Quasi-linear First Order Equation

Now we turn to quasi-linear 1st order PDE

$$a(x,t,u)\, u_t + b(x,t,u)\, u_x = c(x,t,u)$$

We consider the simplest form

$$u_t + a(u)\, u_x = 0, \quad -\infty < x < \infty, \quad t > 0 \qquad (2.1)$$

The simplest $a(u)$ is u.

$$\begin{cases} u_t + u\, u_x = 0, & -\infty < x < \infty, \quad t > 0 \\ u(x,0) = \phi(x) \end{cases} \qquad (2.2)$$

In traffic flow theory we get

$$\rho_t + (Q(\rho))_x = 0 \quad \text{where } \rho = \text{car density} \qquad (2.3)$$

We will derive (2.3) later.

First, let us solve (2.2) and see what difficulties can arise.

As in the method of characteristics, the equation for characteristics is

$$\frac{dx}{dt} = u(x,t) \qquad (2.4)$$

~~where~~

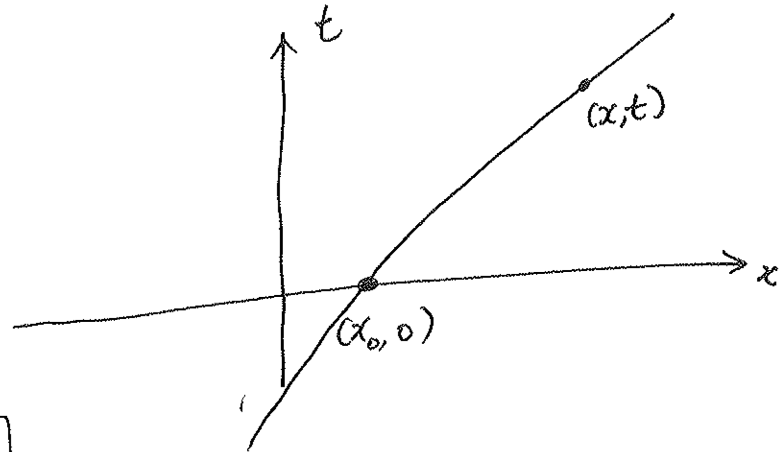
$$\frac{du(x(t),t)}{dt} = 0 \qquad (2.5)$$

In eqn (2.4), $u(x,t)$ is unknown. But from (2.5), we know that $u \equiv$ Constant on each characteristic curve. Thus all characteristic curves are straight lines

Let us try to solve (2.4)–(2.5) with initial condition

$u(x,0) = \phi(x)$

$$\frac{x-x_0}{t-0} = \text{slope } \frac{dx}{dt}$$
$$= u(x,t)$$
$$= u(x_0,0)$$
$$= \phi(x_0)$$



$$\boxed{x - x_0 = t\,\phi(x_0)} \quad \text{--- (2.6)}$$

Solving $x_0 = x_0(x,t)$

Then the general solution is

$$u = \phi(x_0) = \phi(x_0(x,t))$$

The main problem is: is there a unique solution to (2.6)?

This is true if

$\phi(x_0)$ is an increasing function of x_0 ?

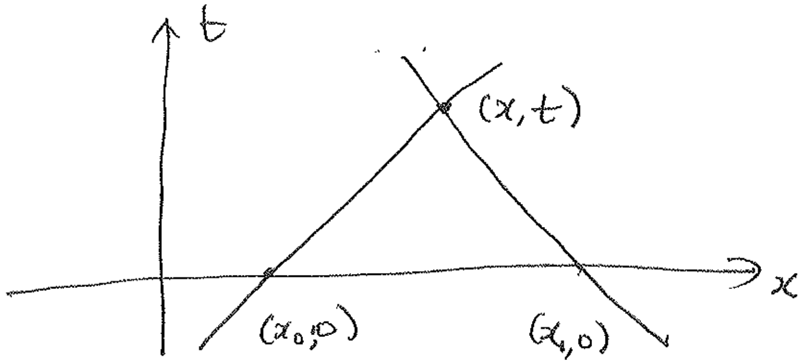
This is called expansive wave or rarefaction wave

Example 1
$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = 4x+1 \end{cases}$$

Solution:
$$\begin{cases} \frac{dx}{dt} = u \\ \frac{du}{dt} = 0 \end{cases} \Rightarrow \quad x - x_0 = t(4x_0+1) \Rightarrow x_0 = \frac{x-t}{4t+1}$$
$$u = 4x_0 + 1 \Rightarrow u = \frac{4(x-t)}{4t+1} + 1$$

What if $\phi(x)$ is not an increasing function?

Scenerio:



$\phi(x_0) > \phi(x_1)$

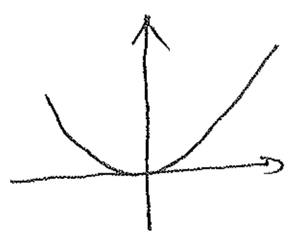
two characteristic curves intersect at the same place!

So at the intersection: $u(x,t)$ is multi-valued?

$$x - x_0 = t\,\phi(x_0)$$
$$x - x_1 = t\,\phi(x_1)$$

This is a "shock" wave.

Example 2: $\begin{cases} u_t + u u_x = 0 \\ u(x,0) = x^2 \end{cases}$



Solution: $\phi(x) = x^2$ is $\nearrow$ when $x \geq 0$.

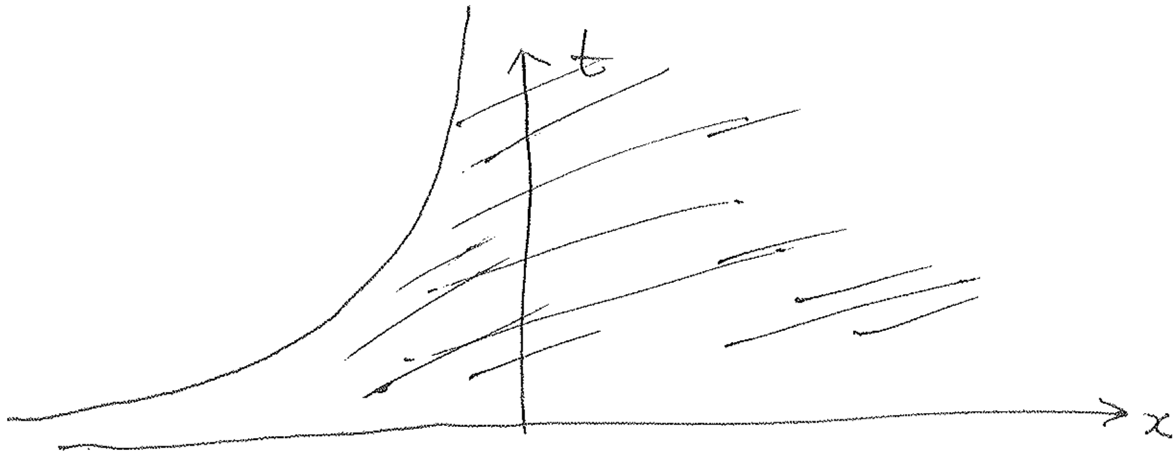
Method of charact.

$$x - x_0 = t\,x_0^2, \quad u = x_0^2$$

$$x_0 = \frac{-1 \pm \sqrt{1+4tx}}{2t} \quad \text{for } t \neq 0$$

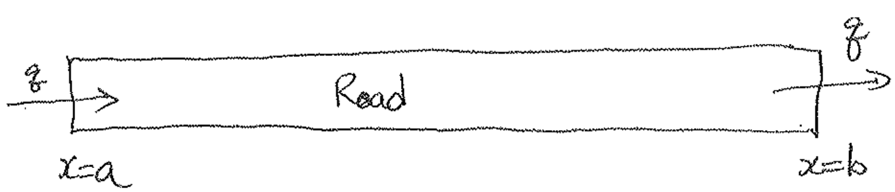
$$u = x_0^2 = \frac{1+2tx \mp \sqrt{1+4tx}}{2t^2}$$

IC: $u(x,t) = \dfrac{1+2tx - \sqrt{1+4tx}}{2t^2}$, defined only when $1+4tx \geq 0$



Problems how to deal with a "shock". We need to return to the physics. We illustrate this problem with the traffic flow problem.

Traffice flow problem:



ρ = density of cars

q = flux of cars.

$$\frac{d}{dt}\int_a^b \rho\, dx = q(a,t) - q(b,t) = \text{Flow in} - \text{Flow out}$$

$N = \int_a^b \rho\, dx$ = number of cars in $a \le x \le b$

$\frac{d}{dt}\int_a^b \rho dx$ = variation of number of cars.

