cdots,\nu; a = 1,\cdots,n+1.\end{cases} \tag{5.36}$$

Recall that $N_\sigma(\phi) = (\sigma+\phi)|\sigma+\phi|^{p-1} - \sigma^p - p\sigma^{p-1}\phi$ and $h = \sigma^p - \sum_{i=1}^{\nu} U_i^p$. Then, (5.36) is equivalent to

$$\phi = A(\phi) =: -\mathcal{L}(N_\sigma(\phi)) - \mathcal{L}(h), \tag{5.37}$$

where $\mathcal{L}$ is defined in Proposition 5.3. We will show that A is a contraction mapping.

First, we claim that $\|N_\sigma(\phi)\|_{**} \leq C_1R^{-4(p-1)}\|\phi\|_*$. In fact, since $|N_\sigma(\phi)| \leq C|\phi|^p \leq C\|\phi\|_*^p W^p$, then

$$\|N_\sigma(\phi)\|_{**} \leq C\|\phi\|_* \sup_{\mathbb{R}^n} W^p(x)V^{-1}(x). \tag{5.38}$$

For the inner weight functions, if $|y_i| \leq R$, we have

$$\frac{(w_i^{\text{in}})^p}{v_i^{\text{in}}} = \left(\frac{\lambda_i^{\frac{n-2}{2}}R^{2-n}}{\langle y_i\rangle^2}\right)^p\left(\frac{\langle y_i\rangle^4}{\lambda_i^{\frac{n+2}{2}}R^{2-n}}\right) = R^{-4}\langle y_i\rangle^{4-2p} \leq R^{-2p}.$$

For the outer weight functions, if $R < |y_i|$, we have

$$\frac{(w_i^{\mathrm{out}})^p}{v_i^{\mathrm{out}}} = \left(\frac{\lambda_i^{\frac{n-2}{2}} R^{-4}}{\langle y_i\rangle^{n-4}}\right)^p \left(\frac{\langle y_i\rangle^{n-2}}{\lambda_i^{\frac{n+2}{2}} R^{-4}}\right) = R^{4(1-p)} \langle y_i\rangle^{n-2-p(n-4)} \leq R^{4(1-p)},$$

since $n-2-p(n-4) < 0$ when $n \geq 7$. One can also prove $(\hat{w}_i^{\mathrm{in}})^2/\hat{v}_i^{\mathrm{in}} + (\hat{w}_i^{\mathrm{out}})^2/\hat{v}_i^{\mathrm{out}} \leq R^{-4}$ in dimension $n = 6$. Thanks to the inequality (5.21), we have

$$W^p V^{-1} \leq R^{-4(p-1)}.$$

Thus, there exists $C_1 = C_1(n, \nu)$ such that

$$\|N_\sigma(\phi)\|_{**} \leq C_1 R^{-4(p-1)} \|\phi\|_*. \tag{5.39}$$

Making C_1 possibly larger, we also have $\|\mathcal{L}(h)\|_* \leq C_1 \|h\|_{**}$ in Proposition 5.3.

Second, it follows from Proposition 3.4 that there exists $C_2 = C_2(n,\nu)$ such that $\|h\|_{**} \leq C_2$.

Now we define the space

$$E = \{u: \ u \in C(\mathbb{R}^n), \|u\|_* \leq C_1C_2 + 1\}.$$

We will show that A is a contraction mapping from E to E. Choosing δ small, then R large such that $R^{-4(p-1)} C_1^2 (C_1C_2+1) \leq 1$, we have

$$\begin{aligned}\|A(\phi)\|_* &\leq C_1 \|N_\sigma(\phi)\|_{**} + C_1\|h\|_{**} \\ &\leq R^{-4(p-1)} C_1^2 (C_1C_2+1) + C_1C_2 \leq C_1C_2 + 1.\end{aligned}$$

Thus, $A(E) \subset E$. Furthermore,

$$\begin{aligned}\|A(\phi_1) - A(\phi_2)\|_* &\leq \|\mathcal{L}(N_\sigma(\phi_1)) - \mathcal{L}(N_\sigma(\phi_2))\|_{**} \\ &\leq C_1 \|N_\sigma(\phi_1) - N_\sigma(\phi_2)\|_{**}.\end{aligned}$$

If $n \geq 6$, then we have the $|N_\sigma(\phi_1) - N_\sigma(\phi_2)| \lesssim |\phi_1 - \phi_2|(|\phi_1|^{p-1} + |\phi_2|^{p-1})$ (see [14, Appendix D]). As a result,

$$|N_\sigma(\phi_1) - N_\sigma(\phi_2)| \leq C\left(\|\phi_1\|_*^{p-1} + \|\phi_2\|_*^{p-1}\right) \|\phi_1 - \phi_2\|_* W^p.$$

Since $W^pV^{-1} \leq R^{-4(p-1)} \ll 1$ if δ small, we get

$$\|A(\phi_1) - A(\phi_2)\|_* \leq \frac{1}{2}\|\phi_1 - \phi_2\|_*.$$

Thus, A is a contraction mapping. It follows from the contraction mapping theorem that there exists a unique $\rho_0 \in E$, such that $\rho_0 = A(\rho_0)$. Moreover, it follows from Proposition 5.3 that $\|\rho_0\|_* \leq C$. □

6. Gradient estimate of the error function

In this section, we will establish L^2 estimates for the $\nabla\rho$, based on the pointwise estimates from the previous section 5.

Proposition 6.1. *Suppose δ is small enough. We have the gradient estimate*

$$\|\nabla \rho_0\|_{L^2} \lesssim \zeta_n(Q). \tag{6.1}$$

Proof. By (5.34), we have ρ_0 satisfies that

$$\Delta\rho_0 + p\sigma^{p-1}\rho_0 + (\sigma^p - \sum_{i=1}^{\nu} U_i^p) + N_\sigma(\rho_0) = \sum_{i,a} c_a^i U_i^{p-1} Z_i^a.$$

Multiplying the above equation by ρ_0 and using the orthogonal condition in (5.34), we get

$$\int |\nabla\rho_0|^2 \lesssim \int \sigma^{p-1}\rho_0^2 + \int |N_\sigma(\rho_0)\rho_0| + \int (\sigma^p - \sum U_i^p)\rho_0. \tag{6.2}$$

