ON THE SHARP CRITICAL MASS THRESHOLD FOR THE 3D PATLAK-KELLER-SEGEL-NAVIER-STOKES SYSTEM VIA COUETTE FLOW

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ABSTRACT. As is well-known, the solution of the Patlak-Keller-Segel system in 3D may blow up in finite time regardless of any initial cell mass. In this paper, we are interested in the suppression of blow-up and the critical mass threshold for the 3D Patlak-Keller-Segel-Navier-Stokes system via the Couette flow $(Ay, 0, 0)$. It is proved that if the Couette flow is sufficiently strong (A is large enough), then the solutions for the system are global in time in the periodic domain $(x, y, z) \in \mathbb{T}^3$ as long as the initial cell mass is less than $16\pi^2$. This result seems to be sharp, since the zero-mode function (the mean value in $x-$direction) of the three dimensional density is a modification of the two-dimensional Keller-Segel equations, whose critical mass in 2D is 8π. Our first key observation is the dissipative decay of $(\widetilde{u}_{2,0}, \widetilde{u}_{3,0})$ (see Lemma 4.3 for more details). Then we combine the quasi-linear method proposed by Wei-Zhang (Comm. Pure Appl. Math., 2021) with the zero-mode estimate of the density by the logarithmic Hardy-Littlewood-Sobolev inequality as in Bedrossian-He (SIAM J. Math. Anal., 2017) or He (Nonlinearity, 2025) to obtain the bounded-ness of the density and the velocity.

Keywords: suppression of blow-up; Patlak-Keller-Segel-Navier-Stokes; stability; Couette flow

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1. Introduction

Consider the following parabolic-elliptic Patlak-Keller-Segel (PKS) system

$$\begin{cases} \partial_t n = \triangle n - \nabla \cdot (n \nabla c), \\ \triangle c + n = 0, \end{cases} \tag{1.1}$$

which is designed to depict the diffusion and chemotactic motion of chemical substances within a population of cells or microorganisms. Patlak made significant contributions in [27], and later, Keller and Segel further advanced it in [23]. Extensive applications have been found across diverse scientific fields, including biology, ecology, and medicine [19]. In the realm of biology, this system provides crucial insights into the complex behavior of cells, such as their migration, aggregation, and diffusion [18].

A well-known characteristic of the PKS system is its critical dependence on spatial dimension. For the one-dimensional PKS system, all its solutions are globally well-posed. When the spacial dimension is higher than one, the solutions of the PKS system (1.1) may blow up in finite time. In the two-dimensional space, the PKS system for both parabolic-elliptic and parabolic-parabolic forms have a critical mass of 8π. In the 2D space, the parabolic-elliptic PKS system is globally well-posed if and only if the total mass $M \leq 8\pi$ by Wei [35], see also Blanchet-Dolbeault-Perthame [3] for $M < 8\pi$. When $M = 8\pi$, the solution may blow up at infinity, see Blanchet-Carrillo-Masmoudi [2] and Davila-del Pino-Dolbeault-Musso-Wei [11] for the blow-up rate at infinity. The parabolic-parabolic PKS model ($\triangle c$ is replaced by $\triangle c - \partial_t c$ in $(1.1)_2$) also has a critical mass of 8π in 2D: if the cell mass $M := ||n_{\rm in}||_{L^1}$ is less than 8π, the solutions of the system are global in time proved by Calvez-Corrias [5]; if the cell mass is greater than 8π, the solutions will blow up in finite time proved by Buseghin-Davila-del Pino-Musso [4], Collot-Ghoul-Masmoudi-Nguyen [7], and Schweyer [28].

When the spatial dimension is higher than two, the PKS system (1.1) becomes supercritical. In this condition, regardless of parabolic-elliptic form or parabolic-parabolic form, the finite-time blow-up may occur for arbitrarily small values of the initial mass. In this time, the solution with any initial mass may blow up in finite time. This behavior has been established in various settings: for the parabolic-elliptic case by Nagai [26] and Souplet-Winkler [31], and for the parabolic-parabolic case by Winkler [37]. Additional results and further developments on this topic can be found in [2, 11, 33] and the related therein.

Generally, the processes involving chemical attraction take place in fluids. As said in [24]: *A natural question is whether the presence of fluid flow can affect singularity formation by mixing the bacteria thus making concentration harder to achieve.*

In this paper, we investigate the suppression of blow-up and the critical mass threshold of the following three-dimensional parabolic-elliptic Patlak-Keller-Segel (PKS) system coupled with

Navier-Stokes (NS) equations in $(x,y,z)\in\mathbb{T}^3$ with $\mathbb{T}=[0,2\pi]$:

$$\begin{cases} \partial_t n + v\cdot\nabla n = \triangle n - \nabla\cdot(n\nabla c), \\ \triangle c + n - \bar{n} = 0, \\ \partial_t v + v\cdot\nabla v + \nabla P = \triangle v + n\nabla\phi, \\ \nabla\cdot v = 0, \end{cases} \tag{1.2}$$

along with initial conditions

$$(n,v)\big|_{t=0} = (n_{\text{in}}, v_{\text{in}}),$$

where n represents the cell density, c denotes the chemoattractant density, and v denotes the velocity of fluid. In addition, $\bar{n}=\frac{1}{|\mathbb{T}|^3}\int_{\mathbb{T}^3} n dxdydz$ denotes the average of n, P is the pressure and ϕ is the given potential function.

Let us by briefly review some results obtained in the 2D case, which implies the 8π critical mass threshold vanishes by the mixing effect of the fluid. For the parabolic-elliptic PKS system of $(1.2)_1-(1.2)_2$, Kiselev-Xu [24] showed no critical mass threshold by suppressing the blow-up by stationary relaxation enhancing flows and time-dependent Yao-Zlatos near-optimal mixing flows in $\mathbb{T}^d$ with $d=2,3$. Bedrossian-He [1] studied the suppression of blow-up by non-degenerate shear flows in $\mathbb{T}^2$ for the 2D parabolic-elliptic case. He [14] investigated the suppression of blow-up for the parabolic-parabolic PKS model near the large strictly monotone shear flow in $\mathbb{T}\times\mathbb{R}$. For the coupled PKS-NS system, Zeng-Zhang-Zi [38] firstly considered the 2D PKS-NS system near the Couette flow in $\mathbb{T}\times\mathbb{R}$, and proved that if the Couette flow is sufficiently strong, the solution stays globally regular. He [15] considered the blow-up suppression for the parabolic-elliptic PKS-NS system in $\mathbb{T}\times\mathbb{R}$ with the coupling of buoyancy effects for a class of initial data with small vorticity. Wang-Wang-Zhang [34] studied the blow-up suppression of 2D supercritical parabolic-elliptic PKS-NS system near the Couette flow in $\mathbb{T}\times\mathbb{R}$. Li-Xiang-Xu [25] suppressed the blow-up for the PKS-NS system via the Poiseuille flow in $\mathbb{T}\times\mathbb{R}$, and proved that the solution is global as long as the Poiseuille flow is enough strong. Furthermore, Cui-Wang [8] considered the blow-up suppression for the PKS-NS system in $\mathbb{T}\times\mathbb{I}$ with Navier-slip boundary condition. In addition, Hu [20] proved that sufficiently large buoyancy can also suppress the blow-up of the PKS system, see also the recent results by Hu-Kiselev-Yao [22] and Hu-Kiselev [21]. The blow-up also can be suppressed by adding some logistic terms, see [32] and the references therein.

In 3D, some results have been made in the study of the corresponding problem. Bedrossian-He [1] studied the suppression of blow-up for the parabolic-elliptic PKS system by shear flows in $\mathbb{T}^3$ and $\mathbb{T}\times\mathbb{R}^2$. Feng-Shi-Wang [13] suppressed the blow-up for the advective Kuramoto-Sivashinsky and the Keller-Segel equations via the planar helical flows. Shi-Wang [30] considered the suppression effect of the Couette-Poiseuille flow $(z,z^2,0)$ in $\mathbb{T}^2\times\mathbb{R}$, and Deng-Shi-Wang [12] proved the Couette flow with a sufficiently large amplitude prevents the blow-up of solutions in the whole space for exponential decay data. For the parabolic-parabolic PKS system, He [16] introduced a family of time-dependent alternating shear flows in the domain $\mathbb{T}^3$, and proved the solution remains globally regular as long as the flow is sufficiently strong. For a time-dependent shear flow, He [17] demonstrated that when the total mass of the cell density is below a specific threshold $(8\pi|\mathbb{T}|)$, the solution remains globally regular in $\mathbb{T}^3$ as long as the flow is sufficiently strong.

For the 3D PKS-NS system, fewer results are obtained. The first three authors [9] considered the PKS system coupled with the linearized NS equations near the Couette flow in $\mathbb{T}\times\mathbb{I}\times\mathbb{T}$, and showed that when A is big enough the solutions are global in time as long as $M<\frac{8\pi}{9}$ and $A(\|u_{2,\rm in}\|_{L^2}+\|u_{3,\rm in}\|_{L^2})\leq C$. Then, the first three authors [10] studied the blow-up suppression and the nonlinear stability of the PKS-NS system for the Couette flow $(Ay,0,0)$ in $\mathbb{T}\times\mathbb{R}\times\mathbb{T}$, and proved that the solutions are global in time as long as the initial cell mass $M<\frac{24}{5}\pi^2$ and initial velocity $A^{\frac13+}\|u_{\rm in}\|_{H^2}\leq C$.

Our main goal is to investigate the suppression of blow-up and obtain the optimal critical mass for the PKS-NS system (1.2) near the 3D Couette flow $(Ay,0,0)$. Introduce a perturbation $u=(u_1,u_2,u_3)$ around the Couette flow $(Ay,0,0)$, which $u(t,x,y,z)=v(t,x,y,z)-(Ay,0,0)$ satisfying $u\big|_{t=0}=u_{\rm in}=(u_{1,\rm in},u_{2,\rm in},u_{3,\rm in})$. Assume $\phi=x$, then we rewrite the system (1.2) into

$$\begin{cases}\partial_t n+Ay\partial_x n+u\cdot\nabla n-\triangle n=-\nabla\cdot(n\nabla c),\\ \triangle c+n-\bar{n}=0,\\ \partial_t u+Ay\partial_x u+\begin{pmatrix}Au_2\\0\\0\end{pmatrix}-\triangle u+u\cdot\nabla u+\nabla P^{N_1}+\nabla P^{N_2}=\begin{pmatrix}n\\0\\0\end{pmatrix},\\ \nabla\cdot u=0,\end{cases}\tag{1.3}$$

where the pressure P^{N_1} and P^{N_2} are determined by

$$\begin{aligned}\triangle P^{N_1}&=-2A\partial_x u_2+\partial_x n,\\ \triangle P^{N_2}&=-{\rm div}\ (u\cdot\nabla u).\end{aligned}\tag{1.4}$$

To estimate non-zero norms more conveniently, we introduce the vorticity $\omega_2=\partial_z u_1-\partial_x u_3$ and $\triangle u_2$, satisfying

$$\partial_t\omega_2+Ay\partial_x\omega_2+A\partial_z u_2-\triangle\omega_2=\partial_z n-\partial_z(u\cdot\nabla u_1)+\partial_x(u\cdot\nabla u_3)$$

and

$$\partial_t\triangle u_2+Ay\partial_x\triangle u_2-\triangle^2u_2=-\partial_y\partial_x n-(\partial_x^2+\partial_z^2)(u\cdot\nabla u_2)+\partial_y[\partial_x(u\cdot\nabla u_1)+\partial_z(u\cdot\nabla u_3)].$$

After the time rescaling $t\mapsto\frac{t}{A}$, we get

$$\begin{cases}\partial_t n+y\partial_x n+\frac1A u\cdot\nabla n-\frac1A\triangle n=-\frac1A\nabla\cdot(n\nabla c),\\ \triangle c+n-\bar{n}=0,\\ \partial_t\omega_2+y\partial_x\omega_2-\frac1A\triangle\omega_2+\partial_z u_2=-\frac1A\partial_z(u\cdot\nabla u_1)+\frac1A\partial_x(u\cdot\nabla u_3)+\frac1A\partial_z n,\\ \partial_t\triangle u_2+y\partial_x\triangle u_2-\frac1A\triangle(\triangle u_2)=-\frac1A\partial_y\partial_x n-\frac1A(\partial_x^2+\partial_z^2)(u\cdot\nabla u_2)\\ \qquad\qquad+\frac1A\partial_y\left[\partial_x(u\cdot\nabla u_1)+\partial_z(u\cdot\nabla u_3)\right],\\ \nabla\cdot u=0.\end{cases}\tag{1.5}$$

Before stating the result, we need to define the following modes

$$P_0 f=f_0=\frac{1}{|\mathbb{T}|}\int_{\mathbb{T}}f(t,x,y,z)dx,\ \text{and}\ P_{\neq}f=f_{\neq}=f-f_0.$$

Throughout this paper, f_0 and $f_{\neq}$ respectively represent the zero and non-zero modes of f.

Our main result is stated as follows.

Theorem 1.1. *Assume that* $0 < n_{\rm in}(x,y,z) \in H^2(\mathbb{T}^3)$ *and* $u_{\rm in}(x,y,z) \in H^2(\mathbb{T}^3)$. *There exist a sufficiently small positive constant* ϵ *depending on* $\|n_{\rm in}\|_{H^2(\mathbb{T}^3)}$ *and* $\|(u_{\rm in})_{\neq}\|_{H^2(\mathbb{T}^3)}$, *and a positive constant* A_1 *depending on* $\|n_{\rm in}\|_{H^2(\mathbb{T}^3)}$ *and* $\|u_{\rm in}\|_{H^2(\mathbb{T}^3)}$, *such that if* $A \geq A_1$,

$$\|(u_{2,\rm in})_0\|_{H^2} + \|(u_{3,\rm in})_0\|_{H^1} \leq \epsilon \tag{1.6}$$

and

$$M = \int_{\mathbb{T}^3} n_{\rm in} dxdydz < 16\pi^2. \tag{1.7}$$

Then the solution of (1.5) *is global in time.*

Remark 1.1. *For the three-dimensional space, the index* $16\pi^2$ *in* (1.7) *seems to be the sharp threshold for initial cell mass. Recall that* n_0 *satisfies (see also* (6.16)*)*

$$\partial_t n_0 = \frac{1}{A}\triangle n_0 - \frac{1}{A}\nabla\cdot(n_0\nabla c_0) - \frac{1}{A}\nabla\cdot(n_{\neq}\nabla c_{\neq})_0 - \frac{1}{A}(u_0\cdot\nabla n_0) - \frac{1}{A}(u_{\neq}\cdot\nabla n_{\neq})_0. \tag{1.8}$$

When the velocity u *vanishes and* $n(t,x,y,z) = n(t,y,z)$, *which does not depend on the variable* x. *Then* (1.8) *is reduced to*

$$\partial_t n_0 = \frac{1}{A}\triangle n_0 - \frac{1}{A}\nabla\cdot(n_0\nabla c_0) + \text{"good terms"}, \tag{1.9}$$

which is similar to the 2D Keller-Segel equations and the critical mass is $\int_{\mathbb{T}^2} n_0 dydz = 8\pi = \frac{1}{2\pi}\int_{\mathbb{T}^3} ndxdydz$. *It implies that the critical mass threshold for the initial cell mass to the system* (1.8) *is just* $16\pi^2$. *A new observation is the logarithmic Hardy-Littlewood-Sobolev inequality and the dissipative decay of the* $(\widetilde{u}_{2,0}, \widetilde{u}_{3,0})$ *(see Lemma 4.3 in Section 4.2 for more details) to estimate the norm of* $\|n_0\|_{L^2}$.

Remark 1.2. *The small constant* ϵ *in* (1.6) *depends on* $\|n_{\rm in}\|_{H^2(\mathbb{T}^3)}$ *and* $\|(u_{\rm in})_{\neq}\|_{H^2(\mathbb{T}^3)}$, *which can be found in* (6.8), *that is* $\epsilon m^{\frac{1}{3}} E_3^{\frac{5}{3}} \leq C$. *Moreover, the estimate of* E_3 *is decided by* (6.14), (7.4) *and Corollary 8.1.*

Remark 1.3. *In a very recent work [10], the first three authors investigated the suppression of blow-up and the nonlinear stability of the PKS-NS system via the 3D Couette flow in a different domain of* $\mathbb{T}\times\mathbb{R}\times\mathbb{T}$ *and* $\phi = y$. *When* A *is big enough, as long as*

$$\begin{cases} A^{\frac{1}{3}+}(\|u_{\rm in}\|_{H^2(\mathbb{T}\times\mathbb{R}\times\mathbb{T})} + \|(n_{\rm in})_{(0,\neq)}\|_{L^2(\mathbb{T}\times\mathbb{R}\times\mathbb{T})}) \leq C, \\ M = \int_{\mathbb{T}\times\mathbb{R}\times\mathbb{T}} n_{\rm in} dxdydz < \frac{24}{5}\pi^2, \end{cases}$$

the solutions for the PKS-NS system are global in time, where a new energy estimate for $\frac{\partial_z}{\sqrt{1-\triangle}} n_{(0,\neq)}$ *was introduced to control the density. It's still unknown whether the threshold for initial cell mass can be* $16\pi^2$, *since it is difficult to apply the logarithmic Hardy-Littlewood-Sobolev inequality without the dissipative decay of* $(\widetilde{u}_{2,0}, \widetilde{u}_{3,0})$ *at this time.*

Remark 1.4. *According to the standard arguments, the result of local well-posedness of the system (1.5) exists, which can be refered to [10, 21, 37], and we omitted it.*

Here are some notations used in this paper.

Notations:

- For given $f(t,x,y,z)$, the Fourier transform can be defined by
$$f(t,x,y,z)=\sum_{k_1,k_2,k_3\in\mathbb{Z}}\widehat{f}_{k_1,k_2,k_3}(t)e^{i(k_1x+k_2y+k_3z)},$$
where $\widehat{f}_{k_1,k_2,k_3}(t)=\frac{1}{|\mathbb{T}|^3}\int_{\mathbb{T}^3}f(t,x,y,z)e^{-i(k_1x+k_2y+k_3z)}dxdydz$.
- Especially, we use $u_{j,0}$, and $u_{j,\neq}$ to represent the zero mode and non-zero mode of the velocity $u_j(j=1,2,3)$, respectively. Similarly, we use $\omega_{2,0}$ and $\omega_{2,\neq}$ to represent the zero mode and non-zero mode of the vorticity ω_2, respectively.
- The norm of the L^p space and the time-space norm $\|f\|_{L^qL^p}$ are defined as $\|f\|_{L^p(\mathbb{T}^3)}=\left(\int_{\mathbb{T}^3}|f|^pdxdydz\right)^{\frac{1}{p}}$, and $\|f\|_{L^qL^p}=\left\|\|f\|_{L^p(\mathbb{T}^3)}\right\|_{L^q(0,t)}$. Moreover, $\langle\cdot,\cdot\rangle$ denotes the standard L^2 scalar product. For simplicity, we write $\|f\|_{L^p(\mathbb{T}^3)}$ as $\|f\|_{L^p}$.
- For $a>0$, we define the norms
$$\|f\|_{X_a}^2=\|\mathrm{e}^{aA^{-\frac13}t}f\|_{L^\infty L^2}^2+\|\mathrm{e}^{aA^{-\frac13}t}\nabla\triangle^{-1}\partial_xf\|_{L^2L^2}^2+\frac{\|\mathrm{e}^{aA^{-\frac13}t}f\|_{L^2L^2}^2}{A^{\frac13}}+\frac{\|\mathrm{e}^{aA^{-\frac13}t}\nabla f\|_{L^2L^2}^2}{A},$$
$$\|f\|_{Y_0}^2=\|f\|_{L^\infty L^2}^2+\frac{1}{A}\|\nabla f\|_{L^2L^2}^2.$$
- We sometimes denote the partial derivatives ∂_x, ∂_y and ∂_z by ∂_1, ∂_2 and ∂_3, respectively.
- In this paper, unless otherwise specified, we use the Einstein summation convention.
- The total mass $\|n(t)\|_{L^1}$ is denoted by M, and let $m:=\|n_0\|_{L^1}$. Clearly,
$$M:=\|n(t)\|_{L^1}=\|n_{\rm in}\|_{L^1},$$
$$m:=\|(n_{\rm in})_0\|_{L^1}=\frac{\|n\|_{L^1}}{|\mathbb{T}|}=\frac{M}{|\mathbb{T}|}.$$
- Throughout this paper, we denote C by a positive constant independent of A, t and the initial data, and it may be different from line to line.

2. Key ideas and proof of Theorem 1.1

2.1. Fourier analysis. For the given function f, by Fourier series, we get
$$f(t,x,y,z)=\sum_{k_1,k_2,k_3\in\mathbb{Z}}\widehat{f}_{k_1,k_2,k_3}(t)\mathrm{e}^{i(k_1x+k_2y+k_3z)},$$
where $\widehat{f}_{k_1,k_2,k_3}(t)=\frac{1}{|\mathbb{T}|^3}\int_{\mathbb{T}^3}f(t,x,y,z)\mathrm{e}^{-i(k_1x+k_2y+k_3z)}dxdydz$.

According to the frequency k_1, f can be decomposed into
$$f=f_{\neq}+f_0,$$
where
$$f_{\neq}(t,y,z)=\sum_{k_2,k_3\in\mathbb{Z},k_1\neq0}\widehat{f}_{k_1,k_2,k_3}(t)\mathrm{e}^{i(k_1x+k_2y+k_3z)},\ \ f_0(t,y,z)=\sum_{k_2,k_3\in\mathbb{Z}}\widehat{f}_{0,k_2,k_3}(t)\mathrm{e}^{i(k_2y+k_3z)}.$$
In fact, f_0 is the zero mode, and $f_{\neq}$ is the non-zero mode.

Furthermore, according to the frequency k_3, the zero mode f_0 can be decomposed into two parts $f_0 = \overline{f}_0 + \widetilde{f}_0$ satisfying

$$\begin{aligned}
&\overline{f}_0(t) = \widehat{f}_{0,0,0}(t) = \frac{1}{|\mathbb{T}|^3}\int_{\mathbb{T}^3} f(t,x,y,z)dxdydz,\\
&\widetilde{f}_0(t,y,z) = \sum_{k_2^2+k_3^2\neq 0} \widehat{f}_{0,k_2,k_3}(t)\mathrm{e}^{i(k_2y+k_3z)},
\end{aligned} \tag{2.1}$$

where $\overline{f}_0(t)$ is called the average of f in the periodic domain $\mathbb{T}^3$.