This equation express conservation of cars

Let $b = a + \Delta a$ and let $\Delta a \to 0$

$$\frac{1}{\Delta a}\frac{d}{dt}\int_a^{a+\Delta a} \rho \, dx = \frac{q(a,t) - q(a+\Delta a, t)}{\Delta t}$$

Now let $\Delta a \to 0$. Then $\frac{d}{dt}\left(\frac{1}{\Delta a}\int_a^{a+\Delta a} \rho \, dx\right) \to \frac{d\rho}{dt}(a,t)$

Thus

$$\partial_t \rho = - \partial_x q$$

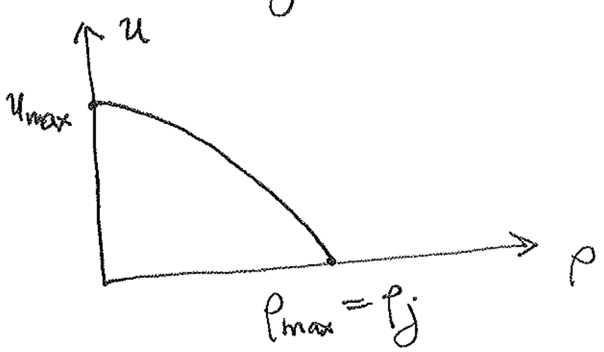
Hence

$$\boxed{\rho_t + q_x = 0}$$

Now the flow rate q is

$q = \rho u$, ρ = density, u = velocity field

We will assume that $u = u(\rho)$ so that cars respond only to local density. Moreover we assume u has the shape

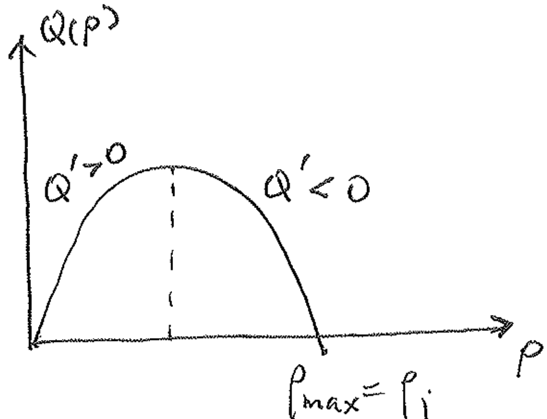


$u'(\rho) < 0$

$\rho_{max} = \rho_j$ = gridlock density

$U(0) = U_{max}$: light traffic

Therefore the flow $q = \rho u(\rho) \equiv Q(\rho)$ has the form



$$Q'(\rho) = c(\rho)$$

The flow rate is maximum at some intermediate value in $0 < \rho < \rho_j$.

Therefore, we have

$$\rho_t + [Q(\rho)]_x = 0 \quad \text{or}$$

$$\rho_t + c(\rho)\rho_x = 0, \quad \text{where } c(\rho) = Q'(\rho)$$

A simple choice would be

$$Q(\rho) = U_{max}\, \rho \left(1 - \frac{\rho}{\rho_j}\right)$$

Notice that $c(\rho) = Q'(\rho) = U_{max}\left(1 - \frac{2\rho}{\rho_j}\right)$ and our model

is simply

$$\begin{cases} \rho_t + u_{max}\left(1 - \frac{2\rho}{\rho_j}\right)\rho_x = 0 \\ \rho(x,0) = \rho_0(x) \end{cases}$$

where $c'(\rho) < 0$, $c = 0$ when $\rho = \frac{\rho_j}{2}$

A SIMPLE MODEL PROBLEM

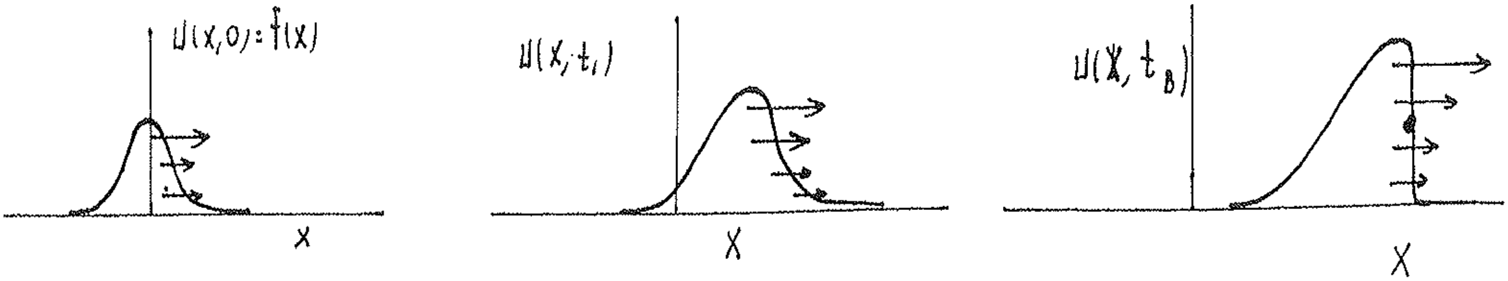
CONSIDER $u_t + u u_x = 0 \quad -\infty < x < \infty, \ t > 0$

$u(x, 0) = f(x)$

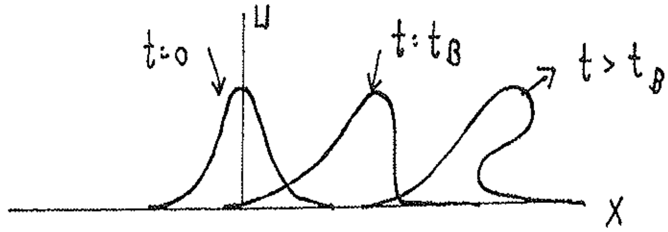
$c(u) = u$

$c'(u) = 1$

Higher values of u move faster.



NOTICE THAT THE WAVE STEEPENS FROM THE FRONT AND WILL EVENTUALLY BREAK.



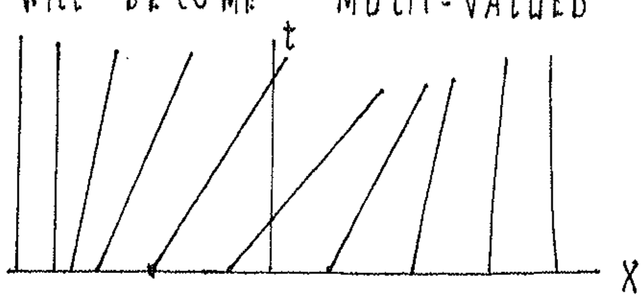
NOW WE ANALYZE BY THE METHOD OF CHARACTERISTICS:

ON $dx/dt = u$, $\quad x = \xi$ when $t = 0$ $\qquad -\infty < \xi < \infty$ parametrizes the x-axis.

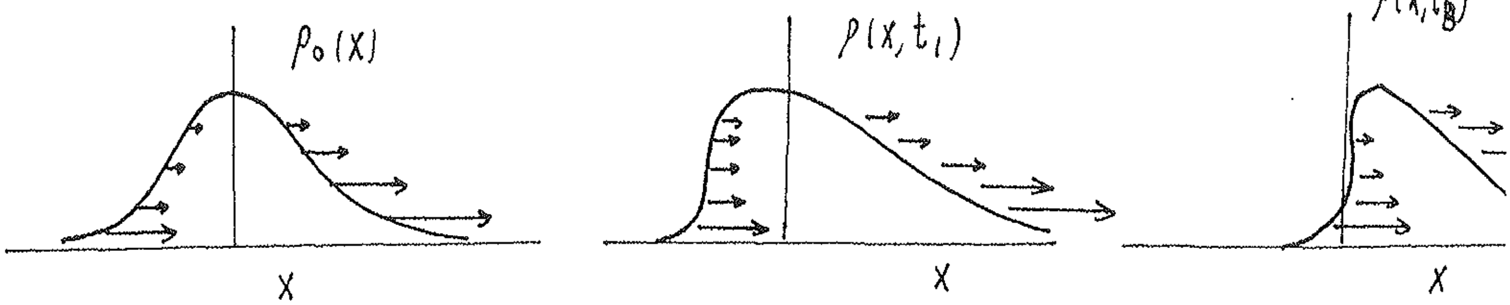
THEN $du/dt = 0$, $\quad u = f(\xi)$ when $t = 0$

WE OBTAIN $\left.\begin{array}{l} x = f(\xi)\, t + \xi \\ u = f(\xi) \end{array}\right\} \longrightarrow u = f(x - ut)$

WHEN $f(x)$ IS BELL-SHAPED IT IS CLEAR THAT THE CHARACTERISTICS WILL CROSS AT SOME TIME t_B AND THE SOLUTION WILL BECOME MULTI-VALUED



SINCE $c'(\rho) < 0$ FOR $\rho > 0$, THIS MEANS THAT LARGER VALUES OF ρ WILL HAVE SLOWER SPEEDS. HENCE A WAVE WILL STEEPEN UP IN THE BACK AS t-INCREASES



AT $t = t_B$ THE WAVE "BREAKS" AND HAS AN INFINITE SLOPE AT SOME VALUE OF X. AT THIS POINT THE SOLUTION FIRST BECOMES MULTI-VALUED.

MODEL OF TRAFFIC FLOW

- neglects lane changes
- drivers respond only to local density rather than changes in density.
- does not take into account lag time in reacting to traffic.

IF DRIVERS REACT TO THE DENSITY AND THE GRADIENT OF DENSITY THEN
$$q(x,t) = Q[\rho(x,t)] - \nu \rho_x(x,t) \qquad \nu > 0$$

IN THIS CASE $\rho_t + q_x = 0$ BECOMES $\rho_t + [Q(\rho)]_x = \nu \rho_{xx}$, WHICH IS A DIFFUSION EQUATION.

