LONG-TIME DYNAMICS FOR THE ENERGY CRITICAL HEAT EQUATION IN $\mathbb{R}^5$

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Abstract. We investigate the long-time behavior of global solutions to the energy critical heat equation in $\mathbb{R}^5$

$$\begin{cases} \partial_t u = \Delta u + |u|^{\frac{4}{3}} u & \text{in } \mathbb{R}^5 \times (t_0, \infty), \\ u(\cdot, t_0) = u_0 & \text{in } \mathbb{R}^5. \end{cases}$$

For t_0 sufficiently large, we show the existence of positive solutions for a class of initial value $u_0(x) \sim |x|^{-\gamma}$ as $|x| \to \infty$ with $\gamma > \frac{3}{2}$ such that the global solutions behave asymptotically

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^5)} \sim \begin{cases} t^{-\frac{3(2-\gamma)}{2}} & \text{if } \frac{3}{2} < \gamma < 2 \\ (\ln t)^{-3} & \text{if } \gamma = 2 \\ 1 & \text{if } \gamma > 2 \end{cases} \quad \text{for } t > t_0,$$

which is slower than the self-similar time decay $t^{-\frac{3}{4}}$. These rates are inspired by Fila-King [9, Conjecture 1.1].

1. Introduction and main results

Consider the semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + |u|^{p-1} u, & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(\cdot, 0) = u_0, & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

with $p > 1$. It corresponds to the negative L^2-gradient flow of the associated energy functional

$$\frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1},$$

which is decreasing along classical solutions.

Equation (1.1) has been widely studied since Fujita's celebrated work [12]. The Fujita equation looks rather simple, but extensively rich and sophisticated phenomena arise, and these are intimately related to the power nonlinearity in a rather precise manner. For instance, the Fujita exponent p_F, the Sobolev exponent p_S defined respectively as

$$p_F := 1 + \frac{2}{n}, \qquad p_S = \begin{cases} \frac{n+2}{n-2} & \text{for } n \geq 3 \\ \infty & \text{for } n = 1, 2 \end{cases}$$

play an important role in (1.1) concerning singularity formation, long-time dynamics, and many others, and they have been studied intensively in innumerable literature. It is well known that (1.1) possesses a global nontrivial solution $u \geq 0$ if and only if $p > p_F$. Whether or not the steady states exist greatly affects the dynamical behavior of (1.1). The stationary equation of (1.1) does not have positive classical solutions if and only if $p < p_S$ (see [15] and [2] for instance). For $p = p_S$, up to translations and dilations, the positive steady state to the Yamabe problem is the well known Aubin-Talenti bubble

$$U(x) = \alpha_n (1 + |x|^2)^{-\frac{n-2}{2}}, \quad \alpha_n = [n(n-2)]^{\frac{n-2}{4}}.$$

Such a profile is commonly used when investigating the mechanism of singularity formation for (1.1) with critical exponent $p = p_S$. On the other hand, Liouville type theorems for the heat flow (1.1) and their applications have also been thoroughly investigated. In the subcritical case $p < p_S$, Poláčik and Quittner [22] proved the nonexistence of positive radially symmetric bounded entire solution, and they showed that the global nonnegative radial solution of (1.1) decays to 0 uniformly as $t \to \infty$. Poláčik, Quittner and Souplet [23] developed a general scheme connecting parabolic Liouville type theorems and universal estimates of solutions. Recently in [27], Quittner proved the optimal Liouville theorems without extra symmetry nor decay assumptions on the solutions for $1 < p < p_S$ and showed that the nonnegative global solution of (1.1) must decay to 0 as $t \to \infty$.

This paper aims to understand possible long-time dynamics for global solutions of (1.1) with $p = p_S$ in $\mathbb{R}^5$. Here we call a solution global if its maximal existence time is infinity. The long-time behavior for the solution of (1.1) is partially motivated by the study of threshold solutions. For any nonnegative, smooth function $\phi(x)$ with $\phi \not\equiv 0$, let us define

$$\alpha^* = \alpha^*(\phi) := \sup\{\alpha > 0 : \ T_{\max}(\alpha\phi) = \infty\},$$

and $u^* := u(x, t; \alpha^*\phi)$ is called the threshold solution associated with ϕ. Roughly speaking, the threshold solution lies on the borderline between global solutions and those that blow up in finite time since for $\alpha \gg \alpha^*$, the nonlinearity dominates the Laplacian and vice versa. At the threshold level, the dynamics for u^* in the pointwise sense might be global and bounded, global and unbounded, or blow up in finite time. Any of these might happen depending on the power nonlinearity and the domain. We refer the readers to Ni-Sacks-Tavantzis [20], Lee-Ni [19], Galaktionov-Vázquez [14], Poláčik [21], Quittner [26], and the monograph by Quittner and Souplet [28] and their references for comprehensive studies and descriptions of threshold solutions. On the other hand, the global decaying threshold and non-threshold solutions of Fujita equation have been studied extensively, see [10, 11, 16–19, 23–25, 30, 31] and the references therein.

In [18], Kawanago gave a complete description of the asymptotic behavior of the positive solution in the case $p_F < p < p_S$. Specially, $\|u(\cdot, t; \alpha^*\phi)\|_{L^\infty} \sim t^{-\frac{1}{p-1}}$ for $t > 1$. The spatial decay of initial value plays an important role in the long-time behavior of solutions and threshold solutions of (1.1). For $p \ge p_S$, under the assumption that the initial value u_0 is radial, positive, continuous, and

$$\lim_{|x|\to\infty} u_0(x)|x|^{\frac{2}{p-1}} = 0,$$

Quittner [25, Theorem 1.2] showed that there are no global positive radial solutions with self-similar time decay $t^{-\frac{1}{p-1}}$. From this point, for $p = p_S$, Fila and King [9] predicted formally, via matched asymptotics, the possible decaying/growing rate (in time) of threshold solutions to (1.1) with the radial initial value u_0 satisfying

$$\lim_{r\to\infty} r^\gamma u_0(r) = A \ \text{ for some } \ A > 0 \ \text{ and } \ \gamma > \frac{n-2}{2}. \tag{1.2}$$

They conjectured that the threshold solution u of (1.1) with initial value u_0 should satisfy

$$\lim_{t\to\infty} \frac{\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)}}{\varphi(t; n, \gamma)} = C$$

for some positive constant C depending on n and u_0, where $\varphi(t; n, \gamma)$ is given as:

	$\frac{n-2}{2} < \gamma < 2$	$\gamma = 2$	$\gamma > 2$
$n = 3$	$t^{\frac{\gamma-1}{2}}$	$t^{\frac{1}{2}}(\ln t)^{-1}$	$t^{\frac{1}{2}}$
$n = 4$	$t^{-\frac{2-\gamma}{2}}\ln t$	1	$\ln t$
$n = 5$	$t^{-\frac{3(2-\gamma)}{2}}$	$(\ln t)^{-3}$	1
$n \geq 6$	1	1	1

Table 1. Fila-King [9, Conjecture 1.1]

The case $\gamma > 1,\ n = 3$ was answered affirmatively by del Pino, Musso and Wei [7], where the infinite time blow-up solutions were constructed by the gluing method. The infinite time blow-up solutions are also called grow-up/growing solutions in some literature. The case $\gamma > 2,\ n = 4$ was solved in [32] recently. Due to the intimate connection with the critical Fujita equation in $\mathbb{R}^4$, the trichotomy dynamics of the 1-equivariant harmonic map heat flow was studied in [33]. See also Galaktionov-King [13], Cortázar-del Pino-Musso [3], del Pino-Musso-Wei-Zheng [8] (sign-changing solutions), and Ageno-del Pino [1] for their counterparts in the case of the bounded domain, where the Dirichlet boundary plays a significant role in determining the blow-up dynamics.

This paper addresses the case for $n = 5$ in Table 1. We first introduce some notations that we will use throughout the paper.

Notations:

- We write $a \lesssim b$ (respectively $a \gtrsim b$) if there exists a constant $C > 0$ independent of t_0 such that $a \leq Cb$ (respectively $a \geq Cb$). Set $a \sim b$ if $b \lesssim a \lesssim b$. Denote $f_1 = O(f_2)$ if $|f_1| \lesssim f_2$.
- For any $x \in \mathbb{R}^n$ with $|x| = \big(\sum_{i=1}^{n} x_i^2\big)^{1/2}$, the Japanese bracket denotes $\langle x\rangle = \sqrt{|x|^2+1}$.
- For any $c \in \mathbb{R}$, we use the notation $c-$ (respectively $c+$) to denote a constant less (respectively greater) than c and can be chosen arbitrarily close to c.
- $\eta(x)$ is a smooth cut-off function satisfying $\eta(x) = 1$ for $|x| \leq 1$, $\eta(x) = 0$ for $|x| \geq 2$, and $0 \leq \eta(x) \leq 1$ for all $x \in \mathbb{R}^n$.

The main theorem is stated below.

Theorem 1.1. *Consider*

$$\partial_t u = \Delta u + |u|^{\frac{4}{3}} u \quad in \quad \mathbb{R}^5 \times (t_0, \infty). \tag{1.3}$$

Given constants $\gamma > \frac{3}{2}$ and $D_0,\ D_1$ satisfying $0 < D_0 \leq D_1 < 2D_0$, for t_0 sufficiently large, then there exists a positive solution u of the form

$$u = 15^{\frac{3}{4}}\mu^{-\frac{3}{2}}\left(1 + \left|\frac{x-\xi}{\mu}\right|\right)^{-\frac{3}{2}} \eta\left(\frac{x-\xi}{\sqrt{t}}\right) + O\left(t^{-\frac{\tilde{\gamma}}{2}} R^5 \ln^2 R\right), \quad R = \ln\ln t, \tag{1.4}$$

where $\tilde{\gamma} = \min\{\gamma,\ 3-\}$, $\mu = \mu(t)$, $\xi = \xi(t) \in C^1[t_0, \infty)$ *satisfy*

$$\mu \sim \begin{cases} t^{2-\gamma}, & \gamma < 2 \\ \ln^2 t, & \gamma = 2 \\ 1, & \gamma > 2 \end{cases}, \qquad |\xi| \lesssim R^{-\frac{7}{4}} \begin{cases} t^{2-\gamma}, & \gamma < 2 \\ \ln^2 t, & \gamma = 2 \\ 1, & \gamma > 2. \end{cases} \tag{1.5}$$

In particular, $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^5)} = 15^{\frac{3}{4}} \mu^{-\frac{3}{2}} \left(1 + O\left(t^{\max\{3-2\gamma, -\frac{\tilde{\gamma}}{2}\}} \ln^4 t\right)\right)$*. Moreover, the initial value satisfies*

$$u(x, t_0) = (4\pi t_0)^{-\frac{5}{2}} \int_{\mathbb{R}^5} e^{-\frac{|x-z|^2}{4t_0}} \psi_0(z) dz \qquad \text{for } |x| > 4t_0^{\frac{1}{2}}$$

with an arbitrary function $\psi_0(x)$ *satisfying* $D_0 \langle x \rangle^{-\gamma} \le \psi_0(x) \le D_1 \langle x \rangle^{-\gamma}$*. Furthermore, if* $D_0 = D_1$*, we have*

$$\lim_{|x| \to \infty} \langle x \rangle^\gamma u(x, t_0) = D_0. \tag{1.6}$$

Remark 1.1.

- *The restriction* $D_1 < 2D_0$ *is due to a technical reason in the derivation process of* (5.15) *for the case* $\gamma \le 2$.
- *The scaling rate/dynamics* μ *is derived by balancing the heat flow of the initial value and the Aubin-Talenti bubble via the orthogonal condition* (3.1).
- *Consider* $\partial_t u = \partial_{rr} u + \frac{n-1}{r} \partial_r u + u^{\frac{n+2}{n-2}}$, $r > 0$, $t > 0$ *with* $n \in (4, 6)$*. It is possible to deduce similar results by redoing the construction process.*
- *The scaling rate with logarithmic correction* $t^{k_1} (\ln t)^{k_2} (\ln \ln t)^{k_3} \cdots$ *for some* $k_i \in \mathbb{R}$, $i \in \mathbb{Z}_+$ *with finite multiplicity can be expected when we take the initial value of the form* $u_0(x) \sim \langle x \rangle^{\gamma_1} \langle \ln \langle x \rangle \rangle^{\gamma_2} \langle \ln \langle \ln \langle x \rangle \rangle \rangle^{\gamma_3} \cdots$ *for some* $\gamma_i \in \mathbb{R}$, $i \in \mathbb{Z}_+$.

