A LANE-EMDEN SYSTEM OF FREE BOUNDARY TYPE: EXISTENCE, UNIQUENESS AND MONOTONICITY OF SOLUTIONS

DANIELE BARTOLUCCI, YEYAO HU, ALEKS JEVNIKAR, JUNCHENG WEI, AND WEN YANG

Abstract. We consider a Hamiltonian system of free boundary type, showing first uniform bounds and existence of solutions and of the free boundary. Then, for any smooth and bounded domain, we prove uniqueness of positive solutions in a suitable interval and show that the associated energies and boundary values have a monotonic behavior. Some consequences are discussed about the parametrization of the unbounded Rabinowitz continuum for a class of superlinear strongly coupled elliptic systems.

Keywords: Free boundary problems, Hamiltonian elliptic systems, bifurcation analysis, existence, uniqueness, monotonicity.

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open and bounded domain of class C^3, we are concerned with the following constrained Hamiltonian system of free boundary type,

$$\begin{cases} -\Delta v_1 = \lambda (v_2)_+^{p_2} \quad \text{in } \Omega \\ -\Delta v_2 = \lambda (v_1)_+^{p_1} \quad \text{in } \Omega \\ -\int\limits_{\partial\Omega} \frac{\partial v_1}{\partial \nu} = 1 = -\int\limits_{\partial\Omega} \frac{\partial v_2}{\partial \nu} \\ v_1 = \alpha_1, \quad v_2 = \alpha_2 \quad \text{on } \partial\Omega \end{cases} \qquad (\mathbf{F})_\lambda$$

2020 *Mathematics Subject classification:* 35J57, 35B30, 35B32, 35R35.

D.B. is partially supported by MIUR Excellence Department Project MatMod@TOV awarded to the Department of Mathematics, Univ. of Rome Tor Vergata, by PRIN 2022 " *Variational and Analytical aspects of Geometric PDEs*" and by E.P.G.P. Project funded by Univ. of Rome Tor Vergata 2024. A.J. is partially supported by PRIN 2022 "*Pattern formation in nonlinear phenomena*". D.B. and A.J. are members of the INDAM Research Group GNAMPA. Y.H. is supported by NSFC China No. 12471205. J.W. is partially supported by GRF fund of RGC of Hong Kong entitled "New frontiers in singularity formations of nonlinear partial differential equations." W.Y. is supported by the National Key R&D Program of China 2022YFA1006800, NSFC China No. 12171456, NSFC China No. 12271369, FDCT No. 0070/2024/RIA1, Start-up Research Grant No. SRG2023-00067-FST, Multi-Year Research Grant No. MYRG-GRG2024-00082-FST and UMDF-TISF/2025/006/FST. .

for the unknowns $\alpha_i \in \mathbb{R}$, and $v_i \in C^{2,r}(\overline{\Omega})$, $i = 1, 2$, for some fixed $r \in (0, 1)$. Here, $(v)_+$ is the positive part of v, ν is the exterior unit normal, $\lambda > 0$ and

$$\frac{1}{p_1+1} + \frac{1}{p_2+1} > \frac{N-2}{N-1}, \quad p_i \in (0, +\infty),\, i = 1, 2. \tag{1.1}$$

The relevance of the limiting hyperbola in (1.1) for classical Hamiltonian elliptic systems of Lane-Emden type was first noticed in [34], see [20] for more details. Putting,

$$p_N = \begin{cases} +\infty, \ N = 2 \\ \frac{N}{N-2}\,, \ N \geq 3, \end{cases}$$

we remark that, as far as $N \geq 3$, $p_1 = p_N = p_2$ satisfy $\frac{1}{p_1+1} + \frac{1}{p_2+1} = \frac{N-2}{N-1}$.
To simplify the exposition, by a suitable scaling of λ we assume that $|\Omega| = 1$.

The system $\mathbf{(F)}_\lambda$ is a vectorial generalization of the classical ([35, 36, 9]) "scalar" free boundary problem which is obtained in the particular case $p_1 = p_2$, $\alpha_1 = \alpha_2$, $v_1 = v_2$, whose study is motivated by Tokamak's plasma physics ([23, 39]). In the scalar case p_N turns out to be a natural critical exponent see [9, 28].
Another motivation to purse the analysis of $\mathbf{(F)}_\lambda$ is to find a parametrization of solutions of the Hamiltonian strongly coupled elliptic system,

$$\begin{cases} -\Delta u_1 = \mu_2(1+u_2)^{p_2} & \text{in } \Omega \\ -\Delta u_2 = \mu_1(1+u_1)^{p_1} & \text{in } \Omega \\ u_1 > 0, \quad u_2 > 0 & \text{in } \Omega \\ u_1 = 0, \quad u_2 = 0 & \text{on } \partial\Omega. \end{cases} \qquad \mathbf{(H)}$$

For $\mu_1 = \mu_2$ the existence of an unbounded continuum of "scalar" ($u_1 = u_2$) solutions of $(\mathbf{H})$ follows from the classical result in [31]. The analysis of Hamiltonian elliptic systems is a classical subject and we refer to [20] for a comprehensive introduction about this topic, see also [33] and references therein. Minimal solutions branches (in the sense of Crandall-Rabinowitz ([18])) and multiplicity results for general systems including $(\mathbf{H})$ has been described in [12, 13], see also [29] and references therein.
However our main motivation comes from the fact that we are not aware of any result either about uniqueness and qualitative behavior of branches of solutions of $\mathbf{(F)}_\lambda$ or just about the qualitative behavior of <u>non minimal</u> solutions of $(\mathbf{H})$. Remark that if $p_1 \neq p_2$ then $\mathbf{(F)}_\lambda$ has no scalar solutions. In particular concerning $(\mathbf{H})$, inspired by recent results in [3], [4], [5], we look for integral quantities naturally arising from $\mathbf{(P)}_\lambda$ to describe the monotonic behavior of the solutions.

First of all let us consider the auxiliary problem,

$$\begin{cases} -\Delta\psi_1 = (\alpha_2 + \lambda\psi_2)_+^{p_2} & \text{in } \Omega \\[2mm] -\Delta\psi_2 = (\alpha_1 + \lambda\psi_1)_+^{p_1} & \text{in } \Omega \\[2mm] \int\limits_{\Omega} (\alpha_2 + \lambda\psi_2)_+^{p_2} = 1 = \int\limits_{\Omega} (\alpha_1 + \lambda\psi_1)_+^{p_1} \\[2mm] \psi_i = 0 \quad \text{on } \partial\Omega, \quad i = 1, 2, \\[2mm] \alpha_i \in \mathbb{R}, \quad i = 1, 2, \end{cases} \qquad \mathbf{(P)}_\lambda$$

for the unknowns $\alpha_i \in \mathbb{R}$ and $\psi_i \in C_0^{2,r}(\overline{\Omega})$, $i = 1, 2$. Here $\lambda \geq 0$, (p_1, p_2) satisfy (1.1) and, for some fixed $r \in (0, 1)$, we set,

$$C_0^{2,r}(\overline{\Omega}) = \{\psi \in C^{2,r}(\overline{\Omega}) \,:\, \psi = 0 \text{ on } \partial\Omega\},\ C_{0,+}^{2,r}(\overline{\Omega}) = \{\psi \in C_0^{2,r}(\overline{\Omega}) \,:\, \psi > 0 \text{ in } \Omega\}.$$

Here and in the rest of this paper we refer to solutions of $(\mathbf{P})_\lambda$ of this sort as classical solutions. Interestingly enough, problem $(\mathbf{P})_\lambda$ seems to be of independent interest as it defines the stationary solutions in the study of two species chemiotaxis models with nonlinear diffusion recently pursued in [14], see also Remark 1.1 below.
For $\lambda \geq 0$ fixed it is useful to denote a solution of $(\mathbf{P})_\lambda$ by $\boldsymbol{\alpha}_\lambda = (\alpha_{1,\lambda}, \alpha_{2,\lambda})$, $\boldsymbol{\psi}_\lambda = (\psi_{1,\lambda}, \psi_{2,\lambda})$, and define positive/non negative solutions as follows,

Definition. *We say that* $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ *is a positive/non negative solution of* $(\mathbf{P})_\lambda$ *if* $\alpha_{i,\lambda} > 0/\alpha_{i,\lambda} \geq 0$*,* $i = 1, 2$*, respectively.*

Clearly, if $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ is a positive solution, then by the strong maximum principle $\psi_{i,\lambda} > 0$ in Ω, $i = 1, 2$. Remark that for $\lambda > 0$, $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ is a solution of $(\mathbf{P})_\lambda$ if and only if $((\alpha_{1,\lambda}, \alpha_{2,\lambda}), (v_{1,\lambda}, v_{2,\lambda}))$ solves $(\mathbf{F})_\lambda$, with $v_{i,\lambda} = \alpha_{i,\lambda} + \lambda\psi_{i,\lambda}$, $i = 1, 2$.

We point out that, since $|\Omega| = 1$ and $\lambda \geq 0$ by assumption, then if $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ is a non negative solution of $(\mathbf{P})_\lambda$ then necessarily,

$$\alpha_{i,\lambda} \leq 1, \; i = 1, 2,$$

and the equality $\alpha_{1,\lambda} = 1 = \alpha_{2,\lambda}$ holds if and only if $\lambda = 0$. We will frequently use this fact without further comments. Actually, if $\lambda = 0$, then $(\mathbf{P})_\lambda$ takes the form

$$\begin{cases} -\Delta\psi_{1,0} = 1 \quad \text{in } \Omega \\ -\Delta\psi_{2,0} = 1 \quad \text{in } \Omega \\ \psi_{1,0} = 0 = \psi_{1,0} \quad \text{on } \partial\Omega \end{cases}$$

and admits a unique (in fact scalar) solution $(\boldsymbol{\alpha}_0, \boldsymbol{\psi}_0) = ((1, 1), (G[1], G[1]))$, where we define,

$$G[\rho](x) = \int_\Omega G_\Omega(x, y)\rho(y)\, dy, \; x \in \Omega.$$

Here G_Ω is the Green function of $-\Delta$ with Dirichlet boundary conditions on Ω. Obviously, to say that $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ is a solution of $(\mathbf{P})_\lambda$ is the same as to say that $\boldsymbol{\psi}_\lambda = (G[\rho_{2,\lambda}], G[\rho_{1,\lambda}])$ and $\int_\Omega \rho_{1,\lambda} = 1 = \int_\Omega \rho_{2,\lambda}$, where, unless otherwise specified, we set

$$\rho_{i,\lambda} = (\alpha_{i,\lambda} + \lambda\psi_{i,\lambda})_+^{p_i}, \quad i = 1, 2.$$

By a standard fixed point argument (see Appendix A) it can be shown that, for any $\lambda > 0$ small enough, there exists at least one solution of $(\mathbf{P})_\lambda$ and in particular that $\alpha_i > \frac{1}{3}$, $i = 1, 2$ for any such solution of $(\mathbf{P})_\lambda$. Also, by a well known argument in [9], one could prove the existence of at least one solution for any $\lambda > 0$ as far as $p_i < p_N$, $i = 1, 2$.
By using the weak Young inequality ([25]), we refine here the variational argument in [9], see section 4, to come up with at least one solution of $(\mathbf{P})_\lambda$ whenever (p_1, p_2) satisfy (1.1). Remark that, still as far as (1.1) is satisfied, we can prove that if $(\psi_{1,\lambda}, \psi_{2,\lambda}) \in W_0^{2,p_2}(\Omega) \times W_0^{2,p_1}(\Omega)$ is just assumed to be a strong solution of $(\mathbf{P})_\lambda$, then $(\psi_{1,\lambda}, \psi_{2,\lambda}) \in C_0^{2,r_0}(\Omega) \times C_0^{2,r_0}(\Omega)$ and satisfy to some uniform bound for bounded λ, see Lemma 2.1 below. These uniform estimates seems to be new and in particular are crucial for our purposes. Also, at least in the scalar case $p_1 = p = p_2$, $\psi_1 = \psi_2$, they are sharp since in fact, if $p \geq p_N$, it is well known ([37],[38]) that solutions may blow up for λ large enough. It is also not too difficult to prove that our variational functional (see (4.3) below) is in fact not anymore coercive as far as $p_1 = p = p_2$ and $p \geq p_N$.

Remark 1.1. *After the completion of this work, we came to learn about the recent reference* [14] *where essentially the same variational functional is analyzed* (see (4.3) below) *on the whole* *space* $\mathbb{R}^N$. *It seems to be an interesting open problem to extend our uniform estimates* (Lemma 2.1 below) *to the larger region defined as follows,*

$$p_1\left(p_2-\frac{2}{N-2}\right)<p_N \quad or \quad p_2\left(p_1-\frac{2}{N-2}\right)<p_N,$$

as far as $N\geq 3$. *Although not explicitly used in* [14], *these inequalities follow just by considering the range of parameters pursued therein. Also, we do not exclude that some arguments in* [14] *could be used to come up with the existence of solutions of* $\mathbf{(P)}_\lambda$ *in this larger region. Remark that the intersection point of these two hyperbolas is the symmetric boundary point* (p_N,p_N) *in* (1.1).
More in general, it could be interesting to investigate the relevance of the well known critical hyperbolas pushed forward in [16] *and* [17] *for problem* $\mathbf{(P)}_\lambda$.

However, our main concern is about uniqueness and qualitative behavior of $(\boldsymbol{\alpha}_\lambda,\boldsymbol{\psi}_\lambda)$ depending on λ. This is not trivial for two reasons. First of all we have in principle four unknowns to control, which are $\alpha_{i,\lambda}$ and $\psi_{i,\lambda}$, $i=1,2$. On the other side, due to the constraints in $\mathbf{(P)}_\lambda$, as recently proved in ([5]) in the scalar case $\psi_\lambda=\psi_{1,\lambda}=\psi_{2,\lambda}$, $\alpha_\lambda=\alpha_{1,\lambda}=\alpha_{2,\lambda}$, for $\lambda>0$ small enough α_λ is strictly decreasing while, by standard arguments ([18]), for fixed $\alpha_\lambda=\alpha$ and disregarding the constraints in $\mathbf{(P)}_\lambda$, then ψ_λ is strictly increasing for any λ small enough. Actually the same monotonicity property holds, of course at fixed (α_1,α_2) and for λ small enough, for $\psi_{1,\lambda}$ and $\psi_{2,\lambda}$, due to well known results about the maximum principle for cooperative elliptic systems ([22]). Therefore, unlike classical scalar problems ([18]) there is a competition between the monotonic behavior of $(\alpha_{i,\lambda}+\lambda\psi_{i,\lambda})$ as a function of λ.
This is why we do not adopt classical maximum principles based argument but rather rely on ideas recently pursued in [5, 7], see also [2, 4, 8] where different class of problems are considered. We will prove existence, uniqueness and monotonicity via a refined dual spectral formulation suitable to analyze positive solutions of the constrained problem $\mathbf{(P)}_\lambda$. The first eigenvalue in this spectral setting is denoted by $\sigma_1(\boldsymbol{\alpha}_\lambda,\boldsymbol{\psi}_\lambda)$, see section 3. In particular, we will prove the monotonicity of two naturally defined variational quantities associated to $\mathbf{(P)}_\lambda$ (see section 4), which are the energy,

$$E_\lambda:=\int_\Omega \rho_{1,\lambda}G[\rho_{2,\lambda}]\equiv\int_\Omega \rho_{i,\lambda}\psi_{i,\lambda}=\int_\Omega(\nabla\psi_{1,\lambda},\nabla\psi_{2,\lambda}),\ i=1,2,$$

and the free energy,

$$F_\lambda=\frac{1}{r_1}\int_\Omega(\rho_{1,\lambda})^{r_1}+\frac{1}{r_2}\int_\Omega(\rho_{2,\lambda})^{r_2}-\lambda\int_\Omega\rho_{1,\lambda}G[\rho_{2,\lambda}],$$

where $r_i=1+\frac{1}{p_i}$, $i=1,2$. Furthermore, we can prove the monotonicity of the linear combination

$$\frac{p_1\alpha_{1,\lambda}}{p_1+1}+\frac{p_2\alpha_{2,\lambda}}{p_2+1}.$$

Set $\mathbf{p}=(p_1,p_2)$ and

$$\lambda^*(\Omega,\mathbf{p})=\sup\{\lambda>0\,:\,\sigma_1(\boldsymbol{\alpha}_\lambda,\boldsymbol{\psi}_\lambda)>0,\ \alpha_{i,\mu}>0,\ i=1,2,\text{ for any solution of }\mathbf{(P)}_\mu,\ \forall\,\mu<\lambda\}.$$

It can be shown, see Lemma A.1 in Appendix A and Proposition 3.4 in section 3, that $\lambda^*(\Omega,\mathbf{p})$ is well defined and strictly positive and our first task is to prove that $\lambda^*(\Omega,\mathbf{p})<+\infty$. More exactly our first result is about the existence of a free boundary in the interior of Ω for solutions of $\mathbf{(F)}_\lambda$ with (p_1,p_2) satisfying (1.1). The point here is that one would like to know whether or not, for a fixed λ, $\min\{\alpha_{1,\lambda},\alpha_{2,\lambda}\}$ is negative, which implies in particular that at least one among $\Omega_{i,-}:=\{x\in\Omega\,:\,v_i<0\}$, $i=1,2$ is not empty. In the scalar case this problem for $p=1$ is fully understood, see [9, 30, 36], while, for $p>1$, the existence of a multiply connected free boundary has been proved in [38] for λ large and under some assumptions about the existence of non degenerate critical points of a suitably defined Kirchoff-Routh type functional. Still for λ large, but only for $N=2$ and for domains with non trivial topology, a similar result has been obtained in [27]. Other sufficient conditions for the existence of solutions with $\alpha < 0$ has been found in [1], which however assume the nonlinearity v_+^p to be replaced by $g_+(x,v)$ satisfying $g(x,t) \geq ct$, for some $c > 0$, which therefore does not fit our scalar problem. More recently it has been shown in [3] that there are no scalar solutions with $\alpha_\lambda \geq 0$ for λ large. We generalize that result here to the case of the cooperative and strongly coupled systems of free boundary type $(\mathbf{F})_\lambda$.

Theorem 1.2. *Let (p_1, p_2) satisfy (1.1). Then we have:*

(a) (Existence) *For any $\lambda > 0$ there exists at least one solution of $(\mathbf{F})_\lambda$.*

(b) (Existence of free boundary) *Suppose either $N = 2$ or $N \geq 3$ and Ω convex. Then, there exists $\overline{\lambda} = \overline{\lambda}(\Omega, \mathbf{p}, N) > 0$ depending only on $\mathbf{p}, N$ and Ω such that if $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ is a non negative solution of $(\mathbf{P})_\lambda$, then $\lambda \leq \overline{\lambda}$. In particular, for any $\lambda > \overline{\lambda}$ we have $\min\{\alpha_{1,\lambda}, \alpha_{2,\lambda}\} < 0$ for any solution of $(\mathbf{F})_\lambda$.*

As mentioned above the existence part in (a) follows by a refinement of the variational argument for the scalar case about $(\mathbf{P})_\lambda$ provided in [9]. We prove (b) by a blow up argument, which is a well known tool in the study of a priori estimates for Lane-Emden systems, see [20] and references therein. However the situation here is slightly different from standard models, which is why we provide a self contained proof.

We denote by $\mathcal{G}(\Omega)$ the set of solutions of $(\mathbf{P})_\lambda$ for $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$. Our next aim is to show uniqueness of these solutions, see Theorem 1.3 below. Observe that, under the assumption of Theorem 1.2-(b), we have $\lambda^*(\Omega, \mathbf{p}) < +\infty$. Actually, since we do not expect uniqueness for any $\lambda \in (0, +\infty)$, we believe Theorem 1.2-(b) holds true also for non-convex domains with $N \geq 3$.
Let $B_r = \{x \in \mathbb{R}^N : |x| < r\}$ with volume $|B_r|$, $\mathbb{D}_N$ be the N-dimensional ball of unit volume and let us denote by,

$$\Lambda(\Omega, t) = \inf_{w \in H_0^1(\Omega), w \not\equiv 0} \frac{\int_\Omega |\nabla w|^2}{\left(\int_\Omega |w|^t\right)^{\frac{2}{t}}}, \tag{1.2}$$

which provides the best constant in the Sobolev embedding $\|w\|_p \leq S_p(\Omega)\|\nabla w\|_2$, $S_p(\Omega) = \Lambda^{-2}(\Omega, p)$, $p \in [1, 2p_N)$. Let us set

$$\sigma_{1,*} = \liminf_{\lambda \to \lambda^*(\Omega, \mathbf{p})^-} \sigma_1(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda), \quad \alpha_{i,*} = \liminf_{\lambda \to \lambda^*(\Omega, \mathbf{p})^-} \alpha_{i,\lambda},\ i = 1, 2.$$

For the sake of clarity, we point out that if a map $\mathcal{M}$ from an interval $[a, b] \subset \mathbb{R}$ to a Banach space X is said to be real analytic, then it is understood that $\mathcal{M}$ can be extended in an open neighborhood of a and b where it admits a power series expansion, totally convergent in the X-norm.
Here $\|\cdot\|_{i,\lambda}$ stands for the weighted norms naturally associated to the problem, see section 3 for definitions.

