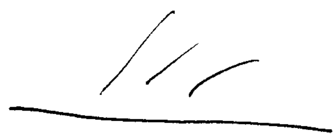
In chapters 1-5, we have studied

$u_t - k u_{xx} = 0$ (heat eqn, parabolic)

$u_{tt} - c^2 u_{xx} = 0$ (wave eqn, hyperbolic)

Method

① Formula on  , $-\infty < x < +\infty$

② Reflection  , $0 < x < +\infty$

③ Separation of Variables  , $0 < x < l$

From now on, we concentrate on Laplace equation:

$$\Delta u = f$$

$$\Delta u = \begin{cases} u_{xx}, & n=1 \\ u_{xx}+u_{yy}, & n=2 \\ u_{xx}+u_{yy}+u_{zz}, & n=3 \end{cases}$$

6.1 Laplace Equation

$$(1) \quad \begin{cases} \Delta u = f & \text{in } D \\ \text{B.C.} & \text{on } \partial D \end{cases}$$

$$\begin{cases} \text{Dirichlet BC} & u = g \text{ on } \partial D \\ \text{Neumann BC} & \frac{\partial u}{\partial n} = g \text{ on } \partial D \\ \text{Robin BC} & \frac{\partial u}{\partial n} + a u = g \text{ on } \partial D \end{cases}$$

Uniqueness by "Energy method":

Thm: (1) For Dirichlet BC, the solution is unique

(2) For Neumann BC, the solution is unique, up to a constant

(3) For Robin BC, the solution is unique, if $a(x) \geq 0$ and $a(x) \not\equiv 0$.

Proof: We only prove (3). The other two can be proved similarly.

(3): Let $u_1(x)$, $u_2(x)$ satisfy

$$\begin{cases} \Delta u_1 = f & \text{in } D \\ \frac{\partial u_1}{\partial n} + a(x) u_1 = g & \text{on } \partial D \end{cases} \qquad \begin{cases} \Delta u_2 = f & \text{in } D \\ \frac{\partial u_2}{\partial n} + a(x) u_2 = g & \text{on } \partial D \end{cases}$$

Let $u(x) = u_1(x) - u_2(x)$. Then

$$(2) \begin{cases} \Delta u(x) = \Delta u_1 - \Delta u_2 = f - f = 0 & \text{in } D \\ \frac{\partial u}{\partial n} + a u = \frac{\partial u_1}{\partial n} + a u_1 - \left(\frac{\partial u_2}{\partial n} + a u_2\right) = g - g = 0 & \text{on } \partial D \end{cases}$$

Now consider

$$E(u) = \frac{1}{2} \int_D |\nabla u|^2$$

Identity: $(\Delta u) u = \nabla (u \nabla u) - |\nabla u|^2 \quad (\nabla^2 = \Delta)$

$$0 = \int_D (\Delta u) u = \int_D \nabla (u \nabla u) - \int_D |\nabla u|^2$$

$$\overset{\text{divergence}}{\underset{\text{Theorem}}{=}} \int_{\partial D} u \nabla u \cdot n - \int_D |\nabla u|^2$$

$$= \int_{\partial D} u \frac{\partial u}{\partial n} - \int_D |\nabla u|^2$$

So $\displaystyle\int_D |\nabla u|^2 = \int_{\partial D} u \frac{\partial u}{\partial n} = \int_{\partial D} u(-a(x)\, u)$

$$= - \int_{\partial D} a(x)\, u^2 =$$

If $a(x) \geq 0$, LHS ≥ 0, RHS ≤ 0 $\Rightarrow$

$$\int_D |\nabla u|^2 = \int_{\partial D} a(x)\, u^2 = 0$$

$\Rightarrow$ $|\nabla u| = 0$ in D, $a(x)\, u^2 = 0$ on ∂D

$\Rightarrow$ $u \equiv$ Constant, Constant $= 0$

Thus $u \equiv 0$. This proves that $u_1 = u_2$

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Main Problem: Existence (very difficult)

