FROM

EQUATION

$u_y = 0$

iable coefficient (y). We shall illus
ric method somewhat like Example
directional derivative in the direc
in the xy plane with $(1, y)$ as tang
heir equations are

$= \frac{y}{1}$

Ce^x.

istic curves of the PDE (4). As C
ine perfectly without intersecting.
because

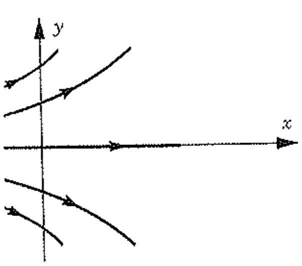
$e^x \frac{\partial u}{\partial y} = u_x + yu_y = 0.$

) is independent of x. Putting $y = C$

$u(0, e^{-x}y)$.

$= f(e^{-x}y)$

where again f is an arbitrary func
isily checked by differentiation using
ietrically, the "picture" of the solu
aracteristic curve in Figure 3.



ure 3

Example 2.

Find the solution of (4) that satisfies the auxiliary condition $u(0, y) = y^3$. Indeed, putting $x = 0$ in (7), we get $y^3 = f(e^{-0}y)$, so that $f(y) = y^3$. Therefore, $u(x, y) = (e^{-x}y)^3 = e^{-3x}y^3$. □

Example 3.

Solve the PDE

$$\boxed{u_x + 2xy^2u_y = 0.} \tag{8}$$

The characteristic curves satisfy the ODE $dy/dx = 2xy^2/1 = 2xy^2$. To solve the ODE, we separate variables: $dy/y^2 = 2x\,dx$; hence $-1/y = x^2 - C$, so that

$$y = (C - x^2)^{-1}. \tag{9}$$

These curves are the characteristics. Again, $u(x, y)$ is a constant on each such curve. (Check it by writing it out.) So $u(x, y) = f(C)$, where f is an arbitrary function. Therefore, the general solution of (8) is obtained by solving (9) for C. That is,

$$\boxed{u(x, y) = f\left(x^2 + \frac{1}{y}\right).} \tag{10}$$

Again this is easily checked by differentiation, using the chain rule: $u_x = 2x \cdot f'(x^2 + 1/y)$ and $u_y = -(1/y^2) \cdot f'(x^2 + 1/y)$, whence $u_x + 2xy^2u_y = 0$. □

In summary, the geometric method works nicely for any PDE of the form $a(x, y)u_x + b(x, y)u_y = 0$. It reduces the solution of the PDE to the solution of the ODE $dy/dx = b(x, y)/a(x, y)$. If the ODE can be solved, so can the PDE. Every solution of the PDE is constant on the solution curves of the ODE.

Moral Solutions of PDEs generally depend on arbitrary functions (instead of arbitrary constants). You need an auxiliary condition if you want to determine a unique solution. Such conditions are usually called *initial* or *boundary conditions*. We shall encounter these conditions throughout the book.

EXERCISES

1. Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.
2. Solve the equation $3u_y + u_{xy} = 0$. (*Hint*: Let $v = u_y$.)

3. Solve the equation $(1 + x^2)u_x + u_y = 0$. Sketch some of the characteristic curves.
4. Check that (7) indeed solves (4).
5. Solve the equation $xu_x + yu_y = 0$.
6. Solve the equation $\sqrt{1 - x^2}u_x + u_y = 0$ with the condition $u(0, y) = y$.
7. (a) Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$.
   (b) In which region of the xy plane is the solution uniquely determined?
8. Solve $au_x + bu_y + cu = 0$.
9. Solve the equation $u_x + u_y = 1$.
10. Solve $u_x + u_y + u = e^{x+2y}$ with $u(x, 0) = 0$.
11. Solve $au_x + bu_y = f(x, y)$, where $f(x, y)$ is a given function. If $a \neq 0$, write the solution in the form

$$u(x, y) = (a^2 + b^2)^{-1/2} \int_L f\, ds + g(bx - ay),$$

where g is an arbitrary function of one variable, L is the characteristic line segment from the y axis to the point (x, y), and the integral is a line integral. (*Hint:* Use the coordinate method.)