Theorem 1.3. *Let (p_1, p_2) satisfy (1.1). Then we have:*

1. (Uniqueness) *For any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$ there exists a unique solution $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ of $(\mathbf{P})_\lambda$ and $\mathcal{G}(\Omega)$ is a real analytic simple curve of positive solutions $[0, \lambda^*(\Omega, \mathbf{p})) \ni \lambda \mapsto (\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$. As $\lambda \to 0^+$ we have,*
$\boldsymbol{\alpha}_\lambda = (1, 1) + \mathrm{O}(\lambda)$, $\boldsymbol{\psi}_\lambda = (\psi_{1,0}, \psi_{2,0}) + \mathrm{O}(\lambda)$, $E_\lambda = E_0(\Omega) + \mathrm{O}(\lambda)$, *where,*
$$E_0(\Omega) = \int_\Omega \int_\Omega G_\Omega(x, y)\, dxdy \leq E_0(\mathbb{D}_N) = \frac{|B_1|^{-\frac{2}{N}}}{2(N+2)}.$$
2. (Monotonicity) *For any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$ it holds,*
$$\frac{dF_\lambda}{d\lambda} < 0,\ \frac{dE_\lambda}{d\lambda} \geq 0,\ \frac{d}{d\lambda}\left(\frac{p_1 \alpha_{1,\lambda}}{p_1 + 1} + \frac{p_2 \alpha_{2,\lambda}}{p_2 + 1}\right) < 0. \tag{1.3}$$
Moreover,
$$\frac{dE_\lambda}{d\lambda} \geq p_1 \|[\psi_{1,\lambda}]_{1,\lambda}\|_{1,\lambda}^2 + p_2 \|[\psi_{2,\lambda}]_{2,\lambda}\|_{2,\lambda}^2.$$

3. (Spectral estimates) *If either* $\sigma_{1,*}=0$ *or if* $\alpha_{i,*}=0$, $i=1,2$, *then*

$$\lambda^*(\Omega,\mathbf{p})\geq \frac{1}{p_2}\Lambda(\Omega,2p_2).$$

Clearly $E_0(\Omega)$ is just the torsional rigidity of Ω. The above theorem holds for any smooth and bounded domain, in any dimension and for any subcritical (in the sense of (1.1)) exponent. Remark that the result is sharp in the scalar case $p=p_1=p_2$, $\alpha_\lambda=\alpha_{1,\lambda}=\alpha_{2,\lambda}$, $\psi_\lambda=\psi_{1,\lambda}=\psi_{2,\lambda}$, where we have that $\alpha_\lambda>0$ and $\sigma_1(\alpha_\lambda,\psi_\lambda)>0$ for any $\lambda<\frac{1}{p}\Lambda(\Omega,2p)$, see [5]. In particular, it has been shown in [5] that if $p<p_N$ there exists a positive solution for any $\lambda<\frac{1}{p}\Lambda(\Omega,2p)$, then by Theorem 1.3 we immediately deduce the following corollary about the case $1\leq p_1=p_2<p_N$.

Corollary 1.4. *Let* $1\leq p_1=p_2<p_N$. *For any* $\lambda\in[0,\lambda^*(\Omega,\mathbf{p}))$ *the unique solution of* $(\mathbf{P})_\lambda$ *is the scalar solution* $\alpha_\lambda=\alpha_{1,\lambda}=\alpha_{2,\lambda}$, $\psi_\lambda=\psi_{1,\lambda}=\psi_{2,\lambda}$.

We still do not know whether or not $\sigma_{1,*}=0$. However, as far as $p_1\neq p_2$, it seems that $\alpha_{i,*}=0$, $i=1,2$ is not a natural assumption, actually a more reasonable guess is that, in general, if $\sigma_{1,*}>0$, then either $\alpha_{1,*}=0,\alpha_{2,*}>0$ or $\alpha_{1,*}>0,\alpha_{2,*}=0$. This is interesting since in this case we come up with the parametrization of an unbounded branch of solutions of $(\mathbf{H})$.
For any classical solution $\mathbf{u}=(u_1,u_2)$ of $(\mathbf{H})$ for some $\boldsymbol{\mu}=(\mu_1,\mu_2)\in([0,+\infty))^2$ we define,

$$\gamma(\boldsymbol{\mu},\mathbf{u})=\frac{p_1}{p_1+1}\frac{1}{\|1+u_1\|_{p_1}}+\frac{p_2}{p_2+1}\frac{1}{\|1+u_2\|_{p_2}},$$

$$E(\boldsymbol{\mu},\mathbf{u})=\frac{1}{2\mu_1\mu_2}\int_\Omega\frac{\mu_1(1+u_1)^{p_1}u_1+\mu_2(1+u_2)^{p_2}u_2}{\|1+u_1\|_{p_1}^{p_1}\|1+u_2\|_{p_2}^{p_2}},$$

$$F(\boldsymbol{\mu},\mathbf{u})=\gamma(\boldsymbol{\mu},\mathbf{u})+\frac{p_1p_2-1}{(p_2+1)(p_1+1)}E(\boldsymbol{\mu},\mathbf{u}).$$

Then we have,

Theorem 1.5. *Let* (p_1,p_2) *satisfy* (1.1) *and* $\mathcal{G}(\Omega)$ *be the set of unique solutions* $(\boldsymbol{\alpha}_\lambda,\boldsymbol{\psi}_\lambda)$ *of* $(\mathbf{P})_\lambda$ *for* $\lambda\in[0,\lambda^*(\Omega,\mathbf{p}))$. *Then*

$$\mathbf{u}_\lambda=(u_{1,\lambda},u_{2,\lambda})=\left(\frac{\lambda}{\alpha_{1,\lambda}}\,\psi_{1,\lambda},\frac{\lambda}{\alpha_{2,\lambda}}\,\psi_{2,\lambda}\right),$$

is a solution of $(\mathbf{H})$ *with,*

$$\boldsymbol{\mu}_\lambda=(\mu_{1,\lambda},\mu_{2,\lambda})=\left(\lambda\,\frac{\alpha_{1,\lambda}^{p_1}}{\alpha_{2,\lambda}},\lambda\,\frac{\alpha_{2,\lambda}^{p_2}}{\alpha_{1,\lambda}}\right),$$

and for any $\lambda\in[0,\lambda^*(\Omega,\mathbf{p}))$ *it holds,*

$$\frac{dF}{d\lambda}(\boldsymbol{\mu}_\lambda,\mathbf{u}_\lambda)<0,\quad \frac{dE}{d\lambda}(\boldsymbol{\mu}_\lambda,\mathbf{u}_\lambda)\geq 0,\quad \frac{d\gamma}{d\lambda}(\boldsymbol{\mu}_\lambda,\mathbf{u}_\lambda)<0. \tag{1.4}$$

In particular, if $\alpha_{1,*}=0,\alpha_{2,*}>0$ *then* $(\boldsymbol{\mu}_\lambda,\mathbf{u}_\lambda)$ *is a real analytic and unbounded curve and, possibly along a subsequence, we have that*

$$\mu_{1,\lambda}\to 0,\quad \mu_{2,\lambda}\to+\infty,\quad \|1+u_{1,\lambda}\|_{p_1}\to+\infty.$$

Moreover, under the assumption of Theorem 1.2-(b), we have $\frac{u_{1,\lambda}}{\|1+u_{1,\lambda}\|_{p_1}}\to\lambda^*\psi_{1,*}$, $\frac{u_{2,\lambda}}{\|1+u_{2,\lambda}\|_{p_2}}\to\lambda^*\psi_{2,*}$, *with convergence in* $C^2(\overline{\Omega})$ *where* $(\psi_{1,*},\psi_{2,*})$ *is a solution of* $(\mathbf{P})_\lambda$ *with* $\lambda=\lambda^*=\lambda^*(\Omega,\mathbf{p})$, $\alpha_1=0$ *for some* $\alpha_2>0$. *The conclusion is analogous in the case* $\alpha_{1,*}>0,\alpha_{2,*}=0$.

Concerning Theorem 1.3, due to the competition between the monotonic behavior of $\alpha_{i,\lambda}$ and $\psi_{i,\lambda}$, it seems difficult to attack the problem by arguments based on the standard maximum principle. On the other side, the fact that E_λ, F_λ are monotonic increasing could be deduced for variational solutions of $(\mathbf{P})_\lambda$ once we know the uniqueness of solutions. However this is not enough to claim the smoothness of E_λ or the monotonicity/smoothness of $\left(\frac{p_1\alpha_{1,\lambda}}{p_1+1}+\frac{p_2\alpha_{2,\lambda}}{p_2+1}\right)$. The problem is more subtle for E_λ since it seems that variational arguments do not yield any information in this case.

Therefore, the proof of Theorem 1.3 relies on the interplay between a refined bifurcation analysis and the variational formulation of $(\mathbf{P})_\lambda$.
The crucial point is the set up of a sort of dual "Hamiltonian" spectral theory for the linearized operator of $(\mathbf{P})_\lambda$, see the definition (3.9) of $\mathbf{L}_\lambda=(L_{1,\lambda},L_{2,\lambda})$ and the related eigenvalues equation (3.10) in section 3. Remark that the eigenvalue equation is a non standard one, which is why we refer to it as an "Hamiltonian" eigenvalue problem. The use of the operator $\mathbf{L}_\lambda$ is rather delicate also because it arises as the linearization of a vectorial constrained problem ($\int_\Omega \rho_{i,\lambda}=1, i=1,2$) with respect to $(\boldsymbol{\alpha}_\lambda,\boldsymbol{\psi}_\lambda)$, which yields a non-local problem. As a consequence it is not true in general, neither for scalar solutions, that its first eigenvalue, which we denote by $\sigma_1(\boldsymbol{\alpha}_\lambda,\boldsymbol{\psi}_\lambda)$, is simple and neither that if $\sigma_1(\boldsymbol{\alpha}_\lambda,\boldsymbol{\psi}_\lambda)$ is positive then the maximum principle holds. For example this is exactly what happens in the scalar case for $\lambda=0$ on $\mathbb{D}_2$, where $\sigma_1(\alpha_0,\psi_0)$ can be evaluated explicitly (see [4]) and one finds that $\sigma_1(\alpha_0,\psi_0)=\lambda^{(2,0)}(\mathbb{D}_2)\simeq\pi(3,83)^2$ has three eigenfunctions, two of which indeed change sign. Here $\lambda^{(2,0)}(\Omega)$ is the first non vanishing eigenvalue of $-\Delta$ on Ω on the space of $H^1(\Omega)$ vanishing mean functions with constant boundary trace. See also [6] for a related results.

However, if for a positive solution $(\boldsymbol{\alpha}_\lambda,\boldsymbol{\psi}_\lambda)$ with $\lambda\geq 0$ it holds $0\notin\sigma(\mathbf{L}_\lambda)$, where $\sigma(\mathbf{L}_\lambda)$ stands for the spectrum of $\mathbf{L}_\lambda$, then by the real analytic implicit function theorem ([10]) the set of solutions of $(\mathbf{P})_\lambda$ is locally a real analytic curve of positive solutions. In particular a real analytic curve of positive solutions exists around $(\boldsymbol{\alpha}_0,\boldsymbol{\psi}_0)$. Since solutions of $(\mathbf{P})_\lambda$ are uniformly bounded (see Lemma 2.1) then for any $\lambda<\lambda^*(\Omega,\mathbf{p})$ there exists a unique solution and these solutions form a real analytic curve which we denoted by $\mathcal{G}(\Omega)$. An a priori bound from below far away from zero for $\sigma_1(\boldsymbol{\alpha}_\lambda,\boldsymbol{\psi}_\lambda)$ for positive solutions can be derived at this stage, see Proposition 3.4. An estimate about the range where the $\alpha_{i,\lambda}$, $i=1,2$ may possibly vanish at the same time follows as well, see Proposition 3.5. At this point, since we know that F_λ, E_λ and $\alpha_{i,\lambda}$ are real analytic as functions of λ as far as $\lambda<\lambda^*(\Omega,\mathbf{p})$, then, by the variational characterization of $(\boldsymbol{\alpha}_\lambda,\boldsymbol{\psi}_\lambda)$, we deduce the monotonicity properties of F_λ, E_λ and $\alpha_{i,\lambda}$. The estimate about the derivative of E_λ requires a more careful analysis of the Fourier expansion of $\frac{dE_\lambda}{d\lambda}$ in terms of the "Hamiltonian" Fourier basis, see section 4.
It would be interesting to find a fourth monotonic quantity naturally associated to the problem, for example the self-interaction energy

$$E_{\lambda,s}:=\frac{1}{2}\int_\Omega \rho_{2,\lambda}G[\rho_{2,\lambda}]+\frac{1}{2}\int_\Omega \rho_{1,\lambda}G[\rho_{1,\lambda}].$$

However, it seems not easy to catch the qualitative behavior of $E_{\lambda,s}$ and we postpone this problem to a future work.

This paper is organized as follows. In section 2 we discuss about the existence of the free boundary and prove Theorem 1.2. In section 3 we set up the spectral and bifurcation analysis with the needed spectral estimates. In section 4 we prove existence of variational solutions, uniqueness and monotonicity, which yield the proof of Theorem 1.3 and, as a corollary, that of Theorem 1.5. The proof of the existence of solutions and of the positivity of $\alpha_{i,\lambda}$ for λ small is discussed in Appendix A.

2. Existence of the free boundary

For later purposes, see either Theorem 4.2 below, we prove a regularity result of independent interest, showing that if $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ is a solution of $\mathbf{(P)}_\lambda$ such that $(\psi_{1,\lambda}, \psi_{2,\lambda}) \in W^{2,p_2}_0(\Omega) \times W^{2,p_1}_0(\Omega)$ is just assumed to be a strong solution of $\mathbf{(P)}_\lambda$, then $(\psi_{1,\lambda}, \psi_{2,\lambda}) \in C^{2,r_0}_0(\Omega) \times C^{2,r_0}_0(\Omega)$ and is uniformly bounded in $C^{2,r_0}_0(\Omega) \times C^{2,r_0}_0(\Omega)$, as far as λ is bounded as well. Indeed we have,

Lemma 2.1. *Let $\mathbf{p} = (p_1, p_2)$ satisfy (1.1). For any $\overline{\lambda} > 0$ there exists a positive constant $C_1 = C_1(r, \Omega, \overline{\lambda}, \mathbf{p}, N)$ depending only on Ω, $\overline{\lambda}$, $\mathbf{p}$, N and $r \in (0,1)$ such that $\|\psi_{i,\lambda}\|_{C^{2,r_0}_0(\overline{\Omega})} \leq C_1$, $i = 1,2$ for any strong solution $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ of $\mathbf{(P)}_\lambda$ with $\lambda \in [0, \overline{\lambda}]$, where $r_0 = \min\{p_1, p_2, r\}$.*

Proof. To simplify the notations we set $(\alpha_i, \psi_i) = (\alpha_{i,\lambda}, \psi_{i,\lambda})$. Since Ω is of class C^3, by standard elliptic estimates and a bootstrap argument it is enough to prove that either ψ_1 or ψ_2 is bounded. We prove only the case $N \geq 3$ which is more delicate.

Let (p_1, p_2) satisfy(1.1) and assume w.l.o.g. that $p_1 < p_N$. Since $\int_\Omega (\alpha_i + \lambda\psi_i)_+^{p_i} = 1$, $i = 1,2$, then it is well known ([26]) that for any $t \in [1, \frac{N}{N-1})$ there exists $C = C(t, N, \Omega)$ such that $\|\psi_i\|_{W^{1,t}_0(\Omega)} \leq C(t, N, \Omega)$, $i = 1,2$ for any solution of $\mathbf{(P)}_\lambda$. Thus, by the Sobolev inequality, for any $1 \leq s < \frac{N}{N-2}$ we have $\|\psi_i\|_{L^s(\Omega)} \leq C(s, N, \Omega)$, $i = 1,2$ for some $C(s, N, \Omega)$. By the maximum principle $\psi_i \geq 0$, $i = 1,2$. Thus, either $\alpha_i > 0$ and then $\int_\Omega (\alpha_i + \lambda\psi_i) = 1$ and (recall $|\Omega| = 1$) $\alpha_i \leq 1$, or $\alpha_i < 0$ and then $(\alpha_i + \psi_i)_+ \leq \psi_i$. Since $p_1 < p_N$, then for any $1 < m < \frac{p_N}{p_1}$, $\|(\alpha_1 + \lambda\psi_1)^{p_1}\|_{L^m(\Omega)} \leq C(p_1, N, \lambda, s, \Omega)$.
From now on we suppress the indications of the properties of the various constants involved in the estimates, being understood that C is just a suitable uniform constant which do not depend by the solutions.

By standard elliptic theory we have that $\|\psi_2\|_{W^{2,m}_0(\Omega)} \leq C$, for any $m < \frac{p_N}{p_1}$ and by the Sobolev embedding either $p_1 \leq \frac{2}{N-2}$ and then $\|\psi_2\|_{L^r(\Omega)} \leq C$ for any $r \geq 1$ and then the desired conclusion follows in a standard way by a bootstrap argument, or $\frac{2}{N-2} < p_1 < p_N$ and then $\|\psi_2\|_{L^r(\Omega)} \leq C$ for any $r < \frac{N}{p_1 - \frac{2}{N-2}}$. By (1.1) we have

$$p_2 < \frac{p_1 + N}{p_1(N-2) - 1},$$

and observe that, putting $p_1 = \frac{x}{N-2}$, $x \in (2, N)$,

$$m_{N,p_1} := \frac{\frac{N}{p_1 - \frac{2}{N-2}}}{\frac{p_1+N}{p_1(N-2)-1}} = \frac{N(N-2)}{p_1+N}\frac{p_1(N-2)-1}{p_1(N-2)-2} = (N-2)f_N(x),$$

where $f_N(x) = \frac{x-1}{(x-2)(1+a_N x)}$, $a_N = \frac{1}{N(N-2)}$. Therefore, since f_N is decreasing, we see that m_{N,p_1} is monotonic decreasing, with $m_{N,p_1} \to +\infty$ as $p_1 \to \left(\frac{2}{N-2}\right)^+$, $m_{N,p_1} \to (N-2)^+$ as $p_1 \to \left(\frac{N}{N-2}\right)^-$.
In particular $\|(\alpha_2 + \lambda\psi_2)^{p_2}\|_{L^n(\Omega)} \leq C$ for any $n < m_{N,p_1}$, whence by standard elliptic theory $\|\psi_1\|_{W^{2,n}_0(\Omega)} \leq C$, for any $n < m_{N,p_1}$ and either $m_{N,p_1} \geq \frac{N}{2}$ and then $\|\psi_1\|_{L^r(\Omega)} \leq C$ for any $r \geq 1$ and then the desired conclusion follows in a standard way by a bootstrap argument, or $m_{N,p_1} < \frac{N}{2}$ and then $\|\psi_1\|_{L^r(\Omega)} \leq C$ for any $r < \frac{N}{N - 2m_{N,p_1}}$. On the other side, as deduced above, we have that $m_{N,p_1} \geq (N-2)$ whence in particular $m_{N,p_1} \geq \frac{N}{2}$, as far $N \geq 4$ and we are just left with the case $N = 3$ which requires a different argument.
With the notations adopted above, we have that $\|(\alpha_2 + \lambda\psi_2)^{p_2}\|_{L^n(\Omega)} \leq C$ for any $n < m^{(1)}_{3,p_1}$, where

$$m^{(1)}_{3,p_1} = \frac{1}{1 + \frac{p_1}{3}}\frac{p_1 - 1}{p_1 - 2} \quad \text{and} \quad 1 < m^{(1)}_{3,p_1} < \frac{3}{2}.$$