For $p > p_F$, Lee and Ni [19, Theorem 3.8] gave positive global solutions of (1.1) with the decay rate

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \sim t^{-k} \quad \text{for any} \quad k \in \left[\frac{1}{p-1}, \ \frac{n}{2}\right].$$

In particular, for $n = 5$ and $p = p_S$, $k \in \left[\frac{3}{4}, \frac{5}{2}\right]$.

Theorem 1.1 implies a direct consequence that somewhat expands the picture of global dynamics of positive solutions in the critical case $p = p_S$ in $\mathbb{R}^5$ with algebraic decay rate:

Corollary 1.1. *For* $n = 5$, $p = \frac{7}{3}$*, for all* $k \in [0, \frac{5}{2}]$*, there exists a global positive solution of* (1.1) *with the rate* $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^5)} \sim t^{-k}$ *as* $t \to \infty$.

The construction of Theorem 1.1 is done by the gluing method recently developed in [3, 6]. It is a rather versatile and systematic tool that can be used to investigate the singularity formation for various evolution PDEs, and we refer to [3–7, 29] and the references therein.

The rest of this paper is devoted to the proof of Theorem 1.1.

2. Approximate solutions and the gluing system

Consider the critical heat equation

$$\partial_t u = \Delta u + |u|^{\frac{4}{n-2}} u \quad \text{in} \quad \mathbb{R}^n \times (t_0, \infty). \tag{2.1}$$

The unique positive solution (up to translations and dilations) of the stationary equation $\Delta u + u^{\frac{n+2}{n-2}} = 0$, is given by the Aubin-Talenti solution

$$U(x) = \alpha_n (1 + |x|^2)^{-\frac{n-2}{2}}, \quad \alpha_n = [n(n-2)]^{\frac{n-2}{4}}.$$

The corresponding linearized operator $\Delta + \frac{n+2}{n-2}U^{\frac{4}{n-2}}$ has bounded kernels

$$Z_i(x) = \partial_{x_i}U(x), \quad i = 1, \cdots, n, \quad Z_{n+1}(x) = \frac{n-2}{2}U(x) + x \cdot \nabla U(x).$$

The leading term of the solution to (2.1) is taken as the following form

$$u_1(x,t) = \mu^{-\frac{n-2}{2}}U(y)\,\eta(\tilde{y}) + \Psi_0(x,t), \quad \text{where } \ y := \frac{x-\xi}{\mu}, \quad \tilde{y} := \frac{x-\xi}{\sqrt{t}},$$

$\mu = \mu(t) > 0,\ \xi = \xi(t) \in C^1[t_0, \infty)$ will be determined later, and

$$\Psi_0(x,t) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|x-z|^2}{4t}} \psi_0(z) dz,$$

where $D_0\langle x\rangle^{-\gamma} \le \psi_0(x) \le D_1 \langle x\rangle^{-\gamma}$ with some constants $0 < D_0 \le D_1$. Obviously, $\Psi_0 > 0$ and

$$\partial_t \Psi_0 = \Delta \Psi_0, \quad \Psi_0(\cdot, 0) = \psi_0.$$

We first give a lemma concerning a precise estimate related to Ψ_0.

Lemma 2.1. *Given $n > 0$, $\gamma \in \mathbb{R}$, $t \ge 1$, then*

$$(4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4t}} \langle y\rangle^{-\gamma} dy = v_{n,\gamma}(t)(C_{n,\gamma} + g_{n,\gamma}(t)), \tag{2.2}$$

where

$$v_{n,\gamma}(t) = \begin{cases} t^{-\frac{\gamma}{2}}, & \gamma < n \\ t^{-\frac{n}{2}} \ln(1+t), & \gamma = n \\ t^{-\frac{n}{2}}, & \gamma > n, \end{cases} \tag{2.3}$$

$$C_{n,\gamma} = \begin{cases} (4\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|z|^2}{4}} |z|^{-\gamma} dz, & \gamma < n \\ (4\pi)^{-\frac{n}{2}} \frac{1}{2} \left|S^{n-1}\right|, & \gamma = n \ , \\ (4\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \langle y\rangle^{-\gamma} dy, & \gamma > n \end{cases} \qquad g_{n,\gamma}(t) = O\left(\begin{cases} t^{-1}, & \gamma < n-2 \\ t^{-1}\langle \ln t\rangle, & \gamma = n-2 \\ t^{\frac{\gamma-n}{2}}, & n-2 < \gamma < n \\ (\ln(1+t))^{-1}, & \gamma = n \\ t^{\frac{n-\gamma}{2}}, & n < \gamma < n+2 \\ t^{-1}\langle \ln t\rangle, & \gamma = n+2 \\ t^{-1}, & \gamma > n+2 \end{cases} \right). \tag{2.4}$$

The proof of Lemma 2.1 is postponed to Appendix A.

Hereafter, we always assume $t_0 \ge 1$ is sufficiently large and $t \ge t_0$. By Lemma 2.1, we have

$$D_0 v_{n,\gamma}(t) \left(C_{n,\gamma} + g_{n,\gamma}(t)\right) \le \Psi_0(0,t) \le D_1 v_{n,\gamma}(t) \left(C_{n,\gamma} + g_{n,\gamma}(t)\right). \tag{2.5}$$

By similar calculation, we have

$$\|\nabla \Psi_0(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{-\frac{1}{2}} v_{n,\gamma}(t). \tag{2.6}$$

By [32, Lemma A.3],

$$\Psi_0(x,t) \lesssim t^{-\frac{\tilde{\gamma}}{2}} \mathbf{1}_{|x| \le t^{\frac{1}{2}}} + |x|^{-\tilde{\gamma}} \mathbf{1}_{|x| > t^{\frac{1}{2}}}, \tag{2.7}$$

where $\tilde{\gamma}$ is defined as

$$\tilde{\gamma} := \min\{\gamma, 3-\}. \tag{2.8}$$

Define the error of f as

$$E[f] := -\partial_t f + \Delta f + |f|^{\frac{4}{n-2}} f.$$

Straightforward computation implies

$$E[u_1] = \mu^{-\frac{n}{2}} \dot{\mu} Z_{n+1}(y) \eta(\tilde{y}) + \mu^{-\frac{n}{2}} \dot{\xi} \cdot (\nabla U)(y)\, \eta(\tilde{y}) + \mathcal{E}_\eta + |u_1|^{\frac{4}{n-2}} u_1 - \mu^{-\frac{n+2}{2}} U(y)^{\frac{n+2}{n-2}} \eta(\tilde{y}),$$

where

$$\mathcal{E}_\eta := \mu^{-\frac{n-2}{2}}U(y)\left(2^{-1}t^{-1}\tilde{y}+t^{-\frac{1}{2}}\dot{\xi}\right)\cdot(\nabla\eta)(\tilde{y})+2\mu^{-\frac{n}{2}}t^{-\frac{1}{2}}(\nabla U)(y)\cdot(\nabla\eta)(\tilde{y})+\mu^{-\frac{n-2}{2}}t^{-1}U(y)(\Delta\eta)(\tilde{y}). \tag{2.9}$$

We look for an exact solution u of (2.1) in the form

$$u = u_1 + \psi(x,t) + \mu^{-\frac{n-2}{2}}\phi\left(\frac{x-\xi}{\mu},t\right)\eta_R, \quad \eta_R := \eta\left(\frac{x-\xi}{\mu R}\right), \quad R = R(t) = \ln\ln t. \tag{2.10}$$

We make the ansatz

$$2\mu R \le \sqrt{t}/9. \tag{2.11}$$

Direct calculation deduces that

$$\begin{aligned}
E[u] = &\left(\mu^{-\frac{n}{2}}\dot{\mu}Z_{n+1}(y)+\mu^{-\frac{n}{2}}\dot{\xi}\cdot(\nabla U)(y)\right)\eta(\tilde{y})+\mathcal{E}_\eta+\mu^{-\frac{n+2}{2}}U(y)^{\frac{n+2}{n-2}}\left(\eta(\tilde{y})^{\frac{n+2}{n-2}}-\eta(\tilde{y})\right)\\
&+\mathcal{N}[\psi,\phi,\mu,\xi]+\frac{n+2}{n-2}\mu^{-2}U(y)^{\frac{4}{n-2}}\eta(\tilde{y})^{\frac{4}{n-2}}\left(\Psi_0+\psi+\mu^{-\frac{n-2}{2}}\phi(y,t)\eta_R\right)\\
&-\partial_t\psi+\Delta\psi-\mu^{-\frac{n-2}{2}}\partial_t\phi(y,t)\eta_R+\mu^{-\frac{n+2}{2}}\Delta_y\phi(y,t)\eta_R+\Lambda_1[\phi,\mu,\xi]+\Lambda_2[\phi,\mu,\xi],
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1[\phi,\mu,\xi] := &\ \mu^{-\frac{n+2}{2}}R^{-2}\phi(y,t)(\Delta\eta)(\frac{y}{R})+2\mu^{-\frac{n+2}{2}}R^{-1}\nabla_y\phi(y,t)\cdot(\nabla\eta)(\frac{y}{R})\\
&+\mu^{-\frac{n-2}{2}}\phi(y,t)(\nabla\eta)(\frac{y}{R})\cdot\left(\frac{\dot{\xi}}{\mu R}+\frac{y}{R}\frac{\partial_t(\mu R)}{\mu R}\right),
\end{aligned} \tag{2.12}$$

$$\Lambda_2[\phi,\mu,\xi] := \mu^{-\frac{n}{2}}\dot{\mu}\left(\frac{n-2}{2}\phi(y,t)+y\cdot\nabla_y\phi(y,t)\right)\eta_R+\mu^{-\frac{n}{2}}\dot{\xi}\cdot\nabla_y\phi(y,t)\eta_R, \tag{2.13}$$

$$\mathcal{N}[\psi,\phi,\mu,\xi] := |u|^{\frac{4}{n-2}}u-\mu^{-\frac{n+2}{2}}U(y)^{\frac{n+2}{n-2}}\eta(\tilde{y})^{\frac{n+2}{n-2}}-\frac{n+2}{n-2}\mu^{-2}U(y)^{\frac{4}{n-2}}\eta(\tilde{y})^{\frac{4}{n-2}}\left(\Psi_0+\psi+\mu^{-\frac{n-2}{2}}\phi(y,t)\eta_R\right). \tag{2.14}$$

In order to make $E[u]=0$, it suffices to solve the following gluing system.

The outer problem:

$$\partial_t\psi = \Delta\psi+\mathcal{G}[\psi,\phi,\mu,\xi] \quad \text{in } \mathbb{R}^n\times(t_0,\infty), \quad \psi(\cdot,t_0)=0 \quad \text{in } \mathbb{R}^n, \tag{2.15}$$

where

$$\begin{aligned}
\mathcal{G}[\psi,\phi,\mu,\xi] := &\ \Lambda_1[\phi,\mu,\xi]+\Lambda_2[\phi,\mu,\xi]+\left(\mu^{-\frac{n}{2}}\dot{\mu}Z_{n+1}(y)+\mu^{-\frac{n}{2}}\dot{\xi}\cdot(\nabla U)(y)\right)\eta(\tilde{y})(1-\eta_R)+\mathcal{E}_\eta\\
&+\mu^{-\frac{n+2}{2}}U(y)^{\frac{n+2}{n-2}}\left(\eta(\tilde{y})^{\frac{n+2}{n-2}}-\eta(\tilde{y})\right)+\mathcal{N}[\psi,\phi,\mu,\xi]+\frac{n+2}{n-2}\mu^{-2}U(y)^{\frac{4}{n-2}}\eta(\tilde{y})^{\frac{4}{n-2}}(\Psi_0+\psi)(1-\eta_R);
\end{aligned} \tag{2.16}$$

The inner problem:

$$\mu^2\partial_t\phi = \Delta_y\phi+\frac{n+2}{n-2}U(y)^{\frac{4}{n-2}}\phi+\mathcal{H}[\psi,\mu,\xi] \qquad \text{for } t>t_0, \quad y\in B_{4R(t)}, \tag{2.17}$$

where

$$\mathcal{H}[\psi,\mu,\xi] := \mu\dot{\mu}Z_{n+1}(y)+\mu\dot{\xi}\cdot(\nabla U)(y)+\frac{n+2}{n-2}\mu^{\frac{n-2}{2}}U(y)^{\frac{4}{n-2}}\left(\Psi_0(\mu y+\xi,t)+\psi(\mu y+\xi,t)\right). \tag{2.18}$$