Method: Method of Separation of Variables

6.2. Rectangles & Cubes

Separation of Variables

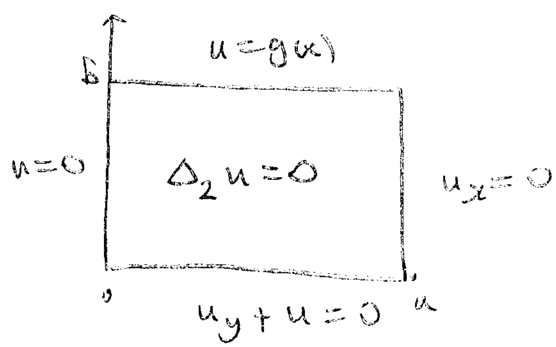
(1) look for separated functions (determined by the domain)
substitute into eqn. Use BCs to obtain eigen. values & ODE

(2) Solve. eigenvalue problem & ODEs

(3) Sum ups use B.Cs. to get the coefficients

Example 1:

$$D = [0, a] \times [0, b]$$



Step 1: $u(x) = X(x)\,Y(y) \Rightarrow$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X_x(a) = 0 \end{cases} \qquad \begin{cases} Y''(y) + \lambda Y(y) = 0 \\ Y'(0) + Y = 0 \end{cases}$$

step 2: $\lambda_n = \beta_n^2 = \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{a^2} \quad n = 0, 1, 2, 3, \dots$

$X_n(x) = \sin \frac{(n+\frac{1}{2})\pi x}{a}$

$Y_n(y) = A \cosh \beta_n y + B \sinh \beta_n y$

$Y_n'(0) + Y(0) = 0$

$Y_n(y) = \beta_n \cosh \beta_n y - \sinh \beta_n y$

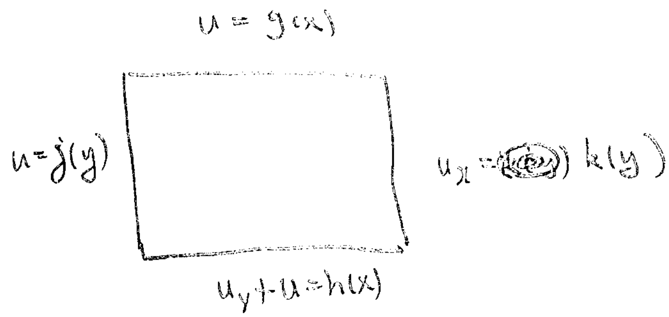
step 3: $u(x, y) = \sum A_n \sin \beta_n x \left(\beta_n \cosh \beta_n y - \sinh \beta_n y \right)$

B.C.S

$$g(x) = \sum A_n(\beta_n \cosh \beta_n 0 - \sinh \beta_n 0)\sin \beta_n x$$

$$A_n = \frac{(\beta_n \cosh \beta_n b - \sinh \beta_n b)}{\int_0^a \sin^2 \beta_n x} \cdot \int_0^a g(x)\sin\beta_n x \, dx$$

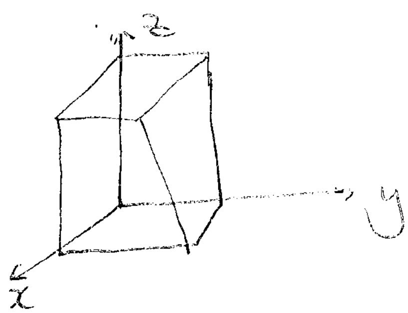
Ex. 2.



$$u = u_1 + u_2 + u_3 + u_4$$

$n = 3$.

Ex. 3. $\Delta_3 u = u_{xx} + u_{yy} + u_{zz} = 0$ in D



$D = \{0 < x < \pi,\ 0 < y < \pi,\ 0 < z < \pi\}$

$u(\pi, y, z) = g(y, z)$

$u(0, y, z) = u(x, 0, z) = u(x, \pi, z) = u(x, y, 0) = u(x, y, \pi) = 0$

$u(x, y, z) = X(x)\,Y(y)\,Z(z)$.