Assume that f satisfying

$$\begin{cases} \partial_t f + y\partial_x f - \frac{1}{A}\triangle f = 0,\\ f|_{t=0} = f_{\mathrm{in}}. \end{cases}$$

The function f is divided into three different modes:

$$f(t,x,y,z) = f_{\neq}(t,x,y,z) + \widetilde{f}_0(t,y,z) + \overline{f}_0(t).$$

Then we consider the dynamics for different modes. The non-zero mode $f_{\neq}$ experiences the enhanced dissipation [2]:

$$\|f_{\neq}(t)\|_{L^2} \leq C\mathrm{e}^{aA^{-\frac{1}{3}}t}\|(f_{\mathrm{in}})_{\neq}\|_{L^2}.$$

The non-average part $\widetilde{f}_0$ experiences the heat dissipation with

$$\|\widetilde{f}_0(t)\|_{L^2} \leq \mathrm{e}^{-\frac{t}{2A}}\|\widetilde{f_{\mathrm{in}}}\|_{L^2}.$$

The average part $\overline{f}_0(t)$ is a constant satisfying

$$\overline{f}_0(t) = \overline{(f_{\mathrm{in}})_0}.$$

What we need to emphasize is that the decomposition of different modes and the application of the corresponding dynamics are the keys to research the blow-up suppression problem in the periodic domain $\mathbb{T}^3$.

2.2. **The decomposition of the first component of the velocity.** The first component $u_{1,0}$ is affected by the 3D lift-up effect and is the worst part (see Section 2.1 in [6]). Therefore, we must handle it more carefully and meticulously.

Recall that $u_{1,0}$ satisfies

$$\partial_t u_{1,0} - \frac{1}{A}\triangle u_{1,0} + u_{2,0} = -\frac{u_{2,0}\partial_y u_{1,0} + u_{3,0}\partial_z u_{1,0}}{A} - \frac{(u_{\neq}\cdot\nabla u_{1,\neq})_0}{A} + \frac{n_0}{A}.$$

First of all, we decompose $u_{1,0}$ into $u_{1,0}(t,y,z) = \mathbf{G}_1(t,y,z) + \mathbf{B}_1(t,y,z) + \mathbf{B}_2(t,y,z)$, satisfying

$$\begin{aligned}
&\partial_t\mathbf{G}_1 - \frac{1}{A}\triangle\mathbf{G}_1 = -\frac{u_{2,0}\partial_y\mathbf{G}_1 + u_{3,0}\partial_z\mathbf{G}_1}{A} - \frac{(u_{\neq}\cdot\nabla u_{1,\neq})_0}{A},\\
&\partial_t\mathbf{B}_1 - \frac{1}{A}\triangle\mathbf{B}_1 = -\frac{u_{2,0}\partial_y\mathbf{B}_1 + u_{3,0}\partial_z\mathbf{B}_1}{A} + \frac{n_0}{A},\\
&\partial_t\mathbf{B}_2 - \frac{1}{A}\triangle\mathbf{B}_2 + u_{2,0} = -\frac{u_{2,0}\partial_y\mathbf{B}_2 + u_{3,0}\partial_z\mathbf{B}_2}{A},
\end{aligned} \tag{2.2}$$

along with the initial conditions

$$\mathbf{G}_1|_{t=0} = (u_{1,\mathrm{in}})_0, \quad \mathbf{B}_1|_{t=0} = 0, \quad \mathbf{B}_2|_{t=0} = 0.$$

In this way, $\mathbf{G}_1(t,y,z)$ is a good term satisfying the following energy estimate

$$\|\mathbf{G}_1\|^2_{L^\infty H^1} + \frac{1}{A}\|\nabla \mathbf{G}_1\|^2_{L^2 H^1} \leq C(\|(u_{1,\mathrm{in}})_0\|^2_{H^1} + \epsilon^2).$$

In addition, according to the frequency k_3, we decompose $\mathbf{B}_1(t,y,z)$ and $\mathbf{B}_2(t,y,z)$ into two parts

$$\begin{aligned} \mathbf{B}_1(t,y,z) &= \overline{\mathbf{B}}_1(t) + \widetilde{\mathbf{B}}_1(t,y,z), \\ \mathbf{B}_2(t,y,z) &= \overline{\mathbf{B}}_2(t) + \widetilde{\mathbf{B}}_2(t,y,z), \end{aligned}$$

satisfying

$$\begin{cases} \partial_t \widetilde{\mathbf{B}}_1(t,y,z) - \frac{1}{A}\triangle \widetilde{\mathbf{B}}_1(t,y,z) = -\frac{1}{A}(u_{2,0}\partial_y \widetilde{\mathbf{B}}_1 + u_{3,0}\partial_z \widetilde{\mathbf{B}}_1) + \frac{1}{A}\widetilde{n}_0(t,y,z), \\ \partial_t \overline{\mathbf{B}}_1(t) = \frac{1}{A}\overline{n}_0 = \frac{M}{A|\mathbb{T}|^3}, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \partial_t \widetilde{\mathbf{B}}_2(t,y,z) - \frac{1}{A}\triangle \widetilde{\mathbf{B}}_2(t,y,z) + \widetilde{u}_{2,0}(t,y,z) = -\frac{1}{A}(u_{2,0}\partial_y \widetilde{\mathbf{B}}_2 + u_{3,0}\partial_z \widetilde{\mathbf{B}}_2), \\ \partial_t \overline{\mathbf{B}}_2(t) + \overline{u}_{2,0}(t) = 0. \end{cases} \tag{2.4}$$

Lastly, we rewrite above decompositions into

$$\begin{aligned} \mathbf{U}_1(t,y,z) &= \mathbf{G}_1(t,y,z) + \widetilde{\mathbf{B}}_1(t,y,z), \\ \mathbf{U}_2(t,y,z) &= \widetilde{\mathbf{B}}_2(t,y,z) + \overline{\mathbf{B}}_2(t) + \overline{\mathbf{B}}_1(t), \end{aligned}$$

satisfying $\mathbf{U}_1(t,y,z) + \mathbf{U}_2(t,y,z) = u_{1,0}(t,y,z)$.

In this way, we decompose the velocity $u_{1,0}$ into the good part $\mathbf{U}_1(t,y,z)$ and the bad part $\mathbf{U}_2(t,y,z)$ satisfying

$$\begin{cases} \partial_t \mathbf{U}_1 - \frac{1}{A}\triangle \mathbf{U}_1 = -\frac{1}{A}(u_{2,0}\partial_y \mathbf{U}_1 + u_{3,0}\partial_z \mathbf{U}_1) + \frac{1}{A}\widetilde{n}_0(t,y,z) - \frac{(u_{\neq}\cdot\nabla u_{1,\neq})_0}{A}, \\ \mathbf{U}_1|_{t=0} = (u_{1,\mathrm{in}})_0, \end{cases}$$

and

$$\begin{cases} \partial_t \mathbf{U}_2 - \frac{1}{A}\triangle \mathbf{U}_2 + u_{2,0} = -\frac{1}{A}(u_{2,0}\partial_y \mathbf{U}_2 + u_{3,0}\partial_z \mathbf{U}_2) + \frac{M}{A|\mathbb{T}|^3}, \\ \mathbf{U}_2|_{t=0} = 0. \end{cases}$$

For the good velocity $\mathbf{U}_1$, the H^1 norm is enough for us to finish all calculations, and it has a good relation with the cell density n_0 by

$$\|\mathbf{U}_1\|_{L^\infty H^1} \leq C(\|(u_{1,\mathrm{in}})_0\|_{H^1} + \|n_0\|_{L^\infty L^2} + \epsilon).$$

For the bad velocity $\mathbf{U}_2$, when it satisfies $\frac{\|\triangle \mathbf{U}_2\|_{L^\infty H^2}}{A} + \|\partial_t \mathbf{U}_2\|_{L^\infty L^\infty} < c$ for some small c, the operator $\mathcal{L}_V$ can be regarded as perturbation of $\mathcal{L}$, which allow us to finish the time-space estimates for the zero modes. Here, $\mathcal{L}_V$ and $\mathcal{L}$ can be found in (5.10).

2.3. **The construction of energy functional.** Firstly, we introduce the energy functional

$$
\begin{aligned}
E_{1,1}(t) &= \|u_{2,0}\|_{Y_0} + \|u_{3,0}\|_{Y_0} + \|\nabla u_{2,0}\|_{Y_0} + \|\nabla u_{3,0}\|_{Y_0} + \|\triangle u_{2,0}\|_{Y_0} \\
&\quad + \|\min\{(A^{-\frac{2}{3}} + A^{-1}t)^{\frac{1}{2}}, 1\}\triangle u_{3,0}\|_{Y_0}, \\
E_{1,2}(t) &= A^{-1}(\|\triangle \mathbf{U}_2\|_{L^\infty H^2} + A^{-\frac{1}{2}}\|\nabla\triangle \mathbf{U}_2\|_{L^2H^2}) + \|\partial_t \mathbf{U}_2\|_{L^\infty H^2}, \\
E_{2,1}(t) &= \|\partial_x^2 n_{\neq}\|_{X_b}, \\
E_{2,2}(t) &= \|\triangle u_{2,\neq}\|_{X_a} + \|\partial_x \omega_{2,\neq}\|_{X_a} + A^{-\frac{1}{3}}\left(\|\partial_y \omega_{2,\neq}\|_{X_a} + \|\partial_z \omega_{2,\neq}\|_{X_a}\right), \\
E_3(t) &= \|n\|_{L^\infty L^\infty}, \\
E_4(t) &= \|\partial_x^2 u_{2,\neq}\|_{X_b} + \|\partial_x^2 u_{3,\neq}\|_{X_b},
\end{aligned}
$$

where a and b are given positive constants with $0 < a < b < 2a$. Moreover, we denote

$$E_1(t) = E_{1,1}(t) + E_{1,2}(t), \quad E_2(t) = E_{2,1}(t) + E_{2,2}(t).$$

To close the energy, when estimating the nonlinear interactions between the non-zero modes of the velocity of $\|e^{2aA^{-\frac{1}{3}}t}\nabla(u_{\neq}\cdot\nabla u_{3,\neq})\|_{L^2L^2}$, it is required that the degree of A must be strictly less than $\frac{5}{3}$ (see Lemma 3.2). To this end, we introduce the first auxiliary energy

$$E_{5,1}(t) = A^{-\frac{2}{3}}\|\triangle u_{3,\neq}\|_{X_b}.$$

Besides, to estimate E_4 and handle the nonlinear interaction between the bad component $\mathbf{U}_2$ of $u_{1,0}$ and the non-zero mode $u_{\neq}$, we also need the second auxiliary energy

$$E_{5,2}(t) = \sum_{j=2}^{3}\left(\|\partial_x^2 u_{j,\neq}\|_{X_b} + \|\partial_x(\partial_z - \kappa\partial_y)u_{j,\neq}\|_{X_b}\right) + \|\partial_x \nabla W\|_{X_b},$$

where the definition of κ and the expression of W can be found in Section 8.

Let us briefly describe the roles of the above-mentioned energy functionals.

- E_1 is introduced to control the zero modes of the velocity u_0. $E_{1,1}$ is used to control the good velocities $u_{2,0}$ and $u_{3,0}$. $E_{1,2}$ is used to control the bad part $\mathbf{U}_2$ of the velocity $u_{1,0}$.
- E_2 is introduced to deal with the non-zero mode of the cell density $n_{\neq}$ and the velocity $u_{\neq}$.
- E_3 is introduced to estimate $\|n_0(t)\|_{L^2}$ and deal with the nonlinear interaction items in the density n.
- E_4 is introduced to deal with the nonlinear interaction items with 3D lift-up effect in $u_{\neq}$.

2.4. **Main steps.**

Proof of Theorem 1.1. We prove Theorem 1.1 in the following three steps.

Step 1: Let's designate T as the terminal point of the largest range $[0, T]$ such that the following hypothesis hold

$$E_1(t) \le 2E_1, \quad E_2(t) \le 2E_2, \quad E_3(t) \le 2E_3, \quad E_4(t) \le 2E_4, \quad E_{5,1}(t) \le 2E_5 \tag{2.5}$$

for any $t \in [0, T]$, where E_1, E_2, E_3, E_4 and E_5 are constants independent of t and A, and will be decided during the calculation.

Step 2: We need to prove the following propositions:

Proposition 2.1. *Assume that the initial data* $(n_{\rm in}, u_{\rm in})$ *satisfy the assumptions of Theorem 1.1 and (2.5), there exists a positive constant* A_2 *independent of* A *and* t, *such that if* $A \geq A_2$, *there holds*

$$E_1(t) \leq E_1$$

for all $t \in [0, T]$.

Proposition 2.2. *Assume that the initial data* $(n_{\rm in}, u_{\rm in})$ *satisfy the assumptions of Theorem 1.1 and (2.5), there exists a positive constant* A_5 *independent of* A *and* t, *such that if* $A \geq A_5$, *there holds*

$$E_3(t) \leq E_3$$

for all $t \in [0, T]$.

Proposition 2.3. *Assume that the initial data* $(n_{\rm in}, u_{\rm in})$ *satisfy the assumptions of Theorem 1.1 and (2.5), there exists a positive constant* A_7 *independent of* A *and* t, *such that if* $A \geq A_7$, *there holds*

$$\begin{aligned} E_4(t) + E_{5,1}(t) &\leq E_4 + E_5, \\ E_2(t) &\leq E_2, \end{aligned}$$

for all $t \in [0, T]$.

Step 3: Combining the above propositions with the well-posedness of system (1.5) given by Remark 1.4, and taking $A_1 = \max\{A_2, A_5, A_7\}$, we complete the proof.

□

3. A priori estimates

Firstly, we give some relationships between velocity $u_{\neq}$ and the new vorticity $\omega_{2,\neq}$, which will be frequently used in the later calculations. Since div $u_{\neq} = 0$, the result follows immediately from Fourier series (see Lemma 3.13 in [10]), we omit the proof.

Lemma 3.1. *There holds*

$$\begin{aligned} &\|(\partial_x, \partial_z)\partial_x u_{\neq}\|_{L^2} \leq C(\|\partial_x \omega_{2,\neq}\|_{L^2} + \|\triangle u_{2,\neq}\|_{L^2}), \\ &\|(\partial_x, \partial_z)\partial_y u_{\neq}\|_{L^2} \leq C(\|\partial_y \omega_{2,\neq}\|_{L^2} + \|\triangle u_{2,\neq}\|_{L^2}), \\ &\|(\partial_x, \partial_z)\partial_z u_{\neq}\|_{L^2} \leq C(\|\partial_z \omega_{2,\neq}\|_{L^2} + \|\triangle u_{2,\neq}\|_{L^2}), \\ &\left\|(\partial_x^2, \partial_z^2) u_{3,\neq}\right\|_{L^2} \leq C(\|\partial_x \omega_{2,\neq}\|_{L^2} + \|\triangle u_{2,\neq}\|_{L^2}), \\ &\|(\partial_x, \partial_z)\partial_x \nabla u_{\neq}\|_{L^2} \leq C(\|\partial_x \nabla \omega_{2,\neq}\|_{L^2} + \|\nabla \triangle u_{2,\neq}\|_{L^2}), \\ &\|(\partial_x, \partial_z)\partial_y \nabla u_{\neq}\|_{L^2} \leq C(\|\partial_y \nabla \omega_{2,\neq}\|_{L^2} + \|\nabla \triangle u_{2,\neq}\|_{L^2}). \end{aligned} \tag{3.1}$$

Secondly, we present the nonlinear interactions between the non-zero modes of the velocity, which will often be used to estimate E_1, E_2 and E_4.

Lemma 3.2. *It holds that*

$$\begin{aligned}
\|e^{2aA^{-\frac{1}{3}}t}|u_{\neq}|^2\|^2_{L^2L^2} &\leq CA^{\frac{2}{3}}E_2^4,\\
\|e^{2aA^{-\frac{1}{3}}t}u_{\neq}\cdot\nabla u_{\neq}\|^2_{L^2L^2} &\leq CA^{\frac{2}{3}}E_2^4,\\
\|e^{2aA^{-\frac{1}{3}}t}\partial_x(u_{\neq}\cdot\nabla u_{\neq})\|^2_{L^2L^2} &\leq CA^{\frac{1}{6}+\alpha}E_2^4,\\
\|e^{2aA^{-\frac{1}{3}}t}\nabla(u_{\neq}\cdot\nabla u_{2,\neq})\|^2_{L^2L^2} &\leq CA^{\frac{1}{2}+\frac{2}{3}\alpha}E_2^4,\\
\|e^{2aA^{-\frac{1}{3}}t}\partial_z(u_{\neq}\cdot\nabla u_{3,\neq})\|^2_{L^2L^2} &\leq CA^{\frac{1}{2}+\frac{2}{3}\alpha}E_2^4,\\
\|e^{2aA^{-\frac{1}{3}}t}\partial_z(u_{\neq}\cdot\nabla u_{1,\neq})\|^2_{L^2L^2} &\leq CA^{\frac{4}{3}}E_2^4,\\
\|e^{2aA^{-\frac{1}{3}}t}\nabla(u_{\neq}\cdot\nabla u_{3,\neq})\|^2_{L^2L^2} &\leq C(A^{\frac{7}{6}+\frac{1}{3}\alpha}E_2^2E_5^2+A^{\frac{4}{3}}E_2^4),
\end{aligned}\tag{3.2}$$

where α is a constant with $\alpha\in(\frac{1}{2},\frac{3}{4})$.

Proof. For convenience, we denote by

$$\begin{aligned}
\Gamma_1 &= \|\partial_x\omega_{2,\neq}\|_{L^2}+\|\triangle u_{2,\neq}\|_{L^2},\\
\Gamma_2 &= \|\nabla\omega_{2,\neq}\|_{L^2}+\|\triangle u_{2,\neq}\|_{L^2},\\
\Gamma_3 &= \|\partial_x\nabla\omega_{2,\neq}\|_{L^2}+\|\nabla\triangle u_{2,\neq}\|_{L^2},\\
\Gamma_4 &= \|\partial_x\nabla u_{2,\neq}\|_{L^2}.
\end{aligned}$$

According to the definition of $E_2(t)$ and assumptions (2.5), there holds

$$\|e^{aA^{-\frac{1}{3}}t}\Gamma_1\|_{L^\infty}+\frac{\|e^{aA^{-\frac{1}{3}}t}\Gamma_1\|_{L^2}}{A^{\frac{1}{6}}}+\frac{\|e^{aA^{-\frac{1}{3}}t}(\Gamma_2,\Gamma_3)\|_{L^2}}{A^{\frac{1}{2}}}+\frac{\|e^{aA^{-\frac{1}{3}}t}\Gamma_2\|_{L^\infty}}{A^{\frac{1}{3}}}+\|e^{aA^{-\frac{1}{3}}t}\Gamma_4\|_{L^2}\leq CE_2,$$

which will be frequently used in later calculations.

Estimate of $(3.2)_1$. Using $(A.2)_5$ in Lemma A.3 and $(3.1)_1$ in Lemma 3.1, we have

$$\|u_{\neq}\|^2_{L^\infty_{x,z}L^2_y}\leq C\big(\|\partial_x u_{\neq}\|^{2\alpha}_{L^2}\|u_{\neq}\|^{2-2\alpha}_{L^2}+\|\partial_x\partial_z u_{\neq}\|^{2\alpha}_{L^2}\|u_{\neq}\|^{2-2\alpha}_{L^2}\big)\leq C\Gamma_1^2.$$

Therefore, by $(A.2)_8$ and $(3.1)_2$, we arrive

$$\||u_{\neq}|^2\|^2_{L^2}\leq C\big(\|\partial_x u_{\neq}\|^{2\alpha}_{L^2}\|u_{\neq}\|^{2-2\alpha}_{L^2}+\|\partial_x\partial_z u_{\neq}\|^{2\alpha}_{L^2}\|u_{\neq}\|^{2-2\alpha}_{L^2}\big)\|\nabla u_{\neq}\|_{L^2}\|u_{\neq}\|_{L^2}\leq C\Gamma_1^3\Gamma_2,$$

which indicates that

$$\|e^{2aA^{-\frac{1}{3}}t}|u_{\neq}|^2\|^2_{L^2L^2}\leq CA^{\frac{2}{3}}(\|\partial_x\omega_{2,\neq}\|^4_{X_a}+\|\triangle u_{2,\neq}\|^4_{X_a})\leq CA^{\frac{2}{3}}E_2^4.$$

Estimate of $(3.2)_2$. Using Lemma A.3 and Lemma 3.1 again, there hold

$$\begin{aligned}
&\|u_{2,\neq}\|^2_{L^\infty_{x,y}L^2_z}\leq C\|\partial_x u_{2,\neq}\|_{L^2}\|\partial_x\nabla u_{2,\neq}\|_{L^2}\leq C\Gamma_4^2,\\
&\|u_{\neq}\|^2_{L^\infty_{x,z}L^2_y}\leq C\big(\|\partial_x u_{\neq}\|^2_{L^2}+\|(\partial_x,\partial_z)\partial_x u_{\neq}\|^2_{L^2}\big)\leq C\Gamma_1^2,\\
&\|\partial_y u_{\neq}\|^2_{L^\infty_x L^2_{y,z}}+\|\partial_y u_{\neq}\|^2_{L^\infty_z L^2_{x,y}}\leq C\Gamma_2^2,\\
&\|(\partial_x,\partial_z)u_{\neq}\|^2_{L^\infty_y L^2_{x,z}}\leq C\Gamma_1\Gamma_2.
\end{aligned}$$

Therefore, one gets

$$\begin{aligned}\|u_{\neq}\cdot\nabla u_{\neq}\|_{L^2}^2 \leq&\|u_{1,\neq}\partial_x u_{\neq}\|_{L^2}^2+\|u_{2,\neq}\partial_y u_{\neq}\|_{L^2}^2+\|u_{3,\neq}\partial_z u_{\neq}\|_{L^2}^2\\ \leq&\|u_{\neq}\|_{L^\infty_{x,z}L^2_y}^2\|(\partial_x,\partial_z)u_{\neq}\|_{L^\infty_y L^2_{x,z}}^2+\|u_{2,\neq}\|_{L^\infty_{x,y}L^2_z}^2\|\partial_y u_{\neq}\|_{L^\infty_z L^2_{x,y}}^2\\ \leq&C(\Gamma_1^3\Gamma_2+\Gamma_2^2\Gamma_4^2),\end{aligned}$$

which implies that

$$\|\mathrm{e}^{2aA^{-\frac13}t}u_{\neq}\cdot\nabla u_{\neq}\|_{L^2L^2}^2\leq CA^{\frac23}E_2^4.$$