Here the assumption about p_1 is that $p_1 \in (\overline{p}, 3)$ where $m^{(1)}_{3,\overline{p}} = \frac{3}{2}$ (actually $\overline{p} > \frac{5}{2}$), where we recall that $m^{(1)}_{3,p_1}$ is decreasing in p_1 and $m_{3,3} = 1$. In particular, as mentioned above, by standard elliptic theory $\|\psi_1\|_{W^{2,n}_0(\Omega)} \leq C$ for any $n < m^{(1)}_{3,p_1}$ and by the Sobolev embedding $\|\psi_1\|_{L^s(\Omega)} \leq C$ for any $s < s_2 := \frac{3}{3-2m^{(1)}_{3,p_1}}$. We define

$$\sigma_1 := \frac{s_2}{3} = \frac{1}{3 - 2m^{(1)}_{3,p_1}} > 1, \quad \delta := \sigma_1 - 1,$$

whence $\|(\alpha_2 + \lambda\psi_1)^{p_1}\|_{L^m(\Omega)} \leq C$ for any $m < \frac{s_2}{p_1} = \frac{3}{p_1}\sigma_1$. Therefore, assuming w.l.o.g. that

$$\sigma_1 < \frac{p_1}{2},$$

by standard elliptic theory $\|\psi_2\|_{W^{2,m}_0(\Omega)} \leq C$ for any $m < \frac{3}{p_1}\sigma_1$ and by the Sobolev embedding $\|\psi_2\|_{L^r(\Omega)} \leq C$ for any $r < \frac{3\sigma_1}{p_1 - 2\sigma_1}$. As a consequence $\|(\alpha_2 + \lambda\psi_2)^{p_2}\|_{L^n(\Omega)} \leq C$ for any $n < m^{(2)}_{3,p_1}$ where, by using (1.1) once more, we define,

$$m^{(2)}_{3,p_1} = \frac{3\sigma_1}{p_1 - 2\sigma_1}\frac{p_1 - 1}{p_1 + 3} = \sigma_1\frac{p_1 - 2}{p_1 - 2\sigma_1}m^{(1)}_{3,p_1} > \sigma_1 m^{(1)}_{3,p_1} = \frac{m^{(1)}_{3,p_1}}{3 - 2m^{(1)}_{3,p_1}}.$$

Obviously we can assume w.l.o.g. that $m^{(2)}_{3,p_1} < \frac{3}{2}$, that is, using the last equality,

$$1 < m^{(1)}_{3,p_1} < \frac{9}{8} \text{ and consequently } 1 < \sigma_1 < \frac{4}{3}.$$

Therefore, as above $\|\psi_1\|_{L^s(\Omega)} \leq C$ for any $s < s_3 := \frac{3}{3-2m^{(2)}_{3,p_1}}$ and we define

$$\sigma_2 := \frac{s_3}{s_2} = \frac{3 - 2m^{(1)}_{3,p_1}}{3 - 2m^{(2)}_{3,p_1}}.$$

At this point, since $m^{(2)}_{3,p_1} > \sigma_1 m^{(1)}_{3,p_1}$, elementary arguments show that

$$\sigma_2 > \frac{3 - 2m^{(1)}_{3,p_1}}{3 - 2\sigma_1 m^{(1)}_{3,p_1}} = 1 + \frac{2m^{(1)}_{3,p_1}(\sigma_1 - 1)}{3 - 2\sigma_1 m^{(1)}_{3,p_1}} > 1 + \delta = \sigma_1.$$

Consequently $\|(\alpha_2+\lambda\psi_1)^{p_1}\|_{L^m(\Omega)} \leq C$ for any $m < \frac{s_3}{p_1} = \frac{3}{p_1}\sigma_1\sigma_2 > \frac{3}{p_1}\sigma_1^2$ and by standard elliptic theory and the Sobolev embedding we have that $\|(\alpha_2 + \lambda\psi_2)^{p_2}\|_{L^n(\Omega)} \leq C$ for any $n < m^{(3)}_{3,p_1}$ where, by using (1.1) once more, we define,

$$m^{(3)}_{3,p_1} := \frac{3\sigma_1^2}{p_1 - 2\sigma_1^2}\frac{p_1 - 1}{p_1 + 3} = \sigma_1\frac{3\sigma_1}{p_1 - 2\sigma_1^2}\frac{p_1 - 1}{p_1 + 3} > \sigma_1 m^{(2)}_{3,p_1},$$

where we can assume w.l.o.g. that,

$$\sigma_1^2 < \frac{p_1}{2}.$$

At this point, by induction, it is not too difficult to prove that after a finite number of iterations, either $p_1 - 2\sigma_1^{k-1} \leq 0$ or $m^{(k)}_{3,p_1} = \frac{3\sigma_1^{k-1}}{p_1 - 2\sigma_1^{k-1}}\frac{p_1 - 1}{p_1 + 3}$ will become larger than $\frac{3}{2}$ and the desired conclusion follows, in this case as well. □

Next we present the proof of Theorem 1.2 about a priori estimates and the existence of a free boundary.

Proof. We postpone the proof of (a), i.e. the existence of at least one solution of $\mathbf{(P)}_\lambda$ for any $\lambda > 0$, to Theorem 4.2 in section 4.
Therefore we are going to prove (b). We argue by contradiction and assume that there exists a sequence of non negative solutions $((\alpha_{1,n}, \alpha_{2,n}), (\psi_{1,n}, \psi_{2,n}))$ of $\mathbf{(P)}_\lambda|_{\lambda=\lambda_n}$ such that $\lambda_n \to +\infty$. By Lemma 2.1, for any fixed n it holds $\|\psi_{i,n}\|_\infty \leq C_n$, $i = 1, 2$. Let $m_{i,n} = \sup\limits_{\Omega}(\alpha_{i,n} + \lambda_n \psi_{i,n})$, $i = 1, 2$. We split the proof in various steps.

STEP 1. Along a subsequence we have $\max\{m_{1,n}, m_{2,n}\} \to +\infty$, as $n \to +\infty$.
We will need the following lemma whose proof can be found in [3].

Lemma 2.2. *Let ψ be any solution of*

$$\begin{cases} -\Delta\psi = f & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega \end{cases}$$

where $\int_\Omega f = 1$ and $\int_\Omega |f|^N \leq C$. Then,

$$\int_\Omega |\nabla\psi|^2 \geq c > 0,$$

for some positive constant $c > 0$ depending only by C, N and Ω.

At this point we argue by contradiction and assume that there exists $C > 0$ such that

$$\sup_n \max\{m_{1,n}, m_{2,n}\} \leq C,$$

so that $\sup\limits_n \max\{\|\psi_{1,n}\|_\infty, \|\psi_{2,n}\|_\infty\} \leq \frac{C}{\lambda_n}$. Therefore, along a subsequence we have,

$$\int_\Omega |\nabla\psi_{1,n}|^2 = \int_\Omega (\alpha_{2,n} + \lambda_n\psi_{2,n})^{p_2}\psi_{1,n} \leq \frac{C}{\lambda_n} \to 0, \text{ as } n \to +\infty,$$

which contradicts Lemma 2.2 since $f_{i,n} = (\alpha_{i,n} + \lambda_n\psi_{i,n})^{p_i}$ obviously satisfies the needed assumptions for any $i = 1, 2$.
Therefore, along a subsequence we have $\max\{m_{1,n}, m_{2,n}\} \to +\infty$, as $n \to +\infty$.

STEP 2. Let $x_{i,n}$ be any maximum point of $\psi_{i,n}$, $i = 1, 2$, we prove that

$$\text{dist}(x_{i,n}, \partial\Omega) \geq d_0,$$

for some positive constant $d_0 > 0$.
Suppose first $N = 2$. We argue as in [24] p.223. Let $x_0 \in \partial\Omega$, ν_0 be the outer unit normal at x_0 and $B_r(x_1) \subset \mathbb{R}^N \setminus \overline{\Omega}$ such that $\overline{B}_r(x_1) \cap \partial\Omega = \{x_0\}$. This is always possible since Ω is of class C^3. After a translation and a rotation we can assume w.l.o.g. that $x_1 = 0$, $\nu_0 = (-1, 0, \cdots, 0)$. Also, after a dilation and a suitable scaling of λ_n, we can assume that $r = 1$, whence $x_0 = (1, 0, \cdots, 0)$. At this point if $N = 2$ we define (see [24] p.223) $y = \frac{x}{|x|^2}$ and $u_{i,n}(y) = \psi_{i,n}(x)$, $i = 1, 2$. The image of Ω under this Kelvin transform, say $\widetilde{\Omega}$, lies inside $B_1(0)$ and its closure touches the boundary only at x_0. Also $(u_{1,n}(y), u_{2,n}(y))$ satisfies

$$\begin{cases} -\Delta u_{1,n} = h(y)(\alpha_{2,n} + \lambda_n u_{2,n})^{p_2} & \text{in } \widetilde{\Omega} \\ \\ -\Delta u_{2,n} = h(y)(\alpha_{1,n} + \lambda_n u_{1,n})^{p_1} & \text{in } \widetilde{\Omega} \\ \\ u_{i,n} = 0 \quad \text{on } \partial\widetilde{\Omega}, \quad i = 1, 2, \\ \\ \alpha_{i,n} \geq 0, \quad i = 1, 2, \end{cases}$$

where $h(y) = \frac{1}{|y|^4}$. Let $y = (y_1, y_2)$, clearly $\frac{\partial}{\partial y_1} h(y) < 0$ as far as $y_1 > 0$, whence we can apply the argument of Theorem $2.1'$ in [24], just replacing the classical strong maximum principle and Hopf lemma used there (see Lemma H in [24]) with the corresponding results for cooperative

and strongly coupled linear elliptic systems, see Theorem 2.2 in [19]. Therefore, if $N=2$, as in [24] p.223 we deduce that in a neighborhood of x_0 depending only by the geometry of Ω there are no critical points of $\psi_{i,n}$, $i=1,2$. Since $\partial\Omega$ is compact the desired conclusion follows by a covering argument.
Suppose now $N\geq 3$ and Ω convex. As above, using the results for cooperative and strongly coupled linear elliptic systems we can exploit the argument of Theorem $2.1^{'}$ in [24]. It is well known that the convexity condition ensures then that there are no critical points of $\psi_{i,n}$, $i=1,2$ in a sufficiently small uniform neighborhood of the boundary, see [21] for further details in the scalar case.

STEP 3. We obtain a contradiction assuming that $m_{2,n}\leq m_{1,n}\to+\infty$, as $n\to+\infty$. We will never use the fact that $p_1\leq p_2$, whence a contradiction arises in the same way in the case where, along a subsequence, $m_{1,n}\leq m_{2,n}\to+\infty$, as $n\to+\infty$.
Let $x_{1,n}$ be such that $\psi_{1,n}(x_{1,n})=m_{1,n}$. Then by step 2 we have $x_{i,n}\to\overline{x}\in\Omega$ and after a translation we can assume w.l.o.g. that $x_{1,n}=0$, $\forall n\in\mathbb{N}$. There are only two possibilities: either,
(j) $\sup\limits_n \dfrac{m_{2,n}^{p_2+1}}{m_{1,n}^{p_1+1}}\leq C$, or, passing to a further subsequence if necessary,
(jj) $\dfrac{m_{1,n}^{p_1+1}}{m_{2,n}^{p_2+1}}\to 0$, as $n\to+\infty$.

We discuss (j) first and define $\delta_n^2=\frac{m_{2,n}}{\lambda m_{1,n}^{p_1}}$ and, for $i=1,2$,

$$v_{i,n}(y)=\frac{1}{m_{i,n}}(\alpha_{i,n}+\lambda\psi_{i,n}(\delta_n y)),\quad y\in\Omega_n=\{y\in\mathbb{R}^N\,:\,\delta_n y\in\Omega\}$$

which satisfy

$$\begin{cases} -\Delta v_{1,n}=\frac{m_{2,n}^{p_2+1}}{m_{1,n}^{p_1+1}}\,v_{2,n}^{p_2} \quad \text{in }\ \Omega_n \\ \\ -\Delta v_{2,n}=v_{1,n}^{p_1} \quad \text{in }\ \Omega_n \\ \\ v_{i,n}(y)\leq v_{1,n}(0)=1 \quad \text{in }\ \Omega_n,\quad i=1,2, \\ \\ v_{i,n}(y)\geq \frac{\alpha_{i,n}}{m_{i,n}}\geq 0 \quad \text{in }\ \Omega_n,\quad i=1,2. \end{cases}$$

Since $0\in\Omega$, then for any $R\geq 1$ we have that for any n large enough it holds $B_R(0)\subset\Omega_n$. Along a subsequence we can assume that $\frac{m_{2,n}^{p_2+1}}{m_{1,n}^{p_1+1}}\to\mu\in[0,+\infty)$. Since $\|v_{i,n}\|_{L^\infty(\Omega_n)}\leq C$, then by standard elliptic estimates there exists a subsequence such that $v_{i,n}$, $i=1,2$ converge in $C^2_{\rm loc}(\mathbb{R}^N)$ to v_i, $i=1,2$ which are classical solutions of

$$\begin{cases} -\Delta v_1=\mu\,v_2^{p_2} \quad \text{in }\ \mathbb{R}^N \\ \\ -\Delta v_2=v_1^{p_1} \quad \text{in }\ \mathbb{R}^N \\ \\ 0\leq v_i(y)\leq v_1(0)=1 \quad \text{in }\ \mathbb{R}^N,\quad i=1,2, \end{cases} \tag{2.1}$$

At this point observe that if $\mu=0$ then necessarily $v_1\equiv 1$ in $\mathbb{R}^N$ and then v_2 would solve,

$$\begin{cases} -\Delta v_2=1 \quad \text{in }\ \mathbb{R}^N \\ \\ 0\leq v_2(y)\leq 1 \quad \text{in }\ \mathbb{R}^N,\quad i=1,2. \end{cases}$$

By the maximum principle we would have $v_2(y)\geq\frac{1}{2N}(R^2-|y|^2)$, for any $R>0$ and in particular $v_2(0)\geq\frac{1}{2N}R^2$, for any $R>0$, which is impossible. Therefore $\mu\in(0,+\infty)$ which however is also

impossible since it is well known by the result in [34] that (1.1) implies that the unique solution of (2.1) is $v_i \equiv 0$, $i = 1, 2$.
Therefore (jj) holds and in this case we choose $\delta_n^2 = \frac{m_{1,n}}{\lambda m_{2,n}^{p_2}}$ so that $(v_{1,n}, v_{2,n})$ satisfies

$$\begin{cases} -\Delta v_{1,n} = v_{2,n}^{p_2} \quad \text{in } \Omega_n \\ \\ -\Delta v_{2,n} = \frac{m_{1,n}^{p_1+1}}{m_{2,n}^{p_2+1}} v_{1,n}^{p_1} \quad \text{in } \Omega_n \\ \\ v_{i,n}(y) \le v_{1,n}(0) = 1 \quad \text{in } \Omega_n, \quad i = 1, 2, \\ \\ v_{i,n}(y) \ge \frac{\alpha_{i,n}}{m_{i,n}} \ge 0 \quad \text{in } \Omega_n, \quad i = 1, 2. \end{cases}$$

This case is easily seen to lead to the same situation described above for $\mu = 0$, whence a contradiction arise in this case as well.
As mentioned above, by symmetry, the discussion of the case in which along a subsequence $m_{1,n} \le m_{2,n} \to +\infty$, as $n \to +\infty$, is exactly the same. □

3. Spectral and bifurcation analysis

In this section we develop the spectral and bifurcation analysis for positive solutions of $\mathbf{(P)}_\lambda$ with $\lambda \ge 0$ and (p_1, p_2) satisfying (1.1). From now on and unless otherwise specified, $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ is assumed to be a positive solution of $\mathbf{(P)}_\lambda$.

By the maximum principle $\psi_{i,\lambda} \ge 0$, $i = 1, 2$ in Ω for any solution, whence for non negative solutions $(\alpha_{i,\lambda} \ge 0)$ we have $\alpha_{i,\lambda} + \psi_{i,\lambda} \equiv (\alpha_{i,\lambda} + \psi_{i,\lambda})_+$. Therefore from now on and unless otherwise specified we will denote by,

$$\tau_{i,\lambda} = \lambda p_i,\ \rho_{i,\lambda,\alpha_i}(\psi_i) = (\alpha_i + \lambda\psi_i)^{p_i},\ \rho_{i,\lambda} = (\alpha_{i,\lambda} + \lambda\psi_{i,\lambda})^{p_i},\ i = 1, 2,$$

$$V_{i,\lambda,\alpha_i}(\psi_i) = (\alpha_i + \lambda\psi_i)^{p_i-1} \text{ and } V_{i,\lambda} = (\alpha_{i,\lambda} + \lambda\psi_{i,\lambda})^{p_i-1},$$

where $\alpha_{i,\lambda}, \psi_{i,\lambda}$, $i = 1, 2$ denote the components of a non negative solution $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ of $\mathbf{(P)}_\lambda$ and q_i denotes the conjugate exponent of p_i, that is,

$$\frac{1}{p_i} + \frac{1}{q_i} = 1, \quad i = 1, 2.$$

For $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ a non negative solution of $\mathbf{(P)}_\lambda$ we denote,

$$< \eta >_{i,\lambda} = \frac{\int_\Omega V_{i,\lambda}\eta}{\int_\Omega V_{i,\lambda}} \quad \text{and} \quad [\eta]_{i,\lambda} = \eta - < \eta >_{i,\lambda},\ i = 1, 2,$$

and define,

$$< \eta, \phi >_{i,\lambda} := \int_\Omega V_{i,\lambda}\eta\phi \quad \text{and} \quad \|\phi\|_{i,\lambda}^2 := < \phi, \phi >_{i,\lambda} = \int_\Omega V_{i,\lambda}\phi^2,\ i = 1, 2,$$

where $\{\eta, \phi\} \subset L^2(\Omega)$. For non negative solutions $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ of $\mathbf{(P)}_\lambda$, by the strong maximum principle we have that $\rho_{i,\lambda}$, $i = 1, 2$, is strictly positive in Ω, whence $< \cdot, \cdot >_{i,\lambda}$, $i = 1, 2$ define scalar products on $L^2(\Omega)$ whose norms are denoted by $\| \cdot \|_{i,\lambda}$, $i = 1, 2$. We will also adopt the useful shorthand notation,

$$m_{i,\lambda} = \int_\Omega V_{i,\lambda},\ i = 1, 2.$$

In the sequel we aim to describe possible branches of solutions of $\mathbf{(P)}_\lambda$ around a positive solution, i.e. with $\alpha_{i,\lambda} > 0$, $i = 1, 2$. To this end, it is not difficult to construct an open subset A_Ω of the Banach space of triples $(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi}) \in \mathbb{R} \times \mathbb{R}^2 \times (C^{2,r_0}_0(\overline{\Omega}\,))^2$ such that, on A_Ω, the densities $\rho_{i,\lambda,\alpha_i} = \rho_{i,\lambda,\alpha_i}(\psi_i) = (\alpha_i + \lambda\psi_i)^{p_i}$ are well defined and

$$\alpha_{i,\lambda} + \lambda\psi_{i,\lambda} \geq \frac{\alpha_{i,\lambda}}{2} \quad \text{in } \overline{\Omega}, \quad i = 1, 2, \tag{3.1}$$

in a sufficiently small open neighborhood in A_Ω of any triple of the form $(\lambda, \boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ whenever $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ is a positive ($\alpha_{i,\lambda} > 0$) solution of $\mathbf{(P)}_\lambda$. At this point we introduce the maps,

$$\mathbf{F} : A_\Omega \to (C^r(\overline{\Omega}\,))^2, \quad \mathbf{F}(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi}) := \begin{pmatrix} -\Delta\psi_1 - \rho_{2,\lambda,\alpha_2}(\psi_2) \\ \\ -\Delta\psi_2 - \rho_{1,\lambda,\alpha_1}(\psi_1) \end{pmatrix} \tag{3.2}$$

and

$$\Phi : A_\Omega \to \mathbb{R}^2 \times (C^r(\overline{\Omega}\,))^2, \quad \Phi(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi}) := \begin{pmatrix} \mathbf{F}(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi}) \\ -1 + \int_\Omega \rho_{1,\lambda,\alpha_1}(\psi_1) \\ -1 + \int_\Omega \rho_{2,\lambda,\alpha_2}(\psi_2) \end{pmatrix}, \tag{3.3}$$

and, for positive solutions and for a fixed $(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi}) \in A_\Omega$, their differentials with respect to $(\boldsymbol{\alpha}, \boldsymbol{\psi})$, that is the linear operators,

$$D_{\boldsymbol{\alpha},\boldsymbol{\psi}}\Phi(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi}) : \mathbb{R}^2 \times (C^{2,r}_0(\overline{\Omega}\,))^2 \to \mathbb{R}^2 \times (C^r(\overline{\Omega}\,))^2,$$

which acts as follows on $(\mathbf{s}, \boldsymbol{\phi}) = (s_1, s_2, \phi_1, \phi_2) \in \mathbb{R}^2 \times (C^{2,r}_0(\overline{\Omega}\,))^2$,

$$D_{\boldsymbol{\alpha},\boldsymbol{\psi}}\Phi(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi})[\mathbf{s}, \boldsymbol{\phi}] = \begin{pmatrix} D_{\boldsymbol{\psi}}\mathbf{F}(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi})[\boldsymbol{\phi}] + d_{\boldsymbol{\alpha}}\mathbf{F}(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi})[\mathbf{s}] \\ \\ \int_\Omega \Big(D_{\psi_1}\rho_{1,\lambda,\alpha_1}(\psi_1)[\phi_1] + d_{\alpha_1}\rho_{1,\lambda,\alpha_1}(\psi_1)[s_1] \Big) \\ \\ \int_\Omega \Big(D_{\psi_2}\rho_{2,\lambda,\alpha_2}(\psi_2)[\phi_2] + d_{\alpha_2}\rho_{2,\lambda,\alpha_2}(\psi_2)[s_2] \Big) \end{pmatrix}, \tag{3.4}$$

where we have introduced the differential operators,

$$D_{\boldsymbol{\psi}}\mathbf{F}(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi})[\boldsymbol{\phi}] = \begin{pmatrix} -\Delta\phi_1 - \tau_{2,\lambda}V_{2,\lambda,\alpha_2}(\psi_2)\phi_2 \\ \\ -\Delta\phi_2 - \tau_{1,\lambda}V_{1,\lambda,\alpha_1}(\psi_1)\phi_1 \end{pmatrix}, \quad \boldsymbol{\phi} = (\phi_1, \phi_2) \in (C^{2,r}_0(\overline{\Omega}\,))^2, \tag{3.5}$$

$$D_{\psi_i}\rho_{i,\lambda,\alpha_i}[\phi_i] = \tau_{i,\lambda}V_{i,\lambda,\alpha_i}(\psi_i)\phi_i, \quad \phi_i \in C^{2,r}_0(\overline{\Omega}\,),\ i = 1, 2, \tag{3.6}$$

and

$$d_{\boldsymbol{\alpha}}\mathbf{F}(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi})[\mathbf{s}] = \begin{pmatrix} -p_2V_{2,\lambda,\alpha_2}(\psi_2)s_2 \\ \\ -p_1V_{1,\lambda,\alpha_1}(\psi_1)s_1 \end{pmatrix} \quad \mathbf{s} = (s_1, s_2) \in \mathbb{R}^2, \tag{3.7}$$

$$d_{\alpha_i}\rho_{i,\lambda,\alpha_i}[s_i] = p_iV_{i,\lambda,\alpha_i}(\psi_i)s_i, \quad s_i \in \mathbb{R},\ i = 1, 2, \tag{3.8}$$

where we recall $\tau_{i,\lambda} = \lambda p_i$.
By the construction of A_Ω, see in particular (3.1), relying on known techniques about real analytic functions on Banach spaces ([10]), it can be shown that $\Phi(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi})$ is jointly real analytic in an open neighborhood of A_Ω around any triple of the form $(\lambda, \boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ whenever $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ is a positive solution of $\mathbf{(P)}_\lambda$.