It follows from Proposition 3.4 and Proposition 5.4 that $|\sigma^p - \sum_i U_i^p| \lesssim V(x)$ and $|\rho_0(x)| \lesssim W(x)$. By Lemma 3.7 and $R^{2-n} \approx Q$, we have $\|W\|_{L^{2^*}} \lesssim \zeta_n(Q)$ and $\|W\|_{L^{2^*}}\|V\|_{L^{(2^*)'}} \lesssim \zeta_n(Q)^2$. Therefore

$$\int \sigma^{p-1}\rho_0^2 \lesssim \int \sigma^{p-1}W^2 \lesssim \|\sigma\|_{L^{2^*}}^{p-1}\|W\|_{L^{2^*}}^2 \lesssim \zeta_n(Q)^2,$$

$$\int (\sigma^p - \sum_{i=1}^{\nu} U_i^p)\rho_0 \lesssim \int VW \lesssim \|W\|_{L^{2^*}}\|V\|_{L^{(2^*)'}} \lesssim \zeta_n(Q)^2,$$

$$\int |N_\sigma(\rho_0)\rho_0| \lesssim \int |\rho_0|^{p+1} \lesssim \int W^{p+1} \lesssim \zeta_n(Q)^{p+1}.$$

Plugging in the above inequalities to (6.2), the proof is complete. □

Now consider $\rho_1 = \rho - \rho_0$. Recall ρ satisfies (2.4) and ρ_0 satisfies (5.34). Thus ρ_1 solves

$$\begin{cases} \Delta\rho_1 + [(\sigma+\rho_0+\rho_1)^p - (\sigma+\rho_0)^p] + \sum_{i,a} c_a^j U_j^{p-1} Z_j^a + f = 0, \\ \int U_i^{p-1} Z_i^a \rho_1 = 0 \quad i = 1, \cdots, \nu;\ a = 1, \cdots, n+1. \end{cases} \tag{6.3}$$

Here the notation x^p means $x|x|^{p-1}$ for any x. We do not know whether $\sigma+\rho_0+\rho_1$ is positive everywhere or not. For rigorous reasons, one needs to write $(\sigma+\rho_0+\rho_1)|\sigma+\rho_0+\rho_1|^{p-1}$. However, we abuse the notation here and adopt it for the rest of this paper to save some space.

Since (1.14) and $\int \nabla\rho_0 \cdot \nabla Z_i^a = p\int U_i^{p-1}\rho_0 Z_i^a = 0$, we see that $\rho_1 = \rho - \rho_0$ is also orthogonal to Z_i^a in $\dot{H}^1$ for any $1 \le i \le \nu$ and $a = 1, \cdots, n+1$. Now we decompose

$$\rho_1 = \sum_{i=1}^{\nu} \beta_i U_i + \rho_2, \tag{6.4}$$

with

$$\beta_i = \int \nabla\rho_1 \cdot \nabla U_i. \tag{6.5}$$

Then ρ_2 satisfies that

$$\int \nabla\rho_2 \cdot \nabla U_i = 0 = \int \nabla\rho_2 \cdot \nabla Z_i^a, \tag{6.6}$$

for all $i = 1, \cdots, \nu$ and $a = 1, \cdots, n+1$. The definition of ρ_2 is intended to provide a second variation estimate in the following lemma.

Lemma 6.2. *If δ is small enough, then ρ_2 satisfies*

$$\|\nabla \rho_2\|_{L^2} \lesssim \sum_{i=1}^{\nu} |\beta_i| + \|f\|_{H^{-1}}.$$

Consequently, $\|\nabla \rho_1\|_{L^2} \lesssim \sum_{i=1}^{\nu} |\beta_i| + \|f\|_{H^{-1}}$.

Proof. Multiplying (6.3) by ρ_2 and integrating by parts, we get

$$\int |\nabla \rho_2|^2 = \int [(\sigma + \rho_0 + \rho_1)^p - (\sigma + \rho_0)^p]\rho_2 + \int \rho_2 f.$$

Here we have used the orthogonal condition (6.6). Using the elementary inequality

$$|(\sigma + \rho_0 + \rho_1)^p - (\sigma + \rho_0)^p - p(\sigma + \rho_0)^{p-1}\rho_1| \lesssim |\rho_1|^p, \tag{6.7}$$

we have

$$\int |\nabla \rho_2|^2 \le p \int |\sigma + \rho_0|^{p-1} |\rho_1 \rho_2| + C \int |\rho_1|^p |\rho_2| + \int |\rho_2 f|. \tag{6.8}$$

Let us estimate each term on the RHS. Denote $\mathcal{B} = \sum_{i=1}^{\nu} |\beta_i|$. The second and last ones are easy to be controlled as follows.

$$\begin{aligned} &\int |\rho_1|^p |\rho_2| \le \|\rho_1\|_{L^{2^*}}^p \|\rho_2\|_{L^{2^*}} \lesssim (\mathcal{B} + \|\nabla \rho_2\|_{L^2})^p \|\nabla \rho_2\|_{L^2}, \\ &\int |\rho_2 f| \lesssim \|\rho_2\|_{\dot{H}^1} \|f\|_{H^{-1}}. \end{aligned} \tag{6.9}$$

The first one on the RHS of (6.8) is a bit more difficult to estimate. First, notice the decomposition of ρ_1 in (6.4), we have $|\rho_1| \le C\mathcal{B}\sigma + |\rho_2|$,

$$p \int |\sigma + \rho_0|^{p-1} |\rho_1 \rho_2| \le C\mathcal{B} \int |\sigma + \rho_0|^{p-1} \sigma |\rho_2| + p \int |\sigma + \rho_0|^{p-1} \rho_2^2.$$

By Hölder's inequality and Sobolev inequality,

$$\begin{aligned} \int |\sigma + \rho_0|^{p-1} \sigma |\rho_2| &\lesssim \|\sigma + \rho_0\|_{L^{2^*}}^{p-1} \|\sigma\|_{L^{2^*}} \|\rho_2\|_{L^{2^*}} \\ &\lesssim \|\nabla \sigma + \nabla \rho_0\|_{L^2}^{p-1} \|\nabla \rho_2\|_{L^2} \lesssim \|\nabla \rho_2\|_{L^2}. \end{aligned} \tag{6.10}$$

We used (6.1) and the fact that $\|U_i\|_{\dot{H}^1}$ and $\|U_i\|_{L^{2^*}}$ are some dimensional constants for all $1 \le i \le \nu$.