Estimate of $(3.2)_3$. For $j\in\{1,3\}$, by Lemma A.3 and Lemma 3.1, we have

$$\|u_{j,\neq}\|_{L^\infty_{x,z}L^2_y}^2+\|\partial_x u_{j,\neq}\|_{L^\infty_z L^2_{x,y}}^2\leq C\|\partial_x(\partial_x,\partial_z)u_{j,\neq}\|_{L^2}^2\leq C\Gamma_1^2$$

and

$$\|\partial_x\partial_j u_{\neq}\|_{L^\infty_y L^2_{x,z}}^2+\|\partial_j u_{\neq}\|_{L^\infty_{x,y}L^2_z}^2\leq C\|\partial_x\partial_j\nabla u_{\neq}\|_{L^2}\|\partial_x\partial_j u_{\neq}\|_{L^2}\leq C\Gamma_1\Gamma_3,$$

which indicate that

$$\|\partial_x(u_{j,\neq}\partial_j u_{\neq})\|_{L^2}^2\leq C\Gamma_1^3\Gamma_3. \tag{3.3}$$

Using $(A.2)_1$, $(A.2)_4$ and $(A.2)_7$, there hold

$$\|\partial_x u_{2,\neq}\|_{L^\infty_{x,y}L^2_z}^2+\|u_{2,\neq}\|_{L^\infty}^2\leq C\|\partial_x\nabla u_{2,\neq}\|_{L^2}^{3-2\alpha}\|\nabla\triangle u_{2,\neq}\|_{L^2}^{2\alpha-1}\leq C\Gamma_4^{3-2\alpha}\Gamma_3^{2\alpha-1}$$

and

$$\|\partial_y u_{\neq}\|_{L^\infty_z L^2_{x,y}}^2\leq C\|(\partial_x,\partial_z)\partial_y u_{\neq}\|_{L^2}^2\leq C\Gamma_2^2,$$

which show that

$$\|\partial_x(u_{2,\neq}\partial_y u_{\neq})\|_{L^2}^2\leq C\Gamma_2^2\Gamma_3^{2\alpha-1}\Gamma_4^{3-2\alpha}. \tag{3.4}$$

Combining (3.3) with (3.4), we conclude that

$$\|\mathrm{e}^{2aA^{-\frac13}t}\partial_x(u_{\neq}\cdot\nabla u_{\neq})\|_{L^2L^2}^2\leq CA^{\frac16+\alpha}E_2^4.$$

Estimate of $(3.2)_4$. By Lemma A.3 and Lemma 3.1, direct calculations show that

$$\begin{aligned}&\|u_{\neq}\|_{L^\infty}^2\leq C(\|\partial_x\omega_{2,\neq}\|_{L^2}+\|\triangle u_{2,\neq}\|_{L^2})(\|\nabla\omega_{2,\neq}\|_{L^2}+\|\triangle u_{2,\neq}\|_{L^2})\leq C\Gamma_1\Gamma_2,\\ &\|\nabla u_{\neq}\|_{L^\infty_z L^2_{x,y}}^2+\|\nabla u_{\neq}\|_{L^\infty_x L^2_{y,z}}^2\leq C\Gamma_2^2,\\ &\|\nabla u_{2,\neq}\|_{L^\infty_{x,y}L^2_z}^2\leq C\|\partial_x\nabla u_{2,\neq}\|_{L^2}\|\nabla\triangle u_{2,\neq}\|_{L^2}^{2\alpha-1}\|\triangle u_{2,\neq}\|_{L^2}^{2-2\alpha}\leq C\Gamma_4\Gamma_3^{2\alpha-1}\Gamma_1^{2-2\alpha}.\end{aligned} \tag{3.5}$$

Combining above results with

$$\begin{aligned}\|\nabla(u_{\neq}\cdot\nabla u_{2,\neq})\|_{L^2}^2&\leq C(\|\nabla u_{\neq}\|_{L^\infty_z L^2_{x,y}}^2\|\nabla u_{2,\neq}\|_{L^\infty_{x,y}L^2_z}^2+\|u_{\neq}\|_{L^\infty}^2\|\triangle u_{2,\neq}\|_{L^2}^2)\\ &\leq C(\Gamma_4\Gamma_3^{2\alpha-1}\Gamma_1^{2-2\alpha}\Gamma_2^2+\Gamma_1^3\Gamma_2),\end{aligned}$$

there holds

$$\|\mathrm{e}^{2aA^{-\frac13}t}\nabla(u_{\neq}\cdot\nabla u_{2,\neq})\|_{L^2L^2}^2\leq A^{\frac12+\frac23\alpha}E_2^4.$$

Estimate of $(3.2)_5$. For $j\in\{1,3\}$, using Lemma A.3 and Lemma 3.1, we have

$$\|\partial_z u_{j,\neq}\|_{L^\infty_x L^2_{y,z}}^2\leq C\Gamma_1^2,\quad \|\partial_j u_{3,\neq}\|_{L^\infty_{y,z}L^2_x}^2\leq C\Gamma_1\Gamma_3,$$

which along with $(3.5)_1$ indicates that

$$\|\partial_z(u_{j,\neq}\partial_j u_{3,\neq})\|_{L^2}^2\leq C\Gamma_1^3\Gamma_3. \tag{3.6}$$

For $j=2$, by Lemma 3.1 and Lemma A.3, there holds

$$\begin{aligned}&\|u_{2,\neq}\|_{L^\infty}^2\le C\|\partial_x\nabla u_{2,\neq}\|_{L^2}^{3-2\alpha}\|\triangle u_{2,\neq}\|_{L^2}^{2\alpha-1}\le C\Gamma_4^{3-2\alpha}\Gamma_1^{2\alpha-1},\\&\|\partial_y u_{3,\neq}\|_{L^\infty_zL^2_{x,y}}^2\le C\|(\partial_x,\partial_z)\nabla u_{3,\neq}\|_{L^2}^2\le C\Gamma_2^2,\end{aligned}\tag{3.7}$$

which along with $(3.1)_2$ and $(3.5)_3$ shows that

$$\|\partial_z(u_{2,\neq}\partial_y u_{3,\neq})\|_{L^2}^2\le C(\Gamma_4\Gamma_3^{2\alpha-1}\Gamma_1^{2-2\alpha}\Gamma_2^2+\Gamma_4^{3-2\alpha}\Gamma_1^{2\alpha-1}\Gamma_2^2).\tag{3.8}$$

Combining (3.6) and (3.8), we obtain that

$$\|\mathrm{e}^{2aA^{-\frac13}t}\partial_z(u_{\neq}\cdot\nabla u_{3,\neq})\|_{L^2L^2}^2\le A^{\frac12+\frac23\alpha}E_2^4.$$

Estimate of $(3.2)_6$. According to (3.7) and (3.8), after replacing $u_{3,\neq}$ with $u_{1,\neq}$, we can prove that

$$\|\partial_z(u_{2,\neq}\partial_y u_{1,\neq})\|_{L^2}^2\le C(\Gamma_4\Gamma_3^{2\alpha-1}\Gamma_1^{2-2\alpha}\Gamma_2^2+\Gamma_4^{3-2\alpha}\Gamma_1^{2\alpha-1}\Gamma_2^2).\tag{3.9}$$

Similar to (3.6), for $j\in\{1,3\}$, by using $(3.1)_3$, one deduces

$$\|\partial_z(u_{j,\neq}\partial_j u_{1,\neq})\|_{L^2}^2\le C\Gamma_1\Gamma_2^2\Gamma_3.\tag{3.10}$$

Collecting (3.9) and (3.10), there holds

$$\|\mathrm{e}^{2aA^{-\frac13}t}\partial_z(u_{\neq}\cdot\nabla u_{1,\neq})\|_{L^2L^2}^2\le CA^{\frac43}E_2^4.$$

Estimate of $(3.2)_7$. For $j\in\{1,3\}$, using Lemma A.3 and Lemma 3.1 again, we have

$$\|\partial_j u_{3,\neq}\|_{L^\infty_{y,z}L^2_x}^2\le C\|\partial_z\partial_j\nabla u_{3,\neq}\|_{L^2}\|\partial_z\partial_j u_{3,\neq}\|_{L^2}\le C\Gamma_1\Gamma_3,$$

which along with $(3.5)_1$ and $(3.5)_2$ indicates that

$$\begin{aligned}\|\nabla(u_{j,\neq}\partial_j u_{3,\neq})\|_{L^2}^2&\le C(\|u_{j,\neq}\|_{L^\infty}^2\|\nabla\partial_j u_{3,\neq}\|_{L^2}^2+\|\nabla u_{j,\neq}\|_{L^\infty_xL^2_{y,z}}^2\|\partial_j u_{3,\neq}\|_{L^\infty_{y,z}L^2_x}^2)\\&\le C(\Gamma_1\Gamma_2^3+\Gamma_2^2\Gamma_1\Gamma_3).\end{aligned}\tag{3.11}$$

According to $(3.5)_3$ and (3.7), one obtains

$$\begin{aligned}\|\nabla(u_{2,\neq}\partial_y u_{3,\neq})\|_{L^2}^2&\le C(\|u_{2,\neq}\|_{L^\infty}^2\|\partial_y\nabla u_{3,\neq}\|_{L^2}^2+\|\nabla u_{2,\neq}\|_{L^\infty_{x,y}L^2_z}^2\|\partial_y u_{3,\neq}\|_{L^\infty_zL^2_{x,y}}^2)\\&\le C(\Gamma_4^{3-2\alpha}\Gamma_1^{2\alpha-1}\|\triangle u_{3,\neq}\|_{L^2}^2+\Gamma_4\Gamma_3^{2\alpha-1}\Gamma_1^{2-2\alpha}\Gamma_2^2).\end{aligned}\tag{3.12}$$

Therefore, we infer from (3.11) and (3.12) that

$$\|\mathrm{e}^{2aA^{-\frac13}t}\nabla(u_{\neq}\cdot\nabla u_{3,\neq})\|_{L^2L^2}^2\le C(A^{\frac76+\frac13\alpha}E_2^2E_5^2+A^{\frac43}E_2^4).$$

We finish the proof. □

Next, we give the nonlinear interaction between the zero mode $u_{1,0}$ and the non-zero mode $u_{\neq}$. Recall that the zero mode $u_{1,0}$ is decomposed into two components with $u_{1,0}=\mathbf{U}_1+\mathbf{U}_2$, where $\mathbf{U}_1$ is the good component and $\mathbf{U}_2$ is the bad component.

The following result will be used in estimating the energies $\{\|\triangle u_{2,\neq}\|_{X_a}^2,\|\nabla\omega_{2,\neq}\|_{X_a}^2\}$. Consequently, to close the energy estimates, the degree of A must be strictly less than 1.

Lemma 3.3. *It holds that*

$$\begin{aligned}
&\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\mathbf{U}_1\partial_x\nabla u_{2,\neq}\|_{L^2L^2}^2 \leq CA^{\frac{1}{2}}\|\mathbf{U}_1\|_{L^\infty H^1}^2\|\triangle u_{2,\neq}\|_{X_a}^2,\\
&\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\mathbf{U}_1\partial_x(\partial_x,\partial_z)u_{3,\neq}\|_{L^2L^2}^2 \leq CA^{\frac{2}{3}}\|\mathbf{U}_1\|_{L^\infty H^1}^2(\|\partial_x\omega_{2,\neq}\|_{X_a}^2+\|\triangle u_{2,\neq}\|_{X_a}^2),\\
&\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\mathbf{U}_1\partial_x^2 u_{1,\neq}\|_{L^2L^2}^2 \leq CA^{\frac{2}{3}}\|\mathbf{U}_1\|_{L^\infty H^1}^2(\|\partial_x\omega_{2,\neq}\|_{X_a}^2+\|\triangle u_{2,\neq}\|_{X_a}^2),\\
&\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\mathbf{U}_1\partial_x\partial_z u_{1,\neq}\|_{L^2L^2}^2 \leq CA^{\frac{2}{3}}\|\mathbf{U}_1\|_{L^\infty H^1}^2\left(\|\partial_x\omega_{2,\neq}\|_{X_a}^2+\|\triangle u_{2,\neq}\|_{X_a}^2\right),\\
&\|\mathrm{e}^{aA^{-\frac{1}{3}}t}(\partial_y,\partial_z)\mathbf{U}_1\partial_x u_{\neq}\|_{L^2L^2}^2 \leq CA^{\frac{2}{3}}\|\mathbf{U}_1\|_{L^\infty H^1}^2(\|\partial_x\omega_{2,\neq}\|_{X_a}^2+\|\triangle u_{2,\neq}\|_{X_a}^2),\\
&\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\partial_y\mathbf{U}_1\partial_y u_{2,\neq}\|_{L^2L^2}^2 \leq CA^{\frac{2}{3}}\|\mathbf{U}_1\|_{L^\infty H^1}^2\|\triangle u_{2,\neq}\|_{X_a}^2.
\end{aligned}\tag{3.13}$$

Proof. Using Lemma A.2 and Lemma A.3, we have

$$\begin{aligned}
\|\mathbf{U}_1\partial_x\nabla u_{2,\neq}\|_{L^2}^2 &\leq \|\mathbf{U}_1\|_{L_z^\infty L_y^2}^2\|\partial_x\nabla u_{2,\neq}\|_{L_y^\infty L_{x,z}^2}^2\\
&\leq C\|\mathbf{U}_1\|_{H^1}^2(\|\partial_x\nabla u_{2,\neq}\|_{L^2}\|\partial_x\partial_y\nabla u_{2,\neq}\|_{L^2}+\|\partial_x\nabla u_{2,\neq}\|_{L^2}^2),
\end{aligned}$$

which implies that

$$\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\mathbf{U}_1\partial_x\nabla u_{2,\neq}\|_{L^2L^2}^2 \leq CA^{\frac{1}{2}}\|\mathbf{U}_1\|_{L^\infty H^1}^2\|\triangle u_{2,\neq}\|_{X_a}^2.$$

For $j\in\{1,3\}$, by Lemma A.2, Lemma A.3 and Lemma 3.1, there holds

$$\begin{aligned}
&\|\mathbf{U}_1\partial_x(\partial_x,\partial_z)u_{j,\neq}\|_{L^2}^2\\
&\leq C\|\mathbf{U}_1\|_{H^1}^2\|\partial_x(\partial_x,\partial_z)u_{j,\neq}\|_{L^2}\|\partial_x(\partial_x,\partial_z)\nabla u_{j,\neq}\|_{L^2}\\
&\leq C\|\mathbf{U}_1\|_{H^1}^2(\|\partial_x\omega_{2,\neq}\|_{L^2}+\|\triangle u_{2,\neq}\|_{L^2})(\|\partial_x\nabla\omega_{2,\neq}\|_{L^2}+\|\nabla\triangle u_{2,\neq}\|_{L^2})
\end{aligned}$$

and

$$\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\mathbf{U}_1\partial_x(\partial_x,\partial_z)u_{j,\neq}\|_{L^2L^2}^2 \leq CA^{\frac{2}{3}}\|\mathbf{U}_1\|_{L^\infty H^1}^2(\|\partial_x\omega_{2,\neq}\|_{X_a}^2+\|\triangle u_{2,\neq}\|_{X_a}^2),$$

which give $(3.13)_2$, $(3.13)_3$ and $(3.13)_4$.

Using Lemma A.3, we obtain that

$$\begin{aligned}
\|(\partial_y,\partial_z)\mathbf{U}_1\partial_x u_{\neq}\|_{L^2}^2 &\leq \|\nabla\mathbf{U}_1\|_{L^2}^2\|\partial_x u_{\neq}\|_{L_{y,z}^\infty L_x^2}^2\\
&\leq C\|\nabla\mathbf{U}_1\|_{L^2}^2\|\partial_x(\partial_x,\partial_z)u_{\neq}\|_{L^2}\|\partial_x(\partial_x,\partial_z)\nabla u_{\neq}\|_{L^2}\\
&\leq C\|\nabla\mathbf{U}_1\|_{L^2}^2(\|\partial_x\omega_{2,\neq}\|_{L^2}+\|\triangle u_{2,\neq}\|_{L^2})(\|\partial_x\nabla\omega_{2,\neq}\|_{L^2}+\|\nabla\triangle u_{2,\neq}\|_{L^2}),
\end{aligned}$$

which indicates that

$$\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\nabla\mathbf{U}_1\partial_x u_{\neq}\|_{L^2L^2}^2 \leq CA^{\frac{2}{3}}\|\mathbf{U}_1\|_{L^\infty H^1}^2(\|\partial_x\omega_{2,\neq}\|_{X_a}^2+\|\triangle u_{2,\neq}\|_{X_a}^2).$$

Using Lemma A.3, we obtain that

$$\begin{aligned}
\|\partial_y\mathbf{U}_1\partial_y u_{2,\neq}\|_{L^2}^2 &\leq \|\nabla\mathbf{U}_1\|_{L^2}^2\|\partial_y u_{2,\neq}\|_{L_{y,z}^\infty L_x^2}^2\\
&\leq C\|\nabla\mathbf{U}_1\|_{L^2}^2\|\triangle u_{2,\neq}\|_{L^2}\|\nabla\triangle u_{2,\neq}\|_{L^2}
\end{aligned}$$

and

$$\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\partial_y\mathbf{U}_1\partial_y u_{2,\neq}\|_{L^2L^2}^2 \leq CA^{\frac{2}{3}}\|\mathbf{U}_1\|_{L^\infty H^1}^2\|\triangle u_{2,\neq}\|_{X_a}^2.$$

□

The following result is only used to estimate $\|(\partial_y,\partial_z)\omega_{2,\neq}\|^2_{X_a}$. Hence, as long as the degree of A is less than $\frac{5}{3}$, the energy estimates become achievable. Sometimes, we need to use the results of Lemma 4.1, and we will prove them in next section.

Lemma 3.4. *Under the conditions of Theorem 1.1 and the assumptions (2.5), there exists a constant A_3 independent of A and t, such that if $A>A_3$, it holds that*

$$\begin{aligned}&\|\mathrm{e}^{aA^{-\frac{1}{3}}t}(\partial_y,\partial_z)(u_{2,\neq}\nabla\mathbf{U}_1)\|^2_{L^2L^2}\leq C\big(A^{\frac{2}{3}}\|\mathbf{U}_1\|^2_{L^\infty H^1}E_2^2+AE_2^{2\alpha}E_4^{2-2\alpha}\big),\\&\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\partial_z(u_{3,\neq}\nabla\mathbf{U}_1)\|^2_{L^2L^2}\leq C\big(A^{\frac{2}{3}}\|\mathbf{U}_1\|^2_{L^\infty H^1}E_2^2+A^{\frac{4}{3}}E_2^{2\alpha}E_4^{2-2\alpha}\big),\end{aligned}\tag{3.14}$$

where α is a constant with $\alpha\in(\frac{1}{2},\frac{3}{4})$.

Proof. First, for $j\in\{2,3\}$, direct calculations show that

$$\begin{aligned}&\|\partial_j(u_{2,\neq}\nabla\mathbf{U}_1)\|^2_{L^2}\leq C\left(\|\partial_ju_{2,\neq}\nabla\mathbf{U}_1\|^2_{L^2}+\|u_{2,\neq}\partial_j\nabla\mathbf{U}_1\|^2_{L^2}\right),\\&\|\partial_z(u_{3,\neq}\nabla\mathbf{U}_1)\|^2_{L^2}\leq C\left(\|\partial_zu_{3,\neq}\nabla\mathbf{U}_1\|^2_{L^2}+\|u_{3,\neq}\partial_z\nabla\mathbf{U}_1\|^2_{L^2}\right).\end{aligned}\tag{3.15}$$

By Lemma A.3, there hold

$$\|\partial_ju_{2,\neq}\nabla\mathbf{U}_1\|^2_{L^2}\leq C\|\nabla\mathbf{U}_1\|^2_{L^2}\|\triangle u_{2,\neq}\|_{L^2}\|\nabla\triangle u_{2,\neq}\|_{L^2}$$

and

$$\begin{aligned}\|\partial_zu_{3,\neq}\nabla\mathbf{U}_1\|^2_{L^2}&\leq C\|\nabla\mathbf{U}_1\|^2_{L^2}\|\partial_x(\partial_x,\partial_z)u_{3,\neq}\|_{L^2}\|\partial_x(\partial_x,\partial_z)\partial_yu_{3,\neq}\|_{L^2}\\&\leq C\|\mathbf{U}_1\|^2_{H^1}(\|\partial_x\omega_{2,\neq}\|_{L^2}+\|\triangle u_{2,\neq}\|_{L^2})(\|\partial_x\nabla\omega_{2,\neq}\|_{L^2}+\|\nabla\triangle u_{2,\neq}\|_{L^2}),\end{aligned}$$

which indicate that

$$\begin{aligned}&\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\partial_ju_{2,\neq}\nabla\mathbf{U}_1\|^2_{L^2L^2}\leq CA^{\frac{2}{3}}\|\mathbf{U}_1\|^2_{L^\infty H^1}\|\triangle u_{2,\neq}\|^2_{X_a}\leq CA^{\frac{2}{3}}\|\mathbf{U}_1\|^2_{L^\infty H^1}E_2^2,\\&\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\partial_zu_{3,\neq}\nabla\mathbf{U}_1\|^2_{L^2L^2}\leq CA^{\frac{2}{3}}\|\mathbf{U}_1\|^2_{L^\infty H^1}(\|\partial_x\omega_{2,\neq}\|^2_{X_a}+\|\triangle u_{2,\neq}\|^2_{X_a})\leq CA^{\frac{2}{3}}\|\mathbf{U}_1\|^2_{L^\infty H^1}E_2^2.\end{aligned}\tag{3.16}$$

Then, we need to deal with $\|\mathrm{e}^{aA^{-\frac{1}{3}}t}u_{2,\neq}\partial_j\nabla\mathbf{U}_1\|^2_{L^2L^2}$ and $\|\mathrm{e}^{aA^{-\frac{1}{3}}t}u_{3,\neq}\partial_z\nabla\mathbf{U}_1\|^2_{L^2L^2}$. Since it is difficult to estimate $\|\partial_j\nabla\mathbf{U}_1\|_{L^\infty L^2}$, the traditional Sobolev embedding is insufficient to handle $\|\mathrm{e}^{aA^{-\frac{1}{3}}t}u_{2,\neq}\partial_j\nabla\mathbf{U}_1\|^2_{L^2L^2}$ and $\|\mathrm{e}^{aA^{-\frac{1}{3}}t}u_{3,\neq}\partial_z\nabla\mathbf{U}_1\|^2_{L^2L^2}$.