For fixed $\lambda > 0$ the pair $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ is a non negative solution of $\mathbf{(P)}_\lambda$ in the classical sense as defined in the introduction if and only if $\Phi(\lambda, \boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda) = (0, 0, 0, 0)$, and, for positive solutions, we define the linearized operator,

$$\mathbf{L}_\lambda[\boldsymbol{\phi}] = \begin{pmatrix} L_{1,\lambda}[\boldsymbol{\phi}] \\ \\ L_{2,\lambda}[\boldsymbol{\phi}] \end{pmatrix} = \begin{pmatrix} -\Delta\phi_1 - \tau_{2,\lambda}V_{2,\lambda}[\phi_2]_{2,\lambda}, \\ \\ -\Delta\phi_2 - \tau_{1,\lambda}V_{1,\lambda}[\phi_1]_{1,\lambda} \end{pmatrix}. \tag{3.9}$$

We say that $\sigma = \sigma(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda) \in \mathbb{R}$ is an eigenvalue of $\mathbf{L}_\lambda$ if the "Hamiltonian" eigenvalue equation,

$$\begin{cases} -\Delta\phi_1 - \tau_{2,\lambda} V_{2,\lambda}[\phi_2]_{2,\lambda} = \sigma p_2 V_{2,\lambda}[\phi_2]_{2,\lambda}, \\ -\Delta\phi_2 - \tau_{1,\lambda} V_{1,\lambda}[\phi_1]_{1,\lambda} = \sigma p_1 V_{1,\lambda}[\phi_1]_{1,\lambda}, \end{cases} \tag{3.10}$$

admits a non-trivial weak solution $(\phi_1, \phi_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$, which is by definition an eigenfunction of σ.
We show that, with this particular definition, the eigenvalues of $\mathbf{L}_\lambda$ share the usual properties of a general self-adjoint elliptic operator.
Let us define the Hilbert space,

$$\mathbf{Y}_0 := \left\{\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \{(L^2(\Omega))^2, < \cdot, \cdot >_\lambda\} : < \varphi_i >_{i,\lambda} = 0, i = 1, 2\right\}, \tag{3.11}$$

where

$$< \boldsymbol{\varphi}, \boldsymbol{\eta} >_\lambda := p_1 < \varphi_1, \eta_1 >_{1,\lambda} + p_2 < \varphi_2, \eta_2 >_{2,\lambda}, \forall\, (\boldsymbol{\varphi}, \boldsymbol{\eta}) \in (\mathbf{Y}_0)^2$$

and the linear operator $\mathbf{T}_0 : \mathbf{Y}_0 \to \mathbf{Y}_0$, which acts on $\boldsymbol{\varphi} = (\varphi_1, \varphi_2) \in \mathbf{Y}_0$ as follows

$$\mathbf{T}_0(\boldsymbol{\varphi}) := \begin{pmatrix} \tau_{2,\lambda} G[V_{2,\lambda}\varphi_2] - < \tau_{2,\lambda} G[V_{2,\lambda}\varphi_2] >_{1,\lambda} \\ \\ \tau_{1,\lambda} G[V_{1,\lambda}\varphi_1] - < \tau_{1,\lambda} G[V_{1,\lambda}\varphi_1] >_{2,\lambda} \end{pmatrix} \equiv \begin{pmatrix} \tau_{2,\lambda}[G[V_{2,\lambda}\varphi_2]]_{1,\lambda} \\ \\ \tau_{1,\lambda}[G[V_{1,\lambda}\varphi_1]]_{2,\lambda} \end{pmatrix}. \tag{3.12}$$

By standard elliptic theory $G[V_{i,\lambda}\varphi_i] \subset W^{2,2}(\Omega)$, $i = 1, 2$, whence $\mathbf{T}_0$ is compact by the Sobolev embedding. Also for any $(\boldsymbol{\varphi}, \boldsymbol{\eta}) \in (\mathbf{Y}_0)^2$, we have,

$$< \boldsymbol{\eta}, \mathbf{T}_0(\boldsymbol{\varphi}) >_\lambda = p_1 < \eta_1, [\tau_{2,\lambda} G[V_{2,\lambda}\varphi_2]]_{1,\lambda} >_{1,\lambda} + p_2 < \eta_2, [\tau_{1,\lambda} G[V_{1,\lambda}\varphi_1]]_{2,\lambda} >_{2,\lambda} =$$

$$\lambda p_1 p_2 \int_\Omega V_{1,\lambda}\eta_1 [G[V_{2,\lambda}\varphi_2]]_{1,\lambda} + \lambda p_2 p_1 \int_\Omega V_{2,\lambda}\eta_2 [G[V_{1,\lambda}\varphi_1]_{2,\lambda} =$$

$$\lambda p_1 p_2 \int_\Omega V_{1,\lambda}\eta_1 G[V_{2,\lambda}\varphi_2] + \lambda p_1 p_2 \int_\Omega V_{2,\lambda}\eta_2 G[V_{1,\lambda}\varphi_1] =$$

$$\lambda p_1 p_2 \int_\Omega G[V_{1,\lambda}\eta_1] V_{2,\lambda}\varphi_2 + \lambda p_1 p_2 \int_\Omega G[V_{2,\lambda}\eta_2] V_{1,\lambda}\varphi_1 =$$

$$p_2 \int_\Omega [\tau_{1,\lambda} G[V_{1,\lambda}\eta_1]]_{2,\lambda} V_{2,\lambda}\varphi_2 + p_1 \int_\Omega [\tau_{2,\lambda} G[V_{2,\lambda}\eta_2]]_{1,\lambda} V_{1,\lambda}\varphi_1 =$$

$$p_1 < [\tau_{2,\lambda} G[V_{2,\lambda}\eta_2]]_{1,\lambda}, \varphi_1 >_{1,\lambda} + p_2 < [\tau_{1,\lambda} G[V_{1,\lambda}\eta_1]]_{2,\lambda}, \varphi_2 >_{2,\lambda} = < \mathbf{T}_0(\boldsymbol{\eta}), \boldsymbol{\varphi} >_\lambda,$$

which shows that $\mathbf{T}_0$ is also self-adjoint. Remark also that $< \boldsymbol{\varphi}, \mathbf{T}_0(\boldsymbol{\varphi}) >_\lambda > 0$ if $\boldsymbol{\varphi} \neq (0, 0)$, as is readily verified observing that $\psi_i = G[V_{i,\lambda}\varphi_i]$ satisfies $\int_\Omega |\nabla\psi_i|^2 = < \varphi_i, G[V_{i,\lambda}\varphi_i] >_{i,\lambda}$, $i = 1, 2$.

As a consequence, by the spectral Theorem for self-adjoint, compact, linear operators on Hilbert spaces, we have that $\mathbf{Y}_0$ is the Hilbertian direct sum of the eigenfunctions of $\mathbf{T}_0$, which can be represented as follows,

$$\boldsymbol{\varphi}_k = (\varphi_{1,k},\, \varphi_{2,k}), \varphi_{i,k} = [\phi_{i,k}]_{i,\lambda},\; i = 1, 2,\; k \in \mathbb{N} = \{1, 2, \cdots\},$$

$$\mathbf{Y}_0 = \overline{\text{Span}\,\{([\phi_{1,k}]_{1,\lambda}, [\phi_{2,k}]_{2,\lambda}),\; k \in \mathbb{N}\}},$$

for some $(\phi_{1,k}, \phi_{2,k}) \in (H_0^1(\Omega))^2$, $k \in \mathbb{N} = \{1, 2, \cdots\}$. In fact, any eigenfunction $\boldsymbol{\varphi}_k$, whose eigenvalue is $\mu_k \in \mathbb{R} \setminus \{0\}$, satisfies, $\mu_k \boldsymbol{\varphi}_k = \mathbf{T}_0(\boldsymbol{\varphi}_k)$ and, by defining,

$$\frac{\lambda}{\lambda + \sigma_k} = \mu_k \in (0, +\infty), \quad 0 > \mu_1 \geq \mu_2 \geq .... \geq \mu_k \to 0,$$

$$\phi_{1,k} := (\tau_{2,\lambda} + p_2\sigma_k) G[V_{2,\lambda}\varphi_{2,k}],\; \phi_{2,k} := (\tau_{1,\lambda} + p_1\sigma_k) G[V_{1,\lambda}\varphi_{1,k}],$$

it is easy to see that $\boldsymbol{\varphi}_k$ is an eigenfunction of $\mathbf{T}_0$ with eigenvalue μ_k if and only if $\boldsymbol{\phi}_k = (\phi_{1,k}, \phi_{2,k}) \in (H_0^1(\Omega))^2$ weakly solves,

$$\begin{cases} -\Delta\phi_{1,k} = (\tau_{2,\lambda} + \sigma_k p_2) V_{2,\lambda}[\phi_{2,k}]_{2,\lambda}, \\ -\Delta\phi_{2,k} = (\tau_{1,\lambda} + \sigma_k p_1) V_{1,\lambda}[\phi_{1,k}]_{1,\lambda}. \end{cases} \tag{3.13}$$

In particular the first eigenvalue $\sigma_1 = \sigma_1(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ is uniquely defined by the spectral radius of $\mathbf{T}_0$, $r(\mathbf{T}_0) \equiv \mu_1 = \frac{\lambda}{\lambda+\sigma_1}$ where

$$\mu_1 = r(\mathbf{T}_0) = \sup_{\boldsymbol{\varphi}\in \mathbf{Y}_0\setminus\{\mathbf{0}\}} \frac{< \boldsymbol{\varphi}, T_0(\boldsymbol{\varphi}) >_\lambda}{< \boldsymbol{\varphi}, \boldsymbol{\varphi} >_\lambda}.$$

Since $\sigma_1 + \lambda = \frac{\lambda}{\mu_1}$ and since μ_1 is positive, then

$$\lambda + \sigma_1 > 0. \tag{3.14}$$

By the Fredholm alternative, if $0 \notin \{\sigma_j\}_{j\in\mathbb{N}}$, then $\mathbf{I} - \mathbf{T}_0$ is an isomorphism of $\mathbf{Y}_0$ onto itself. Clearly, we can construct an orthonormal base of eigenfunctions $\{\boldsymbol{\phi}_k\}_{k\in\mathbb{N}}$ with respect to the scalar product $< \cdot, \cdot >_\lambda$. However we need a refined property which is the following,

Lemma 3.1. *There exists a complete orthonormal base* $\{\boldsymbol{\varphi}_k\}_{k\in\mathbb{N}}$ *of* $\boldsymbol{Y}_0$*, with the property that* $\varphi_k = ([\phi_{1,k}]_{1,\lambda}, [\phi_{2,k}]_{2,\lambda})$ *satisfies,*

$$< [\phi_{1,k}]_{1,\lambda}, [\phi_{1,j}]_{1,\lambda} >_{1,\lambda} = 0 =< [\phi_{2,k}]_{2,\lambda}, [\phi_{2,j}]_{2,\lambda} >_{2,\lambda}, \quad \forall k \neq j. \tag{3.15}$$

In particular $\{[\phi_{i,k}]\}_{k\in\mathbb{N}}$*,* $i = 1, 2$ *is a complete orthonormal base of*

$$Y_{i,0} := \left\{\varphi_i \in \{L^2(\Omega), < \cdot, \cdot >_{i,\lambda}\} : < \varphi_i >_{i,\lambda} = 0\right\}, i = 1, 2.$$

Proof. If σ_k, σ_j denote the eigenvalues of $\boldsymbol{\phi}_k$, $\boldsymbol{\phi}_j$ respectively, and if $\sigma_k \neq \sigma_j$, then for fixed k we can multiply the first equation in (3.13) by $\phi_{2,j}$, the second by $\phi_{1,j}$ and integrate by parts to conclude that

$$\begin{cases} (\tau_{1,\lambda} + \sigma_j p_1) \int_\Omega V_{1,\lambda}[\phi_{1,j}]_{1,\lambda}[\phi_{1,k}]_{1,\lambda} = (\tau_{2,\lambda} + \sigma_k p_2) \int_\Omega V_{2,\lambda}[\phi_{2,k}]_{2,\lambda}[\phi_{2,j}]_{2,\lambda}, \\ (\tau_{2,\lambda} + \sigma_j p_2) \int_\Omega V_{2,\lambda}[\phi_{2,j}]_{2,\lambda}[\phi_{2,k}]_{2,\lambda} = (\tau_{1,\lambda} + \sigma_k p_1) \int_\Omega V_{1,\lambda}[\phi_{1,k}]_{1,\lambda}[\phi_{1,j}]_{1,\lambda}. \end{cases}$$

Putting

$$x = p_1 \int_\Omega V_{1,\lambda}[\phi_{1,j}]_{1,\lambda}[\phi_{1,k}]_{1,\lambda}, \; y = p_2 \int_\Omega V_{2,\lambda}[\phi_{2,k}]_{2,\lambda}[\phi_{2,j}]_{2,\lambda},$$

this is equivalent to the system

$$\begin{cases} (\lambda + \sigma_j)x - (\lambda + \sigma_k)y = 0 \\ -(\lambda + \sigma_k)x + (\lambda + \sigma_j)y = 0 \end{cases},$$

whose determinant is not zero as far as $\sigma_k \neq \sigma_j$. Therefore in particular

$$< [\phi_{i,j}]_{i,\lambda}[\phi_{i,k}]_{i,\lambda} >_{i,\lambda} = \int_\Omega V_{i,\lambda}[\phi_{i,j}]_{1,\lambda}[\phi_{i,k}]_{i,\lambda} = 0, \; i = 1, 2,$$

whenever $\sigma_j \neq \sigma_k$. If $\sigma_k = \sigma_j$, since the eigenspace is of finite dimension, a standard componentwise orthonormalization argument shows that in fact the basis can be chosen to satisfy (3.15). At this point, since $\{\boldsymbol{\varphi}_k\}_{k\in\mathbb{N}}$ is a complete orthonormal base then also $\{[\phi_{i,k}]_{i,\lambda}\}_{k\in\mathbb{N}}$ must be a complete orthonormal base of $Y_{i,0}$ for any $i = 1, 2$. □

Concerning $D_{\boldsymbol{\alpha},\boldsymbol{\psi}}\Phi(\lambda, \boldsymbol{\alpha}, \boldsymbol{\psi})$ we have,

Proposition 3.2. *For any positive solution* $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ *of* $(\mathbf{P})_\lambda$ *with* $\lambda \geq 0$*, the kernel of* $D_{\boldsymbol{\alpha},\boldsymbol{\psi}}\Phi(\lambda, \boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ *is empty if and only if the system,*

$$\begin{cases} -\Delta\phi_1 - \tau_{2,\lambda}V_{2,\lambda}[\phi_2]_{2,\lambda} = 0, \\ -\Delta\phi_2 - \tau_{1,\lambda}V_{1,\lambda}[\phi_1]_{1,\lambda} = 0, \end{cases} \quad (\phi_1, \phi_2) \in (C_0^{2,r}(\overline{\Omega}))^2 \tag{3.16}$$

admits only the trivial solution, or equivalently, if and only if 0 *is not an eigenvalue of* $\mathbf{L}_\lambda$*.*

Proof. If $(\phi_1, \phi_2) \in (H_0^1(\Omega))^2$ solves (3.16) and since Ω is of class C^3, then by standard elliptic regularity theory and a bootstrap argument we have $(\phi_1, \phi_2) \in (C_0^{2,r}(\overline{\Omega}))^2$. Therefore, in particular 0 is not an eigenvalue of $\mathbf{L}_\lambda$ if and only if (3.16) admits only the trivial solution.
Suppose first that there exists a non-vanishing pair $(\mathbf{s}, \boldsymbol{\phi}) \in \mathbb{R}^2 \times (C_0^{2,r}(\overline{\Omega}))^2$ such that

$$D_{\boldsymbol{\alpha},\boldsymbol{\psi}}\Phi(\lambda, \boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)[\mathbf{s}, \boldsymbol{\phi}] = (\mathbf{0}, \mathbf{0}).$$

Then the second pair of equations in (3.4), that is $\int_\Omega \left(D_\psi \rho_{i,\lambda}[\phi_i] + d_{\alpha_i}\rho_{i,\lambda}[s_i]\right)\Big|_{(\boldsymbol{\alpha},\boldsymbol{\psi})=(\boldsymbol{\alpha}_\lambda,\boldsymbol{\psi}_\lambda)} = 0$, take the form,

$$p_i \int_\Omega (\lambda V_{i,\lambda}\phi_i + V_{i,\lambda} s_i) = 0,\ i = 1, 2$$

which is equivalent to

$$s_i = s_{i,\lambda} = -\lambda < \phi_i >_{i,\lambda}.$$

Substituting this relations into the first pair of equations,

$$\mathbf{L}_\lambda[\boldsymbol{\phi}] = D_{\boldsymbol{\psi}} F(\lambda, \boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)[\boldsymbol{\phi}] + d_{\boldsymbol{\alpha}} F(\lambda, \boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)[\mathbf{s}_\lambda] = 0,$$

we conclude that $\boldsymbol{\phi} = (\phi_1, \phi_2)$ is a non-trivial, classical solution of (3.16).
This shows one part of the claim, while on the other side, if a non-trivial, classical solution of (3.16) exists, then by arguing the other way around, obviously we can find some $(\mathbf{s}, \boldsymbol{\phi}) \neq (\mathbf{0}, \mathbf{0})$ such that $D_{\boldsymbol{\alpha},\boldsymbol{\psi}}\Phi(\lambda, \boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)[\mathbf{s}, \boldsymbol{\phi}] = (\mathbf{0}, \mathbf{0})$, as claimed. □

We state now the result needed to describe branches of solutions of $(\mathbf{P})_\lambda$ at regular points.