12. Show that the new coordinate axes defined by (3) are orthogonal.
13. Use the coordinate method to solve the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2.$$

1.3 FLOWS, VIBRATIONS, AND DIFFUSIONS

The subject of PDEs was practically a branch of physics until the twentieth century. In this section we present a series of examples of PDEs as they occur in physics. They provide the basic motivation for all the PDE problems we study in the rest of the book. We shall see that most often in physical problems the independent variables are those of space x, y, z, and time t.

Example 1. Simple Transport

Consider a fluid, water, say, flowing at a constant rate c along a horizontal pipe of fixed cross section in the positive x direction. A substance, say a pollutant, is suspended in the water. Let $u(x, t)$ be its concentration in grams/centimeter at time t. Then

$$\boxed{u_t + cu_x = 0.} \qquad (1)$$

(That is, the rate of change u_t of concentration is proportional to the gradient u_x. Diffusion is assumed to be negligible.) Solving this equation as in Section 1.2, we find that the concentration is a function of $(x - ct)$

is, the wave behind the shock is "catching up" to the wave ahead of it. Mathematically, this means that on a shock curve we have

$$\boxed{a(u^-) > s > a(u^+).} \tag{18}$$

This is the *entropy criterion* for a solution. Notice that (18) is satisfied for Example 6 but not for Example 7. Therefore, Example 7 is rejected. Finally, the definition of a shock wave is complete. *Along its curves of discontinuity it must satisfy both* (17) *and* (18). For further discussion of shocks, see [Wh] or [Sm]. □

EXERCISES

1. (a) Use direct differentiation to check that (4) solves (3).
   (b) Check directly that (5) solves the initial condition $u(x, 0) = x^3$.
2. Solve $(1 + t)u_t + xu_x = 0$. Then solve it with the initial condition $u(x, 0) = x^5$ for $t > 0$.
3. Solve the nonlinear equation $u_t + uu_x = 0$ with the auxiliary condition $u(x, 0) = x$. Sketch some of the characteristic lines.
4. Sketch some typical characteristic lines for Example 4.
5. Solve $u_t + u^2 u_x = 0$ with $u(x, 0) = 2 + x$.
6. Verify by differentiation that the formula (12) provides a solution of the differential equation (6).
7. Solve $xu_t + uu_x = 0$ with $u(x, 0) = x$. (*Hint:* Change variables $x \mapsto x^2$.)
8. Show that a smooth solution of the problem $u_t + uu_x = 0$ with $u(x, 0) = \cos \pi x$ must satisfy the equation $u = \cos[\pi(x - ut)]$. Show that u ceases to exist (as a single-valued continuous function) when $t = 1/\pi$. (*Hint:* Graph $\cos^{-1} u$ versus $\pi(x - ut)$ as functions of u.)
9. Check by direct differentiation that the formula $u(x, t) = \phi(z)$, where z is given implicitly by $x - z = t\, a(\phi(z))$, does indeed provide a solution of the PDE (1).
10. Solve $u_t + uu_x = 0$ with the initial condition $u(x, 0) = 1$ for $x \le 0$, $1 - x$ for $0 \le x \le 1$, and 0 for $x \ge 1$. Solve it for all $t \ge 0$, allowing for a shock wave. Find exactly where the shock is and show that it satisfies the entropy condition. Sketch the characteristics.
11. Show that (15) is equivalent to the statement

$$\frac{d}{dt}\int_a^b u(x, t)\, dx + A(u(b, t)) - A(u(a, t)) = 0 \qquad \text{for all } a, b.$$

12. Solve $u_t + uu_x = 1$ with $u(x, 0) = x$.

EXERCISES 12.6

12.6.1. Determine the solution $\rho(x,t)$ satisfying the initial condition $\rho(x, 0) = f(x)$ if

*(a) $\frac{\partial \rho}{\partial t} = 0$ (b) $\frac{\partial \rho}{\partial t} = -3\rho + 4e^{7t}$

*(c) $\frac{\partial \rho}{\partial t} = -3x\rho$ (d) $\frac{\partial \rho}{\partial t} = x^2 t\rho$

***12.6.2.** Determine the solution of $\partial\rho/\partial t = \rho$ that satisfies $\rho(x,t) = 1 -$ ...

12.6.3. Suppose $\frac{\partial \rho}{\partial t} + c_0\frac{\partial \rho}{\partial x} = 0$ with c_0 constant.

*(a) Determine $\rho(x,t)$ if $\rho(x,0) = \sin x$.