We introduce the new time variable

$$\tau = \tau(t) := \int_{t_0}^t \mu^{-2}(s)ds + C_\tau t_0\mu^{-2}(t_0), \quad \tau_0 := \tau(t_0), \tag{2.19}$$

with a sufficiently large constant C_τ independent of t_0. Then (2.17) can be rewritten as

$$\partial_\tau\phi = \Delta_y\phi(y,t(\tau))+\frac{n+2}{n-2}U(y)^{\frac{4}{n-2}}\phi(y,t(\tau))+\mathcal{H}[\psi,\mu,\xi](y,t(\tau)) \qquad \text{for } \tau>\tau_0, \quad y\in B_{4R(t(\tau))}. \tag{2.20}$$

3. Formal analysis of μ and ϕ

Hereafter, we take $n=5$. As the leading term of μ, μ_0 is determined by the orthogonal condition

$$\int_{B_{4R}}\left(\mu_0\dot\mu_0 Z_{n+1}(y)+\frac{n+2}{n-2}\mu_0^{\frac{n-2}{2}}U(y)^{\frac{4}{n-2}}\Psi_0(0,t)\right)Z_{n+1}(y)dy=0, \tag{3.1}$$

which is equivalent to

$$\dot\mu_0=A(R)\mu_0^{\frac{n-4}{2}}\Psi_0(0,t), \tag{3.2}$$

where

$$A(R):=-\frac{n+2}{n-2}\frac{\int_{B_{4R}}U(y)^{\frac{4}{n-2}}Z_{n+1}(y)dy}{\int_{B_{4R}}Z_{n+1}^2(y)dy}=\frac{n-2}{2}\frac{\int_{\mathbb{R}^n}U(y)^{\frac{n+2}{n-2}}dy}{\int_{\mathbb{R}^n}Z_{n+1}^2(y)dy}\left(1+O\left(R^{\max\{-2,4-n\}}\right)\right)\sim 1 \tag{3.3}$$

for $t\ge M$ with M sufficiently large, and here we have used

$$\int_{\mathbb{R}^n}U(y)^{\frac{4}{n-2}}Z_{n+1}(y)dy=-\frac{(n-2)^2}{2(n+2)}\int_{\mathbb{R}^n}U(y)^{\frac{n+2}{n-2}}dy.$$

We take a solution of (3.2) as

$$\mu_0(t)=\left(\frac{6-n}{2}\int_M^t A(R(s))\Psi_0(0,s)ds\right)^{\frac{2}{6-n}}. \tag{3.4}$$

By (2.5), for $t_0\ge 9M$ sufficiently large,

$$0<\mu_0(t)\sim\mu_{0*}(t):=\begin{cases}t^{\frac{2-\gamma}{6-n}}, & \gamma<2\\ (\ln t)^{\frac{2}{6-n}}, & \gamma=2\\ 1, & \gamma>2\end{cases},\quad \dot\mu_0(t)\sim\mu_{0*}(t)^{\frac{n-4}{2}}v_{n,\gamma}(t). \tag{3.5}$$

We make the following ansatz about μ:

$$\mu=\mu_0+\mu_1,\quad\text{where}\quad \mu_1=\mu_1(t)\in C^1[t_0,\infty),\quad |\mu_1|\le\mu_0/9,\quad |\dot\mu_1|\le\mu_{0*}^{\frac{n-4}{2}}v_{n,\tilde\gamma}/9, \tag{3.6}$$

which implies $\frac{8}{9}\mu_0\le\mu\le\frac{10}{9}\mu_0$ and the ansatz (2.11) for $\gamma>\frac{3}{2}$ and t_0 sufficiently large.

Recall (2.19), then $\tau(t)$ and t have the following relation

$$\tau(t)\sim\begin{cases}t^{\frac{2+2\gamma-n}{6-n}}, & \frac{n-2}{2}<\gamma<2\\ t(\ln t)^{\frac{4}{n-6}}, & \gamma=2\\ t, & \gamma>2.\end{cases} \tag{3.7}$$

By the ansatz (3.6), roughly speaking, the upper bound of $\mathcal{H}[\psi,\mu,\xi]$ is determined by

$$|\mu\dot\mu Z_{n+1}(y)|+\left|\frac{n+2}{n-2}\mu^{\frac{n-2}{2}}U(y)^{\frac{4}{n-2}}\Psi_0(0,t)\right|\lesssim\mu_{0*}^{\frac{n-2}{2}}v_{n,\tilde\gamma}\langle y\rangle^{\max\{-4,2-n\}}. \tag{3.8}$$

By $\gamma>\frac{n-2}{2}$ and (3.7), we have

$$\mu_{0*}^{\frac{n-2}{2}}v_{n,\tilde\gamma}=\begin{cases}t^{\frac{n-2-2\gamma}{6-n}}, & \gamma<2\\ (\ln t)^{\frac{n-2}{6-n}}t^{-1}, & \gamma=2\\ t^{-\frac{\tilde\gamma}{2}}, & 2<\tilde\gamma<3\end{cases}\sim\tilde v_{n,\tilde\gamma}(\tau(t)),\quad\text{where}\quad \tilde v_{n,\tilde\gamma}(\tau):=\begin{cases}\tau^{-1}, & \frac{n-2}{2}<\gamma<2\\ (\tau\ln\tau)^{-1}, & \gamma=2\\ \tau^{-\frac{\tilde\gamma}{2}}, & 2<\tilde\gamma<3.\end{cases} \tag{3.9}$$

Taking $n=5$, we introduce the norm to measure the right hand side of the inner problem

$$\|f\|_*:=\sup_{\tau>\tau_0,\ y\in B_{2R(t(\tau))}}\left[\tilde v_{5,\tilde\gamma}(\tau)\right]^{-1}\langle y\rangle^3|f(y,\tau)|.$$

The linearized operator $\Delta+\frac{7}{3}U^{\frac{4}{3}}$ has only one positive eigenvalue $\gamma_0>0$ such that

$$\Delta Z_0+\frac{7}{3}U^{\frac{4}{3}}Z_0=\gamma_0 Z_0, \tag{3.10}$$

where the corresponding eigenfunction $Z_0 \in L^\infty(\mathbb{R}^5)$ is radially symmetric and has exponential decay at spatial infinity. The following linear theory of the inner problem in dimension 5 is given by [32, Proposition 7.2] and [3, Proposition 7.1].

Proposition 3.1. *Consider*

$$\begin{cases} \partial_\tau f = \Delta f + \frac{7}{3}U(y)^{\frac{4}{3}} f + h & \text{for } \tau > \tau_0, \quad y \in B_{4R(t(\tau))}, \\ f(y,\tau_0) = e_0 Z_0(y) & \text{for } y \in B_{4R(t(\tau_0))}, \end{cases} \tag{3.11}$$

where h satisfies $\|h\|_* < \infty$ and

$$\int_{B_{4R(t(\tau))}} h(y,\tau) Z_j(y) dy = 0, \quad \forall \tau \in (\tau_0, \infty), \quad j = 1, 2, \cdots, 6, \tag{3.12}$$

then for τ_0 suffciently large, there exists a solution $(f, e_0) = (\mathcal{T}_{\rm in}[h], \mathcal{T}_{e_0}[h])$ as a linear mapping about h, which satisfies the estimates

$$\langle y\rangle |\nabla f| + |f| \lesssim \tilde{v}_{5,\tilde{\gamma}}(\tau) R^5 \ln R \langle y\rangle^{-6} \|h\|_*, \quad |e_0| \lesssim \tilde{v}_{5,\tilde{\gamma}}(\tau_0) R(\tau_0) \|h\|_*.$$

Remark 3.1. *By (2.19), $4R(t(\tau))$ given here behaves like $\ln\ln\tau$ but does not satisfy the assumption for $R(\tau)$ in [32, p.37] accurately. In fact, one can repeat the proof of [32, Proposition 7.2] and [3, Proposition 7.1] to obtain Proposition 3.1.*

By Proposition 3.1 and the convenience for applying the Schauder fixed-point theorem for the inner problem (2.20), we define the norm

$$\|g\|_{\rm in} := \sup_{\tau > \tau_0, y \in B_{2R(t(\tau))}} \Big[\tilde{v}_{5,\tilde{\gamma}}(\tau) R^5(t(\tau)) \ln^2 \left(R(t(\tau))\right)\Big]^{-1} \langle y\rangle^6 \big(\langle y\rangle |\nabla g(y,\tau)| + |g(y,\tau)|\big), \tag{3.13}$$

and we will solve (2.20) in the space

$$B_{\rm in} := \left\{ g(x,\tau) \mid g(\cdot,\tau) \in C^1\left(B_{2R(t(\tau))}\right) \text{ for } \tau > \tau_0, \quad \|g\|_{\rm in} \le 1 \right\}. \tag{3.14}$$

4. Solving the outer problem

Proposition 4.1. *Given $\phi \in B_{\rm in}$, μ_1, $\xi \in C^1[t_0,\infty)$ satisfying*

$$|\mu_1| \le \mu_{0*} R^{-\frac{1}{2}}, \quad |\dot{\mu}_1| \le \mu_{0*}^{\frac{1}{2}} v_{5,\tilde{\gamma}} R^{-\frac{1}{2}}, \quad |\xi| \le \mu_{0*} R^{-\frac{1}{2}}, \quad |\dot{\xi}| \le \mu_{0*}^{\frac{1}{2}} v_{5,\tilde{\gamma}} R^{-\frac{3}{2}}, \tag{4.1}$$

then for t_0 sufficiently large, there exists a unique solution $\psi = \psi[\phi, \mu_1, \xi]$ for the outer problem (2.15) with $n = 5$, which satisfies the following estimates:

$$|\psi| \lesssim v_{5,\tilde{\gamma}} R^{-1} \ln^2 R \left(\mathbf{1}_{|x| \le \sqrt{t}} + t|x|^{-2} \mathbf{1}_{|x| > \sqrt{t}}\right), \tag{4.2}$$

$$\|\nabla \psi(\cdot,t)\|_{L^\infty(\mathbb{R}^5)} \lesssim v_{5,\tilde{\gamma}} \mu_{0*}^{-1} R^{-2} \ln^2 R. \tag{4.3}$$

Proof. It suffices to find a fixed point for the following mapping

$$\psi = \mathcal{T}_5^{\rm out}\left[\mathcal{G}\left[\psi, \phi, \mu, \xi\right]\right],$$

where $\mathcal{G}\left[\psi, \phi, \mu, \xi\right]$ is given in (2.16), and

$$\mathcal{T}_5^{\rm out}[f] := \int_{t_0}^t \int_{\mathbb{R}^5} \left[4\pi(t-s)\right]^{-\frac{5}{2}} e^{-\frac{|x-z|^2}{4(t-s)}} f(z,s) dz ds.$$

In this proof, we always assume t_0 is sufficiently large and $\int_{t_2}^{t_1} \cdots ds = 0$ if $t_1 \le t_2$. Obviously, (4.1) implies the ansatz (3.6) as well as (2.11). Combining these with (3.5), we see that there exists a constant $C_\mu > 9$ sufficiently large such that

$$9 C_\mu^{-1} \mu_{0*} < \mu < C_\mu \mu_{0*}/9. \tag{4.4}$$

In what follows, [32, Lemma A.1, Lemma A.2] will be used repetitively to estimate $\mathcal{T}_5^{\text{out}}[\cdot]$.