Lemma 3.3. *Let $(\boldsymbol{\alpha}_{\lambda_0}, \boldsymbol{\psi}_{\lambda_0})$ be a positive solution of $(\mathbf{P})_\lambda$ with $\lambda = \lambda_0 \geq 0$.*
If 0 is not an eigenvalue of $\mathbf{L}_{\lambda_0}$, then:
(i) $D_{\boldsymbol{\alpha},\boldsymbol{\psi}}\Phi(\lambda_0, \boldsymbol{\alpha}_{\lambda_0}, \boldsymbol{\psi}_{\lambda_0})$ is an isomorphism;
(ii) There exists an open neighborhood $\mathcal{U} \subset A_\Omega$ of $(\lambda_0, \boldsymbol{\alpha}_{\lambda_0}, \boldsymbol{\psi}_{\lambda_0})$ such that the set of solutions of $(\mathbf{P})_\lambda$ in $\mathcal{U}$ is a real analytic curve of positive solutions $J \ni \lambda \mapsto (\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda) \in B$, for suitable neighborhoods J of λ_0 and B of $(\boldsymbol{\alpha}_{\lambda_0}, \boldsymbol{\psi}_{\lambda_0})$ in $(0, +\infty)^2 \times (C^{2,r}_{0,+}(\overline{\Omega}))^2$.
(iii) In particular if $(\boldsymbol{\alpha}_{\lambda_0}, \boldsymbol{\psi}_{\lambda_0}) = (\boldsymbol{\alpha}_0, \boldsymbol{\psi}_0)$, then $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda) = (\boldsymbol{\alpha}_0, \boldsymbol{\psi}_0) + \mathrm{O}(\lambda)$ as $\lambda \to 0$.

Proof. By the construction of A_Ω, the map F as defined in (3.2) is jointly analytic in a suitable neighborhood of $(\lambda, \boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$. As a consequence, whenever (i) holds, then (ii) is an immediate consequence of the real analytic implicit function theorem, see for example Theorem 4.5.4 in [10]. In particular (iii) is a straightforward consequence of (ii). Therefore, we are just left with the proof of (i).
Concerning (i) we observe that, although the differential of the constrained equations (which are the last two differentials in (3.4)), do not define a Fredholm operator (since obviously the dimension of their kernel is not finite dimensional), however a simple inspection shows that in fact $D_{\boldsymbol{\alpha},\boldsymbol{\psi}}\Phi(\lambda_0, \boldsymbol{\alpha}_{\lambda_0}, \boldsymbol{\psi}_{\lambda_0})$ is a Fredholm operator, see for example Lemma 2.4 in [5]. As a consequence (i) follows from Proposition 3.2 and the Fredholm alternative. □

We conclude this section with some spectral estimates about $\sigma_1(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ and $\lambda^*(\Omega, \mathbf{p})$.

Proposition 3.4. *Let (p_1, p_2) satisfy (1.1), assume without loss of generality $p_1 \leq p_2$ and suppose that $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ is a positive solution of $(\mathbf{P})_\lambda$ with $\lambda \leq \frac{1}{p_2}\Lambda(\Omega, 2p_2)$. Then $\sigma_1(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda) > 0$.*

Proof. For $k = 1$ and to ease the notations, in this proof we set $(\phi_1, \phi_2) = (\phi_{1,1}, \phi_{2,1})$, where $(\phi_{1,1}, \phi_{2,1})$ is any eigenvector of σ_1. Clearly we can just consider $\lambda > 0$. We multiply the first equation in (3.13) by ϕ_1, the second by ϕ_2 and deduce that

$$\int_\Omega |\nabla \phi_1|^2 = (\lambda + \sigma_1) p_2 \int_\Omega V_{2,\lambda}[\phi_2]_{2,\lambda}[\phi_1]_{1,\lambda},$$

and,

$$\int_\Omega |\nabla \phi_2|^2 = (\lambda + \sigma_1) p_1 \int_\Omega V_{1,\lambda}[\phi_1]_{1,\lambda}[\phi_2]_{2,\lambda},$$

which in view of (3.14) shows that

$$\int_\Omega V_{2,\lambda}[\phi_2]_{2,\lambda}[\phi_1]_{1,\lambda} > 0, \int_\Omega V_{1,\lambda}[\phi_1]_{1,\lambda}[\phi_2]_{2,\lambda} > 0. \tag{3.17}$$

In particular we readily deduce that

$$\sigma_1 = \frac{N(\phi_1,\phi_2)}{\int_\Omega (p_1 V_{1,\lambda} + p_2 V_{2,\lambda})[\phi_1]_{1,\lambda}[\phi_2]_{2,\lambda}},$$

where

$$N(\phi_1,\phi_2) = \int_\Omega |\nabla \phi_1|^2 + \int_\Omega |\nabla \phi_2|^2 - \lambda p_2 \int_\Omega V_{2,\lambda}[\phi_2]_{2,\lambda}[\phi_1]_{1,\lambda} - \lambda p_1 \int_\Omega V_{1,\lambda}[\phi_2]_{2,\lambda}[\phi_1]_{1,\lambda}.$$

By using $ab \leq \frac{1}{2}(a^2+b^2)$ we see that,

$$N(\phi_1,\phi_2) \geq \frac{1}{2}\sum_{i=1}^{2}\left(\int_\Omega |\nabla \phi_i|^2 - \lambda p_1 \int_\Omega V_{1,\lambda}[\phi_i]^2_{i,\lambda}\right) + \tag{3.18}$$

$$\frac{1}{2}\sum_{i=1}^{2}\left(\int_\Omega |\nabla \phi_i|^2 - \lambda p_2 \int_\Omega V_{2,\lambda}[\phi_i]^2_{i,\lambda}\right) \geq$$

$$\frac{1}{2}\sum_{i=1}^{2}\left(\int_\Omega |\nabla \phi_i|^2 - \lambda p_1 \int_\Omega V_{1,\lambda}\phi_i\right) + \frac{1}{2}\sum_{i=1}^{2}\left(\int_\Omega |\nabla \phi_i|^2 - \lambda p_2 \int_\Omega V_{2,\lambda}\phi_i\right),$$

where the last inequality is an equality if and only if $< \phi_1 >_{1,\lambda} = 0 = < \phi_2 >_{2,\lambda}$.
Next observe that for any $\phi \in H^1_0(\Omega) \setminus \{0\}$ and for any $\{i,j\} \in \{1,2\}$, by the Holder inequality and (1.2) we have,

$$\int_\Omega |\nabla \phi|^2 - \lambda p_i \int_\Omega V_{i,\lambda}\phi^2 \geq \int_\Omega |\nabla \phi|^2 - \lambda p_i \left(\int_\Omega \rho_{i,\lambda}\right)^{\frac{1}{q_i}} \left(\int_\Omega \phi^{2p_i}\right)^{\frac{1}{p_i}} = \tag{3.19}$$

$$\int_\Omega |\nabla \phi|^2 - \lambda p_i \left(\int_\Omega \phi^{2p_i}\right)^{\frac{1}{p_i}} \geq \left(\int_\Omega \phi^{2p_i}\right)^{\frac{1}{p_i}} \left(\Lambda(\Omega,2p_i) - \lambda p_i\right),\ i=1,2. \tag{3.20}$$

Since $|\Omega| = 1$ and $p_1 \leq p_2$, it is well known (see Theorem 3 in [15]) that $\Lambda(\Omega,2p_2) \leq \Lambda(\Omega,2p_1)$ and that the inequality is strict as far as $p_1 < p_2$. Therefore we have proved that if $\lambda \leq \frac{1}{p_2}\Lambda(\Omega,2p_2)$ then $\sigma_1 \geq 0$.
At this point we argue by contradiction and assume that $\sigma_1 = 0$ for some $\lambda \leq \frac{1}{p_2}\Lambda(\Omega,2p_2)$. Following the equality sign in all the inequalities used so far we see that if $\sigma_1 = 0$ then we would necessarily have $\lambda = \frac{1}{p_2}\Lambda(\Omega,2p_2)$, $p_1 = p_2$, $< \phi_1 >_{1,\lambda} = 0 = < \phi_2 >_{2,\lambda}$, and in particular $\phi_1 = \phi_2$ a.e. in Ω. By (3.16) this implies also $V_{1,\lambda} = V_{2,\lambda}$ a.e. in Ω. As a consequence we would also have,

$$0 = p_1\sigma_1 = \frac{\int_\Omega |\nabla \phi_1|^2 - \lambda p_1 \int_\Omega V_{1,\lambda}\phi_1^2}{\int_\Omega V_{1,\lambda}\phi_1^2} > \nu_1 := \inf_{\phi \in H^1_0(\Omega)} \frac{\int_\Omega |\nabla \phi|^2 - \lambda p_1 \int_\Omega V_{1,\lambda}\phi^2}{\int_\Omega V_{1,\lambda}\phi^2},$$

where the strict inequality is due to the fact that any function which attains the inf is a first eigenfunction and consequently does not change sign. On the other side if $\phi \in C^1_0(\overline{\Omega})$, $\phi \not\equiv 0$, then, once more by the Holder inequality and (1.2), we have,

$$\frac{\int_\Omega |\nabla \phi|^2}{\int_\Omega V_{2,\lambda}\phi^2} \geq \frac{1}{\left(\int_\Omega \rho_{2,\lambda}\right)^{\frac{1}{q_2}}} \frac{\int_\Omega |\nabla \phi|^2}{\left(\int_\Omega \phi^{2p_2}\right)^{\frac{1}{p_2}}} = \frac{\int_\Omega |\nabla \phi|^2}{\left(\int_\Omega \phi^{2p_1}\right)^{\frac{1}{p_1}}} \geq \Lambda(\Omega,2p_1)$$

which immediately implies that,

$$\nu_1 \geq \Lambda(\Omega,2p_1) - \lambda p_1 \geq 0,\ \forall\, \lambda p_1 \leq \Lambda(\Omega,2p_1),$$

which is a contradiction to $\nu_1 < 0$. □

Proposition 3.5. *Let (p_1,p_2) satisfy (1.1), assume without loss of generality $p_1 \leq p_2$ and suppose that $\alpha_{i,*} = 0$, $i=1,2$. Then $\lambda^*(\Omega,\mathbf{p}) \geq \frac{1}{p_2}\Lambda(\Omega,2p_2)$.*

Proof. We can assume w.l.o.g. $\lambda^*(\Omega,\mathbf{p})\in(0,+\infty)$. By definition there exists a sequence $\lambda_n\to(\lambda^*(\Omega,\mathbf{p}))^-$, such that $\alpha_{i,\lambda_n}\to 0^+$. By Lemma 2.1 and passing to a further subsequence if necessary we can assume that $u_{i,n}=\lambda_n\psi_{i,\lambda_n}$, $i=1,2$, converge smoothly to u_i, $i=1,2$, which are classical solutions of

$$\begin{cases} -\Delta u_1=\lambda^* u_2^{p_2} \quad \text{in } \Omega \\ -\Delta u_2=\lambda^* u_1^{p_1} \quad \text{in } \Omega \\ u_i\geq 0 \quad \text{in } \Omega, \quad i=1,2 \\ u_i=0 \quad \text{on } \partial\Omega, \quad i=1,2. \end{cases}$$

By the monotonicity of the energy E_λ we have $E_{\lambda^*}\geq E_0(\Omega)$, whence both u_1 and u_2 cannot vanish identically. Let us set $V_i=u_i^{p_i-1}$, $\phi_i=u_i$, $i=1,2$, then we have,

$$\begin{cases} -\Delta \phi_1=\lambda^* V_2\phi_2 \quad \text{in } \Omega \\ -\Delta \phi_2=\lambda^* V_1\phi_1 \quad \text{in } \Omega \\ \phi_i\geq 0 \quad \text{on } \Omega, \quad i=1,2 \\ \phi_i=0 \quad \text{on } \partial\Omega, \quad i=1,2. \end{cases}$$

By the strong maximum principle for cooperative and strongly coupled linear elliptic systems (see Theorem 2.2 in [19]) we have $\phi_i>0$ in Ω, $i=1,2$ and we deduce that,

$$\begin{cases} -\Delta \phi_1\leq\lambda^* p_2V_2\phi_2 \quad \text{in } \Omega \\ -\Delta \phi_2\leq\lambda^* p_1V_1\phi_1 \quad \text{in } \Omega \\ \phi_i=0 \quad \text{on } \partial\Omega, \quad i=1,2, \end{cases}$$

where the equality holds if and only if $p_1=1=p_2$. Multiplying the first equation by ϕ_1, the second by ϕ_2 and integrating by parts we deduce that

$$0\geq Q(\phi_1,\phi_2):=\int_\Omega|\nabla\phi_1|^2+\int_\Omega|\nabla\phi_2|^2-\lambda^*p_2\int_\Omega V_2\phi_2\phi_1-\lambda^*p_1\int_\Omega V_1\phi_2\phi_1. \tag{3.21}$$

By using $ab\leq\frac{1}{2}(a^2+b^2)$ we see that,

$$Q(\phi_1,\phi_2)\geq\frac{1}{2}\sum_{i=1}^{2}\left(\int_\Omega|\nabla\phi_i|^2-\lambda^*p_1\int_\Omega V_1\phi_i^2\right)+\frac{1}{2}\sum_{i=1}^{2}\left(\int_\Omega|\nabla\phi_i|^2-\lambda^*p_2\int_\Omega V_2\phi_i^2\right),$$

and then by using (3.19), (3.20), $|\Omega|=1$, $p_1\leq p_2$ and $\Lambda(\Omega,2p_2)\leq\Lambda(\Omega,2p_1)$ (this is well known, see for example [15]) as above we deduce that that

$$Q(\phi_1,\phi_2)\geq\left(\frac{1}{2}\sum_{i=1}^{2}\int_\Omega\int_\Omega V_1\phi_i^2+\frac{1}{2}\sum_{i=1}^{2}\int_\Omega\int_\Omega V_2\phi_i^2\right)(\Lambda(\Omega,2p_2)-\lambda^*p_2),$$

and we see from (3.21) that $\lambda^*p_2\geq\Lambda(\Omega,2p_2)$, as claimed. $\square$

4. **Existence, uniqueness and monotonicity**

Let $\mathcal{G}(\Omega)$ denote the set of solutions of $\mathbf{(P)}_\lambda$ for $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$.

Lemma 4.1. *Let* (p_1, p_2) *satisfy* (1.1)*. For any* $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$ *there exists one and only one solution* $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ *of* $\mathbf{(P)}_\lambda$*. In particular* $\mathcal{G}(\Omega)$ *is a real analytic simple curve of positive solutions* $[0, \lambda^*(\Omega, \mathbf{p})) \ni \lambda \mapsto (\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$.

Proof. By Lemma A.1 and Proposition 3.4 we have that $\lambda^*(\Omega, \mathbf{p}) > 0$ and there exists a unique solution of $\mathbf{(P)}_\lambda$ for $\lambda \leq \lambda_0$ for some λ_0 small enough. In particular these unique solutions are positive and we can assume, possibly taking a smaller λ_0, that $\lambda^*(\Omega, \mathbf{p}) > \lambda_0$. However by definition we have $\sigma_1(\alpha_\lambda, \psi_\lambda) > 0$ for any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$ and then by Lemma 2.1 and Lemma 3.3 any solution whose λ is less than λ_0 generates a real analytic curve of positive solutions which can be continued to a real analytic curve of positive solutions defined in $[(-\delta, \lambda^*(\Omega, \mathbf{p}))$ for some small $\delta > 0$.
If at any point in $(\lambda_0, \lambda^*(\Omega, \mathbf{p}))$ there exist two solutions, then by definition of $\lambda^*(\Omega, \mathbf{p}$ they would be both positive and each one would generate in the same way another curve of positive solutions in $(-\delta, \lambda^*(\Omega, \mathbf{p}))$, for some small $\delta > 0$. Obviously for both curves at $\lambda = 0$ we have $(\alpha_\lambda, \psi_\lambda) = (\alpha_0, \psi_0)$ which is the unique solution of $\mathbf{(P)}_\lambda$ for $\lambda = 0$. This is obviously impossible since then (α_0, ψ_0) would be a bifurcation point, in contradiction with Lemma 3.3. □

Problem $\mathbf{(P)}_\lambda$ is the Euler-Lagrange equation of the constrained minimization principle **(VP)** below for the densities (ρ_1, ρ_2). As far as $1 \leq p_1 \leq p_2 < p_N$, existence of solutions could be proved by an adaptation of an argument in [9], worked out there for a more general "scalar" problem, based on the theory of conjugate convex function. We adopt here a different argument based on the weak Young inequality ([25]), which yields existence for (p_1, p_2) satisfying (1.1), that is,

$$\frac{1}{p_1+1} + \frac{1}{p_2+1} > \frac{N-2}{N-1}, \quad p_i \in (0, +\infty),\ i = 1, 2, \tag{4.1}$$

whose relevance for elliptic systems of Lane-Emden type was first noticed in [34].

Theorem 4.2. *Let* (p_1, p_2) *satisfy* (1.1)*. For any* $\lambda > 0$ *there exists a solution* $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ *of* $\mathbf{(P)}_\lambda$.

Proof. We discuss only the case $N \geq 3$, the case $N = 2$ is easier. Here $\|\rho\|_p$ denotes the standard $L^p(\Omega)$ norm. We will denote with C various constants depending only by (p_1, p_2), Ω and N.

Let us define $r_i = 1 + \frac{1}{p_i}$, $i = 1, 2$ and

$$\mathcal{P}_{\Omega,i} := \left\{ \rho \in L^{r_i}(\Omega) \,|\, \rho \geq 0 \text{ a.e. in } \Omega,\ \int_\Omega \rho = 1 \right\},\ i = 1, 2.$$

It is readily seen that (1.1) is equivalent to

$$\frac{1}{r_1} + \frac{1}{r_2} < \frac{N}{N-1}, \quad r_i \in (1, +\infty),\ i = 1, 2, \tag{4.2}$$

and for any (r_1, r_2) satisfying (4.2), for any $\lambda > 0$ and

$$(\rho_1, \rho_2) \in \mathcal{P}_\Omega \equiv \mathcal{P}_{\Omega,1} \times \mathcal{P}_{\Omega,2},$$

we define the free energy,

$$J_\lambda(\rho_1, \rho_2) = \tfrac{1}{r_1} \int_\Omega (\rho_1)^{r_1} + \tfrac{1}{r_2} \int_\Omega (\rho_2)^{r_2} - \lambda \int_\Omega \rho_1 G[\rho_2]. \tag{4.3}$$

We split the proof in several steps.
STEP 1. We prove that if (r_1, r_2) satisfies (4.2), then J_λ is coercive for any $\lambda > 0$, that is, if $\lambda > 0$ and $(\rho_{1,n}, \rho_{2,n}) \in \mathcal{P}_{\Omega,1} \times \mathcal{P}_{\Omega,2}$ and $\max\{\|\rho_{1,n}\|_{r_1}, \|\rho_{2,n}\|_{r_2}\} \to +\infty$, then $J_\lambda(\rho_{1,n}, \rho_{2,n}) \to +\infty$,

as $n \to +\infty$.
Let us recall the weak Young inequality ([25]),

$$\int_\Omega \rho_1 G[\rho_2] \le C\|\rho_1\|_{s_1}\|\rho_2\|_{s_2}, \quad \frac{1}{s_1}+\frac{1}{s_2}=\frac{N+2}{N}, \quad s_i \in (0,+\infty). \tag{4.4}$$

Assume for the time being that for any (r_1, r_2) which satisfies (4.2), there exists (s_1, s_2) which satisfies (4.4) and

$$s_i \in (1, r_i),\ i = 1, 2. \tag{4.5}$$

Then by the (4.4) and standard interpolation inequalities (recall $\int_\Omega \rho_i = 1$, $i = 1.2$) we would have that,

$$\int_\Omega \rho_1 G[\rho_2] \le C\|\rho_1\|_{s_1}\|\rho_2\|_{s_2} \le C\|\rho_1\|_{r_1}^{\gamma_1 r_1}\|\rho_2\|_{r_2}^{\gamma_2 r_2}, \tag{4.6}$$

where

$$\gamma_i = (1-\frac{1}{s_i})\frac{1}{r_i-1}, \quad i = 1, 2.$$

Based on (4.6), elementary arguments show that J_λ is coercive for any $\lambda > 0$ as far as $\gamma_1+\gamma_2 < 1$. Therefore we are left with showing that for any (r_1, r_2) which satisfies (4.2), there exists (s_1, s_2) which satisfies (4.4), (4.5) and in particular,

$$(1-\frac{1}{s_1})\frac{1}{r_1-1}+(1-\frac{1}{s_2})\frac{1}{r_2-1} < 1. \tag{4.7}$$