*(b) If $c_0 > 0$, determine $\rho(x,t)$ for $x > 0$ and $t > 0$, where $\rho(x, 0) = f(x)$ and $\rho(0,t) = g(t)$ for $t > 0$.

(c) Show that part (b) cannot be solved if $c_0 < 0$.

***12.6.4.** If $u(\rho) = \alpha + \beta\rho$, determine α and β such that $u(0) = u_{\max}$ and ...

(a) What is the flow as a function of density? Graph the flow as a function of density.

(b) At what density is the flow maximum? What is the corresponding ... What is the maximum flow (called the *capacity*)?

12.6.5. Redo Exercise 12.6.4 if $u(\rho) = u_{\max}\left(1 - \frac{\rho^3}{\rho_{\max}^3}\right)$.

12.6.6. Consider the traffic flow problem

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0.$$

Assume $u(\rho) = u_{\max}(1 - \frac{\rho}{\rho_{\max}})$. Solve for $\rho(x,t)$ if the initial conditions ...

(a) $\rho(x,0) = \rho_{\max}$ for $x < 0$ and $\rho(x,0) = 0$ for $x > 0$. This corresponds to the traffic density that results after an infinite line of stopped traffic ... a red light turning green.

(b) $\rho(x,0) = \begin{cases} \rho_{\max} & x < 0 \\ \frac{\rho_{\max}}{2} & 0 < x < a \\ 0 & x > a \end{cases}$

(c) $\rho(x,0) = \begin{cases} \frac{3\rho_{\max}}{5} & x < 0 \\ \frac{\rho_{\max}}{5} & x > 0 \end{cases}$

12.6.7. Solve the following problems:

(a) $\frac{\partial \rho}{\partial t} + \rho^2\frac{\partial \rho}{\partial x} = 0, \quad \rho(x,0) = \begin{cases} 3 & x < 0 \\ 4 & x > 0 \end{cases}$

(b) $\frac{\partial \rho}{\partial t} + 4\rho\frac{\partial \rho}{\partial x} = 0, \quad \rho(x,0) = \begin{cases} 2 & x < 1 \\ 3 & x > 1 \end{cases}$

(c) $\frac{\partial \rho}{\partial t} + 3\rho\frac{\partial \rho}{\partial x} = 0, \quad \rho(x,0) = \begin{cases} 1 & x < 0 \\ 2 & 0 < x < 1 \\ 4 & x > 1 \end{cases}$

(d) $\frac{\partial \rho}{\partial t} + 6\rho\frac{\partial \rho}{\partial x} = 0$ for $x > 0$ only, $\quad \begin{matrix} \rho(x,0) = 5 & x > 0 \\ \rho(0,t) = 2 & t > 0 \end{matrix}$

12.6.8. Solve subject to the initial condition $\rho(x,0) = f(x)$:

*(a) $\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = e^{-3x}$ (b) $\frac{\partial \rho}{\partial t} + 3x\frac{\partial \rho}{\partial x} = 4$

*(c) $\frac{\partial \rho}{\partial t} + t\frac{\partial \rho}{\partial x} = 5$ (d) $\frac{\partial \rho}{\partial t} + 5t\frac{\partial \rho}{\partial x} = 3\rho$

*(e) $\frac{\partial \rho}{\partial t} - t^2\frac{\partial \rho}{\partial x} = -\rho$ (f) $\frac{\partial \rho}{\partial t} + t^2\frac{\partial \rho}{\partial x} = 0$

*(g) $\frac{\partial \rho}{\partial t} + x\frac{\partial \rho}{\partial x} = t$

12.6.9. Determine a parametric representation of the solution satisfying $\rho(x,0) = f(x)$:

*(a) $\frac{\partial \rho}{\partial t} - \rho^2\frac{\partial \rho}{\partial x} = 3\rho$ (b) $\frac{\partial \rho}{\partial t} + \rho\frac{\partial \rho}{\partial x} = t$

*(c) $\frac{\partial \rho}{\partial t} + t^2\rho\frac{\partial \rho}{\partial x} = -\rho$ (d) $\frac{\partial \rho}{\partial t} + \rho\frac{\partial \rho}{\partial x} = -x\rho$

12.6.10. Solve $\frac{\partial \rho}{\partial t} + t^2\frac{\partial \rho}{\partial x} = 4\rho$ for $x > 0$ and $t > 0$ with $\rho(0,t) = h(t)$ and $\rho(x,0) = 0$.