REMARK 1 TO ELIMINATE THE SOLUTION BEING MULTI-VALUED AFTER THE WAVE BREAKS WE MUST INTRODUCE A SHOCK, i.e. A PROPAGATING DISCONTINUITY, THAT IS INITIATED AT THE BREAKIN

NOW TO FIND THE BREAKING TIME WE SOLVE $x = f(\xi)t + \xi$

FOR $\xi = \xi(x,t)$: SO THAT

$$0 = \xi_t f'(\xi) t + f(\xi) + \xi_t \longrightarrow \xi_t = \frac{-f(\xi)}{1+f'(\xi)t}$$

$$1 = \xi_x f'(\xi) t + \xi_x \longrightarrow \xi_x = \frac{1}{1+f'(\xi)t}$$

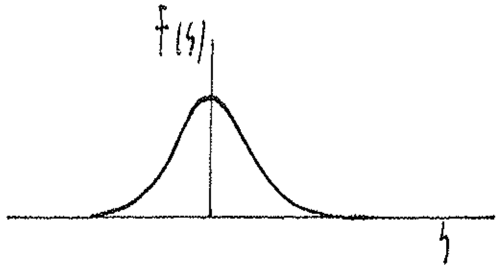
NOW WE CAN'T SOLVE FOR ξ_t, ξ_x when $1+f'(\xi)t = 0$.

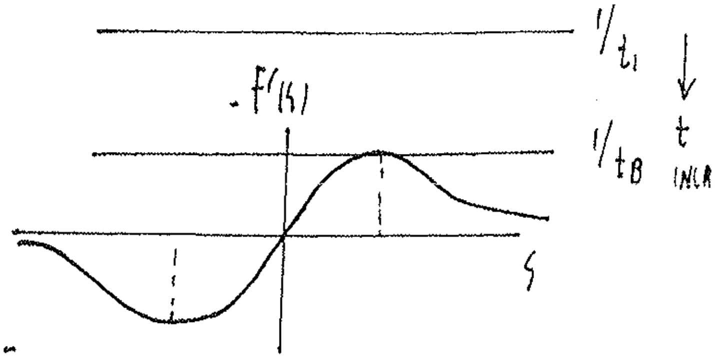
IN ADDITION, WE CALCULATE

$$u_x = f'(\xi)\xi_x = \frac{f'(\xi)}{1+t f'(\xi)} = \infty \quad \text{WHEN} \quad 1+f'(\xi)t = 0.$$

THE MINIMUM VALUE $t = t_B$ FOR WHICH $1 + f'(\xi)t = 0$ IS THE BREAKING TIME WHERE u_x FIRST BECOMES INFINITE.

WE PLOT $\frac{1}{t} = -f'(\xi)$





HENCE $$t_B = \frac{1}{\max\limits_{\xi \in A} |f'(\xi)|} \qquad A = \{\xi \mid f'(\xi) < 0\}$$

<u>EXAMPLE 1</u> LET $f(\xi) = e^{-\xi^2/\sigma^2}$. WE CALCULATE

$$f'(\xi) = -\frac{2\xi}{\sigma^2} e^{-\xi^2/\sigma^2}, \qquad f''(\xi) = \left(-\frac{2}{\sigma^2} + \frac{4\xi^2}{\sigma^4}\right) e^{-\xi^2/\sigma^2}$$

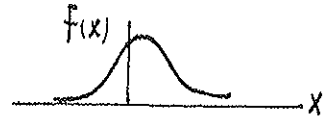
NOW $f''(\xi) = 0$ WHEN $2\xi^2 = \sigma^2$ OR $\xi = \sigma/\sqrt{2}$.

SO $\max |f'(\xi)| = \frac{\sqrt{2}}{\sigma} e^{-1/2}$ $\qquad t_B = \frac{\sigma}{\sqrt{2}} e^{1/2}$ breaking time

REMARK (i) AS THE VARIANCE σ^2 IN THE GAUSSIAN INCREASES TO smoothen $u(x,0)$, THE TIME FOR BREAKING DECREASES

EXAMPLE 2 $u_t + c(u)\, u_x = 0 \qquad c'(u) > 0 \qquad -\infty < x < \infty,\ t > 0$

$u(x,0) = f(x)$



THE CHARACTERISTIC METHOD GIVES

ON $\dfrac{dx}{dt} = c(u) \qquad x(0) = \xi$

THEN $\dfrac{du}{dt} = 0 \qquad u(0) = f(\xi)$

$c'(u) > 0$ IMPLIES THAT THE WAVE STEEPENS UP FROM THE FRONT.

NOW $x = c[f(\xi)]\,t + \xi$

SO TAKING PARTIAL DERIVATIVE WRT x:

$$1 = c'(f(\xi))\, f'(\xi)\, \xi_x\, t + \xi_x$$

SO $$\xi_x = \frac{1}{1 + c'(f(\xi))\, f'(\xi)\, t}$$

AND $$u_x = f'(\xi)\, \xi_x$$

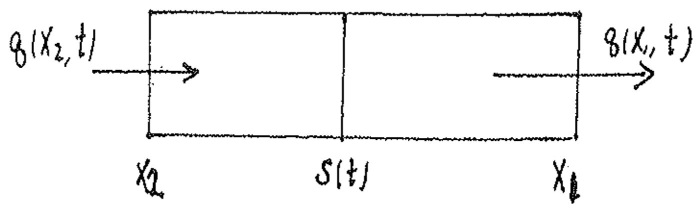
THE BREAKING TIME IS THE SMALLEST VALUE OF t FOR WHICH $1 + c'(f(\xi))\, f'(\xi)\, t = 0$.

THIS VALUE IS $$t_B = \frac{+1}{\max\limits_{\xi \in A} \left[c'(f(\xi))\, f'(\xi) \right]}$$

$A = \{ \xi \mid f'(\xi) < 0$

SHOCK FRONTS (MOVING DISCONTINUITY)

INSTEAD OF ALLOWING A SOLUTION TO BECOME MULTI-VALUED WE WILL INSERT A SHOCK CURVE $x = s(t)$ UPON WHICH THE SOLUTION HAS A JUMP DISCONTINUITY. THIS IS DONE IN SUCH A WAY THAT SOMETHING IS CONSERVED ACROSS THE FRONT.



$$\rho_t + q_x = 0$$

CONSERVATION IS: $$\frac{d}{dt}\int_{x_2}^{x_1} \rho \, dx = q(x_2,t) - q(x_1,t)$$

NOW $$\frac{d}{dt}\left[\int_{x_2}^{s(t)} \rho(x,t)\,dx\right] + \frac{d}{dt}\left[\int_{s(t)}^{x_1} \rho(x,t)\,dx\right] = q(x_2,t) - q(x_1,t).$$

NOW ρ IS DISCONTINUOUS ACROSS $x = s(t)$.

RECALL $$\frac{d}{dt}\int_a^t F(x,t)\,dx = F(t,t) + \int_a^t \frac{\partial}{\partial t} F(x,t)\,dx,$$

FROM CALCULUS.

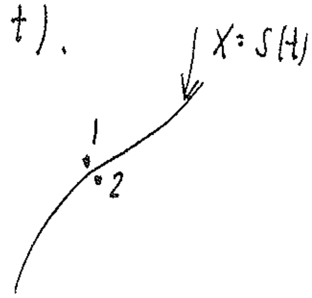
THUS $$\frac{d}{dt}\left(\int_{x_2}^{s(t)} \rho(x,t)\,dx + \int_{s(t)}^{x_1} \rho(x,t)\,dx\right) = \frac{ds}{dt}\rho(s^-,t) + \int_{x_2}^{s(t)} \rho_t\,dx - \frac{ds}{dt}\rho(s^+,t) + \int_{s}^{x_1} \rho_t\,dx$$

NOW IF ρ_t IS BOUNDED ON (x_2, s^-) AND (s^+, x_1) THEN AS $x_2 \to s^-$ AND $x_1 \to s^+$ WE HAVE $\int_{x_2}^{s(t)} \rho_t\,dx + \int_{s(t)}^{x_1} \rho_t\,dx \to 0.$

THEREFORE, WE OBTAIN

$$\lim_{\substack{x_2 \to s^- \\ x_1 \to s^+}} \left[s' \rho(s^-,t) - s' \rho(s^+,t) + \int_{x_2}^{s} \rho_t \, dx + \int_{s}^{x_1} \rho_t \, dx \right] = q(s^-,t) - q(s^+,t)$$

$$\Longrightarrow \qquad s' \left[\rho(s^+,t) - \rho(s^-,t) \right] = q(s^+,t) - q(s^-,t).$$



THIS YIELDS THAT $\quad \dfrac{ds}{dt} = \dfrac{q_2 - q_1}{\rho_2 - \rho_1}$

OR EQUIVALENTLY WITH $[f] = f_2 - f_1$ WE CAN WRITE

$$\frac{ds}{dt} = \frac{[q]}{[\rho]} \qquad \text{(Rankine-Hugoniot formula)}$$

WHEN $\quad \rho_t + \rho \rho_x = 0 \longrightarrow \rho_t + \left(\frac{1}{2} \rho^2 \right)_x = 0 \quad$ OR $\quad q = \frac{1}{2} \rho^2$

WE CALCULATE $\quad \dfrac{ds}{dt} = \dfrac{\frac{1}{2}(\rho_2^2 - \rho_1^2)}{\rho_2 - \rho_1} = \dfrac{1}{2}(\rho_1 + \rho_2) =$ average of ρ on either side of shock.