Recall $\Lambda_1[\phi,\mu,\xi]$ given in (2.12). Using (2.11), (4.1), and $\gamma>\frac{3}{2}$, we have

$$\left|\frac{\dot\xi}{\mu R}\right|+\left|\frac{\partial_t(\mu R)}{\mu R}\right|=\left|\frac{\dot\xi}{\mu R}\right|+\left|\frac{\dot\mu}{\mu}+\frac{\dot R}{R}\right|\lesssim \mu^{-2}R^{-2}.$$

For $\phi\in B_{\text{in}}$, by (3.9),

$$\langle y\rangle|\nabla_y\phi|+|\phi|\lesssim \mu_{0*}^{\frac{3}{2}}v_{5,\tilde\gamma}R^5\ln^2 R\langle y\rangle^{-6}.$$

Thus,

$$|\Lambda_1[\phi,\mu,\xi]|\lesssim \mu_{0*}^{-2}R^{-3}\ln^2 Rv_{5,\tilde\gamma}\mathbf{1}_{R\le|y|\le 2R}\le \mu_{0*}^{-2}R^{-3}\ln^2 Rv_{5,\tilde\gamma}\mathbf{1}_{|x|\le C_\mu\mu_{0*}R}. \tag{4.5}$$

Then

$$\begin{aligned}&\mathcal{T}_5^{\text{out}}\left[\mu_{0*}^{-2}R^{-3}\ln^2 Rv_{5,\tilde\gamma}\mathbf{1}_{|x|\le C_\mu\mu_{0*}R}\right]\lesssim t^{-\frac{5}{2}}e^{-\frac{|x|^2}{16t}}\int_{t_0}^{\frac{t}{2}}\mu_{0*}^3(s)\left(R^2\ln^2 R\right)(s)v_{5,\tilde\gamma}(s)ds\\&+\mu_{0*}^{-2}R^{-3}\ln^2 Rv_{5,\tilde\gamma}\left[(\mu_{0*}R)^2\mathbf{1}_{|x|\le\mu_{0*}R}+|x|^{-3}e^{-\frac{|x|^2}{16t}}(\mu_{0*}R)^5\mathbf{1}_{|x|>\mu_{0*}R}\right]\\&\lesssim w_o(x,t):=v_{5,\tilde\gamma}R^{-1}\ln^2 R\left(\mathbf{1}_{|x|\le\sqrt t}+t|x|^{-2}\mathbf{1}_{|x|>\sqrt t}\right),\end{aligned}$$

where we have used the properties $v_{5,\tilde\gamma}R^{-1}\ln^2 R\gtrsim t^{-\frac{5}{2}+c}$ with a small constant $c>0$ and $\gamma>\frac{3}{2}$ to get the last inequality. Then

$$\left|\mathcal{T}_5^{\text{out}}[\Lambda_1[\phi,\mu,\xi]]\right|\le C_o w_o(x,t)/2$$

with a sufficiently large constant $C_o\ge 2$. For this reason, we define the norm

$$\|f\|_{\text{out}}:=\sup_{t\ge t_0,\ x\in\mathbb{R}^5}(w_o(x,t))^{-1}|f(x,t)|,$$

and the outer problem (2.15) will be solved in the space

$$B_{\text{out}}:=\{f\mid \|f\|_{\text{out}}\le C_o\}. \tag{4.6}$$

Assume $\epsilon_1>0$ is a sufficiently small constant, which can vary from line to line. For $\Lambda_2[\phi,\mu,\xi]$ given in (2.13), we have

$$|\Lambda_2[\phi,\mu,\xi]|\lesssim \mu_{0*}^{-\frac{1}{2}}v_{5,\tilde\gamma}^2R^5\ln^2 R\langle y\rangle^{-6}\mathbf{1}_{|x|\le C_\mu\mu_{0*}R}\lesssim t^{-\epsilon_1}\mu_{0*}^{-2}R^{-3}\ln^2 Rv_{5,\tilde\gamma}\mathbf{1}_{|x|\le C_\mu\mu_{0*}R},$$

where we have used $\gamma>\frac{3}{2}$ and the last term has been handled in (4.5).

By (2.11), (4.1), and $\gamma>\frac{3}{2}$, one has

$$\begin{aligned}&\left|\left(\mu^{-\frac{5}{2}}\dot\mu Z_6(y)+\mu^{-\frac{5}{2}}\dot\xi\cdot(\nabla U)(y)\right)\eta(\tilde y)(1-\eta_R)\right|+|\mathcal{E}_\eta|+\left|\mu^{-\frac{7}{2}}U(y)^{\frac{7}{3}}\left(\eta(\tilde y)^{\frac{7}{3}}-\eta(\tilde y)\right)\right|\\&+\left|\frac{7}{3}\mu^{-2}U(y)^{\frac{4}{3}}\eta(\tilde y)^{\frac{4}{3}}\Psi_0(1-\eta_R)\right|\\&\lesssim \mu_{0*}^{-2}v_{5,\tilde\gamma}\langle y\rangle^{-3}\mathbf{1}_{\mu R\le|x-\xi|\le 2\sqrt t}+\mu_{0*}^{-\frac{3}{2}}t^{-1}\langle y\rangle^{-3}\mathbf{1}_{\sqrt t\le|x-\xi|\le 2\sqrt t}\\&\lesssim v_{5,\tilde\gamma}\mu_{0*}|x|^{-3}\mathbf{1}_{C_\mu^{-1}\mu_{0*}R\le|x|\le 4\sqrt t}+\mu_{0*}^{\frac{3}{2}}t^{-\frac{5}{2}}\mathbf{1}_{\sqrt t/2\le|x|\le 4\sqrt t},\end{aligned}$$

and their convolutions can be estimated as

$$\begin{aligned}&\mathcal{T}_5^{\text{out}}\left[v_{5,\tilde\gamma}\mu_{0*}|x|^{-3}\mathbf{1}_{C_\mu^{-1}\mu_{0*}R\le|x|\le 4\sqrt t}\right]\lesssim t^{-\frac{5}{2}}e^{-\frac{|x|^2}{16t}}\int_{t_0}^{\frac{t}{2}}v_{5,\tilde\gamma}(s)\mu_{0*}(s)sds\\&+v_{5,\tilde\gamma}\mu_{0*}\left((\mu_{0*}R)^{-1}\mathbf{1}_{|x|\le\mu_{0*}R}+|x|^{-1}\mathbf{1}_{\mu_{0*}R<|x|\le\sqrt t}+t|x|^{-3}e^{-\frac{|x|^2}{16t}}\mathbf{1}_{|x|>\sqrt t}\right)\lesssim(\ln R)^{-2}\omega_o(x,t);\\&\mathcal{T}_5^{\text{out}}\left[\mu_{0*}^{\frac{3}{2}}t^{-\frac{5}{2}}\mathbf{1}_{\sqrt t/2\le|x|\le 4\sqrt t}\right]\lesssim t^{-\frac{5}{2}}e^{-\frac{|x|^2}{16t}}\int_{t_0}^{\frac{t}{2}}\mu_{0*}^{\frac{3}{2}}(s)ds+\mu_{0*}^{\frac{3}{2}}t^{-\frac{3}{2}}\left(\mathbf{1}_{|x|\le\sqrt t}+t^{\frac{3}{2}}|x|^{-3}e^{-\frac{|x|^2}{16t}}\mathbf{1}_{|x|>\sqrt t}\right)\lesssim t^{-\epsilon_1}\omega_o(x,t),\end{aligned}$$

where in the last step, we have used the property $\tilde\gamma<3$ in (2.8) and $\gamma>\frac{3}{2}$.

For any $\psi_1, \psi_2 \in B_{\mathrm{out}}$, we have

$$\left|\mu^{-2} U(y)^{\frac{4}{3}} \eta(\tilde{y})^{\frac{4}{3}} (\psi_1 - \psi_2)(1-\eta_R)\right| \lesssim \mu_{0*}^{-2} \langle y\rangle^{-4} \|\psi_1 - \psi_2\|_{\mathrm{out}} w_o(x,t) \mathbf{1}_{\mu R \le |x-\xi| \le 2\sqrt{t}}$$
$$\lesssim v_{5,\tilde{\gamma}} R^{-1} \ln^2 R \mu_{0*}^2 |x|^{-4} \mathbf{1}_{C_\mu^{-1}\mu_{0*} R \le |x| \le 4\sqrt{t}} \|\psi_1 - \psi_2\|_{\mathrm{out}}.$$

Then

$$\mathcal{T}_5^{\mathrm{out}} \left[v_{5,\tilde{\gamma}} R^{-1} \ln^2 R \mu_{0*}^2 |x|^{-4} \mathbf{1}_{C_\mu^{-1}\mu_{0*} R \le |x| \le 4\sqrt{t}} \right] \lesssim t^{-\frac{5}{2}} e^{-\frac{|x|^2}{16t}} \int_{t_0}^{\frac{t}{2}} (v_{5,\tilde{\gamma}} R^{-1} \ln^2 R)(s) \mu_{0*}^2(s) s^{\frac{1}{2}} ds$$
$$+ v_{5,\tilde{\gamma}} R^{-1} \ln^2 R \mu_{0*}^2 \left(\mu_{0*}^{-2} R^{-2} \mathbf{1}_{|x|\le\sqrt{t}} + |x|^{-3} e^{-\frac{|x|^2}{16t}} t^{\frac{1}{2}} \mathbf{1}_{|x|>\sqrt{t}} \right) \lesssim R^{-2} w_o(x,t),$$

where we have used $\gamma > \frac{3}{2}$ in the last step.

For $\mathcal{N}[\psi,\phi,\mu,\xi]$ defined in (2.14), given any $\psi \in B_{\mathrm{out}}$, we estimate

$$|\mathcal{N}[\psi,\phi,\mu,\xi]| \lesssim \left(\left|\mu^{-\frac{3}{2}} U(y)\eta(\tilde{y})\right|^{\frac{1}{3}} + \left|\Psi_0 + \psi + \mu^{-\frac{3}{2}}\phi(y,t)\eta_R\right|^{\frac{1}{3}} \right) \left|\Psi_0 + \psi + \mu^{-\frac{3}{2}}\phi(y,t)\eta_R\right|^2$$
$$\lesssim \mu^{-\frac{1}{2}} U(y)^{\frac{1}{3}} \eta(\tilde{y})^{\frac{1}{3}} \left(\Psi_0^2 + \psi^2 + \left|\mu^{-\frac{3}{2}}\phi(y,t)\eta_R\right|^2 \right) + |\Psi_0|^{\frac{7}{3}} + |\psi|^{\frac{7}{3}} + \left|\mu^{-\frac{3}{2}}\phi(y,t)\eta_R\right|^{\frac{7}{3}}$$
$$\lesssim \mu_{0*}^{-\frac{1}{2}} \langle y\rangle^{-1} \mathbf{1}_{|x-\xi|\le 2t^{\frac{1}{2}}} \Big[t^{-\tilde{\gamma}} \mathbf{1}_{|x|\le t^{\frac{1}{2}}} + |x|^{-2\tilde{\gamma}} \mathbf{1}_{|x|>t^{\frac{1}{2}}} + C_o^2 \left(v_{5,\tilde{\gamma}} R^{-1}\ln^2 R\right)^2 \left(\mathbf{1}_{|x|\le t^{\frac{1}{2}}} + \left(t|x|^{-2}\right)^2 \mathbf{1}_{|x|>t^{\frac{1}{2}}} \right)$$
$$+ \left(v_{5,\tilde{\gamma}} R^5 \ln^2 R \langle y\rangle^{-6}\right)^2 \mathbf{1}_{|x-\xi|\le 2\mu R} \Big]$$
$$+ t^{-\frac{7}{3}\frac{\tilde{\gamma}}{2}} \mathbf{1}_{|x|\le t^{\frac{1}{2}}} + |x|^{-\frac{7}{3}\tilde{\gamma}} \mathbf{1}_{|x|>t^{\frac{1}{2}}} + C_o^{\frac{7}{3}} \left(v_{5,\tilde{\gamma}} R^{-1}\ln^2 R\right)^{\frac{7}{3}} \left(\mathbf{1}_{|x|\le t^{\frac{1}{2}}} + \left(t|x|^{-2}\right)^{\frac{7}{3}} \mathbf{1}_{|x|>t^{\frac{1}{2}}} \right)$$
$$+ \left(v_{5,\tilde{\gamma}} R^5 \ln^2 R \langle y\rangle^{-6}\right)^{\frac{7}{3}} \mathbf{1}_{|x-\xi|\le 2\mu R}$$
$$\lesssim \mu_{0*}^{\frac{1}{2}} (|x|+\mu_{0*})^{-1} \mathbf{1}_{|x|\le 4t^{\frac{1}{2}}} \Big[C_o^2 \left(v_{5,\tilde{\gamma}} R^5 \ln^2 R\right)^2 \mathbf{1}_{|x|\le t^{\frac{1}{2}}} + |x|^{-2\tilde{\gamma}} \mathbf{1}_{|x|>t^{\frac{1}{2}}} + C_o^2 \left(v_{5,\tilde{\gamma}} R^{-1}\ln^2 R\right)^2 \left(t|x|^{-2}\right)^2 \mathbf{1}_{|x|>t^{\frac{1}{2}}} \Big]$$
$$+ C_o^{\frac{7}{3}} \left(v_{5,\tilde{\gamma}} R^5 \ln^2 R\right)^{\frac{7}{3}} \mathbf{1}_{|x|\le t^{\frac{1}{2}}} + |x|^{-\frac{7}{3}\tilde{\gamma}} \mathbf{1}_{|x|>t^{\frac{1}{2}}} + C_o^{\frac{7}{3}} \left(v_{5,\tilde{\gamma}} R^{-1}\ln^2 R\right)^{\frac{7}{3}} \left(t|x|^{-2}\right)^{\frac{7}{3}} \mathbf{1}_{|x|>t^{\frac{1}{2}}}$$
$$\lesssim C_o^2 \left(v_{5,\tilde{\gamma}} R^5 \ln^2 R\right)^2 \mu_{0*}^{\frac{1}{2}} (|x|+\mu_{0*})^{-1} \mathbf{1}_{|x|\le t^{\frac{1}{2}}} + C_o^2 \mu_{0*}^{\frac{1}{2}} t^{-\frac{1}{2}-\tilde{\gamma}} \mathbf{1}_{t^{\frac{1}{2}}<|x|\le 4t^{\frac{1}{2}}}$$
$$+ C_o^{\frac{7}{3}} \left(v_{5,\tilde{\gamma}} R^5 \ln^2 R\right)^{\frac{7}{3}} \mathbf{1}_{|x|\le t^{\frac{1}{2}}} + |x|^{-\frac{7}{3}\tilde{\gamma}} \mathbf{1}_{|x|>t^{\frac{1}{2}}} + C_o^{\frac{7}{3}} \left(v_{5,\tilde{\gamma}} R^{-1}\ln^2 R\right)^{\frac{7}{3}} \left(t|x|^{-2}\right)^{\frac{7}{3}} \mathbf{1}_{|x|>t^{\frac{1}{2}}}.$$