According to (2.2) and (2.3), $\mathbf{U}_1$ satisfies

$$\begin{cases}\partial_t\mathbf{U}_1-\frac{1}{A}\triangle\mathbf{U}_1=\frac{1}{A}\widetilde{n}_0-\frac{1}{A}(u_{2,0}\partial_y\mathbf{U}_1+u_{3,0}\partial_z\mathbf{U}_1)-\frac{1}{A}(u_{\neq}\cdot\nabla u_{1,\neq})_0,\\\mathbf{U}_1|_{t=0}=(u_{1,\mathrm{in}})_0.\end{cases}\tag{3.17}$$

Therefore, for the given positive constant ϵ_1, we have

$$\begin{aligned}&\partial_t(\mathrm{e}^{-\epsilon_1A^{-\frac{1}{3}}t}\mathbf{U}_1)+\frac{\epsilon_1\mathrm{e}^{-\epsilon_1A^{-\frac{1}{3}}t}\mathbf{U}_1}{A^{\frac{1}{3}}}-\frac{\mathrm{e}^{-\epsilon_1A^{-\frac{1}{3}}t}\triangle\mathbf{U}_1}{A}\\=&\frac{\mathrm{e}^{-\epsilon_1A^{-\frac{1}{3}}t}\widetilde{n}_0}{A}-\frac{\mathrm{e}^{-\epsilon_1A^{-\frac{1}{3}}t}u_{j,0}\partial_j\mathbf{U}_1}{A}-\frac{\mathrm{e}^{-\epsilon_1A^{-\frac{1}{3}}t}(u_{\neq}\cdot\nabla u_{1,\neq})_0}{A},\end{aligned}$$

where $j \in \{2,3\}$. Multiplying $-\frac{\mathrm{e}^{-\epsilon_1 A^{-\frac13}t}\triangle \mathbf{U}_1}{2}$ on both sides of the above equation, the energy estimate shows that

$$\begin{aligned}&\|\mathrm{e}^{-\epsilon_1 A^{-\frac13}t}\nabla \mathbf{U}_1\|_{L^\infty L^2}^2+\frac{\|\mathrm{e}^{-\epsilon_1 A^{-\frac13}t}\triangle \mathbf{U}_1\|_{L^2L^2}^2}{A}\\ \leq&\|u_{1,\mathrm{in}}\|_{H^1}^2+\frac{C\big(\|\mathrm{e}^{-\epsilon_1 A^{-\frac13}t}\widetilde{n}_0\|_{L^2L^2}^2+\|\mathrm{e}^{-\epsilon_1 A^{-\frac13}t}u_{j,0}\partial_j\mathbf{U}_1\|_{L^2L^2}^2+\|\mathrm{e}^{-\epsilon_1 A^{-\frac13}t}u_{\neq}\cdot\nabla u_{1,\neq}\|_{L^2L^2}^2\big)}{A}.\end{aligned}\tag{3.18}$$

When $A>A_2$, by Lemma A.2 and Lemma 4.1, we get

$$\|u_{2,0}\|_{L^\infty L^\infty}^2+\|u_{3,0}\|_{L^\infty L^\infty}^2\leq C(\|u_{2,0}\|_{L^\infty H^2}^2+\|u_{3,0}\|_{L^\infty H^1}^2)\leq C\epsilon^2,\tag{3.19}$$

which implies that

$$\frac{\|\mathrm{e}^{-\epsilon_1 A^{-\frac13}t}u_{j,0}\partial_j\mathbf{U}_1\|_{L^2L^2}^2}{A}\leq\frac{C\epsilon^2\|\mathrm{e}^{-\epsilon_1 A^{-\frac13}t}\triangle \mathbf{U}_1\|_{L^2L^2}^2}{A}.\tag{3.20}$$

Using $(3.2)_2$ and (3.20), we infer from (3.18) that

$$\|\mathrm{e}^{-\epsilon_1 A^{-\frac13}t}\nabla \mathbf{U}_1\|_{L^\infty L^2}^2+\frac{\|\mathrm{e}^{-\epsilon_1 A^{-\frac13}t}\triangle \mathbf{U}_1\|_{L^2L^2}^2}{A}\leq C\Big(\|u_{1,\mathrm{in}}\|_{H^1}^2+\frac{\|\mathrm{e}^{-\epsilon_1 A^{-\frac13}t}\widetilde{n}_0\|_{L^2L^2}^2}{A}+\frac{E_2^4}{A^{\frac13}}\Big).$$

Due to

$$\|\widetilde{n}_0\|_{L^2}^2\leq\|n_0\|_{L^2}^2\leq C\|n\|_{L^\infty L^\infty}\|n_0\|_{L^1}\leq CE_3M$$

and

$$\|\mathrm{e}^{-\epsilon_1 A^{-\frac13}t}\|_{L^2(0,T)}^2\leq\frac{A^{\frac13}}{2\epsilon_1},$$

there holds

$$\|\mathrm{e}^{-\epsilon_1 A^{-\frac13}t}\nabla \mathbf{U}_1\|_{L^\infty L^2}^2+\frac{\|\mathrm{e}^{-\epsilon_1 A^{-\frac13}t}\triangle \mathbf{U}_1\|_{L^2L^2}^2}{A}\leq C\Big(\|u_{1,\mathrm{in}}\|_{H^1}^2+\frac{E_3M}{\epsilon_1A^{\frac23}}+\frac{E_2^4}{A^{\frac13}}\Big).\tag{3.21}$$

When $A\geq\max\{A_2,\epsilon_1^{-\frac32}M^{\frac32}E_3^{\frac32},E_2^{12}\}=:A_3$, by $(A.2)_3$, (3.21) and Lemma 3.1, we have

$$\begin{aligned}\|\mathrm{e}^{aA^{-\frac13}t}u_{2,\neq}\partial_j\nabla\mathbf{U}_1\|_{L^2L^2}^2&\leq C\|\triangle u_{2,\neq}\|_{X_a}^{2\alpha}\|u_{2,\neq}\|_{X_b}^{2-2\alpha}\|\mathrm{e}^{-2(1-\alpha)(b-a)A^{-\frac13}t}\triangle\mathbf{U}_1\|_{L^2L^2}^2\\&\leq CAE_2^{2\alpha}E_4^{2-2\alpha},\end{aligned}\tag{3.22}$$

where α is a constant satisfying $\alpha\in\left(\frac12,\frac34\right)$. By $(A.2)_3$ and Lemma 3.1, there holds

$$\begin{aligned}\|u_{3,\neq}\|_{L^\infty_{y,z}L^2_x}^2&\leq C\|\nabla(\partial_x,\partial_z)u_{3,\neq}\|_{L^2}\|(\partial_x,\partial_z)u_{3,\neq}\|_{L^2}^{2\alpha-1}\|u_{3,\neq}\|_{L^2}^{2-2\alpha}\\&\leq C(\|\partial_y\omega_{2,\neq}\|_{L^2}+\|\triangle u_{2,\neq}\|_{L^2})(\|\partial_x\omega_{2,\neq}\|_{L^2}+\|\triangle u_{2,\neq}\|_{L^2})^{2\alpha-1}\|u_{3,\neq}\|_{L^2}^{2-2\alpha},\end{aligned}$$

which along with (3.21) implies that

$$\begin{aligned}\|\mathrm{e}^{aA^{-\frac13}t}u_{3,\neq}\partial_z\nabla\mathbf{U}_1\|_{L^2L^2}^2\leq& C(\|\partial_y\omega_{2,\neq}\|_{X_a}+\|\triangle u_{2,\neq}\|_{X_a})(\|\partial_x\omega_{2,\neq}\|_{X_a}+\|\triangle u_{2,\neq}\|_{X_a})^{2\alpha-1}\\&\cdot\|\partial_x^2u_{3,\neq}\|_{X_b}^{2-2\alpha}\|\mathrm{e}^{-2(1-\alpha)(b-a)A^{-\frac13}t}\triangle\mathbf{U}_1\|_{L^2L^2}^2\\ \leq& CA^{\frac43}E_2^{2\alpha}E_4^{2-2\alpha}.\end{aligned}\tag{3.23}$$

Collecting (3.15), (3.16), (3.22) and (3.23), we finish the proof.

□

Furthermore, we show the the nonlinear interactions between the bad component $\mathbf{U}_2$ of $u_{1,0}$ and the non-zero mode $u_{\neq}$.

Lemma 3.5. *For $j \in \{2,3\}$, there holds*

$$\begin{aligned}
&\|e^{aA^{-\frac{1}{3}}t}\nabla\mathbf{U}_2\partial_x u_{j,\neq}\|^2_{L^2L^2} \le CAE_1^2E_4^{\frac{1}{2}}(t)(\|\partial_x\omega_{2,\neq}\|_{X_a}+\|\triangle u_{2,\neq}\|_{X_a})^{\frac{3}{2}},\\
&\|e^{aA^{-\frac{1}{3}}t}\partial_z\mathbf{U}_2\partial_x^2 u_{1,\neq}\|^2_{L^2L^2} \le CA^{\frac{17}{12}}E_1^2E_4^{\frac{1}{4}}(\|\partial_x\omega_{2,\neq}\|_{X_a}+\|\triangle u_{2,\neq}\|_{X_a})^{\frac{7}{4}},\\
&\|e^{aA^{-\frac{1}{3}}t}\partial_j\mathbf{U}_2\partial_x\partial_j u_{2,\neq}\|^2_{L^2L^2} \le CA^{\frac{4}{3}}E_1^2E_4^{\frac{1}{2}}\|\triangle u_{2,\neq}\|^{\frac{3}{2}}_{X_a},\\
&\|e^{aA^{-\frac{1}{3}}t}\partial_z(u_{2,\neq}\nabla\mathbf{U}_2)\|^2_{L^2L^2} \le CAE_1^2E_4^{\frac{1}{2}}\|\triangle u_{2,\neq}\|^{\frac{3}{2}}_{X_a},\\
&\|e^{aA^{-\frac{1}{3}}t}\partial_z(u_{3,\neq}\nabla\mathbf{U}_2)\|^2_{L^2L^2} \le CE_1^2A^{\frac{4}{3}}E_4(\|\partial_x\omega_{2,\neq}\|_{X_a}+\|\triangle u_{2,\neq}\|_{X_a}).
\end{aligned}\tag{3.24}$$

Proof. First of all, there holds

$$\|\partial_j\mathbf{U}_2\|_{H^1} \le \int_0^t \|\partial_s\partial_j\mathbf{U}_2(s)\|_{H^1}ds \le CE_1 t.$$

For the given positive constant ϵ_2, since $\lim_{t\to\infty} A^{-\frac{1}{3}}te^{-\epsilon_2A^{-\frac{1}{3}}t}=0$, there holds

$$\left\|e^{-\epsilon_2A^{-\frac{1}{3}}t}\|\partial_j\mathbf{U}_2\|_{H^1}\right\|_{L^\infty_t} \le CA^{\frac{1}{3}}E_1.\tag{3.25}$$

By Lemma A.2 and Lemma A.3, we have

$$\|\nabla\mathbf{U}_2\partial_x u_{j,\neq}\|^2_{L^2} \le \|\nabla\mathbf{U}_2\|^2_{L^\infty_yL^2_z}\|\partial_x u_{j,\neq}\|^2_{L^\infty_zL^2_{x,y}} \le C\|\nabla\mathbf{U}_2\|^2_{H^1}\|\partial_x u_{j,\neq}\|^{\frac{1}{2}}_{L^2}\|\partial_x\partial_z u_{j,\neq}\|^{\frac{3}{2}}_{L^2},$$

which along with (3.25) and Lemma 3.1 indicates that

$$\begin{aligned}
\|e^{aA^{-\frac{1}{3}}t}\nabla\mathbf{U}_2\partial_x u_{j,\neq}\|^2_{L^2L^2} &\le C\int_0^t e^{\frac{a-b}{2}A^{-\frac{1}{3}}s}\|\nabla\mathbf{U}_2\|^2_{H^1}e^{\frac{b+3a}{2}A^{-\frac{1}{3}}s}\|\partial_x^2 u_{j,\neq}\|^{\frac{1}{2}}_{L^2}\|\partial_x\partial_z u_{j,\neq}\|^{\frac{3}{2}}_{L^2}ds\\
&\le CAE_1^2E_4^{\frac{1}{2}}(t)(\|\partial_x\omega_{2,\neq}\|_{X_a}+\|\triangle u_{2,\neq}\|_{X_a})^{\frac{3}{2}}.
\end{aligned}$$

Similarly, one obtains that

$$\begin{aligned}
&\|\partial_z\mathbf{U}_2\partial_x^2 u_{1,\neq}\|^2_{L^2} \le \|\partial_z\mathbf{U}_2\|^2_{L^\infty_zL^2_y}\|\partial_x^2 u_{1,\neq}\|^2_{L^\infty_yL^2_{x,z}} \le C\|\partial_z\mathbf{U}_2\|^2_{H^1}\|\partial_x^2 u_{1,\neq}\|_{L^2}\|\partial_x^2\nabla u_{1,\neq}\|_{L^2}\\
&\le C\|\partial_z\mathbf{U}_2\|^2_{H^1}\|\partial_x^2 u_{1,\neq}\|^{\frac{1}{4}}_{L^2}(\|\partial_x\omega_{2,\neq}\|_{L^2}+\|\triangle u_{2,\neq}\|_{L^2})^{\frac{3}{4}}(\|\partial_x\nabla\omega_{2,\neq}\|_{L^2}+\|\nabla\triangle u_{2,\neq}\|_{L^2}).
\end{aligned}$$

By the divergence-free property

$$\partial_x^2 u_{1,\neq} = -\partial_x(\partial_y u_{2,\neq}+\partial_z u_{3,\neq}),$$

we can prove the second result.

Using Lemma A.2 and Lemma A.3 again, by Hölder's inequality, there is

$$\begin{aligned}
\|\partial_j\mathbf{U}_2\partial_x\partial_j u_{2,\neq}\|^2_{L^2} &\le C\|\partial_j\mathbf{U}_2\|^2_{H^1}\|\partial_x\partial_j u_{2,\neq}\|_{L^2}\|\partial_x\partial_j\nabla u_{2,\neq}\|_{L^2}\\
&\le C\|\partial_j\mathbf{U}_2\|^2_{H^1}\|\partial_x^2 u_{2,\neq}\|^{\frac{1}{2}}_{L^2}\|\triangle u_{2,\neq}\|^{\frac{1}{2}}_{L^2}\|\partial_x\partial_j\nabla u_{2,\neq}\|_{L^2},
\end{aligned}$$

which implies that

$$\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\partial_j \mathbf{U}_2\partial_x\partial_j u_{2,\neq}\|_{L^2L^2}^2 \leq CA^{\frac{4}{3}}E_1^2E_4^{\frac{1}{2}}\|\triangle u_{2,\neq}\|_{X_a}^{\frac{3}{2}}.$$

According to Lemma A.2 and Lemma A.3, by Hölder's inequality, we get

$$\begin{aligned}\|\partial_z(u_{2,\neq}\nabla \mathbf{U}_2)\|_{L^2}^2 &\leq C\|\nabla \mathbf{U}_2\|_{H^1}^2\|\nabla u_{2,\neq}\|_{L^2}\|\triangle u_{2,\neq}\|_{L^2}\\ &\leq C\|\nabla \mathbf{U}_2\|_{H^1}^2\|u_{2,\neq}\|_{L^2}^{\frac{1}{2}}\|\triangle u_{2,\neq}\|_{L^2}^{\frac{3}{2}},\end{aligned}$$

which indicates that

$$\begin{aligned}&\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\partial_z(u_{2,\neq}\nabla \mathbf{U}_2)\|_{L^2L^2}^2\\ \leq\ & C\big\|\mathrm{e}^{\frac{a-b}{4}A^{-\frac{1}{3}}t}\|\nabla \mathbf{U}_2\|_{H^1}\big\|_{L_t^\infty}^2\|\mathrm{e}^{bA^{-\frac{1}{3}}t}u_{2,\neq}\|_{L^2L^2}^{\frac{1}{2}}\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\triangle u_{2,\neq}\|_{L^2L^2}^{\frac{3}{2}} \leq CE_1^2E_4^{\frac{1}{2}}A\|\triangle u_{2,\neq}\|_{X_a}^{\frac{3}{2}}.\end{aligned}$$

Using Lemma A.3, there holds

$$\begin{aligned}\|\partial_z u_{3,\neq}\|_{L_y^\infty L_{x,z}^2}^2 &\leq C\|\partial_z\partial_y u_{3,\neq}\|_{L^2}\|\partial_z u_{3,\neq}\|_{L^2}\\ &\leq C\|u_{3,\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_z^2 u_{3,\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_y u_{3,\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_y\partial_z^2 u_{3,\neq}\|_{L^2}^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}\|u_{3,\neq}\|_{L_{y,z}^\infty L_x^2}^2 &\leq C\|(\partial_x,\partial_z)\partial_y u_{3,\neq}\|_{L^2}\|(\partial_x,\partial_z)u_{3,\neq}\|_{L^2}\\ &\leq C\|u_{3,\neq}\|_{L^2}^{\frac{1}{2}}\|(\partial_x^2,\partial_z^2)u_{3,\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_y u_{3,\neq}\|_{L^2}^{\frac{1}{2}}\|\partial_y(\partial_x^2,\partial_z^2)u_{3,\neq}\|_{L^2}^{\frac{1}{2}}.\end{aligned}$$

Using Lemma 3.1, we get

$$\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\partial_z(u_{3,\neq}\nabla \mathbf{U}_2)\|_{L^2L^2}^2 \leq CE_1^2A^{\frac{4}{3}}E_4(\|\partial_x\omega_{2,\neq}\|_{X_a}+\|\triangle u_{2,\neq}\|_{X_a}).$$

□

Lastly, we show the the nonlinear interaction between the zero modes $\{u_{2,0}, u_{3,0}\}$ and the non-zero mode $u_{\neq}$. The proof can be found in Lemma 3.16 of [10], and we omit it.

Lemma 3.6. *For $j\in\{2,3\}$, it holds that*

$$\begin{aligned}&\|\mathrm{e}^{aA^{-\frac{1}{3}}t}u_{j,0}(\partial_x,\partial_z)\nabla u_{\neq}\|_{L^2L^2}^2 \leq CA(\|u_{2,0}\|_{L^\infty H^2}^2+\|u_{3,0}\|_{L^\infty H^1}^2)(\|\partial_x\omega_{2,\neq}\|_{X_a}^2+\|\triangle u_{2,\neq}\|_{X_a}^2),\\ &\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\partial_z\nabla u_{j,0}\cdot u_{\neq}\|_{L^2L^2}^2 \leq CA^{\frac{2}{3}}(\|u_{2,0}\|_{L^\infty H^2}^2+\|u_{3,0}\|_{L^\infty H^1}^2)(\|\partial_x\omega_{2,\neq}\|_{X_a}^2+\|\triangle u_{2,\neq}\|_{X_a}^2),\\ &\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\partial_z u_{j,0}\nabla u_{\neq}\|_{L^2L^2}^2 \leq CA(\|u_{2,0}\|_{L^\infty H^2}^2+\|u_{3,0}\|_{L^\infty H^1}^2)(\|\partial_x\omega_{2,\neq}\|_{X_a}^2+\|\triangle u_{2,\neq}\|_{X_a}^2),\\ &\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\partial_y u_{3,0}(\partial_x,\partial_z)u_{2,\neq}\|_{L^2L^2}^2 \leq CA^{\frac{1}{3}}\left(\|u_{2,0}\|_{L^\infty H^2}^2+\|u_{3,0}\|_{L^\infty H^1}^2\right)\|\triangle u_{2,\neq}\|_{X_a}^2.\end{aligned}$$

4. Estimates for the zero modes of velocity $E_1(t)$: Proof of Proposition 2.1

In this section, we give some estimates for the zero modes of the velocity, which will be used in estimating the zero mode of the density and the non-zero modes.

4.1. **Energy estimates for $u_{2,0}$ and $u_{3,0}$.** We recall that

$$\begin{cases} \partial_t u_{2,0} - \frac{1}{A}\triangle u_{2,0} + \frac{1}{A}(u\cdot\nabla u_2)_0 + \frac{1}{A}\partial_y P_0^{N_1} + \frac{1}{A}\partial_y P_0^{N_2} = 0, \\ \partial_t u_{3,0} - \frac{1}{A}\triangle u_{3,0} + \frac{1}{A}(u\cdot\nabla u_3)_0 + \frac{1}{A}\partial_z P_0^{N_1} + \frac{1}{A}\partial_z P_0^{N_2} = 0, \\ \partial_y u_{2,0} + \partial_z u_{3,0} = 0, \end{cases} \tag{4.1}$$

where

$$\begin{cases} \triangle P^{N_1} = -2A\partial_x u_2 + \partial_x n, \\ \triangle P^{N_2} = -\text{div } (u\cdot\nabla u). \end{cases}$$

Lemma 4.1. *Under the conditions of Theorem 1.1 and the assumptions (1.6) and (2.5), there exists a positive constant $A_{2,3}$ independent of A and t, such that if $A \geq A_{2,3}$, it holds that*

$$\begin{aligned} &\|u_{2,0}\|_{Y_0} + \|u_{3,0}\|_{Y_0} \leq C\epsilon, \\ &\|\nabla u_{2,0}\|_{Y_0} + \|\nabla u_{3,0}\|_{Y_0} \leq C\epsilon, \\ &\|\triangle u_{2,0}\|_{Y_0} \leq C\epsilon, \\ &\|\min\{(A^{-\frac{2}{3}} + A^{-1}t)^{\frac{1}{2}}, 1\}\triangle u_{3,0}\|_{Y_0} \leq C\epsilon. \end{aligned} \tag{4.2}$$

Proof. **Estimate of** $(4.2)_1$. Due to $\nabla\cdot u_0 = 0$, we have

$$< \partial_y P_0^{N_k}, u_{2,0} > + < \partial_z P_0^{N_k}, u_{3,0} >= - < P_0^{N_k}, \partial_y u_{2,0} + \partial_z u_{3,0} >= 0, \text{ for } k \in \{1,2\}$$

$$< u_0\cdot\nabla u_{2,0}, u_{2,0} > + < u_0\cdot\nabla u_{3,0}, u_{3,0} >= 0.$$

When $A \geq \max\{1, \epsilon^{-6}E_2^{12}\} =: A_{2,1}$, using Lemma 3.2 and assumption (1.6), energy estimates of (4.1) show that

$$\begin{aligned} &\|u_{2,0}\|^2_{L^\infty L^2} + \|u_{3,0}\|^2_{L^\infty L^2} + \frac{1}{A}\left(\|\nabla u_{2,0}\|^2_{L^2L^2} + \|\nabla u_{3,0}\|^2_{L^2L^2}\right) \\ \leq&\|(u_{2,\text{in}})_0\|^2_{L^2} + \|(u_{3,\text{in}})_0\|^2_{L^2} + \frac{C\||u_{\neq}|^2\|^2_{L^2L^2}}{A} \leq \|(u_{2,\text{in}})_0\|^2_{L^2} + \|(u_{3,\text{in}})_0\|^2_{L^2} + \frac{CE_2^4}{A^{\frac{1}{3}}} \leq C\epsilon^2. \end{aligned}$$

Estimate of $(4.2)_2$. Multiplying $2\triangle u_{2,0}$ on $(4.1)_1$ and $2\triangle u_{3,0}$ on $(4.1)_2$, energy estimates give that

$$\begin{aligned} &\frac{d}{dt}\left(\|\nabla u_{2,0}\|^2_{L^2} + \|\nabla u_{3,0}\|^2_{L^2}\right) + \frac{1}{A}\left(\|\triangle u_{2,0}\|^2_{L^2} + \|\triangle u_{3,0}\|^2_{L^2}\right) \\ \leq& C\frac{\|u_0\cdot\nabla u_{2,0}\|^2_{L^2} + \|u_0\cdot\nabla u_{3,0}\|^2_{L^2} + \|u_{\neq}\cdot\nabla u_{\neq}\|^2_{L^2}}{A}, \end{aligned} \tag{4.3}$$

where we use $< \partial_y P_0^{N_k}, \triangle u_{2,0} > + < \partial_z P_0^{N_k}, \triangle u_{3,0} >= 0$ for $k \in \{1,2\}$.