Obviously there is no loss of generality in assuming

$$r_2 \ge r_1.$$

Observe that, as far as,

$$(r_1, r_2) \in (1, \frac{N}{2}] \times [\frac{N}{2}, +\infty) \bigcup \{r_2 \ge r_1 : [\frac{N}{2}, +\infty) \times [\frac{N}{2}, +\infty)\},$$

then putting $\frac{1}{s_2} = \frac{2}{N} + \varepsilon$ for some small enough $\varepsilon > 0$, from (4.4) we would have $\frac{1}{s_1} = 1 - \varepsilon$ and then

$$1 < s_2 < \frac{N}{2} \le r_2, \quad 1 < s_1 = \frac{1}{1-\varepsilon} < r_1.$$

Thus (4.5) is satisfied. On the other side, since here $r_2 \ge \frac{N}{2}$, we also have that,

$$(1-\frac{1}{s_1})\frac{1}{r_1-1}+(1-\frac{1}{s_2})\frac{1}{r_2-1} = \frac{\varepsilon}{r_1-1}+(1-\frac{2}{N}-\varepsilon)\frac{1}{r_2-1} =$$

$$\varepsilon(\frac{1}{r_1-1}-\frac{1}{r_2-1})+\frac{N-2}{N}\frac{1}{r_2-1} \le \varepsilon(\frac{1}{r_1-1}-\frac{1}{r_2-1})+\frac{2}{N} < 1,$$

for any $\varepsilon > 0$ small enough, showing that (4.7) is satisfied as well.
Therefore we are left with showing that (s_1, s_2) which satisfies (4.4), (4.5) and (4.7) exists in the region

$$\frac{1}{r_1}+\frac{1}{r_2} < \frac{N}{N-1}, \quad r_1 \in (r_N^*, \frac{N}{2}), \quad r_1 \le r_2 \le \frac{N}{2}, \tag{4.8}$$

where $r_N^* = \frac{N(N-1)}{N^2-N+2}$ is the intersection of $\frac{1}{r_1}+\frac{1}{r_2} < \frac{N}{N-1}$ with the line $r_2 = \frac{N}{2}$. Remark that the lowest possible value of r_2 in this region is the one at the corner point where $r_2 = r_1 = 2\frac{N-1}{N}$. Put

$$\frac{1}{s_1} = \frac{1}{r_1} + \varepsilon,$$

which in view of (4.4) implies that

$$1 - \frac{1}{s_2} = \frac{1}{r_1} - \frac{2}{N} + \varepsilon.$$

Then, in view of (4.2), and since $r_2 - 1 \geq r_1 - 1 \geq 2\frac{N-1}{N} - 1 = \frac{N-2}{N}$, concerning (4.7) we have,

$$
\begin{aligned}
&(1-\frac{1}{s_1})\frac{1}{r_1-1} + (1-\frac{1}{s_2})\frac{1}{r_2-1} = \\
&\varepsilon(\frac{1}{r_2-1} - \frac{1}{r_1-1}) + \frac{1}{r_1} + (\frac{1}{r_1} - \frac{2}{N})\frac{1}{r_2-1} \leq \\
&\frac{1}{r_1} + (\frac{1}{r_1} + \frac{1}{r_2} - 1 - \frac{2}{N})\frac{1}{r_2-1} + (1-\frac{1}{r_2})\frac{1}{r_2-1} = \\
&\frac{1}{r_1} + \frac{1}{r_2} + (\frac{1}{r_1} + \frac{1}{r_2} - 1 - \frac{2}{N})\frac{1}{r_2-1} < \\
&\frac{1}{r_1} + \frac{1}{r_2} + (\frac{1}{N-1} - \frac{2}{N})\frac{1}{r_2-1} \leq \\
&\frac{1}{r_1} + \frac{1}{r_2} + \frac{2-N}{N(N-1)}\frac{N}{N-2} = \\
&\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{N-1} < 1.
\end{aligned}
$$

Therefore we are left with showing that if (r_1, r_2) satisfies (4.8), then we can find (s_1, s_2) which satisfies (4.5) as well as $\frac{1}{s_1} = \frac{1}{r_1} + \varepsilon$, $1 - \frac{1}{s_2} = \frac{1}{r_1} - \frac{2}{N} + \varepsilon$ for some $\varepsilon > 0$. We split the discussion in three regions which could be possibly empty for some $N \geq 3$.

We recall that the symmetric point of the critical hyperbola in (4.4) is just $s_1 = \frac{2N}{N+2} = s_2$ and that

$$
\frac{1}{r_1} + \frac{1}{r_2} < \frac{N}{N-1} < \frac{N+2}{N}. \tag{4.9}
$$

We start with the domain,

$$
\Omega_1 = \{\, r_1 \in [2\frac{N-1}{N}, \frac{N}{2}],\, r_1 \leq r_2 \leq \frac{N}{2}\}.
$$

If $(r_1, r_2) \in \Omega_1$, since $r_1 \in [2\frac{N-1}{N}, \frac{N}{2}]$, then for some $\sigma_\varepsilon \to 0$, as $\varepsilon \to 0^+$, we have that if $\frac{1}{s_1} = \frac{1}{r_1} + \varepsilon$ then $r_1 > s_1 > 2\frac{N-1}{N} - \sigma_\varepsilon > \frac{2N}{N+2}$, whence in particular

$$
s_2 < \frac{2N}{N+2} < 2\frac{N-1}{N} \leq r_2,
$$

where we use (4.9). Therefore (4.5) is satisfied and then we are done with Ω_1. Next we consider the case,

$$
\Omega_2 = \{\frac{1}{r_1} + \frac{1}{r_2} < \frac{N}{N-1},\, r_1 \in [\frac{2N}{N+2}, 2\frac{N-1}{N}),\, r_2 \leq \frac{N}{2}\}.
$$

If $(r_1, r_2) \in \Omega_2$, since $r_1 \in [\frac{2N}{N+2}, 2\frac{N-1}{N})$, then for some $\sigma_\varepsilon \to 0$, $h_\varepsilon \to 0$, as $\varepsilon \to 0^+$, we have that if $\frac{1}{s_1} = \frac{1}{r_1} + \varepsilon$ then $r_1 > s_1 \geq r_1 - \sigma_\varepsilon 2\frac{2N}{N+2} - \sigma_\varepsilon$, whence in particular

$$
s_2 < \frac{2N}{N+2} + h_\varepsilon < 2\frac{N-1}{N} \leq r_2,
$$

where we use again (4.9). Therefore (4.5) is satisfied and then we are done with Ω_2 as well. At last we discuss the case,

$$
\Omega_3 = \{\frac{1}{r_1} + \frac{1}{r_2} < \frac{N}{N-1},\, r_1 \in [r^*_{N}, \frac{2N}{N+2}),\;\, r_2 \leq \frac{N}{2}\}.
$$

If $(r_1, r_2) \in \Omega_3$, since $r_1 \in [r^*_{N}, \frac{2N}{N+2})$, then for some $\sigma_\varepsilon \to 0$, $h_\varepsilon \to 0$, as $\varepsilon \to 0^+$, we have that if $\frac{1}{s_1} = \frac{1}{r_1} + \varepsilon$ then $r_1 > s_1 > 2\frac{2N}{N+2} - \sigma_\varepsilon$, whence in particular

$$
s_2 < 2\frac{N-1}{N} + h_\varepsilon < r^{**}_{N} \leq r_2,
$$

where r^{**}_{N} is the intersection of $\frac{1}{r_1} + \frac{1}{r_2} < \frac{N}{N-1}$ with the line $r_1 = \frac{N}{2}$. Therefore (4.5) is satisfied in this case as well, which concludes the proof of the claim of STEP 1.

STEP 2. We prove the existence of the minimum of J_λ by the direct method.
Let $(\rho_{1,n},\rho_{2,n})$ be a minimizing sequence, by STEP 1 $\|\rho_{1,n}\|_{r_1}$ and $\|\rho_{2,n}\|_{r_2}$ are bounded and we assume that $\rho_{2,n}$ weakly converges in $L^{r_2}(\Omega)$ and $\rho_{1,n}$ weakly converges in $L^{r_1}(\Omega)$ to some $(\rho_{1,\lambda},\rho_{2,\lambda})$. Clearly $\frac{1}{r_1}\int_\Omega(\rho_1)^{r_1}+\frac{1}{r_2}\int_\Omega(\rho_2)^{r_2}$ is convex whence lowersemicontinuous with respect to the weak topology in $L^{r_1}(\Omega)\times L^{r_2}(\Omega)$. We will conclude the proof showing that, possibly along a subsequence, we have

$$\lim_{n\to+\infty}\int_\Omega \rho_{1,n}G[\rho_{2,n}]=\int_\Omega \rho_{1,\lambda}G[\rho_{2,\lambda}].$$

Indeed, writing

$$\int_\Omega \rho_{1,n}G[\rho_{2,n}]-\rho_{1,\lambda}G[\rho_{2,\lambda}]=\int_\Omega \rho_{1,n}(G[\rho_{2,n}-\rho_{2,\lambda}])+\int_\Omega(\rho_{1,n}-\rho_{1,\lambda})G[\rho_{2,\lambda}]=$$

$$\int_\Omega \rho_{1,n}(G[\rho_{2,n}-\rho_{2,\lambda}])+\int_\Omega \rho_{2,\lambda}(G[\rho_{1,n}-\rho_{1,\lambda}]),$$

it is enough to prove that the embeddings of $W^{2,r_2}(\Omega)$ in $L^{r_1'}(\Omega)$ and of $W^{2,r_1}(\Omega)$ in $L^{r_2'}(\Omega)$ are compact, where r_i' is the exponent conjugate to r_i. Clearly it is just enough to prove the former. By (1.1) we readily deduce that,

$$r_2>\frac{N-r_1-1}{N(r_1-1)+1},$$

whence by the Sobolev embedding we find that $W^{2,r_2}(\Omega)$ is compactly embedded in $W^{1,t}(\Omega)$ for any

$$t<t_2=r_1'\frac{N^2-N}{N^2-N+r_1'}=r_1'-\frac{r_1'}{N^2-N+r_1'},$$

where we remark that $N^2-N+r_1'>r_1'>0$. Thus, again by the Sobolev embedding, we find that $W^{2,r_2}(\Omega)$ is compactly embedded in $L^k(\Omega)$ for any

$$k<k_2=\frac{Nr_1'\left(1-\frac{1}{N^2-N+r_1'}\right)}{N-r_1'\left(1-\frac{1}{N^2-N+r_1'}\right)},$$

where we assume without loss of generality that $N-r_1'\left(1-\frac{1}{N^2-N+r_1'}\right)>0$. Therefore it is enough to prove that,

$$\frac{Nr_1'\left(1-\frac{1}{N^2-N+r_1'}\right)}{N-r_1'\left(1-\frac{1}{N^2-N+r_1'}\right)}>r_1',$$

which, after a straightforward evaluation, takes the form,

$$N^2-\frac{1}{r_1}N+\frac{1}{r_1-1}>0.$$

The determinant of this polynomial is $\frac{1}{r_1^2}-\frac{4}{r_1-1}$ which is readily seen to be always negative. Thus we have proved the existence of at least one minimizer of J_λ.

STEP 3 We prove that any minimizer $(\rho_{1,\lambda},\rho_{2,\lambda})$ defines a solution of $\mathbf{(P)}_\lambda$.
Let $\Omega_+=\{x\in\Omega\,:\,\rho_{2,\lambda}>0,\text{ a.e.}\}$ and $\Omega_0=\{x\in\Omega\,:\,\rho_{2,\lambda}=0,\text{ a.e.}\}$. Since $\int_\Omega\rho_{2,\lambda}=1$, then $0<|\Omega_+|\le|\Omega|$. For any $n\in\mathbb{N}$ and for any variation of the form $(\rho_{1,\lambda},\rho_{2,\lambda}+\varepsilon\eta)$, with

$\text{supp}(\eta) \subseteq \{x \in \Omega_+ : \rho_{2,\lambda} > \frac{1}{n}, \text{ a.e.}\}$, $\eta \in L^\infty(\Omega)$ and $\int_\Omega \eta = 0$, by using the minimality of $(\rho_{1,\lambda}, \rho_{2,\lambda})$ we have,

$$\int\limits_{\{x\in\Omega_+:\rho_{2,\lambda}>\frac{1}{n}\}} \left((\rho_{2,\lambda})^{\frac{1}{p_2}} - \lambda G[\rho_{1,\lambda}]\right)\eta \geq o(1), \text{ as } \varepsilon \to 0.$$

Since this is true for any such η with $\int_\Omega \eta = 0$ and any $n \in \mathbb{N}$, then we have that,

$$(\rho_{2,\lambda}(x))^{\frac{1}{p_2}} - \lambda G[\rho_{1,\lambda}](x) = \alpha_2 \text{ a.e. in } \Omega_+, \tag{4.10}$$

for a suitable constant $\alpha_2 \in \mathbb{R}$. Next, let χ_A denote the characteristic function of the set A, and assume that $|\Omega_0| > 0$. For any variation of the form $(\rho_{1,\lambda}, \rho_{2,\lambda} + \varepsilon\eta)$, with

$$\eta = \varphi\chi_{\Omega_0} - \left(\int\limits_{\Omega_0}\varphi\right)\frac{\chi_{\Omega_+}}{|\chi_{\Omega_+}|}, \quad \eta \in L^\infty(\Omega), \ \varphi \geq 0,$$

we have,

$$\int\limits_{\Omega_0}\left(-\alpha_2 - \lambda G[\rho_{1,\lambda}]\right)\varphi \geq o(1), \text{ as } \varepsilon \to 0.$$

Therefore we conclude that,

$$\alpha_2 + \lambda G[\rho_{1,\lambda}](x) \leq 0 \text{ a.e. in } \Omega_0, \tag{4.11}$$

and in particular that $\psi_{2,\lambda} = G[\rho_{1,\lambda}] \in W_0^{2,r_1}(\Omega)$ is a strong solution of the first equation in $\mathbf{(P)}_\lambda$ with $\psi_{1,\lambda} = G[\rho_{2,\lambda}] \in W_0^{2,r_2}(\Omega)$, where $(\alpha_2 + \lambda\psi_{2,\lambda})_+^{p_2} \equiv (\alpha_2 + \lambda\psi_{2,\lambda})^{p_2}$ as far as $|\Omega_0| = 0$. The same argument shows that $\psi_{1,\lambda}$ is a strong solution of the second equation in $\mathbf{(P)}_\lambda$.

At last, by Lemma 2.1 any strong solution determined in STEP 3 is a classical solution. This fact concludes the proof. □

Theorem 4.3. *For any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$ F_λ is real analytic, decreasing with $\frac{dF_\lambda}{d\lambda} < 0$, and concave and we have $F_\lambda' = -E_\lambda$. In particular $\frac{dE_\lambda}{d\lambda} \geq 0$ and $\frac{d}{d\lambda}\left(\frac{p_1\alpha_{1,\lambda}}{p_1+1} + \frac{p_2\alpha_{2,\lambda}}{p_2+1}\right) < 0$ for any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$.*

Proof. We first observe that, by the uniqueness in Lemma 4.1, we have that F_λ is the same as $F(\lambda)$ in **(VP)**. It is useful at this stage to introduce the vectorial density,

$$\underline{\rho} = (\rho_1, \rho_2)$$

and the corresponding entropy,

$$\mathcal{S}(\underline{\rho}) = \tfrac{1}{r_1}\int_\Omega (\rho_1)^{r_1} + \tfrac{1}{r_2}\int_\Omega (\rho_2)^{r_2},$$

and energy

$$\mathcal{E}(\underline{\rho}) = \int_\Omega \rho_1 G[\rho_2],$$

whence $J_\lambda(\underline{\rho}) = \mathcal{S}(\underline{\rho}) - \lambda\mathcal{E}(\underline{\rho})$.
By Lemma 4.1 F_λ is real analytic. If $\lambda_2 > \lambda_1 \geq 0$, then $J_{\lambda_2}(\underline{\rho}) < J_{\lambda_1}(\underline{\rho})$, whence $F_{\lambda_2} \leq F_{\lambda_1}$. Thus F_λ is decreasing for $\lambda > 0$. Moreover, setting $\lambda = t\lambda_1 + (1-t)\lambda_2$, $t \in [0,1]$, and letting $\underline{\rho}_\lambda$ be any minimizer of J_λ, we find that,

$$F_\lambda = \mathcal{S}(\underline{\rho}_\lambda) - \lambda\mathcal{E}(\underline{\rho}_\lambda) = -\mathcal{S}(\underline{\rho}_\lambda)(t + (1-t)) - (t\lambda_1 + (1-t)\lambda_2)\mathcal{E}(\underline{\rho}_\lambda) =$$

$$tJ_{\lambda_1}(\underline{\rho}_\lambda) + (1-t)J_{\lambda_2}(\underline{\rho}_\lambda) \geq tF_{\lambda_1} + (1-t)F_{\lambda_2}.$$

Therefore F_λ is concave with $\frac{d^2F_\lambda}{d\lambda^2} \leq 0$.

At this point we use a well known trick about canonical variational principles (see for example [11]). Let $\lambda_1 \neq \lambda_2$ in $[0, \lambda^*(\Omega, \mathbf{p}))$ and let $\underline{\rho}_1, \underline{\rho}_2$ be the minimizers of $J_{\lambda_1}, J_{\lambda_2}$ respectively. Clearly we have

$$F_{\lambda_1} \leq \mathcal{S}(\underline{\rho}_2) - \lambda_1\mathcal{E}(\underline{\rho}_2) = F_{\lambda_2} - (\lambda_1 - \lambda_2)\mathcal{E}(\underline{\rho}_2), \tag{4.12}$$

$$F_{\lambda_2} \leq \mathcal{S}(\underline{\rho}_1) - \lambda_2\mathcal{E}(\underline{\rho}_1) = F_{\lambda_1} - (\lambda_2 - \lambda_1)\mathcal{E}(\underline{\rho}_1). \tag{4.13}$$

and we deduce from (4.12), (4.13) that

$$-\mathcal{E}(\underline{\rho}_1) \leq \frac{F_{\lambda_1} - F_{\lambda_2}}{\lambda_1 - \lambda_2} \leq -\mathcal{E}(\underline{\rho}_2) \quad \text{if } \lambda_1 > \lambda_2,$$

$$-\mathcal{E}(\underline{\rho}_2) \leq \frac{F_{\lambda_2} - F_{\lambda_1}}{\lambda_2 - \lambda_1} \leq -\mathcal{E}(\underline{\rho}_1) \quad \text{if } \lambda_2 > \lambda_1.$$

By Lemma 4.1, as $\lambda_2 \to \lambda_1$ we have $\underline{\rho}_2 \to \underline{\rho}_1$ smoothly, whence

$$\frac{dF_{\lambda_1}}{d\lambda} = -\mathcal{E}(\underline{\rho}_1) \equiv - \left.E_\lambda\right|_{\lambda=\lambda_1}.$$

Remark that $E_\lambda > 0$ in $[0, \lambda^*(\Omega, \mathbf{p}))$, whence $\frac{dF_\lambda}{d\lambda} < 0$. In particular, since F_λ is real analytic and concave, then $\frac{dE_\lambda}{d\lambda} = -\frac{d^2F_\lambda}{d\lambda^2} \geq 0$, for any $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$.
At this point observe that,

$$F_\lambda = \frac{1}{r_1}\int_\Omega (\rho_{1,\lambda})^{r_1} + \frac{1}{r_2}\int_\Omega (\rho_{2,\lambda})^{r_2} - \lambda\int_\Omega \rho_{1,\lambda} G[\rho_{2,\lambda}] =$$

$$\frac{1}{r_1}\int_\Omega \rho_{1,\lambda}(\alpha_{1,\lambda} + \lambda_1\psi_{1,\lambda}) + \frac{1}{r_2}\int_\Omega \rho_{2,\lambda}(\alpha_2 + \lambda_2\psi_{2,\lambda}) - \lambda\int_\Omega \rho_{1,\lambda} G[\rho_{2,\lambda}] =$$

$$\frac{\alpha_{1,\lambda}}{r_1} + \frac{\lambda}{r_1}\int_\Omega \rho_{1,\lambda} G[\rho_{2,\lambda}] + \frac{\alpha_{2,\lambda}}{r_2} + \frac{\lambda}{r_2}\int_\Omega \rho_{2,\lambda} G[\rho_{1,\lambda}] - \lambda\int_\Omega \rho_{1,\lambda} G[\rho_{2,\lambda}] =$$

$$\frac{p_1\alpha_{1,\lambda}}{p_1+1} + \frac{p_2\alpha_{2,\lambda}}{p_2+1} + \frac{p_1p_2 - 1}{(p_2+1)(p_1+1)}E_\lambda.$$

By Lemma 4.1, $\alpha_{i,\lambda}$, $i = 1, 2$ are real analytic, and then we deduce that,

$$\frac{d}{d\lambda}\left(\frac{p_1\alpha_{1,\lambda}}{p_1+1} + \frac{p_2\alpha_{2,\lambda}}{p_2+1}\right) = \frac{dF_\lambda}{d\lambda} - \frac{p_1p_2 - 1}{(p_2+1)(p_1+1)}\frac{dE_\lambda}{d\lambda} < 0,$$

where the strict inequality follows from $\frac{dF_\lambda}{d\lambda} < 0$. □

Finally we prove the estimate about the derivative of E_λ.