Second, it follows from the second variation estimate (for instance, see [4, Prop 3.1] and [22, Prop 3.10]) and the orthogonal condition (6.6) of ρ_2 that there exists a constant $\tilde{c} < 1$ such that

$$p \int \sigma^{p-1} \rho_2^2 \le \tilde{c} \int |\nabla \rho_2|^2.$$

Recall a simple inequality that if $x>0$ and $p\in(1,2]$ then $\big||x+y|^{p-1}-|x|^{p-1}\big|\le C|y|^{p-1}$ for any y. Consequently,

$$p\int|\sigma+\rho_0|^{p-1}\rho_2^2\le\tilde c\int|\nabla\rho_2|^2+C\int|\rho_0|^{p-1}\rho_2^2\le(\tilde c+C\|\nabla\rho_0\|_{L^2}^{p-1})\|\nabla\rho_2\|_{L^2}^2.$$

Combining the above inequality with (6.10), we obtain

$$p\int|\sigma+\rho_0|^{p-1}|\rho_1\rho_2|\le(\tilde c+C\|\nabla\rho_0\|_{L^2}^{p-1})\|\nabla\rho_2\|_{L^2}^2+C\mathcal{B}\|\nabla\rho_2\|_{L^2}. \tag{6.11}$$

By Proposition 6.1, we can make $\|\nabla\rho_0\|_{L^2}\ll 1$. Plugging in (6.9) and (6.11) to (6.8), we obtain

$$\|\nabla\rho_2\|_{L^2}^2\lesssim\mathcal{B}\|\nabla\rho_2\|_{L^2}+(\mathcal{B}+\|\nabla\rho_2\|_{L^2})^p\|\nabla\rho_2\|_{L^2}+\|f\|_{H^{-1}}\|\nabla\rho_2\|_{L^2}.$$

Dividing $\|\nabla\rho_2\|_{L^2}$ on both sides (unless $\rho_2\equiv 0$, when there is nothing to prove), we have

$$\|\nabla\rho_2\|_{L^2}\lesssim\mathcal{B}+(\mathcal{B}+\|\nabla\rho_2\|_{L^2})^p+\|f\|_{H^{-1}}.$$

By (6.5) and Hölder's inequality, $|\beta_i|\lesssim\|\nabla\rho_1\|_{L^2}\le\|\nabla\rho\|_{L^2}+\|\nabla\rho_0\|_{L^2}$. Since $\sigma=\sum_i U_i$ is the best approximation of u, then $\rho=u-\sigma$ satisfies $\|\nabla\rho\|_{L^2}\le\delta$. Taking δ small and using (6.1), we can obtain $\mathcal{B}\ll 1$ and $\|\nabla\rho_2\|_{L^2}\ll 1$. Thus, the above inequality proves the lemma. □

Lemma 6.3. *If δ is small enough, then*

$$|\beta_i|\lesssim Q^2+\|f\|_{H^{-1}},\quad 1\le i\le\nu.$$

Proof. We shall multiply (6.3) by U_k and integrate it. Before that, let us make some preparations. It follows from (6.7) and $|(\sigma+\rho_0)^{p-1}-U_k^{p-1}|\lesssim\sum_{i\neq k}U_i^{p-1}+|\rho_0|^{p-1}$ that

$$\begin{aligned}\Big|\int[(\sigma+\rho_0+\rho_1)^p-(\sigma+\rho_0)^p]U_k-p\int U_k^p\rho_1\Big|&\\ \lesssim\sum_{i\neq k}\int U_i^{p-1}U_k|\rho_1|+\int|\rho_1|^pU_k+\int|\rho_0|^{p-1}|\rho_1|U_k.&\end{aligned}$$

Denote $\mathcal{B}=\sum_{i=1}^{\nu}|\beta_i|$. It follows from Sobolev inequality and Lemma 6.2 that $\|\rho_1\|_{L^{2^*}}\lesssim\|\nabla\rho_1\|_{L^2}\lesssim\mathcal{B}+\|\nabla\rho_2\|_{L^2}\lesssim\mathcal{B}+\|f\|_{H^{-1}}$. By Hölder's inequality, Sobolev inequality, Lemma 6.2 and Lemma A.3, we have

$$\int|\rho_1|^pU_k\lesssim\|\rho_1\|_{L^{2^*}}^p\lesssim\|\nabla\rho_1\|_{L^2}^p\lesssim\mathcal{B}^p+\|f\|_{H^{-1}}^p,$$

$$\int|\rho_0|^{p-1}|\rho_1|U_k\lesssim\|\nabla\rho_0\|_{L^2}^{p-1}\|\rho_1\|_{L^{2^*}}\|U_k\|_{L^{2^*}}\lesssim o(1)\left(\mathcal{B}+\|f\|_{H^{-1}}\right),$$

$$\int U_i^{p-1}U_k|\rho_1|\le\|U_i^{p-1}U_k\|_{L^{(2^*)'}}\|\rho_1\|_{L^{2^*}}\lesssim o(1)\left(\mathcal{B}+\|f\|_{H^{-1}}\right),\quad i\neq k.$$

Here $o(1)$ denotes a quantity that goes to 0 when $\delta \to 0$. Multiplying (6.3) by U_k and integrating it, the above estimates give

$$\int \Delta\rho_1 U_k + pU_k^p\rho_1 \lesssim o(1)\left(\mathcal{B} + \|f\|_{H^{-1}}\right) + \sum_{j,a} |c_a^j| \left|\int U_j^{p-1} Z_j^a U_k\right| + \int |fU_k|.$$

For the LHS, we use integration by parts and (6.4) to get

$$\begin{aligned}\int \Delta\rho_1 U_k + pU_k^p\rho_1 &= (p-1)\int U_k^p\rho_1 \\ &= -(p-1)\int \nabla U_k\cdot\nabla\rho_1 = -(p-1)\beta_k.\end{aligned}$$

For the RHS, we see that $\int U_j^{p-1} Z_j^a U_k = 0$ if $j = k$ and $\int U_j^{p-1} Z_j^a U_k \lesssim \int U_i^p U_k \approx Q$ if $j \neq k$ by Lemma A.3. It follows from Proposition 3.4 and Lemma 5.2 that $|c_a^j| \lesssim Q$. Putting these estimates together, we obtain

$$|\beta_k| \lesssim o(1)\mathcal{B} + Q^2 + \|f\|_{H^{-1}}.$$

Summing over k, we obtain $\mathcal{B} \lesssim Q^2 + \|f\|_{H^{-1}}$ when δ is small. This completes the proof.