THE QUESTION NOW IS HOW DOES ONE FIT SUCH A SHOCK IN A CONCRETE EXPLICIT WAY FOR A GIVEN EXAMPLE?

Example 1 Consider

$$u_t + u u_x = 0$$

$$u(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$$

Solution:

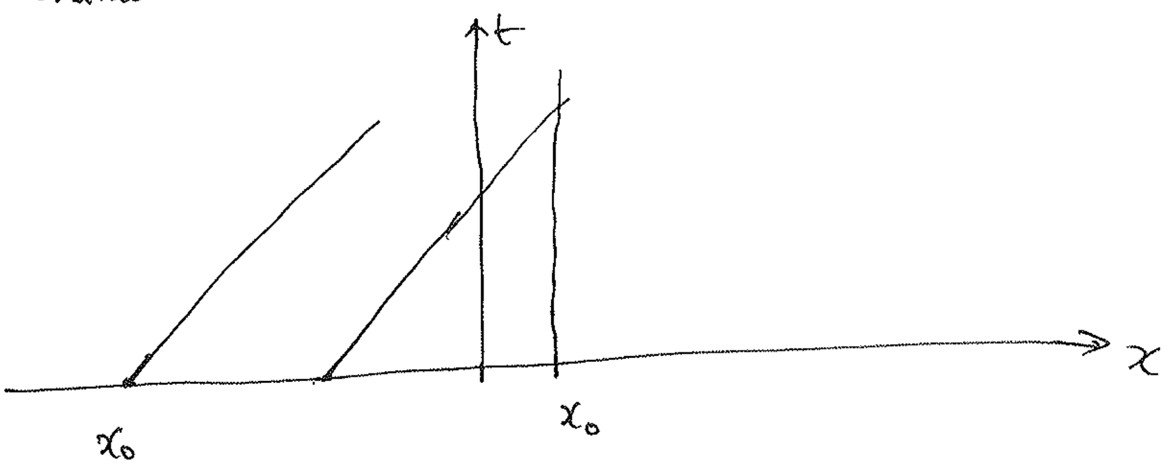
$$\begin{cases} \dfrac{dx}{dt} = u \\ \dfrac{du}{dt} = 0 \end{cases}$$

$$x - x_0 = t\,\phi(x_0)$$

$$u = \phi(x_0)$$

if $x_0 < 0$, then $u = \phi(x_0) = 1$, $x - x_0 = t$

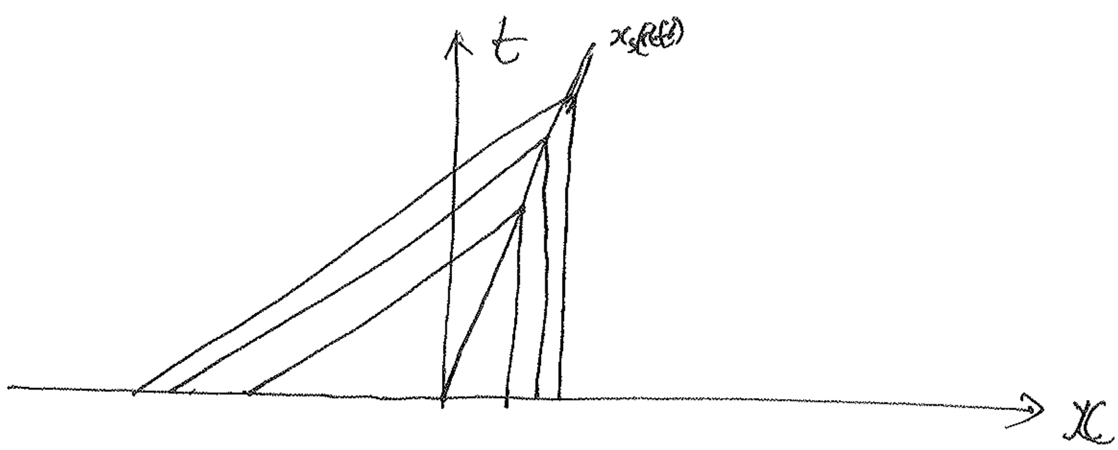
characteristic curve



if $x_0 > 0$, $u = 0$, $x = x_0$

$$\frac{dx_s}{dt} = \frac{[q]}{[\rho]} = \frac{\frac{1}{2}\cdot 0^2 - \frac{1}{2}\cdot 1^2}{0-1} = \frac{1}{2} \qquad x_s(t) = \frac{t}{2}$$

$$x_s(0) = 0$$



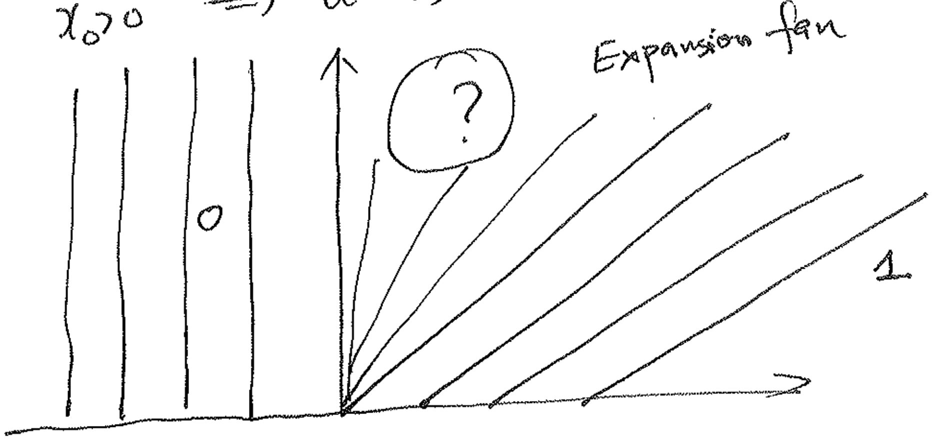
picture of the solution

Example 0. $\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \begin{cases} 0, & x<0 \\ 1, & x>0 \end{cases} \end{cases}$

Sol'n: $x - x_0 = t\,\phi(x_0)$

$x_0 < 0 \Rightarrow u = 0, \quad x = x_0$

$x_0 > 0 \Rightarrow u = 1, \quad x - x_0 = t$

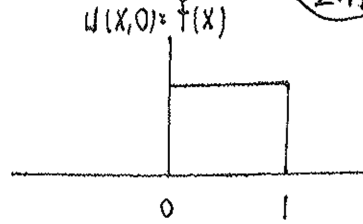


? not defined: Insert an expansion fan $u = U(\frac{x}{t})$

$\Rightarrow U = \frac{x}{t}$

EXAMPLE 2 CONSIDER $u_t + u u_x = 0$

$$u(x,0) = f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



WE FIRST FIT A SHOCK AT THE FRONT END. THE SHOCK FORMS IMMEDIATELY AT $t = 0$.

USING THE METHOD OF CHARACTERISTICS WE GET

$$\text{ON } \frac{dx}{dt} = u \qquad x(0) = \xi, \quad 0 < \xi < 1$$

$$\text{then } \frac{du}{dt} = 0 \qquad u(0) = 1$$

HENCE $x = t + \xi$ AND $u \equiv 1$ WITH $0 < \xi < 1$

THUS ON LEFT-SIDE OF SHOCK THEN $u = 1$ WHILE ON RIGHT SIDE WE HAVE $u = 0$. HENCE WITH $\dfrac{ds_1}{dt} = \dfrac{[q]}{[u]} = \dfrac{1}{2}(u_1 + u_2)$ $\quad q = u^2/2$

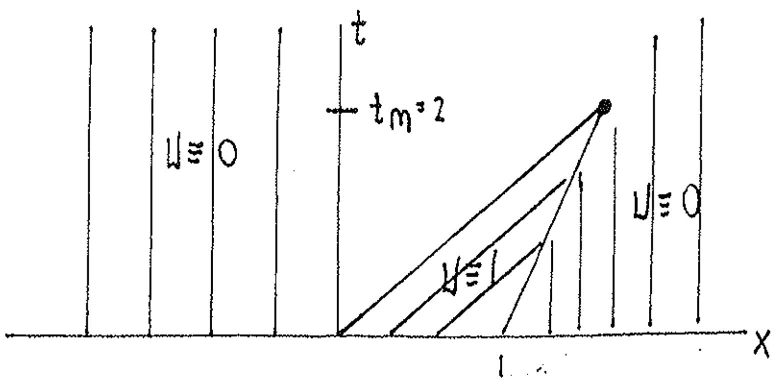
WE GET

$$\frac{ds_1}{dt} = \frac{1}{2}(1 + 0)$$

$$s_1(0) = 1 \qquad \text{(WAVE BREAKS IMMEDIATELY)}$$

THUS $s_1 = \dfrac{t}{2} + 1$. $\qquad s_1(t) = \dfrac{t}{2} + 1$

OUR PICTURE OF SOLUTION SO FAR IS



NOTICE THAT THE CHARACTERISTIC $X = t$ HITS THE SHOCK $S_1 = \frac{t}{2} + 1$ (10)

AT THE TIME $t_m = 2$. AT THIS POINT $S_1(2) = 2$.