For $j \in \{2,3\}$, by Hölder's inequality and Gagliardo-Nirenberg inequality, there holds

$$\begin{aligned} &\|\nabla u_{j,0}\|^2_{L^2} \leq \|u_{j,0}\|_{L^2}\|\triangle u_{j,0}\|_{L^2}, \\ &\|u_0\cdot\nabla u_{j,0}\|^2_{L^2} \leq C\|(u_2,u_3)_0\|_{L^2}\|\nabla(u_2,u_3)_0\|_{L^2}\|\nabla u_{j,0}\|_{L^2}\|\triangle u_{j,0}\|_{L^2} \\ &\qquad + C\|(u_2,u_3)_0\|^2_{L^2}\|\nabla u_{j,0}\|_{L^2}\|\triangle u_{j,0}\|_{L^2}. \end{aligned} \tag{4.4}$$

According to (4.4) and Young's inequality, we rewrite (4.3) into

$$\begin{aligned}\frac{d}{dt}\|\nabla(u_2,u_3)_0\|_{L^2}^2 \leq &-\frac{1}{2A}\left(\frac{\|\nabla u_{2,0}\|_{L^2}^4}{\|u_{2,0}\|_{L^2}^2}+\frac{\|\nabla u_{3,0}\|_{L^2}^4}{\|u_{3,0}\|_{L^2}^2}\right)+\frac{C\|u_{\neq}\cdot\nabla u_{\neq}\|_{L^2}^2}{A}\\ &+C\frac{\|(u_2,u_3)_0\|_{L^2}^2\|\nabla(u_2,u_3)_0\|_{L^2}^4+\|(u_2,u_3)_0\|_{L^2}^4\|\nabla(u_2,u_3)_0\|_{L^2}^2}{A}.\end{aligned} \tag{4.5}$$

Thanks to $(4.2)_1$, by taking ϵ small enough, we infer from (4.5) that

$$\frac{d}{dt}\|\nabla(u_2,u_3)_0\|_{L^2}^2 \leq C\frac{\|u_{\neq}\cdot\nabla u_{\neq}\|_{L^2}^2}{A}+C\frac{\|(u_2,u_3)_0\|_{L^2}^4\|\nabla(u_2,u_3)_0\|_{L^2}^2}{A}. \tag{4.6}$$

From this, along with Lemma 3.2, (1.6) and $(4.2)_1$, when $A\geq A_{2,1}$, one deduces

$$\begin{aligned}\|\nabla u_{2,0}\|_{L^\infty L^2}^2+\|\nabla u_{3,0}\|_{L^\infty L^2}^2 &\leq \|(u_{2,\mathrm{in}})_0\|_{H^1}^2+\|(u_{3,\mathrm{in}})_0\|_{H^1}^2+\frac{C\|u_{\neq}\cdot\nabla u_{\neq}\|_{L^2L^2}^2}{A}+C\epsilon^6\\ &\leq \|(u_{2,\mathrm{in}})_0\|_{H^1}^2+\|(u_{3,\mathrm{in}})_0\|_{H^1}^2+\frac{CE_2^4}{A^{\frac{1}{3}}}+C\epsilon^6\leq C\epsilon^2.\end{aligned} \tag{4.7}$$

By Lemma A.2, $(4.2)_1$ and (4.7), we have

$$\|u_0\cdot\nabla u_{j,0}\|_{L^2L^2}^2\leq C\|(u_2,u_3)_0\|_{L^\infty H^1}^2\|\triangle u_{j,0}\|_{L^2L^2}^2\leq C\epsilon^2\|\triangle u_{j,0}\|_{L^2L^2}^2.$$

Integrating in time for (4.3) and taking ϵ small enough, we obtain

$$\begin{aligned}&\|\nabla u_{2,0}\|_{L^\infty L^2}^2+\|\nabla u_{3,0}\|_{L^\infty L^2}^2+\frac{\|\triangle u_{2,0}\|_{L^2L^2}^2+\|\triangle u_{3,0}\|_{L^2L^2}^2}{A}\\ \leq &C\left(\|(u_{2,\mathrm{in}})_0\|_{H^1}^2+\|(u_{3,\mathrm{in}})_0\|_{H^1}^2+\frac{\|u_{\neq}\cdot\nabla u_{\neq}\|_{L^2L^2}^2}{A}\right)\leq C\epsilon^2.\end{aligned} \tag{4.8}$$

Estimate of $(4.2)_3$. The H^2 energy estimate for $(4.1)_1$ shows that

$$\begin{aligned}&\|\triangle u_{2,0}\|_{L^\infty L^2}^2+\frac{\|\nabla\triangle u_{2,0}\|_{L^2L^2}^2}{A}\\ \leq&\|(u_{2,\mathrm{in}})_0\|_{H^2}^2+\frac{C}{A}\left(\|\nabla(u_0\cdot\nabla u_{2,0})\|_{L^2L^2}^2+\|\nabla(u_{\neq}\cdot\nabla u_{2,\neq})\|_{L^2L^2}^2+\|\triangle P_0^{N_2}\|_{L^2L^2}^2\right).\end{aligned} \tag{4.9}$$

Using Lemma A.2, $(4.2)_{1,2}$ and $\partial_z u_{3,0}=-\partial_y u_{2,0}$, for $j\in\{2,3\}$, there holds

$$\begin{aligned}\frac{\|\nabla(u_0\cdot\nabla u_{j,0})\|_{L^2L^2}^2}{A}\leq&\frac{C}{A}\left(\|u_{2,0}\|_{L^\infty H^2}^2+\|u_{3,0}\|_{L^\infty H^1}^2\right)\left(\|\nabla u_{2,0}\|_{L^2H^1}^2+\|\nabla u_{3,0}\|_{L^2H^1}^2\right)\\ \leq&C\epsilon^4+C\epsilon^2\|\triangle u_{2,0}\|_{L^\infty L^2}^2.\end{aligned} \tag{4.10}$$

Moreover, due to div $(u\cdot\nabla u)=\partial_x(u\cdot\nabla u_1)+\partial_y(u\cdot\nabla u_2)+\partial_z(u\cdot\nabla u_3)$, there holds

$$\begin{aligned}&\|\mathrm{div}\ (u\cdot\nabla u)_0\|_{L^2}^2\leq\|\partial_y(u\cdot\nabla u_2)_0\|_{L^2}^2+\|\partial_z(u\cdot\nabla u_3)_0\|_{L^2}^2\\ \leq&\|\partial_y(u_0\cdot\nabla u_{2,0})\|_{L^2}^2+\|\partial_z(u_0\cdot\nabla u_{3,0})\|_{L^2}^2+\|\partial_y(u_{\neq}\cdot\nabla u_{2,\neq})\|_{L^2}^2+\|\partial_z(u_{\neq}\cdot\nabla u_{3,\neq})\|_{L^2}^2.\end{aligned}$$

From this, along with Lemma 3.2 and (4.10), we have

$$\frac{\|\triangle P_0^{N_2}\|_{L^2L^2}^2}{A}=\frac{\|\mathrm{div}\ (u\cdot\nabla u)_0\|_{L^2L^2}^2}{A}\leq C\left(\epsilon^4+\frac{E_2^4}{A^{\frac{1}{2}-\frac{2}{3}\alpha}}\right)+C\epsilon^2\|\triangle u_{2,0}\|_{L^\infty L^2}^2, \tag{4.11}$$

where $\alpha\in(\frac{1}{2},\frac{3}{4})$ is a constant.

Then using assumption (1.6), (4.10), (4.11) and Lemma 3.2, we get from (4.9) that

$$\|\triangle u_{2,0}\|^2_{L^\infty L^2}+\frac{\|\nabla\triangle u_{2,0}\|^2_{L^2L^2}}{A}\le\epsilon^2+C\left(\epsilon^4+\frac{E_2^4}{A^{\frac12-\frac23\alpha}}\right)+C\epsilon^2\|\triangle u_{2,0}\|^2_{L^\infty L^2}.$$

By taking ϵ small enough, one obtains

$$\|\triangle u_{2,0}\|^2_{Y_0}\le C\epsilon^2$$

provided with $A\ge\max\{A_{2,1},\left(E_2^2\epsilon^{-1}\right)^{\frac{12}{3-4\alpha}}\}=:A_{2.2}$.

Estimate of $(4.2)_4$. Taking H^2 energy estimate for $(4.1)_2$, there holds

$$\frac{d}{dt}\|\triangle u_{3,0}\|^2_{L^2}+\frac{\|\nabla\triangle u_{3,0}\|^2_{L^2}}{A}\le\frac{C}{A}\left(\|\nabla(u_0\cdot\nabla u_{3,0})\|^2_{L^2}+\|\nabla(u_{\neq}\cdot\nabla u_{3,\neq})\|^2_{L^2}+\|\triangle P_0^{N_2}\|^2_{L^2}\right),$$

which follows that

$$\begin{aligned}&\frac{d}{dt}\left(\min\{A^{-\frac23}+A^{-1}t,1\}\|\triangle u_{3,0}\|^2_{L^2}\right)+\frac{\min\{A^{-\frac23}+A^{-1}t,1\}}{A}\|\nabla\triangle u_{3,0}\|^2_{L^2}\\&\le\frac{C}{A}\left(\|\nabla(u_0\cdot\nabla u_{3,0})\|^2_{L^2}+(A^{-\frac23}+A^{-1}t)\|\nabla(u_{\neq}\cdot\nabla u_{3,\neq})\|^2_{L^2}+\|\triangle P_0^{N_2}\|^2_{L^2}+\|\triangle u_{3,0}\|^2_{L^2}\right).\end{aligned}\tag{4.12}$$

Thanks to $(4.2)_3$, (4.10) and (4.11), we get

$$\frac{\|\nabla(u_0\cdot\nabla u_{j,0})\|^2_{L^2L^2}}{A}\le C\epsilon^4,\quad\frac{\|\triangle P_0^{N_2}\|^2_{L^2L^2}}{A}\le C\left(\epsilon^4+\frac{E_2^4}{A^{\frac12-\frac23\alpha}}\right).\tag{4.13}$$

Notice that $\|(A^{-\frac23}+A^{-1}t)\mathrm{e}^{-aA^{-\frac13}t}\|_{L^\infty_t}\le CA^{-\frac23}$, using $(4.2)_2$, (4.13) and Lemma 3.2, one obtains

$$\begin{aligned}&\|\min\{(A^{-\frac23}+A^{-1}t)^{\frac12},1\}\triangle u_{3,0}\|^2_{Y_0}\\&\le\frac{\|(u_{3,\mathrm{in}})_0\|^2_{H^2}}{A^{\frac23}}+C\left(\epsilon^2+\frac{E_2^4}{A^{\frac12-\frac23\alpha}}\right)+\frac{C}{A}\|(A^{-\frac23}+A^{-1}t)\mathrm{e}^{-aA^{-\frac13}t}\|_{L^\infty_t}\|\mathrm{e}^{2aA^{-\frac13}t}\nabla(u_{\neq}\cdot\nabla u_{3,\neq})\|^2_{L^2L^2}\\&\le\frac{\|(u_{3,\mathrm{in}})_0\|^2_{H^2}}{A^{\frac23}}+C\left(\epsilon^2+\frac{E_2^4}{A^{\frac12-\frac23\alpha}}+\frac{E_2^2E_5^2}{A^{\frac12-\frac13\alpha}}\right),\quad\alpha\in\left(\frac12,\frac34\right).\end{aligned}$$

When

$$A\ge\max\{A_{2,2},(\|(u_{3,\mathrm{in}})_0\|_{H^2}\epsilon^{-1})^3,(E_2E_5\epsilon^{-1})^{\frac{12}{3-2\alpha}}\}=:A_{2,3},$$

we infer from the above inequality that

$$\|\min\{(A^{-\frac23}+A^{-1}t)^{\frac12},1\}\triangle u_{3,0}\|^2_{Y_0}\le C\epsilon^2.$$

The proof is complete. □

4.2. **Heat dissipation estimates for $u_{2,0}$ and $u_{3,0}$.** Next, we consider the standard heat equation in $(y,z)\in\mathbb{T}^2$:

$$\partial_tf-\frac{1}{A}\triangle f=g,\quad t\in[0,T],\tag{4.14}$$

where $g(t,y,z)$ is a given function. By performing a decomposition similar to (2.1), we obtain the following heat dissipation estimate for the non-average part $\widetilde{f}=\sum_{k_2^2+k_3^2\neq0}\widehat{f}_{k_2,k_3}(t)\mathrm{e}^{i(k_2y+k_3z)}$.

Lemma 4.2. *Let $f(t,y,z)$ be a solution of (4.14), there holds*

$$\|\mathrm{e}^{\frac{t}{2A}}\widetilde{f}\|_{L^\infty L^2}^2+\frac{1}{A}\|\mathrm{e}^{\frac{t}{2A}}\nabla\widetilde{f}\|_{L^2L^2}^2\leq C\left(\|f_{\mathrm{in}}\|_{L^2}^2+A\|\mathrm{e}^{\frac{t}{2A}}(-\triangle)^{-\frac{1}{2}}\widetilde{g}\|_{L^2L^2}^2\right),$$

where $\widetilde{g}=\sum_{k_2^2+k_3^2\neq0}\widehat{g}_{k_2,k_3}(t)\mathrm{e}^{i(k_2y+k_3z)}$.

Proof. Doing the Fourier transform to (4.14), the Fourier mode $\widetilde{f}$ satisfies

$$\partial_t\widehat{f}_{k_2,k_3}+\frac{k_2^2+k_3^2}{A}\widehat{f}_{k_2,k_3}=\widehat{g}_{k_2,k_3},\quad k_2^2+k_3^2>0. \tag{4.15}$$

The solution of (4.15) is given by

$$\widehat{f}_{k_2,k_3}=\mathrm{e}^{-\frac{k_2^2+k_3^2}{A}t}(\widehat{f}_{\mathrm{in}})_{k_2,k_3}+\int_0^t\mathrm{e}^{\frac{k_2^2+k_3^2}{A}(s-t)}\widehat{g}_{k_2,k_3}(s)ds=:F_{(1)}+F_{(2)}. \tag{4.16}$$

For $F_{(1)}$, due to $k_2^2+k_3^2>0$, there holds $|F_{(1)}|\leq\mathrm{e}^{-\frac{t}{2A}}|\widehat{f}_{\mathrm{in}}|$, which shows that

$$\|\mathrm{e}^{\frac{t}{2A}}F_{(1)}\|_{L^\infty(0,T)}^2\leq|\widehat{f}_{\mathrm{in}}|^2.$$

For $F_{(2)}$, using Hölder's inequality and $k_2^2+k_3^2>0$, we have

$$\begin{aligned}|F_{(2)}|\leq&\|\mathrm{e}^{\frac{k_2^2+k_3^2}{2A}(s-t)}\|_{L^2}\|\mathrm{e}^{\frac{k_2^2+k_3^2}{2A}(s-t)}\widehat{g}_{k_2,k_3}(s)\|_{L^2}\\ \leq&\left(\frac{A}{k_2^2+k_3^2}\right)^{\frac{1}{2}}\|\mathrm{e}^{\frac{k_2^2+k_3^2}{2A}(s-t)}\widehat{g}_{k_2,k_3}(s)\|_{L^2}\leq\left(\frac{A}{k_2^2+k_3^2}\right)^{\frac{1}{2}}\|\mathrm{e}^{\frac{s-t}{2A}}\widehat{g}_{k_2,k_3}(s)\|_{L^2}.\end{aligned}$$

This implies that

$$\|\mathrm{e}^{\frac{t}{2A}}F_{(2)}\|_{L^\infty(0,T)}^2\leq CA(k_2^2+k_3^2)^{-1}\|\mathrm{e}^{\frac{t}{2A}}\widehat{g}_{k_2,k_3}\|_{L^2(0,T)}^2.$$

Collecting the estimates of $F_{(1)},F_{(2)}$, and using the Plancherel's theorem, we get from (4.16) that

$$\begin{aligned}\|\mathrm{e}^{\frac{t}{2A}}\widetilde{f}\|_{L^\infty L^2}^2=&|\mathbb{T}|^2\sum_{k_2^2+k_3^2>0}\|\mathrm{e}^{\frac{t}{2A}}\widehat{f}_{k_2,k_3}(t)\|_{L^\infty(0,T)}^2\\ \leq&|\mathbb{T}|^2\sum_{k_2^2+k_3^2>0}\left(\|\mathrm{e}^{\frac{t}{2A}}F_{(1)}\|_{L^\infty(0,T)}^2+\|\mathrm{e}^{\frac{t}{2A}}F_{(2)}\|_{L^\infty(0,T)}^2\right)\\ \leq&C|\mathbb{T}|^2\sum_{k_2^2+k_3^2>0}\left(|(\widehat{f}_{\mathrm{in}})_{k_2,k_3}|^2+A(k_2^2+k_3^2)^{-1}\|\mathrm{e}^{\frac{t}{2A}}\widehat{g}_{k_2,k_3}\|_{L^2(0,T)}^2\right)\\ \leq&C\left(\|f_{\mathrm{in}}\|_{L^2}^2+A\|\mathrm{e}^{\frac{t}{2A}}(-\triangle)^{-\frac{1}{2}}\widetilde{g}\|_{L^2L^2}^2\right).\end{aligned} \tag{4.17}$$

We also need to estimate $\frac{1}{A}\|\mathrm{e}^{\frac{t}{2A}}\nabla\widetilde{f}\|_{L^2L^2}^2$. Multiplying (4.15) by $\mathrm{e}^{\frac{t}{2A}}$, the energy estimate gives that

$$-\frac{1}{2A}\|\mathrm{e}^{\frac{t}{2A}}\widehat{f}_{k_2,k_3}\|_{L^2}^2+\frac{2(k_2^2+k_3^2)}{3A}\|\mathrm{e}^{\frac{t}{2A}}\widehat{f}_{k_2,k_3}\|_{L^2}^2\leq-\frac{1}{2}\left(|\mathrm{e}^{\frac{t}{2A}}\widehat{f}_{k_2,k_3}|^2\right)|_{t=0}^{t=T}+\frac{CA\|\mathrm{e}^{\frac{t}{2A}}\widehat{g}_{k_2,k_3}\|_{L^2}^2}{k_2^2+k_3^2},$$

which follows that

$$\frac{k_2^2+k_3^2}{A}\|\mathrm{e}^{\frac{t}{2A}}\widehat{f}_{k_2,k_3}\|_{L^2}^2\leq C\left(|(\widehat{f}_{\mathrm{in}})_{k_2,k_3}|^2+A(k_2^2+k_3^2)^{-1}\|\mathrm{e}^{\frac{t}{2A}}\widehat{g}_{k_2,k_3}\|_{L^2}^2\right).$$

From this, along with the Plancherel's theorem, we get

$$\begin{aligned}\frac{1}{A}\|e^{\frac{t}{2A}}\nabla\widetilde{f}\|_{L^2L^2}^2 =&\frac{1}{A}|\mathbb{T}|^2\sum_{k_2^2+k_3^2>0}\|e^{\frac{t}{2A}}(k_2^2+k_3^2)^{\frac{1}{2}}\widehat{f}_{k_2,k_3}\|_{L^2(0,T)}^2\\ \le& C|\mathbb{T}|^2\sum_{k_2^2+k_3^2>0}\left(|(\widehat{f}_{\rm in})_{k_2,k_3}|^2+A(k_2^2+k_3^2)^{-1}\|e^{\frac{t}{2A}}\widehat{g}_{k_2,k_3}\|_{L^2}^2\right)\\ \le& C\left(\|f_{\rm in}\|_{L^2}^2+A\|e^{\frac{t}{2A}}(-\triangle)^{-\frac{1}{2}}\widetilde{g}\|_{L^2L^2}^2\right).\end{aligned}\tag{4.18}$$

Combining (4.17) with (4.18), one obtains

$$\|e^{\frac{t}{2A}}\widetilde{f}\|_{L^\infty L^2}^2+\frac{1}{A}\|e^{\frac{t}{2A}}\nabla\widetilde{f}\|_{L^2L^2}^2\le C\left(\|f_{\rm in}\|_{L^2}^2+A\|e^{\frac{t}{2A}}(-\triangle)^{-\frac{1}{2}}\widetilde{g}\|_{L^2L^2}^2\right).$$

The proof is complete. □

Similarly, we decompose the velocity $u_{2,0}$ and $u_{3,0}$ in the same way of (2.1), satisfying

$$\begin{aligned}u_{2,0}(t,y,z)=\overline{u}_{2,0}(t)+\widetilde{u}_{2,0}(t,y,z),\\ u_{3,0}(t,y,z)=\overline{u}_{3,0}(t)+\widetilde{u}_{3,0}(t,y,z).\end{aligned}\tag{4.19}$$

In this way, combining with Lemma 4.2, we can prove a typical heat dissipation estimates for $\widetilde{u}_{2,0}(t,y,z)$ and $\widetilde{u}_{3,0}(t,y,z)$. Furthermore, for $j\in\{2,3\}$, $\overline{u}_{j,0}(t)$ is a constant satisfying

$$\overline{u}_{j,0}(t)\equiv\frac{1}{|\mathbb{T}|^2}\int_{|\mathbb{T}|^2}(u_{j,\rm in})_0dydz.$$

Lemma 4.3. *Under the assumptions of Lemma 4.1, it holds that*

$$\|e^{\frac{t}{2A}}\widetilde{u}_{2,0}\|_{L^\infty L^2}^2+\|e^{\frac{t}{2A}}\widetilde{u}_{3,0}\|_{L^\infty L^2}^2+\frac{1}{A}\left(\|e^{\frac{t}{2A}}\nabla\widetilde{u}_{2,0}\|_{L^2L^2}^2+\|e^{\frac{t}{2A}}\nabla\widetilde{u}_{3,0}\|_{L^2L^2}^2\right)\le C\epsilon^2.$$