□

Proposition 6.4. *Suppose δ is small enough. We have*

$$\|\nabla\rho_1\|_{L^2} \lesssim Q^2 + \|f\|_{H^{-1}}.$$

Proof. This just follows from Lemma 6.3 and Lemma 6.2 with

$$\|\nabla\rho_1\|_{L^2} \lesssim \sum_i |\beta_i| + \|\nabla\rho_2\|_{L^2}.$$

□

Finally, we can prove the estimates which are used in the proof of the main theorem.

Lemma 6.5. *Suppose δ is small enough. We have*

$$\left|\int \sigma^{p-1}\rho Z_k^{n+1}\right| = o(Q) + \|f\|_{H^{-1}}, \qquad \left|\int |\rho|^p Z_k^{n+1}\right| = o(Q) + \|f\|_{H^{-1}}.$$

Proof. Notice $\rho = \rho_0 + \rho_1$. Then

$$\int \sigma^{p-1}\rho Z_k^{n+1} = \int \sigma^{p-1}\rho_0 Z_k^{n+1} + \int \sigma^{p-1}\rho_1 Z_k^{n+1}. \tag{6.12}$$

By Hölder's inequality, Sobolev inequality and Proposition 6.4, we have

$$\left|\int \sigma^{p-1}\rho_1 Z_k^{n+1}\right| \le \|\rho_1\|_{L^{2^*}} \|\sigma\|_{L^{2^*}}^{p-1} \|Z_k^{n+1}\|_{L^{2^*}} \lesssim \|\nabla\rho_1\|_{L^2} \lesssim Q^2 + \|f\|_{H^{-1}}.$$

It remains to consider the first term on the RHS of (6.12). By the orthogonality condition of ρ_0, similar to (5.29), one has

$$\left|\int \sigma^{p-1}\rho_0 Z_k^{n+1}\right| = \left|\int (\sigma^{p-1} - U_k^{p-1})\rho_0 Z_k^{n+1}\right| = o(Q).$$

Finally, by Hölder's inequality and Sobolev inequality, it follows from Proposition 6.1 and Proposition 6.4 that

$$\left|\int |\rho|^p Z_k^{n+1}\right| \le \|\rho\|_{L^{2^*}}^p \lesssim \|\nabla\rho\|_{L^2}^p \lesssim \|\nabla\rho_0\|_{L^2}^p + \|\nabla\rho_1\|_{L^2}^p \lesssim o(Q) + \|f\|_{H^{-1}}^p.$$

□

7. A SHARP EXAMPLE

In this section, we shall construct an example showing that our quantitative estimate is sharp, i.e., Theorem 1.5. The example is built on two widely separated bubbles with the same height, which is the same as the one in [22]. We obtain refined estimates using the point-wise estimate developed in Proposition 5.4.

Let us consider the two functions $U_1 := U[-Re_1; 1], U_2 := U[Re_1; 1]$ where $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$ and $R \gg 1$. Then the interaction between U_1 and U_2 satisties that $Q \approx R^{2-n} \ll 1$. One can define $\sigma = U_1 + U_2$ and construct norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$ as (3.4) with $y_i = x - (-1)^i Re_1$, $i = 1, 2$.

By Proposition 5.4, choosing R large enough, we can find a solution ρ and a family of scalars (c_a^i) such that

$$\begin{cases} \Delta\rho + (\sigma+\rho)\,|\sigma+\rho|^{p-1} - U_1^p - U_2^p + \sum_{j,a} c_a^j U_j^{p-1} Z_j^a = 0, \\ \int U_j^{p-1} Z_j^a \rho = 0, \quad j = 1, 2;\ a = 1, \cdots, n+1. \end{cases} \tag{7.1}$$

Here $\sigma = U_1 + U_2$, Z_j^a are the corresponding ones in (2.1) for U_1 and U_2. It follows from Lemma 5.2, Proposition 5.4 and Proposition 6.1 that

$$\sum_{j,a} |c_a^j| \lesssim Q, \quad \|\rho\|_* \le C(n, \nu), \quad \|\nabla\rho\|_{L^2} \lesssim \zeta_n(Q). \tag{7.2}$$

Now let $u := U_1 + U_2 + \rho$. Then

$$\Delta u + |u|^{p-1} u = -\sum_{a,j} c_a^j U_j^{p-1} Z_j^a := -f.$$

By the Sobolev embedding, $|Z_j^a| \le U_j$ and (7.2), it is easy to see that

$$\|f\|_{H^{-1}} \lesssim \|f\|_{L^{\frac{2n}{n+2}}} \lesssim \sum_{j,a} |c_a^j|\, \|U_j^p\|_{L^{\frac{2n}{n+2}}} \lesssim Q. \tag{7.3}$$

Lemma 7.1. *For R large enough, one has*

$$\|\nabla\rho\|_{L^2} \gtrsim \zeta_n(\|f\|_{H^{-1}}).$$

Proof. It follows from (7.1) that

$$\Delta\rho + p\sigma^{p-1}\rho + h + N_\sigma(\rho) + f = 0, \tag{7.4}$$

where

$$N_\sigma(\rho) = (\sigma+\rho)|\sigma+\rho|^{p-1} - \sigma^p - p\sigma^{p-1}\rho,$$
$$h = \sigma^p - U_1^p - U_2^p.$$

Using Green's representation, we have

$$\rho(x)=C(n)\int_{\mathbb{R}^n}|x-\xi|^{2-n}(p\sigma^{p-1}\rho(\xi)+h(\xi)+N_\sigma(\rho(\xi))+f(\xi))d\xi.$$

Using (5.39) and Lemma 3.6, we have

$$\left|\int_{\mathbb{R}^n}|x-\xi|^{2-n}N_\sigma(\rho(\xi))d\xi\right|\lesssim\|\rho\|_*R^{-4(p-1)}W(x).$$

Using the estimate of P_1 in Proposition 4.3, we have

$$\left|\int_{\mathbb{R}^n}|x-\xi|^{2-n}\sigma^{p-1}\rho(\xi)d\xi\right|\lesssim\overline{W}(x)+R^{-2}W(x).$$

In fact, in this case of two bubbles, the estimate of P_1 is much simpler than that of Proposition 4.3. We just highlight some details for dimension $n\geq 7$, and omit those of dimension $n=6$. We still adopt the notation $A^{\rm in}_{ii'}$ and $A^{\rm out}_{ii'}$ $(i,i'\in\{1,2\})$ in (4.19). It follows from the proof of Proposition 4.3 that $A^{\rm in}_{ii}\lesssim\overline{W}$ and $A^{\rm out}_{ii}\lesssim\overline{W}$. For the terms when $i'\neq i$, we note $z_{12}=z_{21}\approx 2R$, then $A^{\rm in}_{ii'}+A^{\rm out}_{ii'}\lesssim R^{-2}W$.