NOW IN THE TAIL WE WILL INSERT AN <u>EXPANSION FAN</u>, WHICH SMOOTHES THE INITIAL DISCONTINUITY AT $X = 0$.

THE EXPANSION FAN IS CHARACTERIZED BY A SOLUTION OF THE FORM $u = \mathcal{U}(X/t)$. LET $\Lambda = X/t$

WE CALCULATE

$$u_X = \mathcal{U}'(\Lambda)\,\Lambda_X = \mathcal{U}'(\Lambda)\,1/t$$

$$u_t = \mathcal{U}'(\Lambda)\,\Lambda_t = \mathcal{U}'(\Lambda)\left(-X/t^2\right).$$

HENCE,

$$u_t + u u_X = 0 \longrightarrow \mathcal{U}'(\Lambda)\left(-X/t^2\right) + \mathcal{U}'(\Lambda)\,\mathcal{U}(\Lambda)\left(1/t\right) = 0$$

$$\longrightarrow \mathcal{U}'(\Lambda)\left(-\Lambda + \mathcal{U}(\Lambda)\right) = 0$$

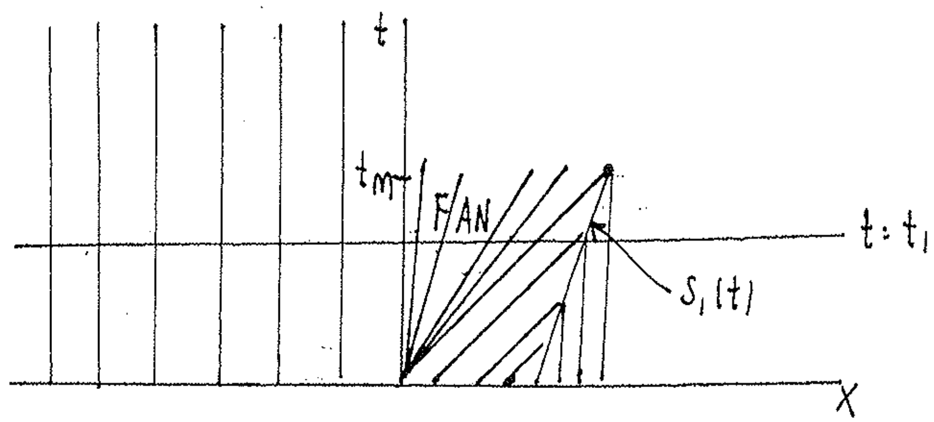
THE SOLUTION IS $\mathcal{U}(\Lambda) = \Lambda$ AND SO

$$u(x,t) = X/t \quad \text{IN THIS REGION.}$$

WE TAKE ANY VALUE u_0 IN $0 < u < 1$. THE EXPANSION FAN IS THE FAMILY OF CURVES

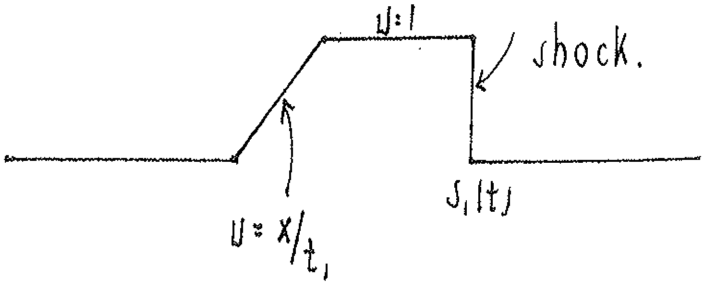
$$X = u_0 t \quad \text{WITH} \quad 0 < u_0 < 1.$$

THIS THEN GIVES THE FOLLOWING PICTURE OF THE CHARACTERISTICS

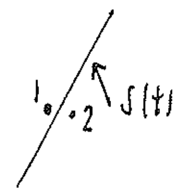


IF WE TAKE A CROSS-SECTION OF t, x plane at $t = t_1 < t_m = 2$

THEN $u(x,t)$ IS AS FOLLOWS



NOTICE THAT, AT $t = t_m = 2$ THE EXPANSION FAN HITS THE SHOCK AND THE EQUATION OF THE SHOCK POSITION CHANGES AFTER $t > t_m = 2$.



NOW FOR $t > 2$ $\quad \dfrac{ds}{dt} = \dfrac{u_1 + u_2}{2}$

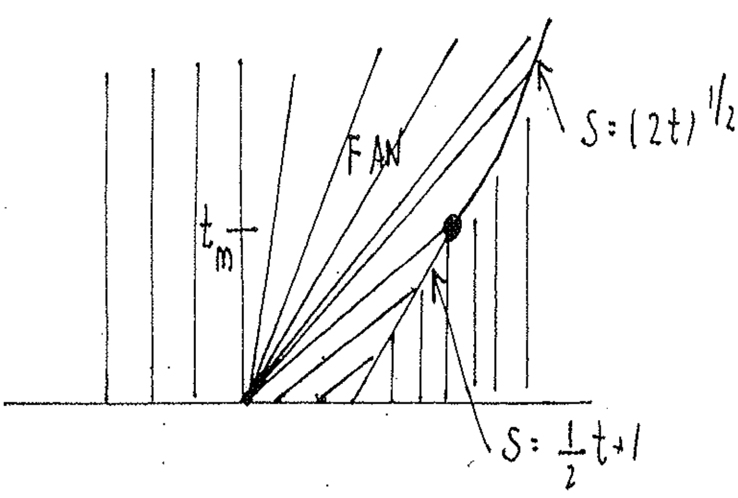
$u_1 = x/t$ EXPANSION FAN $\longrightarrow$ $u_1 = s(t)/t$ ON shock

$u_2 = 0$ ahead OF shock

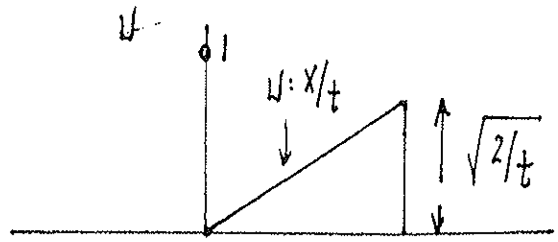
HENCE $\quad \dfrac{ds}{dt} = \dfrac{s}{2t}, \quad s(2) = 2$

WE SOLVE $\quad s^{-1}\,ds = \frac{1}{2}t^{-1}\,dt \quad \log s = \frac{1}{2}\log t + C \longrightarrow s = r\,t^{1/2}$

NOW $s(2) = 2 \rightarrow r = \sqrt{2}$. THUS $\quad s(t) = (2t)^{1/2} \quad t \geq 2$



FOR $t > 2$ THE SOLUTION LOOKS LIKE



EXPANSION FAN REMARK

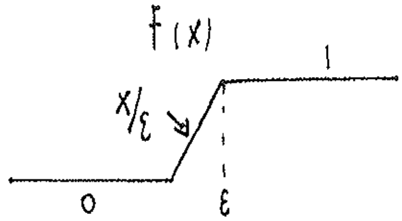
WE RECALL

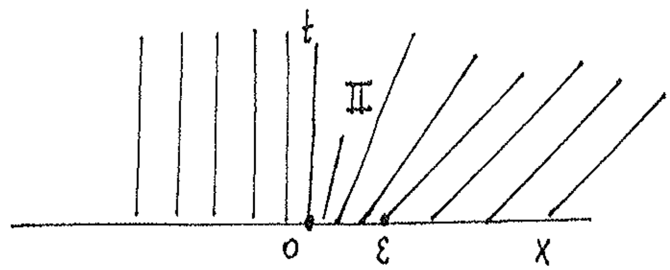
$$u_t + u u_x = 0 \qquad -\infty < x < \infty, \ t > 0$$

$$u(x,0) = f(x)$$

THE INITIAL CONDITION IS $$f(x) = \begin{cases} 0, & x \le 0 \\ x/\varepsilon, & 0 \le x \le \varepsilon \\ 1, & x \ge \varepsilon \end{cases}$$

SO THAT





NOW THE METHOD OF CHARACTERISTICS IN $0 < x < \varepsilon$ GIVES

$$\text{ON} \quad dx/dt = u \qquad x(0) = \Lambda$$

$$\text{Then} \quad du/dt = 0 \qquad u(0) = \Lambda/\varepsilon$$

THIS GIVES $\quad u = \Lambda/\varepsilon \quad$ AND $\quad x = \dfrac{\Lambda}{\varepsilon} t + \Lambda$

HENCE $\quad \Lambda\left[1 + t/\varepsilon\right] = x \longrightarrow \Lambda = \dfrac{x}{1 + t/\varepsilon}$

WE GET $\quad u = \dfrac{\Lambda}{\varepsilon} = \dfrac{x}{\varepsilon + t} \quad$ SOLUTION IN REGION II