Proof. Applying Lemma 4.2 to $(4.1)_{1,2}$, and noting that $\nabla\cdot u=0$, we get

$$\begin{aligned}&\|e^{\frac{t}{2A}}\widetilde{u}_{2,0}\|_{L^\infty L^2}^2+\|e^{\frac{t}{2A}}\widetilde{u}_{3,0}\|_{L^\infty L^2}^2+\frac{1}{A}\left(\|e^{\frac{t}{2A}}\nabla\widetilde{u}_{2,0}\|_{L^2L^2}^2+\|e^{\frac{t}{2A}}\nabla\widetilde{u}_{3,0}\|_{L^2L^2}^2\right)\\ \le& C\Big(\|(u_{2,\rm in})_0\|_{L^2}^2+\|(u_{3,\rm in})_0\|_{L^2}^2+\frac{1}{A}\|e^{\frac{t}{2A}}\widetilde{(u_2u_2)}_0\|_{L^2L^2}^2+\frac{1}{A}\|e^{\frac{t}{2A}}\widetilde{(u_3u_2)}_0\|_{L^2L^2}^2\\ &+\frac{1}{A}\|e^{\frac{t}{2A}}\widetilde{(u_3u_3)}_0\|_{L^2L^2}^2+\frac{1}{A}\|e^{\frac{t}{2A}}\widetilde{P}_0^{N_1}\|_{L^2L^2}^2+\frac{1}{A}\|e^{\frac{t}{2A}}\widetilde{P}_0^{N_2}\|_{L^2L^2}^2\Big)\\ =:& C\left(\|(u_{2,\rm in})_0\|_{L^2}^2+\|(u_{3,\rm in})_0\|_{L^2}^2+I_1+\cdots+I_5\right).\end{aligned}\tag{4.20}$$

For I_1, as $\widetilde{(u_2u_2)}_0=2\overline{u}_{2,0}\widetilde{u}_{2,0}+\widetilde{u}_{2,0}\widetilde{u}_{2,0}+\widetilde{(u_{2,\neq}u_{2,\neq})}_0$, there holds

$$\begin{aligned}I_1\le&\frac{C}{A}\left(\|e^{\frac{t}{2A}}\overline{u}_{2,0}\widetilde{u}_{2,0}\|_{L^2L^2}^2+\|e^{\frac{t}{2A}}\widetilde{u}_{2,0}\widetilde{u}_{2,0}\|_{L^2L^2}^2+\|e^{\frac{t}{2A}}\widetilde{(u_{2,\neq}u_{2,\neq})}_0\|_{L^2L^2}^2\right)\\ =:& I_{11}+I_{12}+I_{13}.\end{aligned}\tag{4.21}$$

Thanks to $\partial_t \overline{u}_{j,0} = 0$ for $j \in \{2, 3\}$, we find

$$|\overline{u}_{j,0}| = |(\overline{u}_{j,\text{in}})_0| \leq \frac{1}{|\mathbb{T}|}\|(u_{j,\text{in}})_0\|_{L^2}. \tag{4.22}$$

From this and Poincaré inequality, I_{11} can be controlled by

$$I_{11} \leq \frac{C}{A}\|(u_{2,\text{in}})_0\|_{L^2}^2\|\mathrm{e}^{\frac{t}{2A}}\nabla\widetilde{u}_{2,0}\|_{L^2L^2}^2.$$

For I_{12}, using Lemma 4.1 and Poincaré inequality, one deduces

$$I_{12} \leq \frac{C}{A}\|\widetilde{u}_{2,0}\|_{L^2L^\infty}^2\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{2,0}\|_{L^\infty L^2}^2 \leq \frac{C}{A}\|\triangle u_{2,0}\|_{L^2L^2}^2\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{2,0}\|_{L^\infty L^2}^2 \leq C\epsilon^2\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{2,0}\|_{L^\infty L^2}^2.$$

When A is sufficiently large and using Lemma 3.2, we arrive

$$I_{13} \leq \frac{C}{A}\|\mathrm{e}^{2aA^{-\frac{1}{3}}t}|u_{\neq}|^2\|_{L^2L^2}^2 \leq \frac{CE_2^4}{A^{\frac{1}{3}}} \leq C\epsilon^2.$$

Collecting $I_{11} - I_{13}$, (4.21) yields that

$$I_1 \leq \frac{C}{A}\|(u_{2,\text{in}})_0\|_{L^2}^2\|\mathrm{e}^{\frac{t}{2A}}\nabla\widetilde{u}_{2,0}\|_{L^2L^2}^2 + C\epsilon^2\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{2,0}\|_{L^\infty L^2} + C\epsilon^2. \tag{4.23}$$

The estimates of I_2 and I_3 are similar to I_1. Note that

$$\widetilde{(u_3u_2)}_0 = \overline{u}_{3,0}\widetilde{u}_{2,0} + \widetilde{u}_{3,0}\overline{u}_{2,0} + \widetilde{u}_{3,0}\widetilde{u}_{2,0} + (\widetilde{u_{3,\neq}u_{2,\neq}})_0,$$

$$\widetilde{(u_3u_3)}_0 = 2\overline{u}_{3,0}\widetilde{u}_{3,0} + \widetilde{u}_{3,0}\widetilde{u}_{3,0} + (\widetilde{u_{3,\neq}u_{3,\neq}})_0.$$

Using (4.22), Lemma 3.2 and Poincaré inequality, we have

$$\begin{aligned} I_2 \leq& \frac{C}{A}\left(\|\mathrm{e}^{\frac{t}{2A}}\overline{u}_{3,0}\widetilde{u}_{2,0}\|_{L^2L^2}^2 + \|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{3,0}\overline{u}_{2,0}\|_{L^2L^2}^2 + \|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{3,0}\widetilde{u}_{2,0}\|_{L^2L^2}^2 + \|\mathrm{e}^{\frac{t}{2A}}(\widetilde{u_{3,\neq}u_{2,\neq}})_0\|_{L^2L^2}^2\right) \\ \leq& \frac{C}{A}\Bigg(\|\overline{u}_{3,0}(t)\|_{L^\infty}^2\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{2,0}\|_{L^2L^2}^2 + \|\overline{u}_{2,0}(t)\|_{L^\infty}^2\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{3,0}\|_{L^2L^2}^2 \\ &+ \|\widetilde{u}_{2,0}\|_{L^2L^\infty}^2\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{3,0}\|_{L^\infty L^2}^2 + \|\mathrm{e}^{2aA^{-\frac{1}{3}}t}|u_{\neq}|^2\|_{L^2L^2}^2\Bigg) \\ \leq& \frac{C}{A}\left(\|(u_{2,\text{in}})_0\|_{L^2}^2 + \|(u_{3,\text{in}})_0\|_{L^2}^2\right)\|\mathrm{e}^{\frac{t}{2A}}\nabla(\widetilde{u}_{2,0}, \widetilde{u}_{3,0})\|_{L^2L^2}^2 + C\epsilon^2\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{3,0}\|_{L^\infty L^2}^2 + C\epsilon^2 \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} I_3 \leq& \frac{C}{A}\left(\|\mathrm{e}^{\frac{t}{2A}}\overline{u}_{3,0}\widetilde{u}_{3,0}\|_{L^2L^2}^2 + \|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{3,0}\widetilde{u}_{3,0}\|_{L^2L^2}^2 + \|\mathrm{e}^{\frac{t}{2A}}(\widetilde{u_{3,\neq}u_{3,\neq}})_0\|_{L^2L^2}^2\right) \\ \leq& \frac{C}{A}\left(\|\overline{u}_{3,0}(t)\|_{L^\infty}^2\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{3,0}\|_{L^2L^2}^2 + \|\widetilde{u}_{3,0}\|_{L^2L^\infty}^2\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{3,0}\|_{L^\infty L^2}^2 + \|\mathrm{e}^{2aA^{-\frac{1}{3}}t}|u_{\neq}|^2\|_{L^2L^2}^2\right) \\ \leq& C\epsilon^2\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{3,0}\|_{L^\infty L^2}^2 + \frac{C}{A}\|(u_{3,\text{in}})_0\|_{L^2}^2\|\mathrm{e}^{\frac{t}{2A}}\nabla\widetilde{u}_{3,0}\|_{L^2L^2}^2 + C\epsilon^2. \end{aligned} \tag{4.25}$$

Taking the Fourier transform for $(1.4)_1$, the Fourier mode $\widehat{P}^{N_1}_{k_1,k_2,k_3}$ follows that

$$-(k_1^2 + k_2^2 + k_3^2)\widehat{P}^{N_1}_{k_1,k_2,k_3} = ik_1(-2A\widehat{u}_{2,k_1,k_2,k_3} + \widehat{n}_{k_1,k_2,k_3}).$$

For the Fourier mode $\widehat{P}^{N_1}_{k_1,k_2,k_3}$ of $\widetilde{P}^{N_1}_0$ with $k_1=0$ and $k_2^2+k_3^2>0$, we have

$$\widehat{P}^{N_1}_{k_1,k_2,k_3}|_{k_1=0,k_2^2+k_3^2>0}=\frac{-ik_1(-2A\widehat{u}_{2,k_1,k_2,k_3}+\widehat{n}_{k_1,k_2,k_3})}{k_1^2+k_2^2+k_3^2}\Big|_{k_1=0,k_2^2+k_3^2>0}=0.$$

This implies that

$$I_4=0. \tag{4.26}$$

It follows from $(1.4)_2$ that

$$\triangle P^{N_2}_0=-\mathrm{div}\ (u\cdot\nabla u)_0=-\partial_y^2(u_2u_2)_0-2\partial_y\partial_z(u_2u_3)_0-\partial_z^2(u_3u_3)_0.$$

Therefore, for I_5, one obtains

$$I_5\leq\frac{C}{A}\left(\|\mathrm{e}^{\frac{t}{2A}}\widetilde{(u_2u_2)_0}\|^2_{L^2L^2}+\|\mathrm{e}^{\frac{t}{2A}}\widetilde{(u_2u_3)_0}\|^2_{L^2L^2}+\|\mathrm{e}^{\frac{t}{2A}}\widetilde{(u_3u_3)_0}\|^2_{L^2L^2}\right)\leq C\left(I_1+I_2+I_3\right). \tag{4.27}$$

Combining (4.23), (4.24), (4.25), (4.26) and (4.27), we get from (4.20) that

$$\begin{aligned}&\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{2,0}\|^2_{L^\infty L^2}+\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{3,0}\|^2_{L^\infty L^2}+\frac{1}{A}\left(\|\mathrm{e}^{\frac{t}{2A}}\nabla\widetilde{u}_{2,0}\|^2_{L^2L^2}+\|\mathrm{e}^{\frac{t}{2A}}\nabla\widetilde{u}_{3,0}\|^2_{L^2L^2}\right)\\ \leq&C\left(\|(u_{2,\mathrm{in}})_0\|^2_{L^2}+\|(u_{3,\mathrm{in}})_0\|^2_{L^2}\right)+C\epsilon^2\left(\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{2,0}\|^2_{L^\infty L^2}+\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{3,0}\|^2_{L^\infty L^2}\right)\\ &+\frac{C}{A}\left(\|(u_{2,\mathrm{in}})_0\|^2_{L^2}+\|(u_{3,\mathrm{in}})_0\|^2_{L^2}\right)\left(\|\mathrm{e}^{\frac{t}{2A}}\nabla\widetilde{u}_{2,0}\|^2_{L^2L^2}+\|\mathrm{e}^{\frac{t}{2A}}\nabla\widetilde{u}_{3,0}\|^2_{L^2L^2}\right)+C\epsilon^2\\ \leq&C\epsilon^2\left(1+\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{2,0}\|^2_{L^\infty L^2}+\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{3,0}\|^2_{L^\infty L^2}\right)+\frac{C\epsilon^2}{A}\left(\|\mathrm{e}^{\frac{t}{2A}}\nabla\widetilde{u}_{2,0}\|^2_{L^2L^2}+\|\mathrm{e}^{\frac{t}{2A}}\nabla\widetilde{u}_{3,0}\|^2_{L^2L^2}\right),\end{aligned} \tag{4.28}$$

where we use assumption (1.6).

Hence, when ϵ is small enough satisfying $\epsilon\leq\frac{1}{\sqrt{2C}}$, we obtain from (4.28) that

$$\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{2,0}\|^2_{L^\infty L^2}+\|\mathrm{e}^{\frac{t}{2A}}\widetilde{u}_{3,0}\|^2_{L^\infty L^2}+\frac{1}{2A}\left(\|\mathrm{e}^{\frac{t}{2A}}\nabla\widetilde{u}_{2,0}\|^2_{L^2L^2}+\|\mathrm{e}^{\frac{t}{A}}\nabla\widetilde{u}_{3,0}\|^2_{L^2L^2}\right)\leq C\epsilon^2.$$

The proof is complete. □

4.3. **Energy estimates for $u_{1,0}$.**

Lemma 4.4. *Under the assumptions of Lemma 4.1, it holds that*

$$\|\mathbf{U}_1\|_{L^\infty H^1}\leq C\left(\|(u_{1,\mathrm{in}})_0\|_{H^1}+\|n_0\|_{L^\infty L^2}+\epsilon\right).$$

Proof. Multiplying $\triangle\widetilde{\mathbf{B}}_1$ on both sides of $(2.3)_1$, and applying Hölder's inequality, the energy estimate shows that

$$\frac{d}{dt}\|\nabla\widetilde{\mathbf{B}}_1\|^2_{L^2}+\frac{1}{A}\|\triangle\widetilde{\mathbf{B}}_1\|^2_{L^2}\leq\frac{C}{A}\left(\|u_{2,0}\|^2_{L^\infty}+\|u_{3,0}\|^2_{L^\infty}\right)\|\nabla\widetilde{\mathbf{B}}_1\|^2_{L^2}+\frac{C}{A}\|\widetilde{n}_0\|^2_{L^2}. \tag{4.29}$$

Using (3.19) and Poincaré inequality

$$\|\nabla\widetilde{\mathbf{B}}_1\|^2_{L^2}\leq C\|\triangle\widetilde{\mathbf{B}}_1\|^2_{L^2},$$

when choosing ϵ is small enough, we infer from (4.29) that

$$\frac{d}{dt}\|\nabla\widetilde{\mathbf{B}}_1\|_{L^2}^2 \leq -\frac{\|\nabla\widetilde{\mathbf{B}}_1\|_{L^2}^2}{2AC} + \frac{C\|\widetilde{n}_0\|_{L^2}^2}{A} \leq -\frac{\|\nabla\widetilde{\mathbf{B}}_1\|_{L^2}^2 - C\|n_0\|_{L^2}^2}{2AC}, \tag{4.30}$$

where we use

$$\|\widetilde{n}_0\|_{L^2} = \|n_0 - \overline{n}_0\|_{L^2} \leq C\|n_0\|_{L^2}.$$

According to (4.30), there holds

$$\|\nabla\widetilde{\mathbf{B}}_1\|_{L^\infty L^2}^2 \leq C\|n_0\|_{L^\infty L^2}^2.$$

Otherwise, there must be a time $t = t^*$, such that

$$\begin{cases} \|\nabla\widetilde{\mathbf{B}}_1\|_{L^2}^2|_{t=t^*} = C\|n_0\|_{L^\infty L^2}^2, \\ \frac{d}{dt}\|\nabla\widetilde{\mathbf{B}}_1\|_{L^2}^2\Big|_{t=t^*} > 0. \end{cases} \tag{4.31}$$

Let $t = t^*$, using $(4.31)_1$, we get from (4.30) that

$$\frac{d}{dt}\|\nabla\widetilde{\mathbf{B}}_1\|_{L^2}^2\Big|_{t=t^*} \leq -\frac{\|\nabla\widetilde{\mathbf{B}}_1\|_{L^2}^2 - C\|n_0\|_{L^2}^2}{2AC}\Big|_{t=t^*} = -\frac{\|n_0\|_{L^\infty L^2}^2 - \|n_0\|_{L^2}^2|_{t=t^*}}{2A} \leq 0. \tag{4.32}$$

A contradiction has arisen between (4.31) and (4.32). This shows that

$$\|\nabla\widetilde{\mathbf{B}}_1\|_{L^\infty L^2}^2 \leq C\|n_0\|_{L^\infty L^2}^2. \tag{4.33}$$

By Poincaré inequality, there holds

$$\|\widetilde{\mathbf{B}}_1\|_{L^2} \leq C\|\nabla\widetilde{\mathbf{B}}_1\|_{L^2}.$$

Then it follows from (4.33) that

$$\|\widetilde{\mathbf{B}}_1\|_{L^\infty H^1} \leq C\|\nabla\widetilde{\mathbf{B}}_1\|_{L^\infty L^2} \leq C\|n_0\|_{L^\infty L^2}. \tag{4.34}$$

Note that the estimate of $\mathbf{G}_1$ is similar to $(4.2)_{1,2}$, so we omit it. For $(2.2)_1$, energy estimates show that

$$\|\mathbf{G}_1\|_{L^\infty H^1}^2 + \frac{1}{A}\|\nabla\mathbf{G}_1\|_{L^2 H^1}^2 \leq C\|(u_{1,\mathrm{in}})_0\|_{H^1}^2 + C\epsilon^2. \tag{4.35}$$

Combining (4.34) with (4.35), we conclude that

$$\|\mathbf{U}_1\|_{L^\infty H^1} \leq C(\|\widetilde{\mathbf{B}}_1\|_{L^\infty H^1} + \|\mathbf{G}_1\|_{L^\infty H^1}) \leq C\left(\|(u_{1,\mathrm{in}})_0\|_{H^1} + \|n_0\|_{L^\infty L^2} + \epsilon\right).$$

The proof is complete. □

Lemma 4.5. *Under the assumptions of Lemma 4.1, there exists a positive constant A_2 independent of A and t, such that if $A \geq A_2$, it holds that*

$$\frac{\|\triangle\mathbf{U}_2\|_{L^\infty H^2}}{A} + \frac{\|\nabla\triangle\mathbf{U}_2\|_{L^2 H^2}}{A^{\frac{3}{2}}} + \|\partial_t\mathbf{U}_2\|_{L^\infty H^2} \leq C\epsilon.$$

Proof. **Estimate of $\widetilde{\mathbf{B}}_2(t, y, z)$.** For $(2.4)_1$, the H^4 energy estimate shows that

$$\frac{\|\triangle^2\widetilde{\mathbf{B}}_2\|_{L^\infty L^2}^2}{A^2} + \frac{\|\nabla\triangle^2\widetilde{\mathbf{B}}_2\|_{L^2 L^2}^2}{A^3} \leq \frac{C}{A}\|\nabla\triangle\widetilde{u}_{2,0}\|_{L^2 L^2}^2 + \frac{C}{A^3}\|\nabla\triangle\big(u_{2,0}\partial_y\widetilde{\mathbf{B}}_2 + u_{3,0}\partial_z\widetilde{\mathbf{B}}_2\big)\|_{L^2 L^2}^2. \tag{4.36}$$

Using Lemma A.2, Lemma 4.1 and Poincaré inequality, one obtains

$$\begin{aligned}\frac{\|\nabla\triangle(u_{2,0}\partial_y\widetilde{\mathbf{B}}_2)\|_{L^2L^2}^2}{A^3}\leq&\frac{C}{A^3}\left(\|\nabla u_{2,0}\|_{L^2H^2}^2\|\partial_y\widetilde{\mathbf{B}}_2\|_{L^\infty H^2}^2+\|u_{2,0}\|_{L^\infty H^2}^2\|\nabla\partial_y\widetilde{\mathbf{B}}_2\|_{L^2H^2}^2\right)\\ \leq&C\epsilon^2\left(\frac{\|\triangle^2\widetilde{\mathbf{B}}_2\|_{L^\infty L^2}^2}{A^2}+\frac{\|\nabla\triangle^2\widetilde{\mathbf{B}}_2\|_{L^2L^2}^2}{A^3}\right),\end{aligned}\tag{4.37}$$

$$\frac{\|u_{3,0}\nabla\triangle\partial_z\widetilde{\mathbf{B}}_2+\nabla u_{3,0}\triangle\partial_z\widetilde{\mathbf{B}}_2\|_{L^2L^2}^2}{A^3}\leq\frac{C}{A^3}\|u_{3,0}\|_{L^\infty H^1}^2\|\partial_z\widetilde{\mathbf{B}}_2\|_{L^2H^4}^2\leq C\epsilon^2\frac{\|\nabla\triangle^2\widetilde{\mathbf{B}}_2\|_{L^2L^2}^2}{A^3},\tag{4.38}$$

and

$$\frac{\|\triangle u_{3,0}\nabla\partial_z\widetilde{\mathbf{B}}_2\|_{L^2L^2}^2}{A^3}\leq\frac{C}{A^3}\|\triangle u_{3,0}\|_{L^2L^2}^2\|\widetilde{\mathbf{B}}_2\|_{L^\infty H^4}^2\leq C\epsilon^2\frac{\|\triangle^2\widetilde{\mathbf{B}}_2\|_{L^\infty L^2}^2}{A^2}.\tag{4.39}$$

Due to $\|\widetilde{\mathbf{B}}_2\|_{H^3}^2\leq\|\widetilde{\mathbf{B}}_2\|_{H^2}\|\widetilde{\mathbf{B}}_2\|_{H^4}$, and

$$\|\widetilde{\mathbf{B}}_2\|_{H^2}\leq\int_0^t\|\partial_s\widetilde{\mathbf{B}}_2(s)\|_{H^2}ds\leq t\|\partial_t\widetilde{\mathbf{B}}_2\|_{L^\infty H^2},$$

there holds

$$\frac{\|\widetilde{\mathbf{B}}_2\|_{H^3}^2}{A^{-1}t}\leq\frac{\|\widetilde{\mathbf{B}}\|_{H^2}\|\widetilde{\mathbf{B}}\|_{H^4}}{A^{-1}t}\leq A\|\partial_t\widetilde{\mathbf{B}}_2\|_{L^\infty H^2}\|\widetilde{\mathbf{B}}_2\|_{H^4}.\tag{4.40}$$

From this, along with $\|\nabla\triangle u_{3,0}\partial_z\widetilde{\mathbf{B}}_2\|_{L^2}^2\leq C\|\nabla\triangle u_{3,0}\|_{L^2}^2\|\partial_z\widetilde{\mathbf{B}}_2\|_{H^2}^2$, we get

$$\begin{aligned}\|\nabla\triangle u_{3,0}\partial_z\widetilde{\mathbf{B}}_2\|_{L^2}^2\leq&C\|\min\{(A^{-\frac{2}{3}}+A^{-1}t)^{\frac{1}{2}},1\}\nabla\triangle u_{3,0}\|_{L^2}^2\left(\frac{\|\widetilde{\mathbf{B}}_2\|_{H^3}^2}{A^{-1}t}+\|\widetilde{\mathbf{B}}_2\|_{H^4}^2\right)\\ \leq&C\|\min\{(A^{-\frac{2}{3}}+A^{-1}t)^{\frac{1}{2}},1\}\nabla\triangle u_{3,0}\|_{L^2}^2\left(A^2\|\partial_t\widetilde{\mathbf{B}}_2\|_{L^\infty H^2}^2+\|\widetilde{\mathbf{B}}_2\|_{H^4}^2\right),\end{aligned}$$

which implies that

$$\frac{\|\nabla\triangle u_{3,0}\partial_z\widetilde{\mathbf{B}}_2\|_{L^2L^2}^2}{A^3}\leq C\epsilon^2\left(\|\partial_t\widetilde{\mathbf{B}}_2\|_{L^\infty H^2}^2+\frac{\|\triangle^2\widetilde{\mathbf{B}}_2\|_{L^\infty L^2}^2}{A^2}\right),\tag{4.41}$$

where we use Lemma 4.1 and Poincaré inequality.