Note that $|f|\lesssim\sum_{j,a}|c^j_a|U_j^p\lesssim R^{2-n}(\langle y_1\rangle^{-n-2}+\langle y_2\rangle^{-n-2})$. Thus, by Lemma A.7, we have

$$\left|\int_{\mathbb{R}^n}|x-\xi|^{2-n}f(\xi)d\xi\right|\lesssim R^{2-n}(\langle y_1\rangle^{2-n}+\langle y_2\rangle^{2-n}).$$

Combining the above three estimates and using the fact that $\overline{W}(x)$ decays faster than $W(x)$, we have

$$\begin{aligned}P(x):&=C(n)\int_{\mathbb{R}^n}|x-\xi|^{2-n}(N_\sigma(\rho)+p\sigma^{p-1}\rho+f)(\xi)d\xi\\&\lesssim R^{2-n}(\chi_{\{|y_1|\leq L\}}+\chi_{\{|y_2|\leq L\}})+L^{-1}W(x)\chi_{\{|y_1|\geq L,|y_2|\geq L\}}\end{aligned}\tag{7.5}$$

for some fixed L large enough to be determined.

We multiply (7.4) by ρ and integrate it by parts to get

$$\int_{\mathbb{R}^n}|\nabla\rho|^2=\int_{\mathbb{R}^n}(p\sigma^{p-1}\rho+h+N_\sigma(\rho))\rho.\tag{7.6}$$

Here the term that involves f vanishes because of the orthogonality condition in (7.1). To get a lower bound of $\|\nabla\rho\|_{L^2}$, we shall throw away the term $\int\sigma^{p-1}\rho^2>0$ and estimate the other two terms in the above identity.

First, since $|\rho|\leq\|\rho\|_*W(x)$ and by (5.39), we have

$$\left|\int_{\mathbb{R}^n}N_\sigma(\rho)\rho\right|\lesssim\|\rho\|_*R^{-4(p-1)}\int_{\mathbb{R}^n}V(x)W(x)dx\lesssim R^{-4(p-1)}\zeta_n(Q)^2.$$

Second, we notice that $\rho(x)=\int_{\mathbb{R}^n}|x-\xi|^{2-n}h(\xi)d\xi+P(x)$, then

$$\begin{aligned}\int_{\mathbb{R}^n}h\rho&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|x-\xi|^{2-n}h(x)h(\xi)dxd\xi+\int_{\mathbb{R}^n}h(x)P(x)dx\\&=:J_1+J_2.\end{aligned}$$

For J_1, denoting $\xi_1 = \xi + Re_1$, we use (3.21) to get

$$\begin{aligned} J_1 &\geq R^{4-2n} \iint_{|y_1|\leq R, |\xi_1|\leq R} |y_1 - \xi_1|^{2-n} \langle y_1 \rangle^{-4} \langle \xi_1 \rangle^{-4} dy_1 d\xi_1 \\ &\gtrsim R^{4-2n} \int_{|\xi_1|\leq R} \langle \xi_1 \rangle^{-6} d\xi_1 \gtrsim \zeta_n(Q)^2. \end{aligned}$$

For J_2, we use $h(x) \leq R^{2-n}\langle y_1 \rangle^{-4}$ on the set $\{|y_1| \leq L\}$ and (7.5) to get

$$\begin{aligned} \left| \int_{\mathbb{R}^n} h(x)P(x)dx \right| &\lesssim \int_{|y_1|\leq L} R^{4-2n}\langle y_1 \rangle^{-4} dx + \int_{|y_2|\leq L} R^{4-2n}\langle y_2 \rangle^{-4} dx \\ &\quad + L^{-1} \int_{|y_1|\geq L, |y_2|\geq L} V(x)W(x) \\ &\lesssim R^{4-2n}L^{n-4} + L^{-1}\zeta_n(Q)^2 \leq \left[\frac{L^{n-4}}{\log R} + \frac{1}{L} \right] \zeta_n(Q)^2. \end{aligned}$$

Plugging in the above three facts to (7.6), we get that there exists a constant $C = C(n, \nu)$ such that

$$\|\nabla \rho\|_{L^2}^2 \gtrsim \zeta_n(Q)^2 \left[1 - CR^{-4(p-1)} - CL^{-1} - C\frac{L^{n-4}}{\log R} \right].$$

Now we choose $L = L(n, \nu)$ large and fix it. Using (7.3) and monotonicity of $\zeta_n(x)$, we have $\zeta_n(\|f\|_{H^{-1}}) \lesssim \zeta_n(Q)$. Taking R large enough, the proof is complete. $\square$

Proof of Theorem 1.5. According to Lemma 7.1, it suffices to show that

$$\inf_{\substack{z_1, z_2 \in \mathbb{R}^n \\ \lambda_1 > 0, \lambda_2 > 0}} \left\| \nabla \left(u - \sum_{j=1,2} U[z_j; \lambda_j] \right) \right\|_{L^2} \gtrsim \|\nabla \rho\|_{L^2}.$$

It is well-known that the minimization problem on the left-hand side can be attained (cf. [4], Lemma A.1) by some

$$\tilde{U}_1 := U[z_1; \lambda_1], \quad \tilde{U}_2 := U[z_2; \lambda_2].$$

Denote $\tilde{\sigma} = \tilde{U}_1 + \tilde{U}_2$ and $\tilde{\rho} = u - \tilde{\sigma}$. We need to show $\|\nabla \tilde{\rho}\|_{L^2} \gtrsim \|\nabla \rho\|_{L^2}$. Since $\tilde{\sigma}$ is the minimizer, then

$$\|\nabla (u - \tilde{\sigma})\|_{L^2} \leq \|\nabla(u - \sigma)\|_{L^2} = \|\nabla \rho\|_{L^2} \lesssim \zeta_n(Q).$$

Recall that $\langle v, w \rangle_{\dot{H}^1} = \int \nabla v \cdot \nabla w$. Hence $\|\sigma - \tilde{\sigma}\|_{\dot{H}^1} \lesssim \zeta_n(Q)$. This implies that (up to some reordering of z_1 and z_2)

$$\lambda_j = 1 + o_R(1), \quad z_1 = -(R + o_R(1))e_1, \quad z_2 = (R + o_R(1))e_1. \tag{7.7}$$