Here, by $\gamma > \frac{3}{2}$, we then estimate their convolutions

$$\mathcal{T}_5^{\mathrm{out}} \left[\left(v_{5,\tilde{\gamma}} R^5 \ln^2 R\right)^2 \mu_{0*}^{\frac{1}{2}} (|x|+\mu_{0*})^{-1} \mathbf{1}_{|x|\le t^{\frac{1}{2}}} \right] \lesssim t^{-\frac{5}{2}} e^{-\frac{|x|^2}{16t}} \int_{t_0}^{\frac{t}{2}} \left(v_{5,\tilde{\gamma}} R^5 \ln^2 R\right)^2 (s) \mu_{0*}^{\frac{1}{2}}(s) s^2 ds$$
$$+ \left(v_{5,\tilde{\gamma}} R^5 \ln^2 R\right)^2 \mu_{0*}^{\frac{1}{2}} \left(t^{\frac{1}{2}} \mathbf{1}_{|x|\le t^{\frac{1}{2}}} + t^2 |x|^{-3} e^{-\frac{|x|^2}{16t}} \mathbf{1}_{|x|>t^{\frac{1}{2}}} \right) \lesssim t^{-\epsilon_1} w_o(x,t);$$

$$\mathcal{T}_5^{\mathrm{out}} \left[\left(v_{5,\tilde{\gamma}} R^5 \ln^2 R\right)^{\frac{7}{3}} \mathbf{1}_{|x|\le t^{\frac{1}{2}}} \right] \lesssim t^{-\frac{5}{2}} e^{-\frac{|x|^2}{16t}} \int_{t_0}^{\frac{t}{2}} \left(v_{5,\tilde{\gamma}} R^5 \ln^2 R\right)^{\frac{7}{3}} (s) s^{\frac{5}{2}} ds$$
$$+ \left(v_{5,\tilde{\gamma}} R^5 \ln^2 R\right)^{\frac{7}{3}} \left(t \mathbf{1}_{|x|\le t^{\frac{1}{2}}} + t^{\frac{5}{2}} |x|^{-3} e^{-\frac{|x|^2}{16t}} \mathbf{1}_{|x|>t^{\frac{1}{2}}} \right) \lesssim t^{-\epsilon_1} w_o(x,t);$$

$$\mu_{0*}^{\frac{1}{2}} t^{-\frac{1}{2}-\tilde{\gamma}} \mathbf{1}_{t^{\frac{1}{2}}<|x|\le 4t^{\frac{1}{2}}} \lesssim |x|^{-\frac{7}{3}\tilde{\gamma}} \mathbf{1}_{|x|>t^{\frac{1}{2}}},$$

where we used the property $\tilde{\gamma} < 3$;

$$\mathcal{T}_5^{\text{out}}\left[|x|^{-\frac{7}{3}\tilde{\gamma}}\mathbf{1}_{|x|>t^{\frac12}}\right] \lesssim \left[t^{-\frac52}\int_{t_0}^{\frac{t}{2}}\begin{cases}0, & \text{if } \frac73\tilde{\gamma}<5\\ \langle \ln\left(ts^{-1}\right)\rangle, & \text{if } \frac73\tilde{\gamma}=5\\ s^{\frac52-\frac76\tilde{\gamma}}, & \text{if } \frac73\tilde{\gamma}>5\end{cases} ds \quad + t^{1-\frac76\tilde{\gamma}}\right]\mathbf{1}_{|x|\le t^{\frac12}}$$

$$+\left[t|x|^{-\frac73\tilde{\gamma}}+t^{-\frac52}e^{-\frac{|x|^2}{16t}}\int_{t_0}^{\frac{t}{2}}\begin{cases}0, & \text{if } \frac73\tilde{\gamma}<5\\ \langle \ln(|x|s^{-\frac12})\rangle, & \text{if } \frac73\tilde{\gamma}=5\\ s^{\frac52-\frac76\tilde{\gamma}}, & \text{if } \frac73\tilde{\gamma}>5\end{cases} ds\right]\mathbf{1}_{|x|>t^{\frac12}} \lesssim t^{-\epsilon_1}w_o(x,t);$$

$$\mathcal{T}_5^{\text{out}}\left[\left(v_{5,\tilde{\gamma}}R^{-1}\ln^2 R\right)^{\frac73}\left(t|x|^{-2}\right)^{\frac73}\mathbf{1}_{|x|>t^{\frac12}}\right] \lesssim \left[t^{-\frac73}\int_{t_0}^{\frac{t}{2}}\left(v_{5,\tilde{\gamma}}R^{-1}\ln^2 R\right)^{\frac73}(s)s^{\frac73}ds + t\left(v_{5,\tilde{\gamma}}R^{-1}\ln^2 R\right)^{\frac73}\right]\mathbf{1}_{|x|\le t^{\frac12}}$$

$$+|x|^{-\frac{14}{3}}\left[t^{\frac{10}{3}}\left(v_{5,\tilde{\gamma}}R^{-1}\ln^2 R\right)^{\frac73}+\int_{t_0}^{\frac{t}{2}}\left(v_{5,\tilde{\gamma}}R^{-1}\ln^2 R\right)^{\frac73}(s)s^{\frac73}ds\right]\mathbf{1}_{|x|>t^{\frac12}} \lesssim t^{-\epsilon_1}w_o(x,t).$$

For any $\psi_1,\psi_2\in B_{\text{out}}$, one has

$$\begin{aligned}
&|\mathcal{N}[\psi_1,\phi,\mu,\xi]-\mathcal{N}[\psi_2,\phi,\mu,\xi]|\\
&=\left|\frac73(\psi_1-\psi_2)\left[\left|\mu^{-\frac32}U(y)\eta(\tilde{y})+\Psi_0+\theta\psi_1+(1-\theta)\psi_2+\mu^{-\frac32}\phi(y,t)\eta_R\right|^{\frac43}-\left|\mu^{-\frac32}U(y)\eta(\tilde{y})\right|^{\frac43}\right]\right|\\
&\lesssim \|\psi_1-\psi_2\|_{\text{out}}w_o(x,t)\Big[\mu^{-\frac12}U(y)^{\frac13}\eta(\tilde{y})^{\frac13}\left(|\Psi_0|+C_ow_o(x,t)+\left|\mu^{-\frac32}\phi(y,t)\eta_R\right|\right)\\
&\quad+|\Psi_0|^{\frac43}+C_o^{\frac43}w_o(x,t)^{\frac43}+\left|\mu^{-\frac32}\phi(y,t)\eta_R\right|^{\frac43}\Big]\\
&\lesssim \|\psi_1-\psi_2\|_{\text{out}}\Big[\mu^{-\frac12}U(y)^{\frac13}\eta(\tilde{y})^{\frac13}\left(|\Psi_0|^2+C_ow_o(x,t)^2+\left|\mu^{-\frac32}\phi(y,t)\eta_R\right|^2\right)\\
&\quad+|\Psi_0|^{\frac73}+C_o^{\frac43}w_o(x,t)^{\frac73}+\left|\mu^{-\frac32}\phi(y,t)\eta_R\right|^{\frac73}\Big],
\end{aligned}$$

which can be handled by the same way for estimating $|\mathcal{N}[\psi,\phi,\mu,\xi]|$.

In sum, for t_0 sufficiently large, $\mathcal{T}_5^{\text{out}}[\mathcal{G}[\psi,\phi,\mu,\xi]]\in B_{\text{out}}$ and is a contraction mapping about ψ, which implies that there exists a unique solution $\psi\in B_{\text{out}}$. Moreover, by $\gamma>\frac32$, we have

$$|\mathcal{G}[\psi,\phi,\mu,\xi]| \lesssim v_{5,\tilde{\gamma}}\mu_{0*}^{-2}R^{-3}\ln^2 R.$$

By the scaling argument, we get (4.3).

□

5. Solving orthogonal equations about μ_1,ξ

For the utilization of Proposition 3.1 for the inner problem, we need to choose suitable μ_1, ξ such that the orthogonal conditions

$$\int_{B_{4R}}\mathcal{H}[\psi,\mu,\xi](y,t)Z_i(y)dy=0,\quad \mu=\mu_0+\mu_1,\quad i=1,\dots,n+1,\quad n=5 \tag{5.1}$$

are satisfied, where $\psi=\psi[\phi,\mu_1,\xi]$ is solved by Proposition 4.1, and $\mathcal{H}[\psi,\mu,\xi]$ is given in (2.18).

Proposition 5.1. *Given $0<D_0\le D_1<2D_0$, for t_0 sufficiently large, then there exists a solution $(\mu_1,\xi)=(\mu_1[\phi],\xi[\phi])$ for (5.1) with $n=5$ satisfying*

$$|\mu_1|\lesssim \mu_{0*}R^{-\frac23},\quad |\dot{\mu}_1|\lesssim \mu_{0*}^{\frac12}v_{5,\tilde{\gamma}}R^{-\frac23},\quad |\xi|\lesssim \mu_{0*}R^{-\frac74},\quad |\dot{\xi}|\lesssim \mu_{0*}^{\frac12}v_{5,\tilde{\gamma}}R^{-\frac74}. \tag{5.2}$$

Proof. First, let us consider the general dimension n. By (2.18), (5.1) is equivalent to

$$\dot{\mu} = -\frac{n+2}{n-2}\left(\int_{B_{4R}} Z_{n+1}^2(y)dy\right)^{-1} \mu^{\frac{n}{2}-2}\int_{B_{4R}} \left(\Psi_0(\mu y+\xi,t)+\psi(\mu y+\xi,t)\right)U(y)^{\frac{4}{n-2}}Z_{n+1}(y)dy, \tag{5.3}$$

$$\dot{\xi} = \vec{\mathcal{S}}[\mu_1,\xi] := \left(\mathcal{S}_1[\mu_1,\xi],\dots,\mathcal{S}_n[\mu_1,\xi]\right), \qquad \text{for } i=1,2,\dots,n, \tag{5.4}$$

$$\begin{aligned}\mathcal{S}_i[\mu_1,\xi] := &-\frac{n+2}{n-2}\left(\int_{B_{4R}} Z_i^2(y)dy\right)^{-1}\mu^{\frac{n}{2}-2}\\ &\times\int_{B_{4R}}\Big[\Psi_0(\mu y+\xi,t)-\Psi_0(0,t)+\psi(\mu y+\xi,t)-\psi(0,t)\Big]U(y)^{\frac{4}{n-2}}Z_i(y)dy,\end{aligned}$$

where we have used the parity of $Z_i(y)$. By $\mu=\mu_0+\mu_1$ and μ_0 satisfying (3.2), we rewrite (5.3) as

$$\dot{\mu}_1+\beta(t)\mu_1 = \mathcal{F}[\mu_1,\xi](t), \tag{5.5}$$

where

$$\begin{aligned}\beta(t) :=& \frac{n+2}{n-2}\left(\int_{B_{4R}} Z_{n+1}^2(y)dy\right)^{-1}\frac{n-4}{2}\mu_0^{\frac{n}{2}-3}\Psi_0(0,t)\int_{B_{4R}}U(y)^{\frac{4}{n-2}}Z_{n+1}(y)dy\\ =& \frac{n-4}{n-6}\frac{A(R)\Psi_0(0,t)}{\int_M^t A(R(s))\Psi_0(0,s)ds}\end{aligned} \tag{5.6}$$

with the application of (3.4) in the last step;

$$\begin{aligned}\mathcal{F}[\mu_1,\xi](t) :=& -\frac{n+2}{n-2}\left(\int_{B_{4R}} Z_{n+1}^2(y)dy\right)^{-1}\Bigg[\mu^{\frac{n}{2}-2}\int_{B_{4R}}\psi(\mu y+\xi,t)U(y)^{\frac{4}{n-2}}Z_{n+1}(y)dy\\ &+\mu^{\frac{n}{2}-2}\int_{B_{4R}}\left(\Psi_0(\mu y+\xi,t)-\Psi_0(0,t)\right)U(y)^{\frac{4}{n-2}}Z_{n+1}(y)dy\\ &+\left(\mu^{\frac{n}{2}-2}-\mu_0^{\frac{n}{2}-2}-\frac{n-4}{2}\mu_0^{\frac{n}{2}-3}\mu_1\right)\Psi_0(0,t)\int_{B_{4R}}U(y)^{\frac{4}{n-2}}Z_{n+1}(y)dy\Bigg].\end{aligned} \tag{5.7}$$