Combining (4.37), (4.38), (4.39) and (4.41), we get from (4.36) that

$$\begin{aligned}&\frac{\|\triangle^2\widetilde{\mathbf{B}}_2\|_{L^\infty L^2}^2}{A^2}+\frac{\|\nabla\triangle^2\widetilde{\mathbf{B}}_2\|_{L^2L^2}^2}{A^3}\\ \leq&C\|\triangle u_{2,0}\|_{Y_0}^2+C\epsilon^2\|\partial_t\widetilde{\mathbf{B}}_2\|_{L^\infty H^2}^2+C\epsilon^2\left(\frac{\|\triangle^2\widetilde{\mathbf{B}}_2\|_{L^\infty L^2}^2}{A^2}+\frac{\|\nabla\triangle^2\widetilde{\mathbf{B}}_2\|_{L^2L^2}^2}{A^3}\right).\end{aligned}\tag{4.42}$$

When ϵ is small enough satisfying $C\epsilon^2\leq\frac{1}{2}$, using Lemma 4.1, (4.42) yields

$$\frac{\|\triangle^2\widetilde{\mathbf{B}}_2\|_{L^\infty L^2}^2}{A^2}+\frac{\|\nabla\triangle^2\widetilde{\mathbf{B}}_2\|_{L^2L^2}^2}{A^3}\leq C\epsilon^2+C\epsilon^2\|\partial_t\widetilde{\mathbf{B}}_2\|_{L^\infty H^2}^2.\tag{4.43}$$

From $(2.4)_1$ and (3.19), direct calculations indicate that

$$\begin{aligned}\|\partial_t\widetilde{\mathbf{B}}_2\|_{L^\infty L^2}\leq&\frac{\|\triangle\widetilde{\mathbf{B}}_2\|_{L^\infty L^2}}{A}+\|\widetilde{u}_{2,0}\|_{L^\infty L^2}+\frac{1}{A}\left(\|u_{2,0}\|_{L^\infty L^\infty}+\|u_{3,0}\|_{L^\infty L^\infty}\right)\|\nabla\widetilde{\mathbf{B}}_2\|_{L^\infty L^2}\\ \leq&\frac{\|\triangle\widetilde{\mathbf{B}}_2\|_{L^\infty L^2}}{A}+\|\widetilde{u}_{2,0}\|_{L^\infty L^2}+C\epsilon\frac{\|\nabla\widetilde{\mathbf{B}}_2\|_{L^\infty L^2}}{A}.\end{aligned}\tag{4.44}$$

Taking $\triangle$ for $(2.4)_1$, there holds

$$\partial_t\triangle\widetilde{\mathbf{B}}_2-\frac{1}{A}\triangle^2\widetilde{\mathbf{B}}_2=-\frac{1}{A}\triangle\left(u_{2,0}\partial_y\widetilde{\mathbf{B}}_2+u_{3,0}\partial_z\widetilde{\mathbf{B}}_2\right)-\triangle\widetilde{u}_{2,0},$$

which follows that

$$\|\partial_t\triangle\widetilde{\mathbf{B}}_2\|^2_{L^\infty L^2}\leq\frac{\|\triangle^2\widetilde{\mathbf{B}}_2\|^2_{L^\infty L^2}}{A^2}+\frac{\|\triangle(u_{2,0}\partial_y\widetilde{\mathbf{B}}_2+u_{3,0}\partial_z\widetilde{\mathbf{B}}_2)\|^2_{L^\infty L^2}}{A^2}+\|\triangle\widetilde{u}_{2,0}\|^2_{L^\infty L^2}.\tag{4.45}$$

Using Lemma A.2, Lemma 4.1 and Poincaré inequality, we have

$$\frac{\|\triangle(u_{2,0}\partial_y\widetilde{\mathbf{B}}_2)\|^2_{L^\infty L^2}}{A^2}\leq\frac{C}{A^2}\|u_{2,0}\|^2_{L^\infty H^2}\|\partial_y\widetilde{\mathbf{B}}_2\|^2_{L^\infty H^2}\leq C\epsilon^2\frac{\|\triangle^2\widetilde{\mathbf{B}}_2\|^2_{L^\infty L^2}}{A^2}\tag{4.46}$$

and

$$\frac{\|u_{3,0}\triangle\partial_z\widetilde{\mathbf{B}}_2+\nabla u_{3,0}\cdot\nabla\partial_z\widetilde{\mathbf{B}}_2\|^2_{L^\infty L^2}}{A^2}\leq\frac{C}{A^2}\|u_{3,0}\|^2_{L^\infty H^1}\|\widetilde{\mathbf{B}}_2\|^2_{L^\infty H^4}\leq C\epsilon^2\frac{\|\triangle^2\widetilde{\mathbf{B}}_2\|^2_{L^\infty L^2}}{A^2}.\tag{4.47}$$

Note that $\|\triangle u_{3,0}\partial_z\widetilde{\mathbf{B}}_2\|^2_{L^2}\leq C\|\triangle u_{3,0}\|^2_{L^2}\|\widetilde{\mathbf{B}}_2\|^2_{H^3}$, by (4.40), and we get

$$\begin{aligned}\|\triangle u_{3,0}\partial_z\widetilde{\mathbf{B}}_2\|^2_{L^2}\leq&C\|\min\{(A^{-\frac{2}{3}}+A^{-1}t)^{\frac{1}{2}},1\}\triangle u_{3,0}\|^2_{L^2}\left(\frac{\|\widetilde{\mathbf{B}}_2\|^2_{H^3}}{A^{-1}t}+\|\triangle\partial_z\widetilde{\mathbf{B}}_2\|^2_{L^2}\right)\\ \leq&C\|\min\{(A^{-\frac{2}{3}}+A^{-1}t)^{\frac{1}{2}},1\}\triangle u_{3,0}\|^2_{L^2}\left(A^2\|\partial_t\widetilde{\mathbf{B}}_2\|^2_{H^2}+\|\triangle^2\widetilde{\mathbf{B}}_2\|^2_{L^2}\right),\end{aligned}$$

which along with Lemma 4.1 implies that

$$\frac{\|\triangle u_{3,0}\partial_z\widetilde{\mathbf{B}}_2\|^2_{L^\infty L^2}}{A^2}\leq C\epsilon^2\left(\|\partial_t\widetilde{\mathbf{B}}_2\|^2_{L^\infty H^2}+\frac{\|\triangle^2\widetilde{\mathbf{B}}_2\|^2_{L^\infty L^2}}{A^2}\right).\tag{4.48}$$

Collecting (4.46), (4.47) and (4.48), we get from (4.45) that

$$\|\partial_t\triangle\widetilde{\mathbf{B}}_2\|^2_{L^\infty L^2}\leq C\frac{\|\triangle^2\widetilde{\mathbf{B}}_2\|^2_{L^\infty L^2}}{A^2}+C\epsilon^2\|\partial_t\widetilde{\mathbf{B}}_2\|^2_{L^\infty H^2}+\|\triangle\widetilde{u}_{2,0}\|^2_{L^\infty L^2}.$$

Combining it with (4.44), Lemma 4.1 and Poincaré inequality, one deduces

$$\|\partial_t\widetilde{\mathbf{B}}_2\|^2_{L^\infty H^2}\leq C\frac{\|\triangle^2\widetilde{\mathbf{B}}_2\|^2_{L^\infty L^2}}{A^2}+C\epsilon^2\|\partial_t\widetilde{\mathbf{B}}_2\|^2_{L^\infty H^2}+C\epsilon^2.\tag{4.49}$$

Substituting (4.43) into (4.49), we have

$$\|\partial_t\widetilde{\mathbf{B}}_2\|^2_{L^\infty H^2}\leq C\epsilon^2\|\partial_t\widetilde{\mathbf{B}}_2\|^2_{L^\infty H^2}+C\epsilon^2,$$

which indicates that

$$\|\partial_t\widetilde{\mathbf{B}}_2\|^2_{L^\infty H^2}\leq C\epsilon^2\tag{4.50}$$

provided with ϵ is sufficiently small satisfying $C\epsilon^2 \le \frac{1}{2}$. By (4.50), we infer from (4.43) that

$$\frac{\|\triangle^2\widetilde{\mathbf{B}}_2\|^2_{L^\infty L^2}}{A^2} + \frac{\|\nabla\triangle^2\widetilde{\mathbf{B}}_2\|^2_{L^2L^2}}{A^3} \le C\epsilon^2. \tag{4.51}$$

For $j \in \{2,3\}$, combining (4.50) with (4.51), and using Poincaré inequality

$$\|\nabla^j\widetilde{\mathbf{B}}_2\|^2_{L^2} \le C\|\nabla^{j+1}\widetilde{\mathbf{B}}_2\|^2_{L^2},$$

one obtains

$$\frac{\|\triangle\widetilde{\mathbf{B}}_2\|_{L^\infty H^2}}{A} + \frac{\|\nabla\triangle\widetilde{\mathbf{B}}_2\|_{L^2H^2}}{A^{\frac{3}{2}}} + \|\partial_t\widetilde{\mathbf{B}}_2\|_{L^\infty H^2} \le C\epsilon. \tag{4.52}$$

Estimate of $\overline{B}_2(t)$. According to $(4.1)_1$, $\overline{u}_{2,0}$ satisfies $\partial_t\overline{u}_{2,0} = 0$, which implies that

$$\overline{u}_{2,0}(t) = (\overline{u}_{2,\mathrm{in}})_0.$$

Therefore, we rewrite $(2.4)_2$ into

$$\partial_t\overline{\mathbf{B}}_2(t) = -\overline{u}_{2,0}(t) = -(\overline{u}_{2,\mathrm{in}})_0.$$

By Hölder's inequality, there holds

$$|\partial_t\overline{\mathbf{B}}_2(t)| \le C\|(u_{2,\mathrm{in}})_0\|_{H^2}. \tag{4.53}$$

Furthermore, $\overline{\mathbf{B}}_2(t)$ only depends on t and does not depend on any spatial variables, and $\partial_t\overline{\mathbf{B}}_2$ is a constant. Using assumption (1.6), we infer from (4.53) that

$$\frac{\|\triangle\overline{\mathbf{B}}_2\|_{L^\infty H^2}}{A} + \frac{\|\nabla\triangle\overline{\mathbf{B}}_2\|_{L^2H^2}}{A^{\frac{3}{2}}} + \|\partial_t\overline{\mathbf{B}}_2\|_{L^\infty H^2} \le C\|(u_{2,\mathrm{in}})_0\|_{H^2} \le C\epsilon. \tag{4.54}$$

Estimate of $\overline{B}_1(t)$. Next, we rewrite $(2.3)_2$ into

$$\partial_t\overline{\mathbf{B}}_1(t) = \frac{M}{A|\mathbb{T}|}.$$

Moreover, $\overline{\mathbf{B}}_1(t)$ only depends on t and does not depend on any spatial variables, and $\partial_t\overline{\mathbf{B}}_1$ is a constant. When $A \ge \max\{\frac{M}{\epsilon}, A_{2,3}\} =: A_2$, we obtain that

$$\frac{\|\triangle\overline{\mathbf{B}}_1\|_{L^\infty H^2}}{A} + \frac{\|\nabla\triangle\overline{\mathbf{B}}_1\|_{L^2H^2}}{A^{\frac{3}{2}}} + \|\partial_t\overline{\mathbf{B}}_1\|_{L^\infty H^2} \le \frac{CM}{A} \le C\epsilon. \tag{4.55}$$

Combining (4.52), (4.54) and (4.55), we conclude that

$$\frac{\|\triangle\mathbf{U}_2\|_{L^\infty H^2}}{A} + \frac{\|\nabla\triangle\mathbf{U}_2\|_{L^2H^2}}{A^{\frac{3}{2}}} + \|\partial_t\mathbf{U}_2\|_{L^\infty H^2} \le C\epsilon.$$

The proof is complete. □

Corollary 4.1. *Under the conditions of Theorem 1.1 and the assumptions (1.6) and (2.5), according to Lemma 4.1 and Lemma 4.5, when $A \ge A_2$, there holds*

$$E_1(t) \le C\epsilon =: E_1.$$

5. Estimates for the non-zero modes

5.1. Energy estimates for $E_{2,1}(t)$.

Lemma 5.1. *Under the conditions of Theorem 1.1, the assumptions (1.6) and (2.5), there exists a positive constant A_3 independent of t and A, such that if $A \geq A_3$, there holds*

$$E_{2,1}(t) \leq C \left(\|(\partial_x^2 n_{\rm in})_{\neq}\|_{L^2} + 1 \right).$$

Proof. According to $(1.5)_1$, the non-zero mode $\partial_x^2 n_{\neq}$ satisfies

$$\partial_t \partial_x^2 n_{\neq} + y \partial_x^3 n_{\neq} - \frac{1}{A} \triangle \partial_x^2 n_{\neq} = -\frac{1}{A} \nabla \cdot \partial_x^2 (un)_{\neq} - \frac{1}{A} \nabla \cdot \partial_x^2 (n \nabla c)_{\neq}. \tag{5.1}$$

Noting that for given functions f and g, we have

$$(fg)_{\neq} = f_0 g_{\neq} + f_{\neq} g_0 + (f_{\neq} g_{\neq})_{\neq}. \tag{5.2}$$

Therefore, by decomposing $u_{1,0} = \mathbf{U}_1 + \mathbf{U}_2$, we rewrite (5.1) into

$$\begin{aligned} \partial_t \partial_x^2 n_{\neq} + \left(y + \frac{\mathbf{U}_2}{A} \right) \partial_x^3 n_{\neq} - \frac{\triangle \partial_x^2 n_{\neq}}{A} =& -\frac{1}{A} \nabla \cdot \partial_x^2 (u_{\neq} n_{\neq})_{\neq} - \frac{1}{A} \nabla \cdot \partial_x^2 (n \nabla c)_{\neq} \\ & - \frac{1}{A} \nabla \cdot (\partial_x^2 u_{\neq} n_0) - \frac{1}{A} \nabla \cdot \partial_x^2 (U_0 n_{\neq}), \end{aligned} \tag{5.3}$$

where $U_0 = (\mathbf{U}_1, u_{2,0}, u_{3,0})$.

Moreover, a careful deformation of the nonlinear interaction term $\partial_x^3 (u_{1,\neq} n_{\neq})_{\neq}$ plays a crucial role. Specifically, it can be expanded as follows:

$$\partial_x^3 (u_{1,\neq} n_{\neq})_{\neq} = (\partial_x^3 u_{1,\neq} n_{\neq})_{\neq} + (u_{1,\neq} \partial_x^3 n_{\neq})_{\neq} + 3 \partial_x (\partial_x u_{1,\neq} \partial_x n_{\neq})_{\neq}.$$

When $A > Cc^{-1}$, it follows from (1.6) and Lemma 4.5 that $\frac{\|\triangle \mathbf{U}_2\|_{L^\infty H^2}}{A} + \|\partial_t \mathbf{U}_2\|_{L^\infty L^\infty} < C\epsilon$. By applying Proposition A.1, we get

$$\begin{aligned} \|\partial_x^2 n_{\neq}\|_{X_b}^2 \leq & C \Big(\|(\partial_x^2 n_{\rm in})_{\neq}\|_{L^2}^2 + \frac{1}{A} \|{\rm e}^{bA^{-\frac13}t} U_0 \partial_x^2 n_{\neq}\|_{L^2L^2}^2 + \frac{1}{A^{\frac53}} \|{\rm e}^{bA^{-\frac13}t} n_0 \partial_x^3 u_{1,\neq}\|_{L^2L^2}^2 \\ & + \frac{1}{A} \|{\rm e}^{bA^{-\frac13}t} n_0 \partial_x^2 (u_2, u_3)_{\neq}\|_{L^2L^2}^2 + \frac{1}{A^{\frac53}} \|{\rm e}^{bA^{-\frac13}t} n_{\neq} \partial_x^3 u_{1,\neq}\|_{L^2L^2}^2 \\ & + \frac{1}{A^{\frac53}} \|{\rm e}^{bA^{-\frac13}t} u_{1,\neq} \partial_x^3 n_{\neq}\|_{L^2L^2}^2 + \frac{1}{A} \|{\rm e}^{bA^{-\frac13}t} \partial_x u_{1,\neq} \partial_x n_{\neq}\|_{L^2L^2}^2 \\ & + \frac{1}{A} \|{\rm e}^{bA^{-\frac13}t} \partial_x^2 \left((u_2, u_3)_{\neq} n_{\neq} \right) \|_{L^2L^2}^2 + \frac{1}{A} \|{\rm e}^{bA^{-\frac13}t} \partial_x^2 (n \nabla c)_{\neq}\|_{L^2L^2}^2 \Big) \\ =: & C \left(\|(\partial_x^2 n_{\rm in})_{\neq}\|_{L^2}^2 + T_{1,1} + T_{1,2} + \cdots + T_{1,8} \right). \end{aligned}$$

Estimate of $T_{1,1}$. By Lemma A.7, we have

$$\|\partial_x^2 n_{\neq}\|_{L^2L^4} \leq C \|\partial_x^2 n_{\neq}\|_{L^2L^2}^{\frac14} \|\nabla \partial_x^2 n_{\neq}\|_{L^2L^2}^{\frac34} \tag{5.4}$$

and

$$\|U_0\|_{L^4} \leq C \|U_0\|_{H^1} \leq C (\|\mathbf{U}_1\|_{H^1} + \|u_{2,0}\|_{H^1} + \|u_{3,0}\|_{H^1}).$$

Then using Lemma 4.1, Lemma 4.4 and $\|n_0\|^2_{L^\infty L^2} \le CME_3$, we obtain

$$\begin{aligned}\|e^{bA^{-\frac{1}{3}}t}U_0\partial_x^2 n_{\neq}\|^2_{L^2L^2} \le &\|U_0\|^2_{L^\infty L^4}\|e^{bA^{-\frac{1}{3}}t}\partial_x^2 n_{\neq}\|^2_{L^2L^4}\\ \le &C\|U_0\|^2_{L^\infty H^1}\|e^{bA^{-\frac{1}{3}}t}\partial_x^2 n_{\neq}\|^{\frac{1}{2}}_{L^2L^2}\|e^{bA^{-\frac{1}{3}}t}\nabla\partial_x^2 n_{\neq}\|^{\frac{3}{2}}_{L^2L^2} \le CA^{\frac{5}{6}}H_2^2E_2^2,\end{aligned}$$

where $H_2 = \|(u_{1,\rm in})_0\|_{H^1} + E_3 + M + 1$.

Estimates of $T_{1,2}$ **and** $T_{1,3}$. Using $\|n_0\|_{L^\infty L^\infty} \le \|n\|_{L^\infty L^\infty} \le 2E_3$ and $\partial_x u_{1,\neq} + \partial_y u_{2,\neq} + \partial_z u_{3,\neq} = 0$, we arrive at

$$\begin{aligned}&\|e^{bA^{-\frac{1}{3}}t}n_0\partial_x^3 u_{1,\neq}\|^2_{L^2L^2} \le CE_3^2\|e^{bA^{-\frac{1}{3}}t}\partial_x^3 u_{1,\neq}\|^2_{L^2L^2} \le CAE_3^2E_4^2,\\ &\|e^{bA^{-\frac{1}{3}}t}n_0\partial_x^2 (u_2,u_3)_{\neq}\|^2_{L^2L^2} \le CE_3^2\|e^{bA^{-\frac{1}{3}}t}\partial_x^2 (u_2,u_3)_{\neq}\|^2_{L^2L^2} \le CA^{\frac{1}{3}}E_3^2E_4^2.\end{aligned} \tag{5.5}$$

Estimate of $T_{1,4}$. Since $\|n_{\neq}\|_{L^\infty L^\infty} \le 2\|n\|_{L^\infty L^\infty} \le 4E_3$, by an argument similar to $(5.5)_1$, one obtains

$$\begin{aligned}\|e^{bA^{-\frac{1}{3}}t}n_{\neq}\partial_x^3 u_{1,\neq}\|^2_{L^2L^2} \le &C\|n\|_{L^\infty L^\infty}\|e^{bA^{-\frac{1}{3}}t}\partial_x^3 u_{1,\neq}\|^2_{L^2L^2}\\ \le &CAE_3^2\|\partial_x^2(u_2,u_3)_{\neq}\|^2_{X_b} \le CAE_3^2E_4^2.\end{aligned}$$

Estimate of $T_{1,5}$. By $(3.5)_1$, we have

$$\|e^{bA^{-\frac{1}{3}}t}u_{1,\neq}\partial_x^3 n_{\neq}\|^2_{L^2L^2} \le \|u_{1,\neq}\|^2_{L^\infty L^\infty}\|e^{bA^{-\frac{1}{3}}t}\partial_x^3 n_{\neq}\|^2_{L^2L^2} \le CA^{\frac{4}{3}}E_2^4.$$

Estimate of $T_{1,6}$. By Lemma A.3 and Lemma 3.1, one obtains that

$$\begin{aligned}&\|e^{bA^{-\frac{1}{3}}t}\partial_x u_{1,\neq}\partial_x n_{\neq}\|^2_{L^2L^2} \le \|e^{aA^{-\frac{1}{3}}t}\partial_x u_{1,\neq}\|^2_{L^\infty_{t,z}L^2_{x,y}}\|e^{aA^{-\frac{1}{3}}t}\partial_x n_{\neq}\|^2_{L^\infty_{x,y}L^2_{t,z}}\\ \le &C\|e^{aA^{-\frac{1}{3}}t}\partial_x\partial_z u_{1,\neq}\|^2_{L^\infty L^2}\|e^{aA^{-\frac{1}{3}}t}\partial_x^2 n_{\neq}\|_{L^2L^2}\|e^{aA^{-\frac{1}{3}}t}\partial_x^2\partial_y n_{\neq}\|_{L^2L^2} \le CA^{\frac{2}{3}}E_2^4.\end{aligned}$$