12.6.11. *Solve $\frac{\partial \rho}{\partial t} + (1+t)\frac{\partial \rho}{\partial x} = 3\rho$ for $t > 0$ and $x > -t/2$ with $\rho(x,0) = f(t)$ for $x > 0$ and $\rho(x,t) = g(t)$ along $x = -t/2$.

12.6.12. Consider (12.6.8) if there is a moving shock x_s such that $a < x_s(t) < b$. By differentiating the integral [with a discontinuous integrand at $x_s(t)$], *derive* (12.6.20).

12.6.13. Suppose that, instead of $u = U(\rho)$, a car's velocity u is

$$u = U(\rho) - \frac{\nu}{\rho}\frac{\partial \rho}{\partial x},$$

where ν is a constant.

(a) What sign should ν have for this expression to be physically reasonable?

(b) What equation now describes conservation of cars?

(c) Assume that $U(\rho) = u_{\max}(1 - \rho/\rho_{\max})$. Derive **Burgers' equation:**

$$\frac{\partial \rho}{\partial t} + u_{\max}\left[1 - \frac{2\rho}{\rho_{\max}}\right]\frac{\partial \rho}{\partial x} = \nu\frac{\partial^2 \rho}{\partial x^2}.$$

12.6.14. Consider Burgers' equation as derived in Exercise 12.6.13. Suppose that a solution exists as a density wave moving without change of shape at velocity $V, \rho(x,t) = f(x - Vt)$.

*(a) What ordinary differential equation is satisfied by f?

(b) Integrate this differential equation once. By graphical techniques show that a solution exists such that $f \to \rho_2$ as $x \to +\infty$ and $f \to \rho_1$ as $x \to -\infty$ only if $\rho_2 > \rho_1$. Roughly sketch this solution. Give a physical interpretation of this result.

*(c) Show that the velocity of wave propagation, V, is the same as the shock velocity separating $\rho = \rho_1$ from $\rho = \rho_2$ (occurring if $\nu = 0$).

12.6.15. Consider Burgers' equation as derived in Exercise 12.6.13. Show of dependent variables

$$\rho = \frac{\nu \rho_{\max}}{u_{\max}} \frac{\phi_x}{\phi},$$

introduced independently by E. Hopf in 1950 and J. D. Cole in Burgers' equation into a diffusion equation, $\frac{\partial \phi}{\partial t} + u_{\max}\frac{\partial \phi}{\partial x} =$ solve the initial value problem $\rho(x,0) = f(x)$ for $-\infty < x <$ [1999] it is shown that this exact solution can be asymptotically using Laplace's method for exponential integrals to show that the solution obtained for $\nu = 0$ using the method of characte dynamics.]

12.6.16. Suppose that the initial traffic density is $\rho(x,0) = \rho_0$ for $x <$ for $x > 0$. Consider the two cases, $\rho_0 < \rho_1$ and $\rho_1 < \rho_0$. For which cases is a density shock necessary? Briefly explain.

12.6.17. Consider a traffic problem, with $u(\rho) = u_{\max}(1 - \frac{\rho}{\rho_{\max}})$. Determ

*(a) $\rho(x,0) = \begin{cases} \frac{\rho_{\max}}{5} & x < 0 \\ \frac{3\rho_{\max}}{5} & x > 0 \end{cases}$ (b) $\rho(x,0) = \begin{cases} \frac{\rho_{\max}}{3} & \\ \frac{2\rho_{\max}}{3} & \end{cases}$

12.6.18. Assume that $u(\rho) = u_{\max}(1 - \frac{\rho^2}{\rho_{\max}^2})$. Determine the traffic de $\rho(x,0) = \rho_1$ for $x < 0$ and $\rho(x,0) = \rho_2$ for $x > 0$.