$\longrightarrow$ EXPANSION FAN REGION $\longleftarrow$

WITH $\varepsilon \ll 1$ THIS BECOMES $\quad u \approx \dfrac{x}{t}$

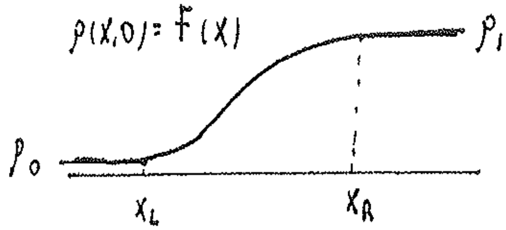
EXAMPLE (CONGESTION AHEAD)

$$\rho_t + [Q(\rho)]_x = 0 \qquad Q(\rho) = u_{MAX}\,\rho\,(1 - \rho/\rho_j)$$

THIS GIVES $\quad \rho_t + c(\rho)\,\rho_x = 0 \qquad c(\rho) = Q'(\rho) = u_{MAX}(1 - 2\rho/\rho_j)$

$$c'(\rho) < 0$$

THE INITIAL CONDITION IS



SUPPOSE $c(\rho_0) > 0$ AND $c(\rho_1) > 0$

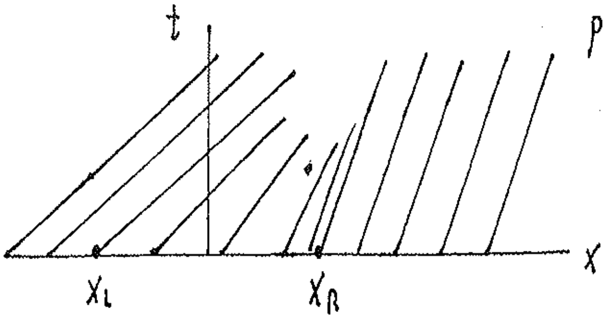
SINCE $c'(\rho) < 0$ THE SPEED DECREASES WITH INCREASING ρ.

THE METHOD OF CHARACTERISTICS GIVES

ON $\quad \dfrac{dx}{dt} = c(\rho) \qquad \left.\begin{array}{l}\rho(0) = F(s)\\ x(0) = s\end{array}\right\}$ parametrizes data curve

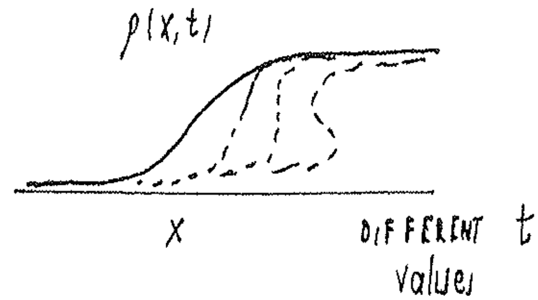
THEN $\quad \dfrac{d\rho}{dt} = 0$

THUS $\quad x = c[F(s)]\,t + s, \quad \rho = F(s)$



CHARACTERISTICS WILL CROSS AND SOLUTION WILL BECOME MULTI-VALUED

THE SOLUTION WILL LOOK LIKE



WHEN IS THE BREAKING TIME?

$$\rho_x = f'(s)\, s_x$$

AND $$1 = c'(f(s))\, f'(s)\, s_x t + s_x$$

SO $$\rho_x = \frac{f'(s)}{c'(f(s))\, f'(s)\, t + 1}$$

BREAKING OCCURS AT THE FIRST $t > 0$ WITH $c'(f(s))\, f'(s)\, t + 1 = 0$

OR $$t = \frac{-1}{c'(f(s))\, f'(s)}$$

HENCE WITH $c'(\ ) < 0$ WE OBTAIN THE BREAKING TIME.

$$t_B = \frac{1}{\max\limits_{s \in A} |c'(f(s))\, f'(s)|} \qquad A = \{ s \mid f'(s) > 0 \}$$

FOR EXAMPLE IF $\rho(x,0) = f(x) = \left(\frac{\rho_0 + \rho_1}{2} \right) + \left(\frac{\rho_1 - \rho_0}{2} \right) \tanh(s/\sigma)$

THEN $$\max_{s \in A} |f'(s)| = f'(0) = \frac{\rho_1 - \rho_0}{2\sigma} \operatorname{sech}^2(0) = \frac{\rho_1 - \rho_0}{2\sigma}.$$

WITH $c'(\rho) = -\frac{2 u_{max}}{\rho_j}$ WE GET $$t_B = \frac{1}{\frac{2 u_{max}}{\rho_j} \frac{(\rho_1 - \rho_0)}{2\sigma}}$$

THIS YIELDS $$t_B = \frac{\sigma \rho_j}{u_{max}(\rho_1 - \rho_0)}$$

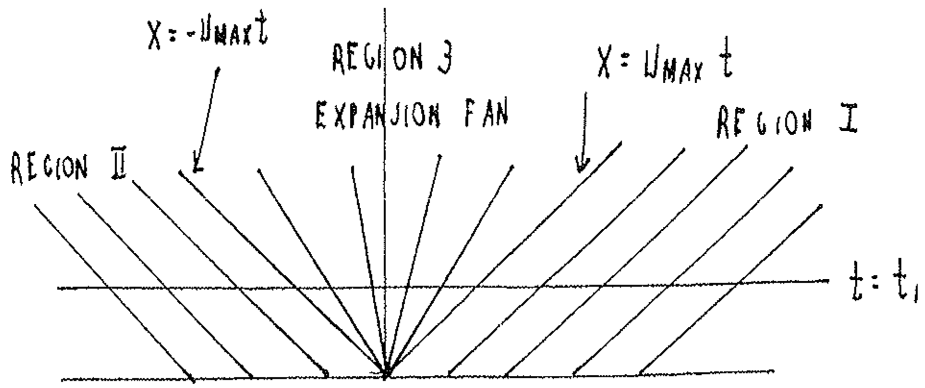
EXAMPLE (AFTER A TRAFFIC LIGHT TURNS GREEN)

THE MODEL IS $\rho_t + [Q(\rho)]_x = 0 \qquad Q(\rho) = U_{MAX}\, \rho\,(1 - \rho/\rho_j)$

THIS YIELDS THAT $\rho_t + C(\rho)\, \rho_x = 0 \qquad C(\rho) = U_{MAX}\,(1 - 2\rho/\rho_j)$

$$\rho(x,0) = \begin{cases} \rho_j, & x < 0 \quad \text{gridlock} \\ 0, & x > 0 \quad \text{NO CARS} \end{cases}$$

USING THE METHOD OF CHARACTERISTICS WE OBTAIN 2 DISTINCT REGIONS, REGIONS I AND II, SEPARATED BY AN EXPANSION FAN REGION, REGION 3.



REGION I ON $dx/dt = U_{MAX}\,(1 - 2\rho/\rho_j)$, $x(0) = \xi \qquad 0 \le \xi \le \infty$

then $d\rho/dt = 0 \qquad , \ \rho(0) = 0$

WE SOLVE THIS TO GET $x = U_{MAX}\, t + \xi$, $\rho \equiv 0$ IN REGION I

REGION II ON $dx/dt = U_{MAX}\,(1 - 2\rho/\rho_j)$, $x(0) = \xi$

then $d\rho/dt = 0 \qquad , \ \rho(0) = \rho_j$

WE GET $\rho = \rho_j$ ON $x = -U_{MAX}\, t + \xi$, $\xi \le 0$

NOW IN THE EXPANSION FAN REGION WE WANT ρ TO BE CONTINUOUS. WE LET $\rho = H(x/t)$ SO THAT

$$\rho_t = -\frac{x}{t^2} H'(x/t) \; , \quad \rho_x = \frac{1}{t} H'(x/t)$$

THEN $$\rho_t + c(\rho)\,\rho_x = 0$$

$$\longrightarrow \quad -\frac{x}{t^2} H'(\Lambda) + c[H(\Lambda)]\frac{1}{t} H'(\Lambda) = H'(\Lambda)\Big[c(H(\Lambda)) - \Lambda \Big] = 0.$$

THIS YIELDS THAT $c[H(\Lambda)] = \Lambda$ WITH $\Lambda = x/t$.

HENCE $$u_{MAX}\left(1 - \frac{2H}{\rho_j}\right) = \Lambda \quad \longrightarrow \quad 1 - \frac{2H}{\rho_j} = \frac{\Lambda}{u_{MAX}}$$

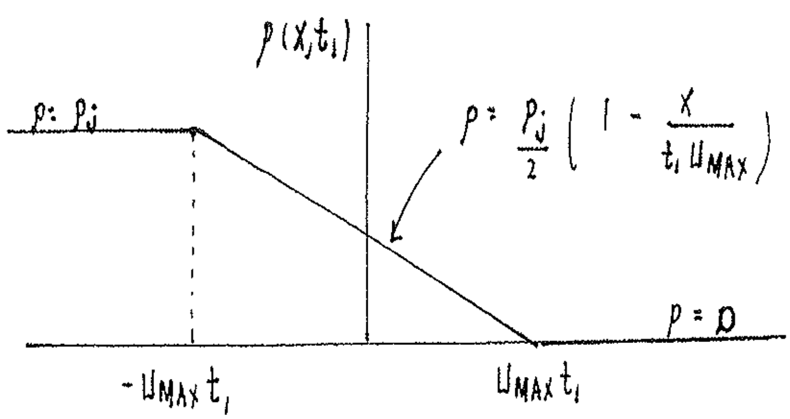
THIS GIVES $$\rho(x,t) = H(x/t) = \frac{\rho_j}{2}\left(1 - \frac{x}{t\,u_{MAX}}\right)$$

THUS THE EXPANSION FAN SOLUTION IS

$$\rho(x,t) = \frac{\rho_j}{2}\left(1 - \frac{x}{t\,u_{MAX}}\right) \quad \text{IN} \quad -u_{MAX}\,t < x < u_{MAX}\,t.$$

HENCE ρ IS CONSTANT ON LINES x/t = CONSTANT.