In order to find a solution (μ_1,ξ) for the system (5.4)-(5.5), it suffices to solve the following fixed point problem about $\dot{\mu}_1,\dot{\xi}$,

$$\begin{aligned}&\dot{\mu}_1 = \mathcal{S}_{n+1}[\mu_1,\xi] := \frac{d}{dt}\left(\int_{\tilde{t}_0}^t \mathcal{F}[\mu_1,\xi](s)e^{\int_t^s\beta(a)da}ds\right) = -\beta(t)\int_{\tilde{t}_0}^t\mathcal{F}[\mu_1,\xi](s)e^{\int_t^s\beta(a)da}ds+\mathcal{F}[\mu_1,\xi](t),\\ &\mu_1=\mu_1[\dot{\mu}_1](t):=\int_{\tilde{t}_0}^t\dot{\mu}_1(a)da,\quad \dot{\xi}=\vec{\mathcal{S}}[\mu_1,\xi],\quad \xi=\xi[\dot{\xi}](t):=\int_{\tilde{t}_0}^t\dot{\xi}(a)da \ \text{ with } \ \tilde{t}_0:=\begin{cases}t_0, & \gamma\le 2\\ \infty, & \gamma>2\end{cases}\end{aligned} \tag{5.8}$$

if these integrals are well-defined.

Hereafter, we take $n=5$. By (2.5) and (3.3), we have

$$\begin{cases}-\frac{D_1}{D_0}\left(1-\frac{\gamma}{2}\right)t^{-1}\left(1+O(R^{-\frac12})\right)\le\beta(t)\le-\frac{D_0}{D_1}\left(1-\frac{\gamma}{2}\right)t^{-1}\left(1+O(R^{-\frac12})\right) & \text{if } \gamma<2,\\ -\frac{D_1}{D_0}(t\ln t)^{-1}\left(1+O(R^{-\frac12})\right)\le\beta(t)\le-\frac{D_0}{D_1}(t\ln t)^{-1}\left(1+O(R^{-\frac12})\right) & \text{if } \gamma=2,\\ \beta(t)\sim -v_{5,\gamma} & \text{if } \gamma>2.\end{cases} \tag{5.9}$$

We will solve the system (5.8) in the space

$$B_{\dot{\mu}_1} := \left\{f\in C[t_0,\infty)\mid \|f\|_{\dot{\mu}_1}\le 1\right\},\quad B_{\dot{\xi}}=\left\{\vec{f}=(f_1,\dots,f_5)\in C[t_0,\infty)\mid \|\vec{f}\|_{\dot{\xi}}\le 1\right\} \tag{5.10}$$

with the norm

$$\|f\|_{\dot{\mu}_1} := \sup_{t\ge t_0}\left(\mu_{0*}^{\frac12}v_{5,\tilde{\gamma}}R^{-\frac23}\right)^{-1}(t)\,|f(t)|,\quad \|\vec{f}\|_{\dot{\xi}} := \sup_{t\ge t_0}\left(\mu_{0*}^{\frac12}v_{5,\tilde{\gamma}}R^{-\frac74}\right)^{-1}(t)|\vec{f}(t)|, \tag{5.11}$$

where

$$\mu_{0*}^{\frac{1}{2}} v_{5,\tilde{\gamma}} = \begin{cases} t^{1-\gamma}, & \gamma < 2 \\ t^{-1}\ln t, & \gamma = 2 \\ t^{-\frac{\tilde{\gamma}}{2}}, & \gamma > 2 \end{cases}.$$

For any $(\dot{\mu}_1, \dot{\xi}) \in B_{\dot{\mu}_1} \times B_{\dot{\xi}}$, it is easy to see that $\int_{\tilde{t}_0}^t \dot{\mu}_1(a)da$ and $\int_{\tilde{t}_0}^t \dot{\xi}(a)da$ in (5.8) are well-defined, and

$$|\mu_1| \lesssim \begin{cases} t^{2-\gamma}R^{-\frac{2}{3}}, & \gamma < 2 \\ (\ln t)^2 R^{-\frac{2}{3}}, & \gamma = 2 \\ t^{1-\frac{\tilde{\gamma}}{2}}R^{-\frac{2}{3}}, & \gamma > 2 \end{cases} \lesssim \mu_{0*}R^{-\frac{2}{3}}, \quad |\xi| \lesssim \mu_{0*}R^{-\frac{7}{4}}. \tag{5.12}$$

Thus, μ_1, $\dot{\mu}_1$, ξ, $\dot{\xi}$ satisfy the assumption (4.1) in Proposition 4.1. By (2.6), (4.3), and $\gamma > \frac{3}{2}$, we get

$$\left|\vec{\mathcal{S}}[\mu_1, \xi]\right| \lesssim \mu^{\frac{1}{2}} \left(\|\nabla_x \Psi_0(\cdot, t)\|_{L^\infty(\mathbb{R}^5)} + \|\nabla_x \psi(\cdot, t)\|_{L^\infty(\mathbb{R}^5)}\right)(|\mu| + |\xi|) \lesssim \mu_{0*}^{\frac{1}{2}} v_{5,\tilde{\gamma}} R^{-2}\ln^2 R. \tag{5.13}$$

Using (4.2), (2.6), (2.5) in order, we have

$$\left|\mu^{\frac{1}{2}} \int_{B_{4R}} \psi(\mu y + \xi, t) U(y)^{\frac{4}{3}} Z_6(y)dy\right| \lesssim \mu_{0*}^{\frac{1}{2}} v_{5,\tilde{\gamma}} R^{-1}\ln^2 R,$$

$$\left|\mu^{\frac{1}{2}} \int_{B_{4R}} (\Psi_0(\mu y + \xi, t) - \Psi_0(0, t)) U(y)^{\frac{4}{3}} Z_6(y)dy\right| \lesssim \mu^{\frac{1}{2}} (\mu + |\xi|) t^{-\frac{1}{2}} v_{5,\gamma} \sim \mu_{0*}^{\frac{3}{2}} t^{-\frac{1}{2}} v_{5,\gamma},$$

$$\left|\left(\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}} - \frac{1}{2}\mu_0^{-\frac{1}{2}}\mu_1\right)\Psi_0(0, t)\int_{B_{4R}} U(y)^{\frac{4}{3}} Z_6(y)dy\right| \lesssim \mu_{0*}^{\frac{1}{2}} v_{5,\gamma} R^{-\frac{4}{3}},$$

which implies

$$|\mathcal{F}[\mu_1, \xi](t)| \lesssim \mu_{0*}^{\frac{1}{2}} v_{5,\tilde{\gamma}} R^{-\frac{3}{4}}. \tag{5.14}$$

Since $0 < D_0 \le D_1 < 2D_0$, there exists $\epsilon_1 > 0$ sufficiently small so that $\frac{D_1}{D_0}(1 + \epsilon_1) < 2$. By taking t_0 sufficiently large, which can depend on γ, and using (5.9), we obtain that for $\gamma < 2$,

$$\left|\int_{\tilde{t}_0}^t \mathcal{F}[\mu_1, \xi](s) e^{\int_t^s \beta(a)da} ds\right| \lesssim \int_{t_0}^t s^{1-\gamma}(\ln\ln s)^{-\frac{3}{4}} e^{\frac{D_1}{D_0}\left(1-\frac{\gamma}{2}\right)(1+\epsilon_1)\int_s^t a^{-1}da} ds$$

$$= t^{\frac{D_1}{D_0}\left(1-\frac{\gamma}{2}\right)(1+\epsilon_1)} \int_{t_0}^t s^{1-\gamma-\frac{D_1}{D_0}\left(1-\frac{\gamma}{2}\right)(1+\epsilon_1)}(\ln\ln s)^{-\frac{3}{4}} ds \lesssim t^{2-\gamma}R^{-\frac{3}{4}};$$

for $\gamma = 2$,

$$\left|\int_{\tilde{t}_0}^t \mathcal{F}[\mu_1, \xi](s) e^{\int_t^s \beta(a)da} ds\right| \lesssim \int_{t_0}^t s^{-1}\ln s(\ln\ln s)^{-\frac{3}{4}} e^{\frac{D_1}{D_0}(1+\epsilon_1)\int_s^t (a\ln a)^{-1}da} ds$$

$$= (\ln t)^{\frac{D_1}{D_0}(1+\epsilon_1)} \int_{t_0}^t s^{-1}(\ln s)^{1-\frac{D_1}{D_0}(1+\epsilon_1)}(\ln\ln s)^{-\frac{3}{4}} ds$$

$$= (\ln t)^{\frac{D_1}{D_0}(1+\epsilon_1)} \int_{\ln t_0}^{\ln t} z^{1-\frac{D_1}{D_0}(1+\epsilon_1)}(\ln z)^{-\frac{3}{4}} dz \lesssim (\ln t)^2 R^{-\frac{3}{4}};$$

for $\gamma > 2$,

$$\left|\int_{\tilde{t}_0}^t \mathcal{F}[\mu_1, \xi](s) e^{\int_t^s \beta(a)da} ds\right| \lesssim \int_t^\infty s^{-\frac{\tilde{\gamma}}{2}} (\ln\ln s)^{-\frac{3}{4}} ds \lesssim t^{1-\frac{\tilde{\gamma}}{2}} R^{-\frac{3}{4}}.$$

Thus, $\int_{\tilde{t}_0}^t \mathcal{F}[\mu_1, \xi](s) e^{\int_t^s \beta(a)da} ds$ is well-defined in (5.8), and

$$\left|\beta(t) \int_{\tilde{t}_0}^t \mathcal{F}[\mu_1, \xi](s) e^{\int_t^s \beta(a)da} ds\right| \lesssim \mu_{0*}^{\frac{1}{2}} v_{5,\tilde{\gamma}} R^{-\frac{3}{4}}. \tag{5.15}$$

Combining (5.13), (5.14) and (5.15), we have

$$|\mathcal{S}_6[\mu_1, \xi]| \lesssim \mu_{0*}^{\frac{1}{2}} v_{5,\tilde{\gamma}} R^{-\frac{3}{4}}, \quad \left|\vec{\mathcal{S}}[\mu_1, \xi]\right| \lesssim \mu_{0*}^{\frac{1}{2}} v_{5,\tilde{\gamma}} R^{-2}\ln^2 R, \tag{5.16}$$

which implies $\big(\mathcal{S}_6, \vec{\mathcal{S}}\big)[\mu_1, \xi] \in B_{\dot\mu_1} \times B_{\dot\xi}$.

For any sequence $(\dot\mu_1^{[j]}, \dot\xi^{[j]})_{j\ge 1} \subset B_{\dot\mu_1} \times B_{\dot\xi}$, denote $\mu_1^{[j]} = \int_{\tilde t_0}^t \dot\mu_1^{[j]}(a)da$, $\xi^{[j]} = \int_{\tilde t_0}^t \dot\xi^{[j]}(a)da$. We set $\tilde{\dot\mu}_1^{[j]} := \mathcal{S}_6[\mu_1^{[j]}, \xi^{[j]}]$, $\tilde{\dot\xi}^{[j]} := \vec{\mathcal{S}}[\mu_1^{[j]}, \xi^{[j]}]$. By the same method for deducing (5.16), we have

$$|\tilde{\dot\mu}_1^{[j]}| \le C_1 \mu_{0*}^{\frac12} v_{5,\tilde\gamma} R^{-\frac34}, \quad |\tilde{\dot\xi}^{[j]}| \le C_1 \mu_{0*}^{\frac12} v_{5,\tilde\gamma} R^{-2}\ln^2 R \quad \text{ for all } \ j\ge 1 \tag{5.17}$$

with a constant $C_1 > 0$ independent of j.