Here $o_R(1)$ means a quantity that goes to 0 when $R \to \infty$. Denote

$$\varepsilon = \sum_{i=1,2} |\lambda_i - 1| + |z_i - (-1)^i Re_1|.$$

It is easy to see that $(z,\lambda)\to U[z,\lambda]$ is a smooth map from $\mathbb{R}^n\times(0,\infty)$ to $\dot H^1(\mathbb{R}^n)$. Using the Taylor expansion, there exist $A_1, A_2\in \dot H^1$ and $\|A_1\|_{\dot H^1}=O(\varepsilon^2)$, $\|A_2\|_{\dot H^1}=O(\varepsilon^2)$ such that

$$\tilde U_1 - U_1 = \sum_{a=1}^n Z_1^a(z_1+Re_1)_a + Z_1^{n+1}(\lambda_1-1)+A_1,$$
$$\tilde U_2 - U_2 = \sum_{a=1}^n Z_2^a(z_2-Re_1)_a + Z_2^{n+1}(\lambda_2-1)+A_2,$$

where Z_1^a, Z_2^a are defined in (2.1) with respect to U_1 and U_2. Consequently, $\|U_1-\tilde U_1\|_{\dot H^1}\approx\varepsilon$, $\|U_2-\tilde U_2\|_{\dot H^1}\approx\varepsilon$, and using Lemma A.5 we get

$$|\langle U_1-\tilde U_1, U_2-\tilde U_2\rangle_{\dot H_1}|\approx o(\varepsilon^2)+\varepsilon^2\sum_{a=1}^{n+1}|\langle Z_1^a, Z_2^a\rangle_{\dot H^1}| = o(\varepsilon^2).$$

Combining the above estimates, we have $\|\nabla(\sigma-\tilde\sigma)\|_{L^2}\approx\varepsilon$ because

$$\begin{aligned}\|\nabla(\sigma-\tilde\sigma)\|_{L^2}^2 =&\|\nabla(U_1-\tilde U_1)\|_{L^2}^2+\|\nabla(U_2-\tilde U_2)\|_{L^2}^2\\ &+2\langle U_1-\tilde U_1, U_2-\tilde U_2\rangle_{\dot H_1}.\end{aligned}$$

By the orthogonality condition in (7.1), we have ρ is orthogonal to Z_1^a and Z_2^a in $\dot H^1$ for $a=1,\cdots,n+1$. Thus

$$\langle\sigma-\tilde\sigma,\rho\rangle_{\dot H^1}=\int\nabla(A_1+A_2)\cdot\nabla\rho\lesssim o(1)\|\nabla(\sigma-\tilde\sigma)\|_{L^2}\|\nabla\rho\|_{L^2}.$$

Since $\tilde\rho=\rho+\sigma-\tilde\sigma$, the above inequality implies that

$$\begin{aligned}\|\nabla\tilde\rho\|_{L^2}^2 &= \|\nabla\rho\|_{L^2}^2+\|\nabla(\sigma-\tilde\sigma)\|_{L^2}^2+2\langle\sigma-\tilde\sigma,\rho\rangle_{\dot H^1}\\ &\gtrsim \|\nabla\rho\|_{L^2}^2+\|\nabla(\sigma-\tilde\sigma)\|_{L^2}^2\ge\|\nabla\rho\|_{L^2}^2.\end{aligned}$$

From this, the assertion follows.

□

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Appendix A. Some useful estimates

This appendix contains some useful estimates involving Aubin-Talenti bubbles and their derivatives. We always denote $U_i=U[z_i,\lambda_i]$ defined in (1.3). See the definition of q_{ij} in (1.5).

Lemma A.1. *Let $\alpha > \beta > 1$ and $\alpha + \beta = 2^*$,*

$$\int U_i^\beta \inf(U_i^\alpha, U_j^\alpha) = O(q_{ij}^{\frac{n}{n-2}} |\log q_{ij}|).$$

Proof. See the proof in [3, E4]. □

Lemma A.2. *For any two bubbles, there exists C_n such that*

$$\int_{\mathbb{R}^n} U_i^p \lambda_j \partial_{\lambda_j} U_j = -C_n \left(q_{ij}^{-\frac{2}{n-2}} - 2\frac{\lambda_i}{\lambda_j} \right) q_{ij}^{\frac{n}{n-2}} + O(q_{ij}^{\frac{n}{n-2}} |\log q_{ij}|).$$

Proof. See the proof in [3, F16]. Moreover, if $\lambda_i \leq \lambda_j$ then RHS $\approx -q_{ij}$ when $q_{ij} \ll 1$. □

Lemma A.3. *Given $n \geq 3$, for any fixed $\varepsilon > 0$ and any non-negative exponents such that $\alpha + \beta = 2^*$, it holds*

$$\int_{\mathbb{R}^n} U_1^\alpha U_2^\beta \approx_{n,\varepsilon} \begin{cases} q_{12}^{\min(\alpha,\beta)} & \text{if } \ |\alpha - \beta| \geq \varepsilon, \\ q_{12}^{\frac{n}{n-2}} |\log q_{ij}| & \text{if } \ \alpha = \beta. \end{cases}$$

Proof. See the proof of proposition B.2 in [22]. □

Lemma A.4. *Given $n \geq 6$, let $\{U_i\}_{i=1}^3$ be three bubbles with δ-interaction, that is $Q := \max\{q_{12}, q_{13}, q_{23}\} \leq \delta$ which is small enough.*

(1) *For $n = 6$, we have*

$$\int_{\mathbb{R}^n} U_1 U_2 U_3 \lesssim Q^{\frac{3}{2}} |\log Q|. \tag{A.1}$$

(2) *For $n \geq 7$, we have*

$$\int_{\mathbb{R}^n} U_1^{p-1} U_2 U_3 \lesssim Q^{\frac{n-1}{n-2}} |\log Q|^{\frac{n-5}{n}}. \tag{A.2}$$

Proof. For $n = 6$, $2^* = 3$, by the Hölder's inequality, we get

$$\int_{\mathbb{R}^n} U_1 U_2 U_3 \leq \left(\int_{\mathbb{R}^n} U_1^{\frac{3}{2}} U_2^{\frac{3}{2}} \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^n} U_1^{\frac{3}{2}} U_3^{\frac{3}{2}} \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^n} U_2^{\frac{3}{2}} U_3^{\frac{3}{2}} \right)^{\frac{1}{3}}.$$

By Lemma A.3, we have

$$\int_{\mathbb{R}^n} U_1 U_2 U_3 \lesssim q_{12}^{\frac{1}{2}} |\log q_{12}|^{\frac{1}{3}} q_{13}^{\frac{1}{2}} |\log q_{13}|^{\frac{1}{3}} q_{23}^{\frac{1}{2}} |\log q_{23}|^{\frac{1}{3}}.$$

Since the function $x^{\frac{1}{2}} |\log x|^{\frac{1}{3}}$ is increasing near 0, choosing δ small, we get (A.1).