WE PLOT THE SOLUTION $\rho(x,t)$ AT $t = t_1$ BY TAKING A HORIZONTAL SLICE IN THE (t,x) PLANE:



NOW CONSIDER THE MOTION OF AN INDIVIDUAL CAR THAT STARTS AT $X = -X_0$ AT TIME $t = 0$.

THEN RECALLING THAT $q = \rho u$ (flow = density × speed), THE ODE IS

$$\frac{dx}{dt} = u[x, t] \qquad X(0) = -X_0.$$

WE HAVE $q = u_{MAX}(\rho - \rho^2/\rho_j)$ SO $u = u_{MAX}(1 - \rho/\rho_j)$

THIS GIVES

$$\begin{cases} \dfrac{dx}{dt} = u_{MAX}\left[1 - \dfrac{\rho(x,t)}{\rho_j}\right] \\ X(0) = -X_0 < 0 \end{cases}$$

IN THE HOMEWORK YOU USE EXPLICIT FORMULAE FOR $\rho(x, t)$ IN REGIONS I, II, AND III, IN ORDER TO FIND THE ODE IN DIFFERENT REGIONS. THE SOLUTION $X = X(t)$ IS THE TRAJECTORY OF AN INDIVIDUAL CAR.

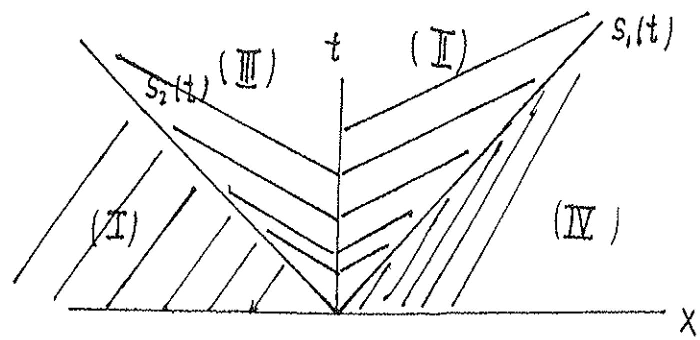
EXAMPLE (AFTER A LIGHT TURNS RED)

WE SUPPOSE THAT THE TRAFFIC FLOW IS

$$\rho_t + U_{MAX}\left(1 - 2\rho/\rho_j\right)\rho_x = 0 \qquad q(\rho) = U_{MAX}\left(\rho - \rho^2/\rho_j\right) \qquad c(\rho) = U_{MAX}\left(1 - 2\rho/\rho_j\right)$$

$$\rho(x,0) = \rho_0, \quad \rho(0^-,t) = \rho_j, \quad \rho(0^+,t) = 0.$$

WE WILL TAKE $\rho_0 < \rho_j/2$.



NOW ON $\dfrac{dx}{dt} = c(\rho)$

THEN $\dfrac{d\rho}{dt} = 0$

IN REGION I $\quad x = \xi < 0 \quad \Big\}$ $\rho = \rho_0$, $\quad x = c(\rho_0)t + \xi \qquad c(\rho_0) < 0$ SINCE $\rho_0 < \rho_j/2$.

IN REGION IV $\quad x = \xi > 0 \quad \Big\}$ $\rho = \rho_0$, $\quad x = c(\rho_0)t + \xi$

REGION II

NOW FOR $x = 0^+$; $t = \tau$ AND $\rho = 0$ parametrizes DATA CURVE

THUS ON $\dfrac{dt}{dx} = \dfrac{1}{c(\rho)}$ THEN $\dfrac{d\rho}{dt} = 0 \longrightarrow t = \dfrac{1}{c(0)}x + \tau$

$\rho = 0$, $c(0) = U_{MAX}$

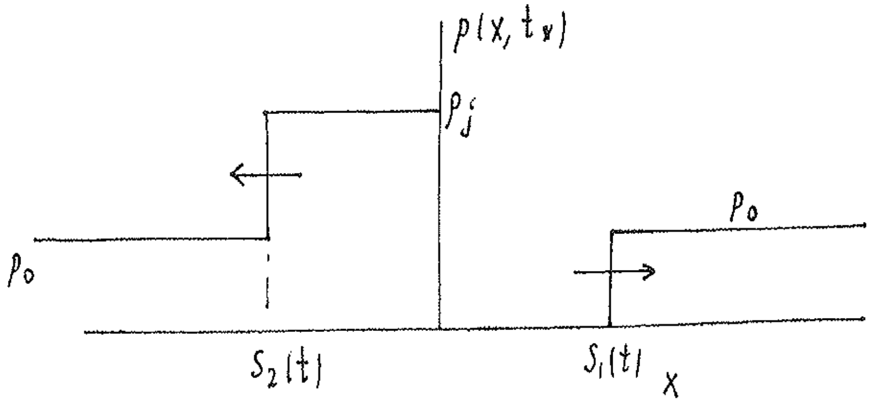
REGION III

NOW FOR $x = 0^-$; $t = \tau$ AND $\rho = \rho_j \longrightarrow t = \dfrac{1}{c(\rho_j)}x + \tau$

$\rho = \rho_j \qquad c(\rho_j) = -U_{MAX}$

WE MUST FIND SHOCKS ACROSS $S_1(t)$ AND $S_2(t)$ AS SHOWN ABOVE.

IF WE TAKE A TIME-SLICE AT $t = t_*$ FIXED THEN



CARS ARE PILING UP AT THE LIGHT.

TO FIND THE SHOCK VELOCITIES

(i) $$\frac{dS_1}{dt} = \frac{q(\rho_0) - q(0)}{\rho_0 - 1} = -u_{MAX} \frac{\rho_0}{\rho_0}(1 - \rho_0/\rho_j) = -u_{MAX}(1 - \rho_0/\rho_j)$$

WE SOLVE WITH $S_1(0) = 0$ TO GET

$$S_1(t) = u_{MAX}\left(1 - \rho_0/\rho_j\right)t.$$

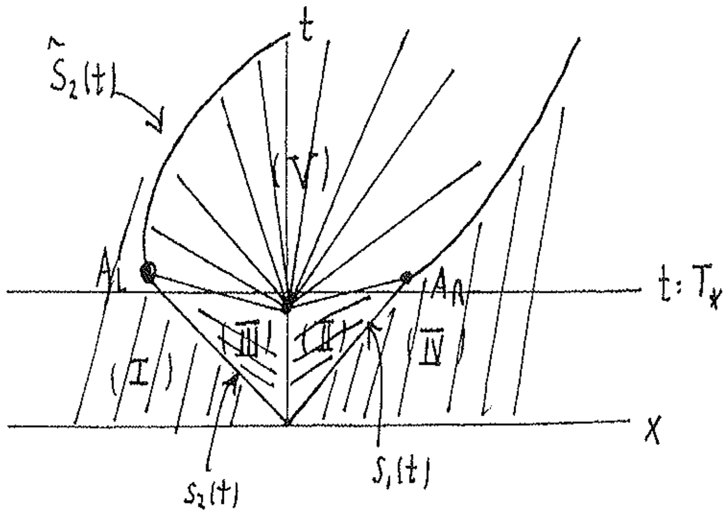
(ii) SIMILARLY $$\frac{dS_2}{dt} = \frac{q(\rho_0) - q(\rho_j)}{\rho_0 - \rho_j} = \frac{u_{MAX}\,\rho_0(1 - \rho_0/\rho_j)}{\rho_0 - \rho_j} = \frac{u_{MAX}\,\rho_0(1 - \rho_0/\rho_j)}{-\rho_j\left(1 - \rho_0/\rho_j\right)}$$

THIS YIELDS, $$\frac{dS_2}{dt} = -\frac{u_{MAX}\,\rho_0}{\rho_j} \qquad S_2(0) = 0$$

HENCE $$S_2 = -\frac{u_{MAX}\,\rho_0}{\rho_j}\,t.$$

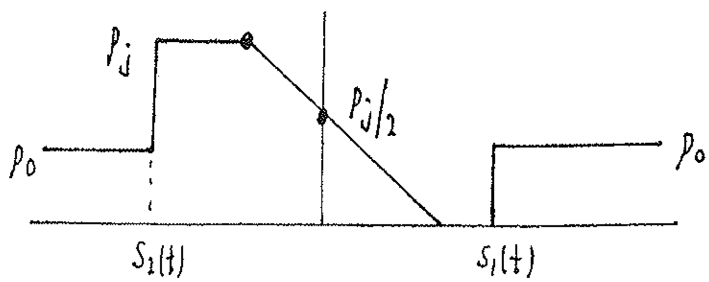
NOW WE ASK WHAT HAPPENS IF THE LIGHT TURNS GREEN AGAIN AT SOME TIME $t = T > 0$?