For any compact subset $K \subset\subset [t_0, \infty)$, by the equation (5.8) and the space-time regularity for the outer solution ψ, for all $j \ge 1$, $\tilde{\dot\mu}_1^{[j]}$ and $\tilde{\dot\xi}^{[j]}$ are uniformly Hölder continuous in K. Since there exist countable compact sets to saturate $[t_0, \infty)$, then up to a subsequence, for any compact set $K \subset\subset [t_0, \infty)$,

$$\tilde{\dot\mu}_1^{[j]} \to g, \quad \tilde{\dot\xi}^{[j]} \to \vec g \ \text{ in } \ L^\infty(K) \ \text{ as } \ j\to\infty$$

for some $g,\ \vec g \in C[t_0, \infty)$. By (5.17), we have

$$|g| \le C_1 \mu_{0*}^{\frac12} v_{5,\tilde\gamma} R^{-\frac34}, \quad |\vec g| \le C_1 \mu_{0*}^{\frac12} v_{5,\tilde\gamma} R^{-2}\ln^2 R.$$

Thus, for any $\epsilon_1 > 0$, there exists t_1 sufficiently large such that for all $j \ge 1$,

$$\sup_{t\ge t_1} \left(\mu_{0*}^{\frac12} v_{5,\tilde\gamma} R^{-\frac23}\right)^{-1}(t) \left|\left(\tilde{\dot\mu}_1^{[j]} - g\right)(t)\right| + \sup_{t\ge t_1} \left(\mu_{0*}^{\frac12} v_{5,\tilde\gamma} R^{-\frac74}\right)^{-1}(t) \left|\left(\tilde{\dot\xi}^{[j]} - \vec g\right)(t)\right| < \epsilon_1.$$

Additionally,

$$\lim_{j\to\infty}\left[\sup_{t_0\le t\le t_1} \left(\mu_{0*}^{\frac12} v_{5,\tilde\gamma} R^{-\frac23}\right)^{-1}(t) \left|\left(\tilde{\dot\mu}_1^{[j]} - g\right)(t)\right| + \sup_{t_0\le t\le t_1} \left(\mu_{0*}^{\frac12} v_{5,\tilde\gamma} R^{-\frac74}\right)^{-1}(t) \left|\left(\tilde{\dot\xi}^{[j]} - \vec g\right)(t)\right|\right] = 0.$$

Consequently, $\lim\limits_{j\to\infty}\left(\|\tilde{\dot\mu}_1^{[j]} - g\|_{\dot\mu_1} + \|\tilde{\dot\xi}^{[j]} - \vec g\|_{\dot\xi}\right) = 0$, which implies $\big(\mathcal{S}_6, \vec{\mathcal{S}}\big)[\mu_1, \xi]$ is a compact mapping on $B_{\dot\mu_1} \times B_{\dot\xi}$.

By the Schauder fixed-point theorem, there exists a solution $(\dot\mu_1, \dot\xi) \in B_{\dot\mu_1} \times B_{\dot\xi}$ for the system (5.8).

□

6. Solving the inner problem

By (5.2), (2.6), (4.2), and (3.9), for $|y| \le 4R$,

$$\left|\mu\dot\xi \cdot (\nabla U)(y) + \frac73 \mu^{\frac32} U(y)^{\frac43}\Big(\Psi_0(\mu y + \xi, t) - \Psi_0(0,t) + \psi(\mu y + \xi, t)\Big)\right|$$
$$\lesssim \mu_{0*}^{\frac32} v_{5,\tilde\gamma} R^{-1}\ln^2 R \langle y\rangle^{-4} \sim \tilde v_{5,\tilde\gamma}(\tau(t)) R^{-1}\ln^2 R\langle y\rangle^{-4}.$$

For brevity, denote $\tilde H[\phi] := \mathcal{H}\left[\psi\left[\phi, \mu_1[\phi], \xi[\phi]\right], \mu_0 + \mu_1[\phi], \xi[\phi]\right]$. From (3.8), we have

$$|\tilde H[\phi]| \lesssim \tilde v_{5,\tilde\gamma}(\tau(t))\langle y\rangle^{-3}. \tag{6.1}$$

By Proposition 5.1, we can apply Proposition 3.1 to the inner problem (2.20), and it suffices to solve the following fixed-point problem

$$\phi = \mathcal{T}_{\rm in}\big[\tilde H[\phi]\big].$$

Indeed, for any $\phi \in B_{\rm in}$, given t_0 (i.e. τ_0) sufficiently large, by Proposition 3.1, we have

$$\langle y\rangle \left|\nabla_y \mathcal{T}_{\rm in}\big[\tilde H[\phi]\big]\right| + \left|\mathcal{T}_{\rm in}\big[\tilde H[\phi]\big]\right| \lesssim \tilde v_{5,\tilde\gamma}(\tau) R^5 \ln R \langle y\rangle^{-6}, \quad \left|\mathcal{T}_{e_0}\big[\tilde H[\phi]\big]\right| \lesssim \tilde v_{5,\tilde\gamma}(\tau_0) R(\tau_0), \tag{6.2}$$

which implies $\mathcal{T}_{\rm in}\big[\tilde H[\phi]\big] \in B_{\rm in}$ in particular.

For any sequence $(\phi_j)_{j\geq 1} \in B_{\rm in}$, denote $\tilde{\phi}_j := \mathcal{T}_{\rm in}\left[\tilde{H}[\phi_j]\right]$, $\tilde{e}_j := \mathcal{T}_{e_0}\left[\tilde{H}[\phi_j]\right]$, which satisfies

$$\begin{cases} \partial_\tau \tilde{\phi}_j = \Delta_y \tilde{\phi}_j + \frac{7}{3}U(y)^{\frac{4}{3}}\tilde{\phi}_j + \tilde{H}[\phi_j] & \text{in } \mathcal{D}_{4R} := \left\{(y,\tau) \mid \tau \in (\tau_0,\infty), \quad y \in B_{4R(t(\tau))}\right\} \\ \tilde{\phi}_j(\cdot,\tau_0) = \tilde{e}_j Z_0 & \text{in } B_{4R(t_0)}. \end{cases}$$

Repeating the process for deducing (6.1) and (6.2), one sees that there exists a constant C_1 independent of j such that

$$|\tilde{H}[\phi_j]| \leq C_1 \tilde{v}_{5,\tilde{\gamma}}(\tau)\langle y\rangle^{-3}, \quad \langle y\rangle \left|\nabla_y \tilde{\phi}_j\right| + \left|\tilde{\phi}_j\right| \leq C_1 \tilde{v}_{5,\tilde{\gamma}}(\tau) R^5 \ln R \langle y\rangle^{-6}, \quad |\tilde{e}_j| \leq C_1 \tilde{v}_{5,\tilde{\gamma}}(\tau_0)R(t_0). \tag{6.3}$$

By the parabolic regularity theory, for any compact set $K \subset\subset \mathcal{D}_{3R} \cup (B_{3R(t_0)} \times \{\tau_0\})$, it holds that $\|\phi_j\|_{C^{1+\ell,\frac{1+\ell}{2}}(K)} \leq C_2$ with a constant C_2 independent of j and a constant $\ell \in (0,1)$. By Arzelà-Ascoli theorem, up to a subsequence, there exists a function g which is C^1 in space, such that

$$\tilde{\phi}_j \to g, \quad \nabla_y \tilde{\phi}_j \to \nabla_y g \ \text{ in } \ L^\infty(K) \ \text{ as } \ j \to \infty.$$

By (6.3), we have

$$\langle y\rangle |\nabla_y g| + |g| \leq C_1 \tilde{v}_{5,\tilde{\gamma}}(\tau) R^5 \ln R \langle y\rangle^{-6} \ \text{ in } \ \mathcal{D}_{3R}.$$

For any $\epsilon_1 > 0$, there exists τ_1 sufficiently large such that

$$\sup_{\tau>\tau_1,\ y\in B_{2R(t(\tau))}} \left(\tilde{v}_{5,\tilde{\gamma}}(\tau)R^5(t(\tau))\ln^2(R(t(\tau)))\right)^{-1}\langle y\rangle^6\left(\langle y\rangle\left|\nabla_y(\tilde{\phi}_j - g)(y,\tau)\right| + |(\tilde{\phi}_j - g)(y,\tau)|\right) < \epsilon_1,$$

and

$$\lim_{j\to\infty}\sup_{\tau_0\leq\tau\leq\tau_1,\ y\in B_{2R(t(\tau))}} \left(\tilde{v}_{5,\tilde{\gamma}}(\tau)R^5(t(\tau))\ln^2(R(t(\tau)))\right)^{-1}\langle y\rangle^6\left(\langle y\rangle\left|\nabla_y(\tilde{\phi}_j - g)(y,\tau)\right| + |(\tilde{\phi}_j - g)(y,\tau)|\right) = 0.$$

Thus $\|\tilde{\phi}_j - g\|_{\rm in} \to 0$, which implies $\mathcal{T}_{\rm in}\left[\tilde{H}[\phi]\right]$ is a compact mapping on $B_{\rm in}$. By the Schauder fixed-point theorem, there exists a solution $\phi \in B_{\rm in}$ and thus the construction is complete.

7. Properties of the solution u

Recall u given in (2.10). By (2.7), ψ given by Proposition 4.1, ϕ solved in $B_{\rm in}$ (see (3.14)), μ_1, ξ given in Proposition 5.1, we have the validity of (1.4). The initial value

$$u(x,t_0) = \mu^{-\frac{3}{2}}(t_0)U\left(\frac{x-\xi(t_0)}{\mu(t_0)}\right)\eta\left(\frac{x-\xi(t_0)}{\sqrt{t_0}}\right) + \Psi_0(x,t_0) + \mu^{-\frac{3}{2}}(t_0)\phi\left(\frac{x-\xi(t_0)}{\mu(t_0)},t_0\right)\eta\left(\frac{x-\xi(t_0)}{\mu(t_0)R(t_0)}\right).$$

Here $\Psi_0 > 0$. Denote $y(t_0) = \frac{x-\xi(t_0)}{\mu(t_0)}$, then

$$U(y(t_0)) - \phi(y(t_0),t_0) \geq 15^{\frac{3}{4}}\langle y(t_0)\rangle^{-3} - C\langle y(t_0)\rangle^{-6}R^5(t_0)\ln^2 R(t_0)\begin{cases} t_0^{3-2\gamma}, & \frac{3}{2}<\gamma<2 \\ t_0^{-1}(\ln t_0)^3, & \gamma = 2 \\ t_0^{-\frac{\tilde{\gamma}}{2}}, & \gamma > 2 \end{cases} > 0$$

for t_0 large enough. Therefore, $u(x,t_0) > 0$, which implies $u > 0$ by the maximum principle. In addition, by Lemma B.1, we get (1.6). Finally, we conclude the proof of Theorem 1.1.

Appendix A. Proof of Lemma 2.1

Proof of Lemma 2.1. For $\gamma < n$, $t \geq 1$,

$$(4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4t}} \langle y\rangle^{-\gamma} dy = (4\pi)^{-\frac{n}{2}} t^{-\frac{\gamma}{2}} \int_{\mathbb{R}^n} e^{-\frac{|z|^2}{4}} \left(|z|^2 + t^{-1}\right)^{-\frac{\gamma}{2}} dz$$
$$= t^{-\frac{\gamma}{2}} (4\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|z|^2}{4}} |z|^{-\gamma} dz + O\Bigg(t^{-\frac{\gamma}{2}} \begin{cases} t^{-1}, & \gamma < n-2 \\ t^{-1}\langle \ln t\rangle, & \gamma = n-2 \\ t^{\frac{\gamma-n}{2}}, & n-2 < \gamma < n \end{cases}\Bigg)$$

since

$$\left| \left(\int_{|z|\leq t^{-\frac{1}{2}}} + \int_{|z|>t^{-\frac{1}{2}}} \right) \int_{\mathbb{R}^n} e^{-\frac{|z|^2}{4}} \left[\left(|z|^2 + t^{-1}\right)^{-\frac{\gamma}{2}} - |z|^{-\gamma} \right] dz \right|$$
$$\lesssim \int_{|z|\leq t^{-\frac{1}{2}}} \begin{cases} t^{\frac{\gamma}{2}}, & \gamma < 0 \\ |z|^{-\gamma}, & \gamma \geq 0 \end{cases} dz + \int_{|z|>t^{-\frac{1}{2}}} e^{-\frac{|z|^2}{4}} |z|^{-\gamma-2} t^{-1} dz$$
$$\lesssim t^{\frac{\gamma-n}{2}} + \begin{cases} t^{-1}, & \gamma < n-2 \\ t^{-1}\langle \ln t\rangle, & \gamma = n-2 \\ t^{\frac{\gamma-n}{2}}, & n-2 < \gamma < n \end{cases} \sim \begin{cases} t^{-1}, & \gamma < n-2 \\ t^{-1}\langle \ln t\rangle, & \gamma = n-2 \\ t^{\frac{\gamma-n}{2}}, & n-2 < \gamma < n. \end{cases}$$