For $n \geq 7$, $2^* = \frac{2n}{n-2}$, let $\alpha = \frac{4n}{5(n-2)}$, $\beta = \frac{6n}{5(n-2)}$, $s_1 = \frac{5}{2n}$ and $s_2 = \frac{n-5}{n}$. By the Hölder's inequality and Lemma A.3, we get

$$\begin{aligned} \int_{\mathbb{R}^n} U_1 U_2 U_3 &\leq \left(U_1^\alpha U_2^\beta \right)^{s_1} \left(U_1^\alpha U_3^\beta \right)^{s_1} \left(U_2^{\frac{2^*}{2}} U_3^{\frac{2^*}{2}} \right)^{s_2} \\ &\lesssim q_{12}^{\frac{2}{n-2}} q_{13}^{\frac{2}{n-2}} q_{23}^{\frac{n-5}{n-2}} |\log q_{23}|^{\frac{n-5}{n}}. \end{aligned}$$

Since the function $x^{\frac{n-5}{n-2}} |\log x|^{\frac{n-5}{n}}$ is increasing near 0, choosing δ small, we get (A.2). □

Lemma A.5. *For the Z_i^a defined in (2.1), there exist some constants $\gamma^a = \gamma^a(n) > 0$ such that*

$$\int U_i^{p-1} Z_i^a Z_i^b = \begin{cases} 0 & \text{if } a \neq b, \\ \gamma^a & \text{if } 1 \leq a = b \leq n+1. \end{cases}$$

If $i \neq j$ and $1 \leq a, b \leq n+1$, we have

$$\left| \int U_i^{p-1} Z_i^a Z_j^b \right| \lesssim q_{ij}.$$

Proof. See the proof in [3, F1-F6]. Moreover, it is known that $\gamma^1 = \cdots = \gamma^n$. □

Lemma A.6. *Suppose $p \in (1, 2]$ and $a_i \geq 0$, then*

$$\left(\sum_{i=1}^{\nu} a_i \right)^p - \sum_{i=1}^{\nu} a_i^p \leq \sum_{i<j} [(a_i + a_j)^p - a_i^p - a_j^p],$$

the equality holds when at most two of a_i are non-zero or $p = 2$.

Proof. It is equivalent to prove

$$f(a_1, a_2, \cdots, a_\nu) = \left(\sum_{i=1}^{\nu} a_i \right)^p + (\nu - 2) \sum_{i=1}^{\nu} a_i^p - \sum_{i<j} (a_i + a_j)^p \leq 0.$$

Denote $a_1 + a_2 = s$. Define $g(x) = f(x, s - x, a_3, \cdots, a_\nu)$. It is easy to see

$$\frac{g''(x)}{p(p-1)} = (\nu - 2)[x^{p-2} + (s - x)^{p-2}] - \sum_{i=3}^{\nu} [(x + a_i)^{p-2} + (s - x + a_i)^{p-2}].$$

Since $p - 2 \leq 0$ and $a_i \geq 0$, then $g''(x) \geq 0$ for $x \in [0, s]$. Since $g(0) = g(s)$, we must have g achieve the maximum at $x = 0$ or s. Therefore $f(a_1, a_2, \cdots, a_\nu) \leq f(0, a_1 + a_2, \cdots, a_\nu)$. Repeating the above process for any pairs, we obtain $f \leq 0$.

If the equality holds, that is, $f(a_1, a_2, \cdots, a_\nu) = 0$, then the above proof shows that either $x = 0$ or s, or $g(x) = g(0) = g(s)$ for $x \in [0, s]$. The first case implies at most one of a_1 and a_2 is non-zero. The second case implies $g''(x) = 0$ for $x \in [0, s]$. It leads to either $a_3 = \cdots = a_\nu = 0$ or $p = 2$. Repeating this process for any pairs, one can get at most two a_i which are non-zero or $p = 2$. □

Lemma A.7. *Denote $\langle y \rangle = \sqrt{1 + |y|^2}$. We have*

$$\int_{\mathbb{R}^n} |y - z|^{2-n} \langle z \rangle^{-\gamma} dz \lesssim \begin{cases} \langle y \rangle^{2-\gamma}, & \text{if } \gamma \in (2, n), \\ \langle y \rangle^{2-n} (1 + \log \langle y \rangle), & \text{if } \gamma = n, \\ \langle y \rangle^{2-n}, & \text{if } \gamma > n, \end{cases} \tag{A.3}$$

Proof. This follows from a simple modification of the proof in [37, Lemma B.2]. □

APPENDIX B. INTEGRAL ESTIMATES REQUIRED IN SECTION 5

This appendix is devoted to computing the integral $\int VU_j$ in Lemma 5.2.

Recall that $U_j(x) = (n(n-2))^{(n-2)/4}\lambda_j^{(n-2)/2}\langle y_j\rangle^{2-n}$ where $y_j = \lambda_j(x-z_j)$. For $i \neq j$, see the definition of R_{ij} in (3.2). To compute integrals in this section, we split the involved integral domain into inner and outer parts where the integrand has a power-like behavior as in the Lemma 3.3.