THEN WE WILL GET A PICTURE IN THE X-t PLANE AS



REGION (I) $\rho = \rho_0$
REGION (IV) $\rho = \rho_0$
REGION (II) $\rho = 0$
REGION (III) $\rho = \rho_j$
REGION (IV) EXPANSION FAN.

AT $t = T_*$ THE SOLUTION LOOK LIKE



AT SOME TIME $t = T_L$ WE LOSE THE REGION WHERE $\rho = \rho_j$
AT SOME OTHER TIME $t = T_R$ WE LOSE THE REGION WHERE $\rho = 0$.

AT $t = T_L$ ALL CARS ARE MOVING. AT $t = T_R$ THE LEAD CAR AFTER THE LIGHT CHANGED CAUGHT UP TO THE PACK.

WE FIRST FIT AN EXPANSION FAN AT $t = T$.

WE LOOK FOR $\rho = F(\Lambda)$ WITH $\Lambda = x/_{t-T}$.

AS USUAL WE OBTAIN $\left(c(F(\Lambda)) - \Lambda \right) F'(\Lambda) = 0$

SO THAT $c[F(\Lambda)] = \Lambda$. WE GET FROM THIS

$u_{MAX} \left(1 - \frac{2F}{\rho_j} \right) = \Lambda$ OR SOLVING FOR $F(\Lambda) = \frac{\rho_j}{2} \left(1 - \frac{x}{(t-T) u_{MAX}} \right)$

FINALLY WE CALCULATE $\hat{S}_2(t)$

$$\frac{d\hat{S}_1}{dt} = \frac{q(\rho_E) - q(\rho_0)}{\rho_E - \rho_0} = \frac{U_{MAX}(\rho_E - \rho_E^2/\rho_j) - U_{MAX}(\rho_0 - \rho_0^2/\rho_j)}{\rho_E - \rho_0}$$

THIS YIELDS THAT $$\frac{d\tilde{S}_2}{dt} = U_{MAX}\frac{\left[(\rho_E - \rho_0) - \frac{1}{\rho_j}(\rho_E - \rho_0)(\rho_E + \rho_0)\right]}{\rho_E - \rho_0}$$

THEREFORE, WE GET $$\frac{d\hat{S}_1}{dt} = U_{MAX}\left(1 - \frac{(\rho_E + \rho_0)}{\rho_j}\right)$$

NOW $$\rho_E + \rho_0 = \frac{\rho_j}{2}\left(1 - \frac{\tilde{S}_2}{(t-T)U_{MAX}}\right) + \rho_0$$

THEN WE OBTAIN: $$\frac{d\hat{S}_1}{dt} = U_{MAX}\left[1 - \frac{1}{2}\left(1 - \frac{\tilde{S}_1}{(t-T)U_{MAX}}\right) - \frac{\rho_0}{\rho_j}\right]$$

$$\tilde{S}_2(T_L) = X_L, \quad T_L = \frac{T\rho_j}{\rho_j - \rho_0}.$$

FIND A TIME $T = T_u$ SUCH THAT $\tilde{S}_2(T_u) = 0$.

THEN $$\frac{d\hat{S}_1}{dt} = a + \frac{b\hat{S}_2}{t-T} \qquad a = U_{MAX}\left(\frac{1}{2} - \frac{\rho_0}{\rho_j}\right)$$

$$b = 1/2$$

WE CAN SOLVE THIS ODE TO OBTAIN

$$\frac{d\hat{S}_1}{dt} - \frac{1}{2(T-t)}\hat{S}_2 = a$$

$$\left[(-T+t)^{+1/2}\tilde{S}_2\right]' = a(-T+t)^{+1/2}$$

THIS GIVES $$\hat{S}_2 = -U_{MAX}\left(1 - \frac{2\rho_0}{\rho_j}\right)(T-t) + c(t-T)^{-1/2}$$

THEREFORE THE EXPANSION FAN SOLUTION IS $\rho = \rho_E$ where

$$\rho_E(x,t) = \frac{\rho_j}{2}\left(1 - \frac{x}{(t-T)u_{MAX}}\right) \quad \text{FOR } -1 \le \frac{x}{(t-T)u_{MAX}} \le 1$$

DETERMINE A_L WE SET $S_2(T_L) = c(\rho_j)(T_L - T)$

THIS GIVES $-\dfrac{u_{MAX}\,\rho_0}{\rho_j} T_L = -u_{MAX}(T_L - T)$

WE SOLVE FOR T_L TO GET $T_L = \dfrac{\rho_j}{\rho_j - \rho_0} T$

AND $X_L = S_2(T_L) = -\dfrac{u_{MAX}\,\rho_0}{\rho_j}\,\dfrac{\rho_j}{\rho_j - \rho_0} T = -\dfrac{u_{MAX}\,\rho_0}{\rho_j - \rho_0} T.$

THE TIME T_L GIVES MINIMUM TIME WHEN CARS ALL START MOVING.

DETERMINE A_R WE SET $S_1(t_R) = c(0)(T_R - T)$ GIVES TIME WHEN LEAD CAR CATCHES UP TO THE PACK

THEN WE OBTAIN $u_{MAX}\left(1 - \rho_0/\rho_j\right) T_R = u_{MAX}(T_R - T)$

THIS GIVES UPON SOLVING FOR T_R: $T_R = T\rho_j/\rho_0$.

$$T_L - T_R = T\rho_j\left(\frac{1}{\rho_j - \rho_0} - \frac{1}{\rho_0}\right) = \frac{T\rho_j}{\rho_0(\rho_j - \rho_0)}\left(2\rho_0 - \rho_j\right) < 0$$

SINCE $\rho_0 < \rho_j/2$

THUS $T_L < T_R$

HENCE THE TIME FOR ALL CARS TO BEGIN MOVING IS LESS THAN THE TIME IT TAKES THE LEAD CAR AFTER THE LIGHT HAS CHANGED TO GREEN TO CATCH UP TO THE PACK.

THEREFORE, IN TERMS OF AN UNKNOWN CONSTANT C.

$$\hat{S}_2 = u_{MAX}\left(1 - \frac{2\rho_0}{\rho_j}\right)(t - T) + C(t-T)^{-1/2}$$

FINALLY $\hat{S}_2(T_L) = X_L \longrightarrow$ DETERMINES C

THEN $\hat{S}_2 = 0$ WHEN $t = T_u$ WHERE

$$u_{MAX}\left(1 - \frac{2\rho_0}{\rho_j}\right)(T_u - T) = -C(T_u - T)^{-1/2}$$

$$\text{OR} \quad (T_u - T)^{3/2} = \frac{-C}{u_{MAX}\left(1 - 2\rho_0/\rho_j\right)}$$

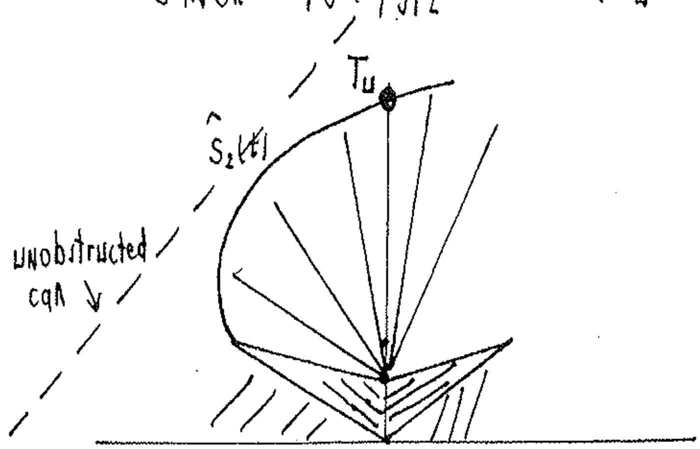
WE OBTAIN $\hat{S}_2(T_L) = X_L$:

$$-\frac{u_{MAX}\,\rho_0}{\rho_j - \rho_0}\,T_L = u_{MAX}\left(1 - \frac{2\rho_0}{\rho_j}\right)(T_L - T) + C(T_L - T)^{-1/2}$$

THIS GIVES $C(T_L - T)^{-1/2} = -\dfrac{u_{MAX}\,\rho_0}{\rho_j - \rho_0}\,T_L - u_{MAX}\left(1 - \dfrac{2\rho_0}{\rho_j}\right)(T_L - T$

AND SO WE CONCLUDE $C < 0$.

SINCE $\rho_0 < \rho_j/2 \longrightarrow (T_u - T)^{3/2} = -\dfrac{C}{u_{MAX}\left(1 - 2\rho_0/\rho_j\right)} > 0$.



SO EVENTUALLY ALL CARS GET THROUGH THE LIGHT AND CARS THAT WERE INITIALLY FAR ENOUGH BACK GO THROUGH UNIMPEDED ---- LINE