Proposition 4.4. *Let* $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda) \in \mathcal{G}(\Omega)$ *be the unique positive solutions of* $(\mathbf{P})_\lambda$ *for* $\lambda \in [0, \lambda^*(\Omega, \mathbf{p}))$. *Then,*

$$\frac{dE_\lambda}{d\lambda} \geq p_1\|[\psi_{1,\lambda}]_{1,\lambda}\|^2_{1,\lambda} + p_2\|[\psi_{2,\lambda}]_{2,\lambda}\|^2_{2,\lambda}. \tag{4.14}$$

Proof. By Lemma 4.1 $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ is a real analytic function of λ and then, by standard elliptic estimates, we have that $\boldsymbol{\eta}_\lambda \in (C^2_0(\overline{\Omega}))^2$ where

$$\boldsymbol{\eta}_\lambda = (\eta_{1,\lambda}, \eta_{2,\lambda}) = \frac{d\boldsymbol{\psi}_\lambda}{d\lambda} = \left(\frac{d\psi_{1,\lambda}}{d\lambda}, \frac{d\psi_{2,\lambda}}{d\lambda}\right),$$

is a classical solution of,

$$\begin{cases} -\Delta\eta_{1,\lambda} = \tau_{2,\lambda}V_{2,\lambda}\eta_{2,\lambda} + p_2V_{2,\lambda}\psi_{2,\lambda} + p_2V_{2,\lambda}\frac{d\alpha_{2,\lambda}}{d\lambda}, \\ -\Delta\eta_{2,\lambda} = \tau_{1,\lambda}V_{1,\lambda}\eta_{1,\lambda} + p_1V_{1,\lambda}\psi_{1,\lambda} + p_1V_{1,\lambda}\frac{d\alpha_{1,\lambda}}{d\lambda}, \end{cases}$$

and $\frac{d\alpha_{i,\lambda}}{d\lambda}$ can be computed by the unit mass constraints in $(\mathbf{P})_\lambda$, that is

$$p_2\frac{d\alpha_{2,\lambda}}{d\lambda}=-\tau_{2,\lambda}<\eta_{2,\lambda}>_{2,\lambda}-p_2<\psi_{2,\lambda}>_{2,\lambda},\; p_1\frac{d\alpha_{1,\lambda}}{d\lambda}=-\tau_{1,\lambda}<\eta_{1,\lambda}>_{1,\lambda}-p_1<\psi_{1,\lambda}>_{1,\lambda}.$$

Therefore, we conclude that $\boldsymbol{\eta}_\lambda$ is a solution of,

$$\begin{cases} -\Delta\eta_{1,\lambda}=\tau_{2,\lambda}V_{2,\lambda}[\eta_{2,\lambda}]_{2,\lambda}+p_2V_{2,\lambda}[\psi_{2,\lambda}]_{2,\lambda}, \\ -\Delta\eta_{2,\lambda}=\tau_{1,\lambda}V_{1,\lambda}[\eta_{1,\lambda}]_{1,\lambda}+p_1V_{1,\lambda}[\psi_{1,\lambda}]_{1,\lambda}. \end{cases} \tag{4.15}$$

Since $E_\lambda=\int_\Omega(\nabla\psi_{1,\lambda},\nabla\psi_{2,\lambda})$, then by using $(\mathbf{P})_\lambda$ and (4.15) we also have,

$$\frac{d}{d\lambda}E_\lambda=\int_\Omega(\nabla\eta_{1,\lambda},\nabla\psi_{2,\lambda})+\int_\Omega(\nabla\psi_{1,\lambda},\nabla\eta_{2,\lambda})=$$
$$\sum_{i=1}^{2}\tau_{i,\lambda}<[\eta_{i,\lambda}]_{i,\lambda}[\psi_{i,\lambda}]_{i,\lambda}>_{i,\lambda}+\sum_{i=1}^{2}p_i\|[\psi_{i,\lambda}]_{i,\lambda}^2\|_{i,\lambda}^2.$$

Let us denote, for $i=1,2$,

$$[\psi_{i,\lambda}]_{i,\lambda}=\sum_{k=1}^{+\infty}\xi_{i,k}[\phi_{i,k}]_{i,\lambda},\quad [\eta_{i,\lambda}]_{i,\lambda}=\sum_{k=1}^{+\infty}\beta_{i,k}[\phi_{i,k}]_{i,\lambda},$$
$$\xi_{i,k}=<[\phi_{i,k}]_{i,\lambda}[\psi_{i,\lambda}]_{i,\lambda}>_{i,\lambda},\quad \beta_{i,k}=<[\phi_{i,k}]_{i,\lambda}[\eta_{i,\lambda}]_{i,\lambda}>_{i,\lambda},$$

the Fourier expansions of $[\psi_{i,\lambda}]_{i,\lambda}$ and $[\eta_{i,\lambda}]_{i,\lambda}$ in $Y_{i,0}$ (see Lemma 3.1), with respect to the normalized eigenfunctions $[\phi_{i,k}]_{i,\lambda}$, satisfying $\|[\phi_{i,k}]_{i,\lambda}\|_{i,\lambda}=1$, $i=1,2$. Then, by Lemma 3.1, we have,

$$\frac{d}{d\lambda}E_\lambda=\tau_{2,\lambda}\sum_{k=1}^{+\infty}\beta_{2,k}\xi_{2,k}+\tau_{1,\lambda}\sum_{k=1}^{+\infty}\beta_{1,k}\xi_{1,k}+p_2\sum_{k=1}^{+\infty}\xi_{2,k}^2+p_1\sum_{k=1}^{+\infty}\xi_{1,k}^2. \tag{4.16}$$

We can now consider $\lambda>0$. On the other side, multiplying the first equation in (4.15) by $\phi_{2,k}$, the second by $\phi_{1,k}$, using (3.13) and integrating by parts, we have,

$$\begin{cases} (\tau_{1,\lambda}+p_1\sigma_k)\beta_{1,k}=\tau_{2,\lambda}\beta_{2,k}+p_2\xi_{2,k}, \\ (\tau_{2,\lambda}+p_2\sigma_k)\beta_{2,k}=\tau_{1,\lambda}\beta_{1,k}+p_1\xi_{1,k}, \end{cases} \tag{4.17}$$

where $\sigma_k=\sigma_k(\boldsymbol{\alpha}_\lambda,\boldsymbol{\psi}_\lambda)$. Since by (3.14) we also have $2\lambda+\sigma_k>\lambda$, then (4.17) admits the unique solution,

$$\beta_{1,k}=\frac{\lambda(p_1\xi_{1,k}+p_2\xi_{2,k})+\sigma_kp_2\xi_{2,k}}{p_1\sigma_k(2\lambda+\sigma_k)},\; \beta_{2,k}=\frac{\lambda(p_1\xi_{1,k}+p_2\xi_{2,k})+\sigma_kp_1\xi_{1,k}}{p_2\sigma_k(2\lambda+\sigma_k)},$$

which we can substitute in (4.16) to deduce that

$$\frac{d}{d\lambda}E_\lambda=\lambda\sum_{k=1}^{+\infty}\frac{\lambda(p_1\xi_{1,k}+p_2\xi_{2,k})(\xi_{1,k}+\xi_{2,k})+2\sigma_k\xi_{1,k}\xi_{2,k}}{\sigma_k(2\lambda+\sigma_k)}+p_2\sum_{k=1}^{+\infty}\xi_{2,k}^2+p_1\sum_{k=1}^{+\infty}\xi_{1,k}^2. \tag{4.18}$$

At this point observe that the equations in $(\mathbf{P})_\lambda$ can be written in the following form,

$$\begin{cases} -\Delta\psi_{1,\lambda}=V_{2,\lambda}(\alpha_{2,\lambda}+\lambda\psi_{2,\lambda}) \quad \text{in } \Omega \\ \\ -\Delta\psi_{2,\lambda}=V_{1,\lambda}(\alpha_{1,\lambda}+\lambda\psi_{1,\lambda}) \quad \text{in } \Omega \\ \\ \int\limits_\Omega V_{i,\lambda}(\alpha_{i,\lambda}+\lambda\psi_{i,\lambda})=1 \quad i=1,2. \end{cases}$$

Therefore we can evaluate

$$\alpha_{i,\lambda}=\frac{1}{m_{i,\lambda}}-\lambda<\psi_{i,\lambda}>_{i,\lambda},\quad i=1,2,$$

and deduce that

$$\begin{cases} -\Delta\psi_{1,\lambda} = (m_{2,\lambda})^{-1}V_{2,\lambda} + \lambda V_{2,\lambda}[\psi_{2,\lambda}]_{2,\lambda} \quad \text{in } \Omega \\ -\Delta\psi_{2,\lambda} = (m_{1,\lambda})^{-1}V_{1,\lambda} + \lambda V_{1,\lambda}[\psi_{1,\lambda}]_{1,\lambda} \quad \text{in } \Omega \\ \psi_{i,\lambda} = 0 \quad \text{on } \partial\Omega, \quad i = 1,2. \end{cases} \tag{4.19}$$

Multiplying the first equation in (4.19) by $\phi_{2,k}$, the second by $\phi_{1,k}$, using (3.13) and integrating by parts, we have,

$$\begin{cases} (\tau_{1,\lambda} + p_1\sigma_k)\xi_{1,k} =< \phi_{2,k} >_{2,\lambda} + \lambda\xi_{2,k}, \\ (\tau_{2,\lambda} + p_2\sigma_k)\xi_{2,k} =< \phi_{1,k} >_{1,\lambda} + \lambda\xi_{1,k}. \end{cases} \tag{4.20}$$

Since $p_i \geq 1$, $i = 1,2$ and since by (3.14) we also have $2\lambda + \sigma_k > \lambda$, then (4.20) admits the unique solution,

$$\xi_{1,k} = \frac{(\tau_{2,\lambda} + p_2\sigma_k) < \phi_{2,k} >_{2,\lambda} + \lambda < \phi_{1,k} >_{1,\lambda}}{p_1p_2(\lambda + \sigma_k)^2 - \lambda^2},$$

$$\xi_{2,k} = \frac{(\tau_{1,\lambda} + p_1\sigma_k) < \phi_{1,k} >_{1,\lambda} + \lambda < \phi_{2,k} >_{2,\lambda}}{p_1p_2(\lambda + \sigma_k)^2 - \lambda^2},$$

and since of course we can assume that $< \phi_{i,k} >_{i,\lambda} \geq 0$, $i = 1,2$, for any $k \in \mathbb{N}$, then we deduce that

$$\xi_{i,k} \geq 0, i = 1,2, \forall\, k \in \mathbb{N}. \tag{4.21}$$

Remark that all the relations above, including (4.18), (4.21), hold just by assuming that $0 \notin \sigma(\mathbf{L}_\lambda)$. However at this point we use the fact that for any $\lambda < \lambda^*(\Omega, \mathbf{p})$ it holds $\sigma_k \geq \sigma_1 > 0$, $\forall\, k \in \mathbb{N}$, which, together with (4.18) and (4.21) implies that

$$\frac{d}{d\lambda}E_\lambda \geq p_2 \sum_{k=1}^{+\infty} \xi_{2,k}^2 + p_1 \sum_{k=1}^{+\infty} \xi_{1,k}^2 = p_2\|[\psi_{2,\lambda}]_{2,\lambda}\|_{2,\lambda}^2 + p_1\|[\psi_{1,\lambda}]_{1,\lambda}\|_{1,\lambda}^2.$$

which is (4.14). □

At last we present the proofs of Theorems 1.3 and 1.5.

The proof of Theorem 1.3. Uniqueness and regularity follow immediately from Lemma 4.1 and then a straightforward evaluation yields the behavior in the claim as $\lambda \to 0$. The inequality about the energy for $\lambda = 0$ is a well known torsional inequality, see [15]. The monotonicity in the claim follows from Theorem 4.3 and Proposition 4.4. Finally, if either $\sigma_{1,*} = 0$ or if $\alpha_{i,*} = 0$, $i = 1,2$, it follows from Propositions 3.4 and 3.5 that $\lambda^*(\Omega, \mathbf{p}) \geq \frac{1}{p_2}\Lambda(\Omega, 2p_2)$. □

The proof of Theorem 1.5. For $(\boldsymbol{\mu}_\lambda, \mathbf{u}_\lambda)$ as defined in the claim, from the constrains in $(\mathbf{P})_\lambda$ we have

$$\alpha_{i,\lambda} = \|1 + u_{i,\lambda}\|_{p_i}^{-1},\; i = 1,2, \tag{4.22}$$

which immediately shows that

$$\frac{p_1\alpha_{1,\lambda}}{p_1+1} + \frac{p_2\alpha_{2,\lambda}}{p_2+1} = \gamma(\boldsymbol{\mu}, \mathbf{u}_\lambda).$$

Next observe that

$$E_\lambda = \frac{1}{2}\int_\Omega \rho_{1,\lambda}\psi_{1,\lambda} + \frac{1}{2}\int_\Omega \rho_{2,\lambda}\psi_{2,\lambda} = \frac{\alpha_{1,\lambda}^{p_1+1}}{2\lambda}\int_\Omega (1+u_1)^{p_1}u_1 + \frac{\alpha_{2,\lambda}^{p_2+1}}{2\lambda}\int_\Omega (1+u_2)^{p_2}u_2,$$

which together with (4.22) and

$$\lambda = \mu_{1,\lambda}\frac{\alpha_{2,\lambda}}{\alpha_{1,\lambda}^{p_1}}, \quad \lambda = \mu_{2,\lambda}\frac{\alpha_{1,\lambda}}{\alpha_{2,\lambda}^{p_2}}, \tag{4.23}$$

yields

$$E_\lambda = \frac{\alpha_{1,\lambda}^{p_1}\alpha_{2,\lambda}^{p_2}}{2\mu_{2,\lambda}}\int_\Omega (1+u_1)^{p_1}u_1 + \frac{\alpha_{2,\lambda}^{p_2}\alpha_{1,\lambda}^{p_1}}{2\mu_{1,\lambda}}\int_\Omega (1+u_2)^{p_2}u_2 = E(\boldsymbol{\mu}_\lambda, \mathbf{u}_\lambda).$$

Moreover, we immediately have $F(\boldsymbol{\mu}_\lambda, \mathbf{u}_\lambda) = F_\lambda$ as well.
Therefore, as far as $\lambda < \lambda^*(\Omega, \mathbf{p})$ the monotonicity properties in the claim follow immediately from Theorem 1.3.
Clearly, as far as $\lambda < \lambda^*(\Omega, \mathbf{p})$, by definition we also have $\sigma_1(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda) > 0$ and $\alpha_{i,\lambda} > 0$, $i = 1, 2$, whence by Lemma 4.1 we see that $(\boldsymbol{\mu}_\lambda, \mathbf{u}_\lambda)$ is a continuous real analytic curve.
Finally, assume that $\alpha_{1,*} = 0$ and $\alpha_{2,*} > 0$. By definition there exists a sequence $\lambda_n \to (\lambda^*(\Omega, \mathbf{p}))^-$, such that $\alpha_{1,\lambda_n} \to 0^+$, $\alpha_{2,\lambda_n} \to \alpha_2 > 0$. The curve is obviously unbounded since by (4.23) we have $\mu_{2,\lambda} \to +\infty$.
Now, under the assumption of Theorem 1.2-(b), we have $\lambda^*(\Omega, \mathbf{p}) < +\infty$ and then by Lemma 2.1 and passing to a further subsequence if necessary we can assume that ψ_{i,λ_n}, $i = 1, 2$ converge smoothly to $\psi_{i,*}$, $i = 1, 2$, which are classical solutions of $(\mathbf{P})_\lambda$ for $\lambda = \lambda^*(\Omega, \mathbf{p})$ and $\alpha_1 = 0$, $\alpha_2 > 0$. Since by definition $\alpha_{i,\lambda_n} u_{i,\lambda_n} = \lambda_n \psi_{i,\lambda_n}$, $i = 1, 2$, then the convergence in the claim follows from (4.22).
The conclusion in case $\alpha_{2,*} = 0$ and $\alpha_{1,*} > 0$ follows in the same way. □

Appendix A. Uniqueness of solutions for λ small

The following lemma is proved by a standard application of the contraction mapping principle and we prove it here just for reader's convenience. We will denote by C_1 the constant in Lemma 2.1 and by C_2, C_3 other positive constants depending only by $r_0, \Omega, \mathbf{p}, N$.

Lemma A.1. *There exists $\lambda_0 > 0$ such that:*
(j) for any $\lambda \in [0, \lambda_0]$ there exist at least one solution $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ of $(\mathbf{P})_\lambda$.
(jj) for any solution of $(\mathbf{P})_\lambda$ we have $\alpha_{i,\lambda} > \frac{1}{3}$, $i = 1, 2$ for any $\lambda \in [0, \lambda_0]$.

Proof. (j)
Putting $u_1 = \lambda\psi_1, u_2 = \lambda\psi_2$ the proof is an immediate consequence of the following lemmas. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\mathbf{u} = (u_1, u_2)$ and let us define,

$$\mathbf{B}_\infty = \left\{ \mathbf{u} \in (L^\infty(\Omega))^2 \,|\, \|\mathbf{u}\|_{L^\infty(\Omega)} := \max_{i=1,2} \|u_i\|_{L^\infty(\Omega)} \le C_1,\ u_i \ge 0, \text{ a.e. in } \Omega,\ i = 1, 2 \right\}$$

where $C_1(r, \Omega, 1, \mathbf{p}, N)$ is the constant obtained in Lemma 2.1 evaluated with $\overline{\lambda} = 1$.

Lemma A.2. *Let (p_1, p_2) satisfy (1.1). There exists $\lambda_0 \in (0, 1]$ such that for any $\lambda \in [0, \lambda_0]$ and for any $\alpha_i \in (-\infty, 1]$ there exists one and only one solution $\mathbf{u} = (u_{1,\lambda,\boldsymbol{\alpha}}, u_{2,\lambda,\boldsymbol{\alpha}}) \in (C_0^{2,r}(\overline{\Omega}))^2$ of the problem*

$$\begin{cases} -\Delta u_1 = \lambda(\alpha_2 + u_2)_+^{p_2} & \text{in } \Omega \\ -\Delta u_2 = \lambda(\alpha_1 + u_1)_+^{p_1} & \text{in } \Omega \\ u_1 = 0 = u_2 & \text{on } \partial\Omega \\ u_i \in B_\infty,\ i = 1, 2. \end{cases} \tag{A.1}$$

Moreover, for fixed $\lambda \in [0, \lambda_0]$, the maps $(-\infty, 1] \ni \alpha_1 \to u_{2,\lambda}[\alpha_1] = u_{2,\lambda,\boldsymbol{\alpha}} \in B_\infty$, $(-\infty, 1] \ni \alpha_2 \to u_{1,\lambda}[\alpha_2] = u_{1,\lambda,\boldsymbol{\alpha}} \in B_\infty$, are continuous and $u_{1,\lambda,\boldsymbol{\alpha}} \equiv 0 \equiv u_{2,\lambda,\boldsymbol{\alpha}}$ if either $\lambda = 0$ or if $\alpha_i \le 0$, $i = 1, 2$.