For $\gamma = n$, $t \geq 1$,

$$(4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4t}} \langle y\rangle^{-n} dy = t^{-\frac{n}{2}} \ln(1+t) \, (4\pi)^{-\frac{n}{2}} \frac{1}{2} \left|S^{n-1}\right| \left(1 + O\left((\ln(1+t))^{-1}\right)\right)$$

since

$$(4\pi t)^{-\frac{n}{2}} \int_{|y|\geq t^{\frac{1}{2}}} e^{-\frac{|y|^2}{4t}} \langle y\rangle^{-n} dy \sim t^{-\frac{n}{2}} \int_{\frac{1}{4}}^{\infty} e^{-z} z^{-1} dz,$$
$$\left| (4\pi t)^{-\frac{n}{2}} \int_{|y|<t^{\frac{1}{2}}} \left(e^{-\frac{|y|^2}{4t}} - 1 \right) \langle y\rangle^{-n} dy \right| \lesssim t^{-\frac{n}{2}} \int_{|y|<t^{\frac{1}{2}}} t^{-1} |y|^2 \langle y\rangle^{-n} dy \sim t^{-\frac{n}{2}},$$
$$(4\pi t)^{-\frac{n}{2}} \int_{|y|<t^{\frac{1}{2}}} \langle y\rangle^{-n} dy = t^{-\frac{n}{2}} (4\pi)^{-\frac{n}{2}} \frac{1}{2} \left|S^{n-1}\right| \int_0^t \left[\frac{z^{\frac{n-2}{2}}}{(1+z)^{\frac{n}{2}}} - \frac{1}{1+z} + \frac{1}{1+z} \right] dz$$
$$= t^{-\frac{n}{2}} (4\pi)^{-\frac{n}{2}} \frac{1}{2} \left|S^{n-1}\right| \left(\ln(1+t) + O(1)\right).$$

For $\gamma > n$, $t \geq 1$,

$$(4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|y|^2}{4t}} \langle y\rangle^{-\gamma} dy = t^{-\frac{n}{2}} (4\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \langle y\rangle^{-\gamma} dy + O\Bigg(t^{-\frac{n}{2}} \begin{cases} t^{\frac{n-\gamma}{2}}, & n < \gamma < n+2 \\ t^{-1}\langle \ln t\rangle, & \gamma = n+2 \\ t^{-1}, & \gamma > n+2 \end{cases}\Bigg)$$

since

$$\left| (4\pi t)^{-\frac{n}{2}} \left(\int_{|y|\leq t^{\frac{1}{2}}} + \int_{|y|>t^{\frac{1}{2}}} \right) \left(e^{-\frac{|y|^2}{4t}} - 1 \right) \langle y\rangle^{-\gamma} dy \right|$$
$$\lesssim t^{-\frac{n}{2}} \left(\int_{|y|\leq t^{\frac{1}{2}}} t^{-1} |y|^2 \langle y\rangle^{-\gamma} dy + \int_{|y|>t^{\frac{1}{2}}} \langle y\rangle^{-\gamma} dy \right) \lesssim t^{-\frac{n}{2}} \begin{cases} t^{\frac{n-\gamma}{2}}, & n < \gamma < n+2 \\ t^{-1}\langle \ln t\rangle, & \gamma = n+2 \\ t^{-1}, & \gamma > n+2. \end{cases}$$

□

Appendix B. The limitation of $\int_{\mathbb{R}^n} e^{-A|x-y|^2}\langle y\rangle^{-b}dy$

Lemma B.1. *For $n>0$, $A>0$, $b\in\mathbb{R}$, we have*

$$\int_{\mathbb{R}^n} e^{-A|x-y|^2}\langle y\rangle^{-b}dy=\langle x\rangle^{-b}\int_{\mathbb{R}^n} e^{-A|z|^2}dz$$
$$+O\bigg(|x|\langle x\rangle^{-b-2}\left(|x|^{n+1}\mathbf{1}_{|x|\le 1}+\mathbf{1}_{|x|>1}\right)+\int_{|z|>\frac{|x|}{2}} e^{-A|z|^2}\left(|z|+\langle x\rangle\right)^{\max\{0,-b\}}dz\bigg).$$

Proof.

$$\int_{\mathbb{R}^n} e^{-A|x-y|^2}\langle y\rangle^{-b}dy=\int_{\mathbb{R}^n} e^{-A|z|^2}\left[\left(1+|x|^2\right)^{-\frac{b}{2}}+\left(1+|x-z|^2\right)^{-\frac{b}{2}}-\left(1+|x|^2\right)^{-\frac{b}{2}}\right]dz.$$

Here, we estimate

$$\begin{aligned}&\left|\int_{|z|\le\frac{|x|}{2}} e^{-A|z|^2}\left[\left(1+|x-z|^2\right)^{-\frac{b}{2}}-\left(1+|x|^2\right)^{-\frac{b}{2}}\right]dz\right|\\ =&\left|-\frac{b}{2}\int_{|z|\le\frac{|x|}{2}} e^{-A|z|^2}\left[1+\theta|x-z|^2+(1-\theta)|x|^2\right]^{-\frac{b}{2}-1}\left(|x-z|-|x|\right)\left(|x-z|+|x|\right)dz\right|\\ \lesssim& |x|\langle x\rangle^{-b-2}\int_{|z|\le\frac{|x|}{2}} e^{-A|z|^2}|z|dz\sim|x|\langle x\rangle^{-b-2}\left(|x|^{n+1}\mathbf{1}_{|x|\le 1}+\mathbf{1}_{|x|>1}\right)\end{aligned}$$

with a parameter $\theta\in[0,1]$;

$$\begin{aligned}&\left|\int_{|z|>\frac{|x|}{2}} e^{-A|z|^2}\left[\left(1+|x-z|^2\right)^{-\frac{b}{2}}-\left(1+|x|^2\right)^{-\frac{b}{2}}\right]dz\right|\\ \lesssim&\begin{cases}\int_{|z|>\frac{|x|}{2}} e^{-A|z|^2}dz, & b\ge 0\\ \int_{|z|>\frac{|x|}{2}} e^{-A|z|^2}\left(|z|^{-b}+\langle x\rangle^{-b}\right)dz, & b<0\end{cases}\sim\int_{|z|>\frac{|x|}{2}} e^{-A|z|^2}\left(|z|+\langle x\rangle\right)^{\max\{0,-b\}}dz.\end{aligned}$$

□

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References

[1] Giacomo Ageno and Manuel del Pino. Infinite time blow-up for the three dimensional energy critical heat equation in bounded domain. *arXiv preprint arXiv:2301.10442*, 2023.

[2] Wen Xiong Chen and Congming Li. Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.*, 63(3):615–622, 1991.

[3] Carmen Cortázar, Manuel del Pino, and Monica Musso. Green's function and infinite-time bubbling in the critical nonlinear heat equation. *J. Eur. Math. Soc. (JEMS)*, 22(1):283–344, 2020.

[4] Juan Davila, Manuel Del Pino,Monica Musso, and Juncheng Wei. Gluing Methods for Vortex Dynamics in Euler Flows. *Arch. Ration. Mech. Anal.*, 235(3):1467–1530, 2020.

[5] Juan Davila, Manuel del Pino, Monica Musso, and Juncheng Wei. Leapfrogging vortex rings for the 3-dimensional incompressible Euler equations. *arXiv preprint arXiv:2207.03263*, 2022.

[6] Juan Dávila, Manuel del Pino, and Juncheng Wei. Singularity formation for the two-dimensional harmonic map flow into S^2. *Invent. Math.*, 219(2):345–466, 2020.

[7] Manuel del Pino, Monica Musso, and Juncheng Wei. Infinite-time blow-up for the 3-dimensional energy-critical heat equation. *Anal. PDE*, 13(1):215–274, 2020.

[8] Manuel del Pino, Monica Musso, Juncheng Wei, and Youquan Zheng. Sign-changing blowing-up solutions for the critical nonlinear heat equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 21:569–641, 2020.

[9] Marek Fila and John R. King. Grow up and slow decay in the critical Sobolev case. *Netw. Heterog. Media*, 7(4):661–671, 2012.

[10] Marek Fila, John R. King, Michael Winkler, and Eiji Yanagida. Linear behaviour of solutions of a superlinear heat equation. *J. Math. Anal. Appl.*, 340(1):401–409, 2008.

[11] Marek Fila, Michael Winkler, and Eiji Yanagida. Slow convergence to zero for a parabolic equation with a supercritical nonlinearity. *Math. Ann.*, 340(3):477–496, 2008.

[12] Hiroshi Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I*, 13:109–124 (1966), 1966.

[13] Victor A. Galaktionov and John R. King. Composite structure of global unbounded solutions of nonlinear heat equations with critical Sobolev exponents. *J. Differential Equations*, 189(1):199–233, 2003.

[14] Victor A. Galaktionov and Juan L. Vazquez. Continuation of blowup solutions of nonlinear heat equations in several space dimensions. *Comm. Pure Appl. Math.*, 50(1):1–67, 1997.

[15] B. Gidas and J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 34(4):525–598, 1981.

[16] Changfeng Gui, Wei-Ming Ni, and Xuefeng Wang. Further study on a nonlinear heat equation. volume 169, pages 588–613. 2001. Special issue in celebration of Jack K. Hale's 70th birthday, Part 4 (Atlanta, GA/Lisbon, 1998).

[17] Otared Kavian. Remarks on the large time behaviour of a nonlinear diffusion equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 4(5):423–452, 1987.

[18] Tadashi Kawanago. Asymptotic behavior of solutions of a semilinear heat equation with subcritical nonlinearity. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 13(1):1–15, 1996.

[19] Tzong-Yow Lee and Wei-Ming Ni. Global existence, large time behavior and life span of solutions of a semilinear parabolic cauchy problem. *Transactions of the American Mathematical Society*, 333(1):365–378, 1992.

[20] Wei-Ming Ni, Paul E. Sacks, and John Tavantzis. On the asymptotic behavior of solutions of certain quasilinear parabolic equations. *J. Differential Equations*, 54(1):97–120, 1984.

[21] P. Poláčik. Threshold solutions and sharp transitions for nonautonomous parabolic equations on $\mathbb{R}^N$. *Arch. Ration. Mech. Anal.*, 199(1):69–97, 2011.

[22] Peter Poláčik and Pavol Quittner. A Liouville-type theorem and the decay of radial solutions of a semilinear heat equation. *Nonlinear Anal.*, 64(8):1679–1689, 2006.

[23] Peter Poláčik, Pavol Quittner, and Philippe Souplet. Singularity and decay estimates in superlinear problems via Liouville-type theorems. II. Parabolic equations. *Indiana Univ. Math. J.*, 56(2):879–908, 2007.

[24] Peter Poláčik and Eiji Yanagida. On bounded and unbounded global solutions of a supercritical semilinear heat equation. *Math. Ann.*, 327(4):745–771, 2003.

[25] Pavol Quittner. The decay of global solutions of a semilinear heat equation. *Discrete Contin. Dyn. Syst.*, 21(1):307–318, 2008.

[26] Pavol Quittner. Threshold and strong threshold solutions of a semilinear parabolic equation. *Adv. Differential Equations*, 22(7-8):433–456, 2017.

[27] Pavol Quittner. Optimal Liouville theorems for superlinear parabolic problems. *Duke Math. J.*, 170(6):1113–1136, 2021.

[28] Pavol Quittner and Philippe Souplet. *Superlinear parabolic problems*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, Cham, 2019. Blow-up, global existence and steady states, Second edition of [MR2346798].

[29] Yannick Sire, Juncheng Wei, and Youquan Zheng. Infinite time blow-up for half-harmonic map flow from $\mathbb{R}$ into $\mathbb{S}^1$. *Amer. J. Math.*, 143(4):1261–1335, 2021.

[30] Ryuichi Suzuki. Asymptotic behavior of solutions of quasilinear parabolic equations with slowly decaying initial data. *Adv. Math. Sci. Appl.*, 9(1):291–317, 1999.

[31] Takashi Suzuki. Semilinear parabolic equation on bounded domain with critical Sobolev exponent. *Indiana Univ. Math. J.*, 57(7):3365–3396, 2008.

[32] Juncheng Wei, Qidi Zhang, and Yifu Zhou. On Fila-King Conjecture in dimension four. *arXiv preprint arXiv:2210.04352*, 2022.

[33] Juncheng Wei, Qidi Zhang, and Yifu Zhou. Trichotomy dynamics of the 1-equivariant harmonic map flow. *arXiv preprint arXiv:2301.09221*, 2023.

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