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$X(0) = Y(0) = Z(0) = Y(\pi) = Z(\pi) = 0$

$Y(y) = \sin my \quad (m = 1, 2, 3, \dots)$ $Z = \sin nz$,

$$X'' = (m^2+n^2)X, \quad X(0)=0$$

$$X(x) = A \sinh\left(\sqrt{m^2+n^2}\, x\right)$$

steps.

$$u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} \sinh\left(\sqrt{m^2+n^2}\, x\right) \sin my \sin nz$$

$$g(y,z) = \sum\sum A_{mn} \sinh\left(\sqrt{m^2+n^2}\, \pi\right) \sin my \sin nz$$

$$A_{mn} = \frac{\qquad}{\int \sin^2 my \sin^2 nz}.$$

Δ in polar coordinates:

$n=2$, $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad r=\sqrt{x^2+y^2}$ $\quad \theta = \arctan\frac{y}{x}$

$$\frac{\partial\theta}{\partial x} = \frac{-\frac{y}{x^2}}{\cancel{\sqrt{}}1+\frac{y^2}{x^2}} = -\frac{y}{\cancel{\sqrt{x^2+y}}}$$

$$= -\frac{1}{\cancel{\sqrt{}}x^2+y^2}\ \frac{y}{x}$$

$$= -\frac{}{r}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x}$$

$$= \frac{x}{r}\frac{\partial u}{\partial x} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} - \frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}$$

$$- \frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta\,2\theta}{r^2}\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial y^2} =$$

$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Radially symmetric fcns: $u = u(r)$ $\qquad u_{rr} + \frac{1}{r}u_r = 0$

$$\Rightarrow \quad (r u_r)_r = 0$$

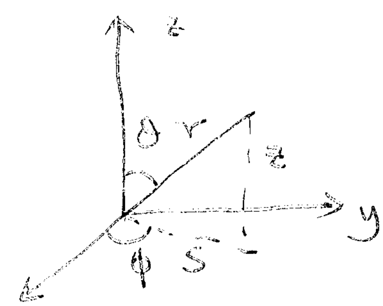
$$u = C_1 \log r + C_2$$

Ex:1. $\begin{cases} \Delta u = 1, & r < a \\ u = 0 & \text{on } r = a \end{cases}$

In R^3, spherical coordinates:

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \end{cases}$$

$$\begin{cases} z = r\sin\theta \\ s = r\cos\theta \end{cases}$$



$\phi \in (0, 2\pi)$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$r = \sqrt{z^2+s^2}$, $\theta = \arctan\frac{z}{s}$

$$\frac{\partial\theta}{\partial s} = \frac{\cos\theta}{r}.$$

$$(x, y, z) \longleftrightarrow (r, \phi, \theta)$$

$$u_{xx} + u_{yy} = u_{ss} + \frac{1}{s}u_s + \frac{1}{s^2} u_{\phi\phi}$$

$$u_{zz} + u_{ss} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r\cos\theta}u_s + \frac{1}{r^2\sin^2\theta}u_{\phi\phi}$$
$$+ \frac{1}{r^2}u_{\theta\theta}$$

$$u_s = u_r \cdot \frac{\partial r}{\partial s} + u_\phi \frac{\partial\phi}{\partial s} + u_\theta \frac{\partial\theta}{\partial s}$$
$$= u_r \cdot \frac{s}{r} + u_\phi \cdot 0 + u_\theta \frac{\cos\theta}{r} = u_r \cdot \frac{s}{r} + u_\theta \cdot \frac{\cos\theta}{r}$$

$$\Delta_3 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[u_{\theta\theta} + \frac{\cos\theta}{\sin\theta}u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}\right]$$

radially symmetric fcns

$$0 = \Delta_3 u = u_{rr} + \frac{2}{r} u_r$$

$$u = -C_1 r^{-1} + C_2$$

Ex. 2.

$$\begin{cases} \Delta u = 0, & \\ u = A & r = a \\ u = B & r = b \end{cases}$$