Proof. First of all, if either $\lambda = 0$ or if $\alpha_i \le 0$, $i = 1, 2$, then $(u_1, u_2) \equiv (0, 0)$ is a solution, whence the last part of the statement will follow immediately from the uniqueness.
For $\lambda_0 \in (0, 1]$ to be fixed later on and for fixed $\lambda \in [0, \lambda_0]$ and $\alpha_i \in (-\infty, 1]$, $i = 1, 2$, we define

$$\mathbf{T}_{\lambda,\boldsymbol{\alpha}}(\mathbf{u}) = \lambda(G[(\alpha_2 + u_2)_+^{p_2}], G[(\alpha_1 + u_1)_+^{p_1}]), \quad u_i \in B_\infty, i = 1, 2.$$

Recall that if $\alpha_i > 0$ then $\alpha_i \le 1$, while if $\alpha_i < 0$ then $(\alpha_i + u_i)_+ \le (u_i)_+$, whence we have,

$$\|\mathbf{T}_{\lambda,\boldsymbol{\alpha}}(\mathbf{u})\|_{L^\infty(\Omega)} := \lambda \max_{i=1,2} \|G[(\alpha_i + u_i)_+^{p_i}]\|_{L^\infty(\Omega)} \le \lambda C_2,$$

and we readily see that $\mathbf{T}_{\lambda,\boldsymbol{\alpha}} : \mathbf{B}_\infty \to \mathbf{B}_\infty$ for any $\lambda \leq \frac{C_1}{C_2}$. Also,

$$\|\mathbf{T}_{\lambda,\boldsymbol{\alpha}}(\mathbf{u}) - \mathbf{T}_{\lambda,\boldsymbol{\alpha}}(\mathbf{v})\|_{L^\infty(\Omega)} \leq \lambda \max_{i=1,2} \|p_i G[(\alpha_i + w_i)_+^{p_i-1}|u_i - v_i|]\|_{L^\infty(\Omega)} \leq$$

$$\lambda \max_{i=1,2} \|p_i G[(\alpha_i + w_i)_+^{p_i-1}\|_{L^\infty(\Omega)} \|u_i - v_i\|_{L^\infty(\Omega)} \leq \lambda C_3 \|\mathbf{u} - \mathbf{v}\|_{L^\infty(\Omega)}$$

where $w_i \in B_\infty$ satisfies $u_i \leq w_i \leq v_i$, $i = 1, 2$.
Therefore, we also have $\|\mathbf{T}_{\lambda,\boldsymbol{\alpha}}(\mathbf{u}) - \mathbf{T}_{\lambda,\boldsymbol{\alpha}}(\mathbf{v})\|_{L^\infty(\Omega)} \leq \frac{1}{2}\|\mathbf{u} - \mathbf{v}\|_{L^\infty(\Omega)}$, for any $\lambda \leq \frac{1}{2C_3}$. As a consequence putting $\lambda_0 = \min\{1, \frac{C_1}{C_2}, \frac{1}{2C_3}\}$, we have that $\mathbf{T}_{\lambda,\boldsymbol{\alpha}}$ is a contraction in $\mathbf{B}_\infty$ for any $\lambda \leq \lambda_0$. Whence, in particular, for any fixed $\boldsymbol{\alpha} \in ((-\infty, 1])^2$, we have that for any $\lambda \in [0, \lambda_0]$ there exists a unique solution of $\mathbf{u} = \mathbf{T}_{\lambda,\boldsymbol{\alpha}}(\mathbf{u})$. The existence and uniqueness claim follows since, by standard elliptic estimates, $(u_{1,\lambda,\boldsymbol{\alpha}}, u_{2,\lambda,\boldsymbol{\alpha}}) \in (C_0^{2,r}(\overline{\Omega}))^2$ solves the problem in the statement of the lemma if and only if $\mathbf{u} \in \mathbf{B}_\infty$ satisfies $\mathbf{u} = \mathbf{T}_{\lambda,\boldsymbol{\alpha}}(\mathbf{u})$.
Concerning the continuity of $(u_{1,\lambda}[\alpha_2], u_{2,\lambda}[\alpha_1]) = (u_{1,\lambda,\boldsymbol{\alpha}}, u_{2,\lambda,\boldsymbol{\alpha}})$ for $\alpha_i \in (-\infty, 1]$, $i = 1, 2$, we observe that if $\boldsymbol{\alpha}_n = (\alpha_{1,n}, \alpha_{1,n}) \to \boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, then

$$\|u_{2,\lambda}[\alpha_{1,n}] - u_{2,\lambda}[\alpha_1]\|_{L^\infty(\Omega)} = \|G[(\alpha_{1,n} + u_{1,\lambda,\boldsymbol{\alpha}_n})_+^{p_1}] - G[(\alpha_1 + u_{1,\lambda,\boldsymbol{\alpha}})_+^{p_1}]\|_{L^\infty(\Omega)} \leq$$

$$\|G[(\alpha_{1,n} + u_{1,\lambda,\boldsymbol{\alpha}_n})_+^{p_1}] - G[(\alpha_{1,n} + u_{1,\lambda,\boldsymbol{\alpha}})_+^{p_1}]\|_{L^\infty(\Omega)} +$$

$$\|G[(\alpha_{1,n} + u_{1,\lambda,\boldsymbol{\alpha}})_+^{p_1}] - G[(\alpha_1 + u_{1,\lambda,\boldsymbol{\alpha}})_+^{p_1}]\|_{L^\infty(\Omega)} \leq$$

$$\lambda C_3 \|u_{1,\lambda}[\alpha_{2,n}] - u_{1,\lambda}[\alpha_2]\|_{L^\infty(\Omega)} + p_1\lambda \|G[(s + u_{1,\lambda,\alpha})_+^{p_1-1}]\|_{L^\infty(\Omega)} |\alpha_{1,n} - \alpha_1| \leq$$

$$\frac{1}{2}\|u_{1,\lambda}[\alpha_{2,n}] - u_{1,\lambda}[\alpha_2]\|_{L^\infty(\Omega)} + \lambda_0 C_3 |\alpha_{1,n} - \alpha_1|.$$

Clearly the same argument applies to the other component and then we deduce that

$$\|u_{2,\lambda}[\alpha_{1,n}] - u_{2,\lambda}[\alpha_1]\|_{L^\infty(\Omega)} + \|u_{1,\lambda}[\alpha_{2,n}] - u_{1,\lambda}[\alpha_2]\|_{L^\infty(\Omega)} \leq 4\lambda_0 C_3 (|\alpha_{1,n} - \alpha_1| + |\alpha_{2,n} - \alpha_2|),$$

which readily implies the claim. □

For fixed $\lambda \in [0, \lambda_0]$ we consider the continuous map

$$((-\infty, 1])^2 \ni \boldsymbol{\alpha} = (\alpha_1, \alpha_2) \to \mathbf{u}_{\boldsymbol{\alpha}} = (u_{1,\lambda}[\alpha_2], u_{2,\lambda}[\alpha_1]) \in (B_\infty)^2,$$

where $u_{1,\lambda}[\alpha_2] = u_{1,\lambda,\boldsymbol{\alpha}}, u_{2,\lambda}[\alpha_1] = u_{2,\lambda,\boldsymbol{\alpha}}$. Then we have,

Lemma A.3. *By taking a smaller λ_0 if necessary, for any fixed $\lambda \in [0, \lambda_0]$ we have:*
(i) The maps $u_{1,\lambda}[\alpha_2], u_{2,\lambda}[\alpha_1]$ are monotonic increasing,

$$u_{1,\lambda}[\alpha_2] \leq u_{1,\lambda}[\beta_2], \quad \forall\, 0 < \alpha_2 < \beta_2 \leq 1, \quad u_{2,\lambda}[\alpha_1] \leq u_{2,\lambda}[\beta_1], \quad \forall\, 0 < \alpha_1 < \beta_1 \leq 1.$$

(ii) There exists at least one $\boldsymbol{\alpha}_\lambda = (\alpha_{1,\lambda}, \alpha_{2,\lambda}) \in ((\frac{1}{3}, 1])^2$ such that,

$$\int_\Omega (\alpha_{1,\lambda} + u_{1,\lambda}[\alpha_{2,\lambda}])^{p_1} = 1 = \int_\Omega (\alpha_{2,\lambda} + u_{2,\lambda}[\alpha_{1,\lambda}])^{p_2}.$$

Proof. (*i*) If $\lambda = 0$ we have $u_{1,0,\boldsymbol{\alpha}} = 0$ for any $\boldsymbol{\alpha}$ and the conclusion is trivial. For any fixed $-\infty < \alpha_2 < \beta_2 \leq 1$ let us set,

$$(w_1, w_2) = (u_{1,\lambda}[\beta_2] - u_{1,\lambda}[\alpha_2], u_{2,\lambda}[\beta_1] - u_{2,\lambda}[\alpha_1]) \in (C_0^{2,r}(\overline{\Omega}))^2,$$

then

$$-\Delta w_1 = \lambda(\beta_2 + u_{2,\lambda}[\beta_1])_+^{p_2} - \lambda(\alpha_2 + u_{2,\lambda}[\alpha_1])_+^{p_2} \geq \lambda(\alpha_2 + u_{2,\lambda}[\beta_1])_+^{p_2} - \lambda(\alpha_2 + u_{2,\lambda}[\alpha_1])_+^{p_2} \geq$$

$$\lambda p_2(\alpha_2 + u_{2,\lambda}[\alpha_1])_+^{p_2-1}(u_{2,\lambda}[\beta_1] - u_{2,\lambda}[\alpha_1]) = \lambda p_2(\alpha_2 + u_{2,\lambda}[\alpha_1])_+^{p_2-1} w_2,$$

by the convexity of $f(t) = (\alpha + t)_+^p$ for $t \in \mathbb{R}$. By applying the same argument to w_2 we deduce that

$$\begin{cases} -\Delta w_1 \geq V_2 w_2 \\ -\Delta w_2 \geq V_1 w_1 \end{cases}, \quad V_1 = \lambda p_1(\alpha_1 + u_{1,\lambda}[\alpha_2])_+^{p_1-1}, \; V_2 = \lambda p_2(\alpha_2 + u_{2,\lambda}[\alpha_1])_+^{p_2-1}.$$

and then, in view of Lemma 2.1 and possibly taking a smaller λ_0, well known results for cooperative elliptic systems ([22]) show that $w_i \geq 0$, $i = 1, 2$, as claimed.

(ii) For $\lambda = 0$ we have $u_{i,0,\boldsymbol{\alpha}} = 0$, $i = 1, 2$ and then necessarily $\alpha_{i,\lambda} = 1$, $i = 1, 2$. For fixed $\lambda \in (0, \lambda_0]$, by Lemma A.2 and (i) the functions

$$g_1(\boldsymbol{\alpha}) = \int_\Omega (\alpha_1 + u_{1,\lambda}[\alpha_2])_+^{p_1}, \quad g_2(\boldsymbol{\alpha}) = \int_\Omega (\alpha_2 + u_{2,\lambda}[\alpha_1])_+^{p_2}, \quad \boldsymbol{\alpha} \in ((-\infty, 1])^2,$$

are continuous as a function of (α_1, α_2) and increasing in α_1 and α_2. Clearly $\|u_{1,\lambda}[\alpha_2]\|_\infty \leq \lambda C_2$, $\|u_{1,\lambda}[\alpha_2]\|_\infty \leq \lambda C_2$ for any $\lambda \leq \lambda_0$, and then, possibly taking λ_0 small enough to guarantee that

$$(2\lambda_0 C_2)^{p_i} \leq \frac{1}{4}, \; i = 1, 2, \tag{A.2}$$

we have,

$$g_1((0, \alpha_2)) = \int_\Omega (u_{1,\lambda}[\alpha_2])^{p_1} \leq \frac{1}{4}, \; \forall \lambda \in (0, \lambda_0], \; \forall \alpha_2 \in (-\infty, 1],$$

$$g_2((\alpha_1, 0)) \leq \int_\Omega (u_{2,\lambda}[\alpha_1])^{p_2} \leq \frac{1}{4}, \; \forall \lambda \in (0, \lambda_0], \; \forall \alpha_1 \in (-\infty, 1],$$

while we also have,

$$g_1((1, \alpha_2)) = \int_\Omega (1 + u_{1,\lambda}[\alpha_2])^{p_1} > 1, \; \forall \lambda \in (0, \lambda_0], \; \forall \alpha_2 \in (-\infty, 1],$$

$$g_2((\alpha_1, 1)) = \int_\Omega (1 + u_{2,\lambda}[\alpha_1])^{p_2} > 1, \; \forall \lambda \in (0, \lambda_0], \; \forall \alpha_1 \in (-\infty, 1].$$

As a consequence, for any $\lambda \in (0, \lambda_0]$ there exists at least one $\boldsymbol{\alpha}_\lambda = (\alpha_{1,\lambda}, \alpha_{2,\lambda})$ such that $g_i(\boldsymbol{\alpha}_\lambda) = 1$, $i = 1, 2$ which, by the monotonicity of g_i, must necessarily satisfy

$$\alpha_{i,\lambda} \in (0, 1), \; i = 1, 2. \tag{A.3}$$

This concludes the proof of Lemma A.1-(j).

Proof of (jj)
Let $(\boldsymbol{\alpha}_\lambda, \boldsymbol{\psi}_\lambda)$ be any solution of $\mathbf{(P)}_\lambda$ for $\lambda \in [0, \lambda_0]$. If $\lambda = 0$ then necessarily $\boldsymbol{\alpha}_\lambda = (1, 1)$. Otherwise let $\lambda \in (0, \lambda_0]$ and observe that $\lambda \boldsymbol{\psi}_\lambda = \mathbf{u}_\lambda$ where $\mathbf{u}_\lambda$ is the unique solution of (A.1) found in (j). As a consequence, by (A.2) and (A.3) we have that,

$$1 = \int_\Omega (\alpha_{1,\lambda} + u_{1,\lambda}[\alpha_{2,\lambda}])_+^{p_1} \leq 2^{p_1} (\alpha_{1,\lambda})_+^{p_1} + (2\lambda C_2)^{p_1} \leq 2^{p_1} \alpha_{1,\lambda}^{p_1} + \frac{1}{4},$$

whence $\alpha_{1,\lambda} \geq (\frac{3}{4})^{\frac{1}{p_1}} \frac{1}{2} > \frac{1}{3}$. □

References

[1] A. Ambrosetti, G. Mancini, *A free boundary problem and a related semilinear equation*, Nonlin. An. **4**(5) (1980), 909-915.

[2] D. Bartolucci, *Global bifurcation analysis of mean field equations and the Onsager microcanonical description of two-dimensional turbulence*, Calc. Var. & P.D.E. **58**:18 (2019).

[3] D. Bartolucci, Y. Hu, A. Jevnikar, W. Yang, *Generic properties of free boundary problems in plasma physics*, Nonlinearity **35** (2022), 411-444.

[4] D. Bartolucci, A. Jevnikar, *On the global bifurcation diagram of the Gel'fand problem*, Analysis & P.D.E. **14** (2021), 2409-2426.

[5] D. Bartolucci, A. Jevnikar, *On the uniqueness and monotonicity of solutions of free boundary problems*, J. Differential Equations **306** (2022), 152-158.

[6] D. Bartolucci, A. Jevnikar, R. Wu, *A Courant nodal domain theorem for linearized mean field type equations*, Commun. Pure Appl. Anal. **22**(9) (2023), 2744-2759.

[7] D. Bartolucci, A. Jevnikar, R. Wu, *Sharp estimates, uniqueness and spikes condensation for superlinear free boundary problems arising in plasma physics*, Preprint (2024).

[8] D. Bartolucci, G. Wolansky, *Maximal entropy solutions under prescribed mass and energy*, J. Differential Equations **268** (2020), 6646-6665.

[9] H. Berestycki, H. Brezis, *On a free boundary problem arising in plasma physics*, Nonlin. An. **4**(3) (1980), 415-436.

[10] B. Buffoni, J. Toland, Analytic Theory of Global Bifurcation, (2003) Princeton Univ. Press.

[11] E. Caglioti, P.L. Lions, C. Marchioro, M. Pulvirenti, *A special class of stationary flows for two dimensional Euler equations: a statistical mechanics description. II*, Comm. Math. Phys. **174** (1995), 229-260.

[12] R.S. Cantrell, C. Cosner, *On the Positone Problem for Elliptic Systems* Indiana Univ. Math. J. **34**, (1985), 517-532.

[13] R.S. Cantrell, C. Cosner, *On the convex case in the Positone Problem for Elliptic Systems* Nonlinear. An. **12**, (1988), 827-853.

[14] J.A. Carrillo, K. Lin, *Sharp conditions on global existence and blow-up in a degenerate two- species and cross-attraction system*, Adv. Nonlinear Analysis **11** (2022) 1-39.

[15] T. Carroll, J. Ratzkin, *Interpolating between torsional rigidity and principal frequency*, J. Math. Anal. Appl. **379** (2011) 818-826.

[16] Ph. Clément, D.G. de Figueiredo, E. Mitidieri, *Positive solutions of semilinear elliptic systems*, Comm. P. D. E. **17** (1992), 923-940.

[17] Ph. Clément, D.G. de Figueiredo, E. Mitidieri, *A priori estimates for positive solutions of semilinear elliptic systems via Hardy-Sobolev inequalities*, Pitman Res. Notes Math. (1996), 73-91.

[18] M. G. Crandall, P. H. Rabinowitz, *Some Continuation and Variational Methods for Positive Solutions of Nonlinear Elliptic Eigenvalue Problems*, Arch. Rat. Mech. An. **58** (1975), 207-218.

[19] L. Damascelli, F. Pacella, *Qualitative properties of positive solutions of semilinear elliptic equations in symmetric domains via the maximum principle*, S.I.A.M. J. Math. Anal. **45** (2013), 1003-1026.

[20] D.G. de Figueiredo, *Semilinear Elliptic Systems: existence, multiplicity, symmetry of solutions*, Handbook of Differential Equations: stationary partial differential equations. Vol. V (2008), 1-48.

[21] D.G. de Figueiredo, P.L. Lions, R.D. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations* Jour. Math. Pure Appl. **(61)** (1982), 41-63.

[22] D.G. de Figueiredo, E. Mitidieri, *Maximum principle for linear elliptic systems*, Rend. Ist. Mat. Trieste (1992), 36-66.

[23] J.P. Freidberg, *Ideal MHD*, Cambridge U.P. (2014).

[24] B. Gidas, W.M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209-243.

[25] E. H. Lieb, M. Loss *Analysis* 2^{nd}-ed., Graduate Studies in Mathematics, A.M.S. 2001.

[26] W. Littman, G. Stampacchia, H. F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Normale Sup. Pisa **17** (1963), 43-77.

[27] Z. Liu, *Multiple solutions for a free boundary problem arising in plasma physics*, Proc. R. S. E. Sect. A **144**(5) (2014), 965-990.

[28] R. Ortega, *Nonexistence of radial solutions of two elliptic boundary value problems*, Proc. Roy. Soc. Edinburgh Sect. A **114**(1-2) (1990), 27-31.

[29] M. Montenegro, *Minimal solutions for a class of elliptic systems*, Bull. London Math. Soc. **37** (2005) 405-416.

[30] J.P. Puel (with A. Damlamian), *A free boundary, nonlinear eigenvalue problem*, in G.M. de La Penha, L.A. Medeiros (eds.), Contemporary Developments in Continuum Mechanics and Partial Differential Equations, North-Holland Publishing Company (1978).

[31] P.H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. **7** (1971), 487-513.

[32] D.G. Schaeffer, *Non-uniqueness in the equilibrium shape of a confined plasma*, Comm. P.D.E. **2**(6) (1977), 587-600.

[33] P. Souplet, *The proof of the Lane-Emden conjecture in four space dimensions*, Adv. Math. **221** (2009), 1409-1427.

[34] M.A.S. Souto, *A priori estimates and existence of positive solutions of nonlinear cooperative elliptic systems*, Diff. Int. Eq. **8** (1995), 1245-1258.

[35] R. Temam, *A non-linear eigenvalue problem: the shape at equilibrium of a confined plasma*, Arch. Rational Mech. An. **60** (1975), 51-73.

[36] R. Temam, *Remarks on a free boundary value problem arising in plasma physics*, Comm. P.D.E. **2** (1977), 563-585.

[37] G. Wang, D. Ye, *On a nonlinear elliptic equation arising in a free boundary problem*, Math. Z. **244** (2003), 531-548.

[38] J. Wei, *Multiple condensations for a nonlinear elliptic equation with sub-critical growth and critical behaviour* Proc. Edinb. Math. Soc. (2), **44**(3) (2001), 631-660.

[39] J. Wesson, *Tokamaks*, Clarendon press, Oxford (2004).

Daniele Bartolucci, Department of Mathematics, University of Rome *"Tor Vergata"*, Via della ricerca scientifica n.1, 00133 Roma.
Email address: bartoluc@mat.uniroma2.it

Yeyao Hu, School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, Hunan 410083, P. R. China.
Email address: huyeyao@gmail.com

Aleks Jevnikar, Department of Mathematics, Computer Science and Physics, University of Udine, Via delle Scienze 206, 33100 Udine, Italy.
Email address: aleks.jevnikar@uniud.it

Juncheng Wei, Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong.
Email address: wei@math.cuhk.edu.hk

Wen Yang, Department of Mathematics, Faculty of Science and Technology, University of Macau, Macau, P.R. China
Email address: wenyang@um.edu.mo