Lemma B.1. *Suppose $n \geq 6$ and $1 \ll R \leq R_{ij}/2$, we have*

$$\int_{|y_i|\leq R} \frac{\lambda_i^{(n+2)/2}R^{2-n}}{\langle y_i\rangle^4}\frac{\lambda_i^{(n-2)/2}}{\langle y_i\rangle^{n-2}}dx \lesssim R^{2-n}, \tag{B.1}$$

$$\int_{|y_i|\geq R} \frac{\lambda_i^{(n+2)/2}R^{-4}}{\langle y_i\rangle^{n-2}}\frac{\lambda_i^{(n-2)/2}}{\langle y_i\rangle^{n-2}}dx \lesssim R^{-n}, \tag{B.2}$$

$$\int_{|y_i|\leq R} \frac{\lambda_i^{(n+2)/2}R^{2-n}}{\langle y_i\rangle^4}\frac{\lambda_j^{(n-2)/2}}{\langle y_j\rangle^{n-2}}dx \lesssim R^{-n}, \tag{B.3}$$

$$\int_{|y_i|\geq R} \frac{\lambda_i^{(n+2)/2}R^{-4}}{\langle y_i\rangle^{n-2}}\frac{\lambda_j^{(n-2)/2}}{\langle y_j\rangle^{n-2}}dx \lesssim R^{2-n}. \tag{B.4}$$

Proof. Recall that $y_i = \lambda_i(x-z_i)$ and $dy_i = \lambda_i^n dx$. (B.1) and (B.2) follow from direct computations. To prove (B.3), we consider the following two cases separately:

Case 1: $\lambda_i \geq \lambda_j$. Obviously, $R \leq \sqrt{\lambda_i/\lambda_j}R_{ij}/2$. By (3.9) in Lemma 3.3,

$$\begin{aligned} R^{2-n}\int_{|y_i|\leq R} \frac{\lambda_j^{(n-2)/2}}{\langle y_j\rangle^{n-2}}\frac{\lambda_i^{(n+2)/2}}{\langle y_i\rangle^4}dx &\lesssim R^{2-n}R_{ij}^{2-n}\int_{|y_i|\leq R}\frac{dy_i}{\langle y_i\rangle^4} \\ &\lesssim R^{-2}R_{ij}^{2-n} \lesssim R^{-n}. \end{aligned} \tag{B.5}$$

Case 2: $\lambda_i \leq \lambda_j$. Let $\{|y_i| \leq R\} = \mathcal{D}_1 \cup \mathcal{D}_2$, where $\mathcal{D}_1 = \{|y_i| \leq R, |y_j| \leq \sqrt{\lambda_j/\lambda_i}R_{ij}/2\}$ and $\mathcal{D}_2 = \{|y_i| \leq R, |y_j| \geq \sqrt{\lambda_j/\lambda_i}R_{ij}/2\}$. By (3.9) in Lemma 3.3 when $\tau = n-2$,

$$\begin{aligned} &R^{2-n}\int_{|y_i|\leq R} \frac{\lambda_j^{(n-2)/2}}{\langle y_j\rangle^{n-2}}\frac{\lambda_i^{(n+2)/2}}{\langle y_i\rangle^4}dx \\ &\lesssim R^{2-n}\left(\frac{\lambda_i}{\lambda_j}\right)^{(n-2)/2}\int_{\mathcal{D}_1} R_{ij}^{-4}\frac{dy_j}{\langle y_j\rangle^{n-2}} + R^{2-n}\int_{\mathcal{D}_2} R_{ij}^{2-n}\frac{dy_i}{\langle y_i\rangle^4} \\ &\lesssim R^{2-n}R_{ij}^{-2} + R^{-2}R_{ij}^{2-n} \lesssim R^{-n}. \end{aligned} \tag{B.6}$$

Combining with (B.5) and (B.6), we obtain (B.3).

To prove (B.4), as before, we consider the following two cases:

Case 1: $\lambda_i \geq \lambda_j$. Let $\{|y_i| \geq R\} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$, where $\mathcal{D}_1 = \{R \leq |y_i| \leq \sqrt{\lambda_i/\lambda_j}R_{ij}/2\}$, $\mathcal{D}_2 = \{|y_i| \geq \sqrt{\lambda_i/\lambda_j}R_{ij}/2, |y_j| \leq \sqrt{\lambda_j/\lambda_i}R_{ij}/2\}$ and

$\mathcal{D}_3 = \{|y_i| \geq \sqrt{\lambda_i/\lambda_j}R_{ij}/2, |y_j| \geq \sqrt{\lambda_j/\lambda_i}R_{ij}/2\}$. By (3.9) in Lemma 3.3, choosing $\tau = n-2$ in $\mathcal{D}_2$ and $\tau = 0$ in $\mathcal{D}_3$,

$$\begin{aligned} R^{-4}\int_{|y_i|\geq R} \frac{\lambda_j^{(n-2)/2}}{\langle y_j\rangle^{n-2}}\frac{\lambda_i^{(n+2)/2}}{\langle y_i\rangle^{n-2}}dx &\lesssim R^{-4}\int_{\mathcal{D}_1} R_{ij}^{2-n}\frac{dy_i}{\langle y_i\rangle^{n-2}} \\ &+ R^{-4}R_{ij}^{2-n}\left(\frac{\lambda_i}{\lambda_j}\right)^2\int_{\mathcal{D}_2}\frac{dy_j}{\langle y_j\rangle^{n-2}} + R^{-4}\left(\frac{\lambda_j}{\lambda_i}\right)^{(n-6)/2}\int_{\mathcal{D}_3}\frac{dy_j}{\langle y_j\rangle^{2n-4}} \\ &\lesssim R^{-4}R_{ij}^{4-n}(\lambda_i/\lambda_j) \leq R^{2-n}. \end{aligned} \tag{B.7}$$

We have used the fact that $R_{ij}^{-2}(\lambda_i/\lambda_j) \leq 1$ in the last step.

Case 2: $\lambda_i \leq \lambda_j$. Let $\mathcal{D}_1 = \{|y_i| \geq R, |y_j| \leq \sqrt{\lambda_j/\lambda_i}R_{ij}/2\}$ and $\mathcal{D}_2 = \{|y_i| \geq R, |y_j| > \sqrt{\lambda_j/\lambda_i}R_{ij}/2\}$. By (3.9) in Lemma 3.3 when $\tau = 0$,

$$\begin{aligned} &R^{-4}\int_{|y_i|\geq R} \frac{\lambda_j^{(n-2)/2}}{\langle y_j\rangle^{n-2}}\frac{\lambda_i^{(n+2)/2}}{\langle y_i\rangle^{n-2}}dx \\ &\lesssim R^{-4}\left(\frac{\lambda_i}{\lambda_j}\right)^2\int_{\mathcal{D}_1} R_{ij}^{2-n}\frac{dy_j}{\langle y_j\rangle^{n-2}} + R^{-4}\left(\frac{\lambda_i}{\lambda_j}\right)^{(n-2)/2}\int_{\mathcal{D}_2}\frac{dy_i}{\langle y_i\rangle^{2n-4}} \\ &\lesssim R^{-4}R_{ij}^{4-n} + R^{-n} \lesssim R^{-n}. \end{aligned} \tag{B.8}$$

Combining with (B.7) and (B.8), we obtain (B.4). □

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