(a) Assume that $\rho_2 > \rho_1$. *(b) Assume that $\rho_2 <$

12.6.19. Solve the following problems (assuming ρ is conserved):

(a) $\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial \rho}{\partial x} = 0,$ $\rho(x,0) = \begin{cases} 4 & x < 0 \\ 3 & x > 0 \end{cases}$

(b) $\frac{\partial \rho}{\partial t} + 4\rho \frac{\partial \rho}{\partial x} = 0,$ $\rho(x,0) = \begin{cases} 3 & x < 1 \\ 2 & x > 1 \end{cases}$

(c) $\frac{\partial \rho}{\partial t} + 3\rho \frac{\partial \rho}{\partial x} = 0,$ $\rho(x,0) = \begin{cases} 4 & x < 0 \\ 2 & 0 < x < 1 \\ 1 & x > 1 \end{cases}$

(d) $\frac{\partial \rho}{\partial t} + 6\rho \frac{\partial \rho}{\partial x} = 0$ for $x > 0$ only, $\begin{array}{ll} \rho(x,0) = 2 & x > 0 \\ \rho(0,t) = 5 & t > 0 \end{array}$

12.6.20. Redo Exercise 12.6.19, assuming that ρ^2 is conserved.

12.6.21. Compare Exercise 12.6.19(a) with 12.6.20(a). Show that the different.

12.6.22. Solve $\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial \rho}{\partial x} = 0$. If a nonuniform shock occurs, give equation. Do you believe the nonuniform shock persists or by a uniform shock?

(a) $\rho(x,0) = \begin{cases} -1 & x < 0 \\ 3 & x > 0 \end{cases}$ (b) $\rho(x,0) = \begin{cases} -2 & x < 0 \\ 1 & x > 0 \end{cases}$

(c) $\rho(x,0) = \begin{cases} 1 & x < 0 \\ -3 & x > 0 \end{cases}$ (d) $\rho(x,0) = \begin{cases} 2 & x < 0 \\ -1 & x > 0 \end{cases}$

12.6.23. Solve $\frac{\partial \rho}{\partial t} - \rho^2 \frac{\partial \rho}{\partial x} = 0$. If a nonuniform shock occurs, give equation. Do you believe the nonuniform shock persists or by a uniform shock?

(a) $\rho(x,0) = \begin{cases} -2 & x < 0 \\ 1 & x > 0 \end{cases}$ (b) $\rho(x,0) = \begin{cases} -1 & x < 0 \\ 4 & x > 0 \end{cases}$

(c) $\rho(x,0) = \begin{cases} 1 & x < 0 \\ -3 & x > 0 \end{cases}$ (d) $\rho(x,0) = \begin{cases} 5 & x < 0 \\ -2 & x > 0 \end{cases}$

12.7 FIRST-ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

12.7.1 Eikonal Equation Derived from the Wave Equation

For simplicity we consider the two-dimensional wave equation

$$\frac{\partial^2 E}{\partial t^2} = c^2\left(\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2}\right). \tag{12.7.1}$$

Plane waves and their reflections were analyzed in Section 4.6. Nearly plane waves exist under many circumstances. If the coefficient c is not constant but varies slowly, then over a few wavelengths the wave sees nearly constant c. However, over long distances (relative to short wavelengths) we may be interested in the effects of variable c. Another situation in which nearly plane waves arise is the reflection of a plane wave by a curved boundary (or reflection and refraction by a curved interface between two media with different indices of refraction). We assume the radius of curvature of the boundary is much longer than typical wavelengths. In many of these situations the temporal frequency ω is fixed (by an incoming plane wave). Thus,

$$E = A(x,y)e^{-i\omega t}, \tag{12.7.2}$$

where that $A(x,y)$ satisfies the **Helmholtz** or **reduced wave equation:**

$$-\omega^2 A = c^2\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right). \tag{12.7.3}$$

Again the temporal frequency ω is fixed (and given), but $c = c(x,y)$ for inhomogeneous media or $c =$ constant for uniform media. In uniform media ($c =$ constant), plane waves of the form $E = A_0 e^{i(k_1 x + k_2 y - \omega t)}$ or

$$A = A_0 e^{i(k_1 x + k_2 y)} \tag{12.7.4}$$

exist if

$$\omega^2 = c^2(k_1^2 + k_2^2). \tag{12.7.5}$$

For nearly plane waves, we introduce the phase $u(x,y)$ of the reduced wave equation:

$$A(x,y) = R(x,y)e^{iu(x,y)}. \tag{12.7.6}$$

The wave numbers k_1 and k_2 for uniform media are usually called p and q, respectively, and are defined by

$$p = \frac{\partial u}{\partial x} \tag{12.7.7}$$

$$q = \frac{\partial u}{\partial y}. \tag{12.7.8}$$