Estimate of $T_{1,7}$. According to (3.5), there holds

$$\|e^{bA^{-\frac{1}{3}}t}(u_2,u_3)_{\neq}\partial_x^2 n_{\neq}\|^2_{L^2L^2} \le \|u_{\neq}\|^2_{L^\infty L^\infty}\|e^{bA^{-\frac{1}{3}}t}\partial_x^2 n_{\neq}\|^2_{L^2L^2} \le CA^{\frac{2}{3}}E_2^4. \tag{5.6}$$

In addition, there also holds

$$\|e^{bA^{-\frac{1}{3}}t}\partial_x^2(u_2,u_3)_{\neq}n_{\neq}\|^2_{L^2L^2} \le \|n_{\neq}\|^2_{L^\infty L^\infty}\|e^{bA^{-\frac{1}{3}}t}\partial_x^2(u_2,u_3)_{\neq}\|^2_{L^2L^2} \le CA^{\frac{1}{3}}E_3^2E_4^2. \tag{5.7}$$

Using Lemma A.3, we get

$$\begin{aligned}\|\partial_x(u_2,u_3)_{\neq}\partial_x n_{\neq}\|^2_{L^2} \le &\|\partial_x u_{\neq}\|^2_{L^\infty_{y,z}L^2_x}\|\partial_x n_{\neq}\|^2_{L^\infty_x L^2_{y,z}}\\ \le &C\|(\partial_x,\partial_z)\partial_x u_{\neq}\|_{L^2}\|(\partial_x,\partial_z)\partial_x u_{\neq}\|_{H^1}\|\partial_x^2 n_{\neq}\|^2_{L^2},\end{aligned}$$

which along with Lemma 3.1 shows that

$$\|e^{bA^{-\frac{1}{3}}t}\partial_x u_{\neq}\partial_x n_{\neq}\|^2_{L^2L^2} \le CA^{\frac{2}{3}}E_2^4. \tag{5.8}$$

It follows from (5.6), (5.7) and (5.8) that

$$\|e^{bA^{-\frac{1}{3}}t}\partial_x^2(u_{\neq}n_{\neq})_{\neq}\|^2_{L^2L^2} \le CA^{\frac{2}{3}}(E_2^4 + E_3^2E_4^2).$$

Estimate of $T_{1,8}$. Due to (5.2), there holds

$$\begin{aligned}&\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x^2(n\nabla c)_{\neq}\|_{L^2L^2}^2\\ \leq &C\left(\|\mathrm{e}^{bA^{-\frac{1}{3}}t}n_0\partial_x^2\nabla c_{\neq}\|_{L^2L^2}^2+\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x^2 n_{\neq}\nabla c_0\|_{L^2L^2}^2+\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x^2(n_{\neq}\nabla c_{\neq})_{\neq}\|_{L^2L^2}^2\right).\end{aligned}\tag{5.9}$$

Applying Lemma A.8, Lemma A.9 and $\|n_0\|_{L^\infty L^\infty}\leq\|n\|_{L^\infty L^\infty}$, we get

$$\begin{aligned}\|\mathrm{e}^{bA^{-\frac{1}{3}}t}n_0\partial_x^2\nabla c_{\neq}\|_{L^2L^2}^2&\leq\|n_0\|_{L^\infty L^\infty}^2\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x^2\nabla c_{\neq}\|_{L^2L^2}^2\\ &\leq CE_3^2\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x^2 n_{\neq}\|_{L^2L^2}^2\leq CA^{\frac{1}{3}}E_2^2E_3^2\end{aligned}$$

and

$$\begin{aligned}\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x^2 n_{\neq}\nabla c_0\|_{L^2L^2}^2\leq&\|\nabla c_0\|_{L^\infty L^\infty}^2\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x^2 n_{\neq}\|_{L^2L^2}^2\leq CA^{\frac{1}{3}}\|n_0-\bar{n}\|_{L^\infty L^3}^2\|\partial_x^2 n_{\neq}\|_{X_b}^2\\ \leq&CA^{\frac{1}{3}}\|n_0\|_{L^\infty L^\infty}^{\frac{4}{3}}\|n_0\|_{L^\infty L^1}^{\frac{2}{3}}\|\partial_x^2 n_{\neq}\|_{X_b}^2\leq CA^{\frac{1}{3}}E_2^2(E_3^2+M^2).\end{aligned}$$

Using (5.4), Lemma A.3 and Lemma A.9, one obtains that

$$\begin{aligned}&\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x^2(n_{\neq}\nabla c_{\neq})_{\neq}\|_{L^2L^2}^2\\ \leq&C\|\nabla c_{\neq}\|_{L^\infty L^4}^2\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x^2 n_{\neq}\|_{L^2L^4}^2+C\|n\|_{L^\infty L^\infty}^2\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x^2\nabla c_{\neq}\|_{L^2L^2}^2\\ &+C\|\partial_x n_{\neq}\|_{L^\infty_{t,x}L^2_{y,z}}^2\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x\nabla c_{\neq}\|_{L^2_{t,x}L^\infty_{y,z}}^2\\ \leq&C\|n_{\neq}\|_{L^\infty L^2}^2\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x^2 n_{\neq}\|_{L^2L^2}^{\frac{1}{2}}\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\nabla\partial_x^2 n_{\neq}\|_{L^2L^2}^{\frac{3}{2}}+C\|n\|_{L^\infty L^\infty}^2\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x^2 n_{\neq}\|_{L^2L^2}^2\\ &+C\|\partial_x^2 n_{\neq}\|_{L^\infty L^2}^2\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x^2 n_{\neq}\|_{L^2L^2}\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\nabla\partial_x^2 n_{\neq}\|_{L^2L^2}\\ \leq&CA^{\frac{5}{6}}E_2^2(E_2^2+E_3^2),\end{aligned}$$

where we use $\|n_{\neq}\|_{L^\infty L^\infty}\leq 2\|n\|_{L^\infty L^\infty}$. Thus, we infer from (5.9) that

$$\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\partial_x^2(n\nabla c)_{\neq}\|_{L^2L^2}^2\leq CA^{\frac{5}{6}}(E_2^2+E_3^2+M^2)E_2^2.$$

In conclusion, we obtain that

$$\|\partial_x^2 n_{\neq}\|_{X_b}^2\leq C\left(\|(\partial_x^2 n_{\mathrm{in}})_{\neq}\|_{L^2}^2+\frac{E_2^4+E_3^4+E_4^4+M^4+H_2^4}{A^{\frac{1}{6}}}\right).$$

When $A\geq\max\{A_2,(E_2^4+E_3^4+M^4+H_2^4+E_4^4)^6\}=:A_3$, one deduces

$$E_{2,1}(t)=\|\partial_x^2 n_{\neq}\|_{X_b}\leq C\left(\|(\partial_x^2 n_{\mathrm{in}})_{\neq}\|_{L^2}+1\right).$$

The proof is complete. □

5.2. **Energy estimates for $E_{2,2}(t)$.**

Lemma 5.2. *Under the conditions of Theorem 1.1 and the assumptions (1.6) and (2.5), there exists a positive constant A_4 independent of t and A, such that if $A\geq A_4$, there holds*

$$E_{2,2}(t)\leq C\left(\|(u_{\mathrm{in}})_{\neq}\|_{H^2}+E_4+1\right).$$

Proof. It follows from (1.5) that

$$\begin{cases} \partial_t\omega_2 + y\partial_x\omega_2 - \frac{1}{A}\triangle\omega_2 + \partial_z u_2 = -\frac{1}{A}\partial_z(u\cdot\nabla u_1) + \frac{1}{A}\partial_x(u\cdot\nabla u_3) + \frac{1}{A}\partial_z n, \\ \partial_t\triangle u_2 + y\partial_x\triangle u_2 - \frac{1}{A}\triangle^2 u_2 = -\frac{1}{A}\partial_y\partial_x n - \frac{1}{A}(\partial_x^2+\partial_z^2)(u\cdot\nabla u_2) \\ \qquad\qquad + \frac{1}{A}\partial_y\left[\partial_x(u\cdot\nabla u_1) + \partial_z(u\cdot\nabla u_3)\right]. \end{cases}$$

For convenience, we denote

$$\begin{aligned} &\mathcal{L} = \partial_t + y\partial_x - \frac{1}{A}\triangle, \\ &\mathcal{L}_V = \partial_t + (y + \frac{\mathbf{U}_2}{A})\partial_x - \frac{1}{A}\triangle. \end{aligned} \tag{5.10}$$

As previously mentioned, we have decomposed the non-zero mode $u_{1,0}$ into two parts $u_{1,0} = \mathbf{U}_1 + \mathbf{U}_2$. According to the nonlinear interaction, we will reformulate the coupled system $\{\partial_x\omega_{2,\neq}, \triangle u_{2,\neq}\}$ and $(\partial_y,\partial_z)\omega_{2,\neq}$.

Using $u_{1,0} = \mathbf{U}_1 + \mathbf{U}_2$ and (5.2), the velocity $\omega_{2,\neq}$ satisfies

$$\begin{aligned} \mathcal{L}_V\omega_{2,\neq} + \partial_z u_{2,\neq} =& \frac{\partial_x\left[(u_{\neq}\cdot\nabla u_{3,\neq})_{\neq} + U_0\cdot\nabla u_{3,\neq} + u_{\neq}\cdot\nabla u_{3,0}\right]}{A} + \frac{\partial_z n_{\neq}}{A} - \frac{\partial_z\mathbf{U}_2\partial_x u_{1,\neq}}{A} \\ &- \frac{\partial_z\left[(u_{\neq}\cdot\nabla u_{1,\neq})_{\neq} + U_0\cdot\nabla u_{1,\neq} + u_{\neq}\cdot\nabla(\mathbf{U}_1+\mathbf{U}_2)\right]}{A}, \end{aligned}$$

where $U_0 = (\mathbf{U}_1, u_{2,0}, u_{3,0})$. Precise calculations show that

$$(\partial_x^2+\partial_z^2)(\mathbf{U}_2\partial_x u_{2,\neq}) = \mathbf{U}_2\partial_x(\partial_x^2+\partial_z^2)u_{2,\neq} + \partial_z(\partial_z\mathbf{U}_2\partial_x u_{2,\neq}) + \partial_z\mathbf{U}_2\partial_x\partial_z u_{2,\neq}. \tag{5.11}$$

Due to div $u_{\neq} = 0$, we have

$$\begin{aligned} &\partial_y\left[\partial_x(\mathbf{U}_2\partial_x u_{1,\neq}) + \partial_z(\mathbf{U}_2\partial_x u_{3,\neq})\right] = \partial_y\left(-\mathbf{U}_2\partial_x\partial_y u_{2,\neq} + \partial_z\mathbf{U}_2\partial_x u_{3,\neq}\right) \\ &= -\mathbf{U}_2\partial_x\partial_y^2 u_{2,\neq} - \partial_y\mathbf{U}_2\partial_x\partial_y u_{2,\neq} + \partial_y(\partial_z\mathbf{U}_2\partial_x u_{3,\neq}). \end{aligned} \tag{5.12}$$

Using (5.11) and (5.12), $\triangle u_{2,\neq}$ satisfies

$$\begin{aligned} \mathcal{L}_V\triangle u_{2,\neq} =& -\frac{\partial_x\partial_y n_{\neq}}{A} - \frac{(\partial_x^2+\partial_z^2)\left[(u_{\neq}\cdot\nabla u_{2,\neq})_{\neq} + U_0\cdot\nabla u_{2,\neq} + u_{\neq}\cdot\nabla u_{2,0}\right]}{A} \\ &+ \frac{\partial_y\partial_x\left[(u_{\neq}\cdot\nabla u_{1,\neq})_{\neq} + U_0\cdot\nabla u_{1,\neq} + u_{\neq}\cdot\nabla(\mathbf{U}_1+\mathbf{U}_2)\right]}{A} - \frac{\partial_j\mathbf{U}_2\partial_x\partial_j u_{2,\neq}}{A} \\ &+ \frac{\partial_y\partial_z\left[(u_{\neq}\cdot\nabla u_{3,\neq})_{\neq} + U_0\cdot\nabla u_{3,\neq} + u_{\neq}\cdot\nabla u_{3,0}\right]}{A} + \frac{\partial_y(\partial_z\mathbf{U}_2\partial_x u_{3,\neq}) - \partial_z(\partial_z\mathbf{U}_2\partial_x u_{2,\neq})}{A}. \end{aligned}$$

Therefore, one obtains that

$$\begin{cases} \mathcal{L}_V\omega_{2,\neq} + \partial_z u_{2,\neq} = \frac{\partial_x\left[(u_{\neq}\cdot\nabla u_{3,\neq})_{\neq} + U_0\cdot\nabla u_{3,\neq} + u_{\neq}\cdot\nabla u_{3,0}\right]}{A} + \frac{\partial_z n_{\neq}}{A} - \frac{\partial_z\mathbf{U}_2\partial_x u_{1,\neq}}{A} \\ \qquad - \frac{\partial_z\left[(u_{\neq}\cdot\nabla u_{1,\neq})_{\neq} + U_0\cdot\nabla u_{1,\neq} + u_{\neq}\cdot\nabla(\mathbf{U}_1+\mathbf{U}_2)\right]}{A}, \\ \mathcal{L}_V\triangle u_{2,\neq} = -\frac{\partial_x\partial_y n_{\neq}}{A} - \frac{(\partial_x^2+\partial_z^2)\left[(u_{\neq}\cdot\nabla u_{2,\neq})_{\neq} + U_0\cdot\nabla u_{2,\neq} + u_{\neq}\cdot\nabla u_{2,0}\right]}{A} \\ \qquad + \frac{\partial_y\partial_x\left[(u_{\neq}\cdot\nabla u_{1,\neq})_{\neq} + U_0\cdot\nabla u_{1,\neq} + u_{\neq}\cdot\nabla(\mathbf{U}_1+\mathbf{U}_2)\right]}{A} - \frac{\partial_j\mathbf{U}_2\partial_x\partial_j u_{2,\neq}}{A} \\ \qquad + \frac{\partial_y\partial_z\left[(u_{\neq}\cdot\nabla u_{3,\neq})_{\neq} + U_0\cdot\nabla u_{3,\neq} + u_{\neq}\cdot\nabla u_{3,0}\right]}{A} + \frac{\partial_y(\partial_z\mathbf{U}_2\partial_x u_{3,\neq}) - \partial_z(\partial_z\mathbf{U}_2\partial_x u_{2,\neq})}{A}. \end{cases} \tag{5.13}$$

Taking ∂_x for $(5.13)_1$ and applying Proposition A.2, we derive

$$\begin{aligned}
&\|\partial_x\omega_{2,\neq}\|_{X_a}^2+\|\triangle u_{2,\neq}\|_{X_a}^2\leq C\Big(\|(u_{\rm in})_{\neq}\|_{H^2}^2+\frac{\|\partial_x n_{\neq}\|_{X_b}^2}{A^{\frac{2}{3}}}+\frac{\|{\rm e}^{aA^{-\frac13}t}\partial_z\mathbf{U}_2\partial_x(u_{2,\neq},u_{3,\neq})\|_{L^2L^2}^2}{A}\Big)\\
&+CA^{-1}\Big(\|{\rm e}^{aA^{-\frac13}t}\partial_x\left[(u_{\neq}\cdot\nabla u_{1,\neq})_{\neq}+U_0\cdot\nabla u_{1,\neq}+u_{\neq}\cdot\nabla(\mathbf{U}_1+\mathbf{U}_2)\right]\|_{L^2L^2}^2\\
&+\|{\rm e}^{aA^{-\frac13}t}(\partial_x,\partial_z)\left[(u_{\neq}\cdot\nabla u_{2,\neq})_{\neq}+U_0\cdot\nabla u_{2,\neq}+u_{\neq}\cdot\nabla u_{2,0}\right]\|_{L^2L^2}^2\\
&+\|{\rm e}^{aA^{-\frac13}t}(\partial_x,\partial_z)\left[(u_{\neq}\cdot\nabla u_{3,\neq})_{\neq}+U_0\cdot\nabla u_{3,\neq}+u_{\neq}\cdot\nabla u_{3,0}\right]\|_{L^2L^2}^2\Big)\\
&+CA^{-\frac53}\Big(\|{\rm e}^{aA^{-\frac13}t}\partial_z\mathbf{U}_2\partial_x^2u_{1,\neq}\|_{L^2L^2}^2+\|{\rm e}^{aA^{-\frac13}t}\partial_j\mathbf{U}_2\partial_x\partial_ju_{2,\neq}\|_{L^2L^2}^2\Big)\\
&=:C\left(\|(u_{\rm in})_{\neq}\|_{H^2}^2+T_{2,1}+T_{2,2}+\cdots+T_{2,7}\right).
\end{aligned}\tag{5.14}$$

Estimates of $T_{2,2}$, $T_{2,6}$ and $T_{2,7}$. By Lemma 3.5, Lemma 4.5 and Young's inequality, we have

$$T_{2,2}+T_{2,6}+T_{2,7}\leq C\frac{E_2^2+E_4^2}{A^{\frac14}}+C\epsilon^2(E_2^2(t)+E_4^2(t)).$$

Estimate of $T_{2,3}$. Using $(3.2)_3$, $(3.13)_{3,5}$, $(3.24)_1$, Lemma 3.6, Lemma 4.1, Lemma 4.4, Lemma 6.4 and Lemma 4.5, there holds

$$T_{2,3}\leq C\frac{E_2^4+E_4^4+H_1^4+1}{A^{\frac56-\alpha}}+C\epsilon^2(E_2^2(t)+E_4^2(t)).$$

Estimate of $T_{2,4}$. Using $(3.2)_4$, $(3.13)_{1,5}$, Lemma 3.6, Lemma 4.4 and Lemma 6.4, we get

$$T_{2,4}\leq C\frac{E_2^4+H_1^4+1}{A^{\frac12-\frac23\alpha}}+C\epsilon^2E_2^2(t).$$

Estimate of $T_{2,5}$. Using $(3.2)_{3,5}$, $(3.13)_{2,5}$, Lemma 3.6, Lemma 4.4 and Lemma 6.4, we arrive

$$T_{2,5}\leq C\frac{E_2^4+H_1^4+1}{A^{\frac12-\frac23\alpha}}+C\epsilon^2E_2^2(t).$$

Therefore, we infer from (5.14) that

$$\|\partial_x\omega_{2,\neq}\|_{X_a}^2+\|\triangle u_{2,\neq}\|_{X_a}^2\leq C\Big(\|(u_{\rm in})_{\neq}\|_{H^2}^2+\frac{E_2^4+E_4^4+H_1^4+1}{A^{\frac12-\frac23\alpha}}+\epsilon^2(E_2^2(t)+E_4^2(t))\Big).\tag{5.15}$$

For $j\in\{2,3\}$, taking ∂_j to $(5.13)_1$, one gets

$$\begin{aligned}
\mathcal{L}_V\partial_j\omega_{2,\neq}+\partial_j\partial_zu_{2,\neq}=&\frac{\partial_j\partial_zn_{\neq}}{A}-\frac{\partial_j(\partial_z\mathbf{U}_2\partial_xu_{1,\neq})}{A}-\left(\partial_jy+\frac{\partial_j\mathbf{U}_2}{A}\right)\partial_x\omega_{2,\neq}\\
&+\frac{\partial_j\partial_x\left[(u_{\neq}\cdot\nabla u_{3,\neq})_{\neq}+U_0\cdot\nabla u_{3,\neq}+u_{\neq}\cdot\nabla u_{3,0}\right]}{A}\\
&-\frac{\partial_j\partial_z\left[(u_{\neq}\cdot\nabla u_{1,\neq})_{\neq}+U_0\cdot\nabla u_{1,\neq}+u_{\neq}\cdot\nabla(\mathbf{U}_1+\mathbf{U}_2)\right]}{A}.
\end{aligned}$$

Applying Proposition A.1 to it, due to

$$\left\|\partial_j y+\frac{\partial_j \mathbf{U}_2}{A}\right\|_{L^\infty L^\infty}\leq C,$$

we obtain that

$$\begin{aligned}\frac{\|\partial_j\omega_{2,\neq}\|^2_{X_a}}{A^{\frac{2}{3}}}\leq& C\left(\frac{\|(u_{\rm in})_{\neq}\|^2_{H^2}}{A^{\frac{2}{3}}}+\|\triangle u_{2,\neq}\|^2_{X_a}+\|\partial_x\omega_{2,\neq}\|^2_{X_a}+\frac{\|\partial_x n_{\neq}\|^2_{X_b}}{A^{\frac{2}{3}}}+T_{3,1}+T_{3,2}\right)\\&+CA^{-\frac{5}{3}}\|{\rm e}^{aA^{-\frac{1}{3}}t}\partial_z\mathbf{U}_2\partial_x u_{1,\neq}\|^2_{L^2L^2},\end{aligned}\tag{5.16}$$

where

$$\begin{aligned}T_{3,1}&=A^{-\frac{5}{3}}\|{\rm e}^{aA^{-\frac{1}{3}}t}\partial_x\left[(u_{\neq}\cdot\nabla u_{3,\neq})_{\neq}+U_0\cdot\nabla u_{3,\neq}+u_{\neq}\cdot\nabla u_{3,0}\right]\|^2_{L^2L^2},\\T_{3,2}&=A^{-\frac{5}{3}}\|{\rm e}^{aA^{-\frac{1}{3}}t}\partial_z\left[(u_{\neq}\cdot\nabla u_{1,\neq})_{\neq}+U_0\cdot\nabla u_{1,\neq}+u_{\neq}\cdot\nabla(\mathbf{U}_1+\mathbf{U}_2)\right]\|^2_{L^2L^2}.\end{aligned}$$

It is obvious that

$$T_{3,1}\leq T_{2,5}\leq C\frac{E_2^4+H_1^4+1}{A^{\frac{1}{2}-\frac{2}{3}\alpha}}+C\epsilon^2E_2^2(t)$$

Using $(3.2)_6$, $(3.13)_5$, (3.14), Lemma 3.6, Lemma 4.4 and Lemma 6.4, we get

$$T_{3,2}\leq C\frac{E_2^4+E_4^4+H_1^4+1}{A^{\frac{1}{2}-\frac{2}{3}\alpha}}+C\epsilon^2E_2^2(t).$$

By using (5.15), it learns from (5.16) that

$$\frac{\|(\partial_y,\partial_z)\omega_{2,\neq}\|^2_{X_a}}{A^{\frac{2}{3}}}\leq C\left(\|(u_{\rm in})_{\neq}\|^2_{H^2}+\frac{E_2^4+E_4^4+H_1^4+1}{A^{\frac{1}{2}-\frac{2}{3}\alpha}}+\epsilon^2E_2^2(t)+\epsilon^2E_4^2(t)\right).\tag{5.17}$$

When ϵ is small satisfying $C\epsilon^2\leq\frac{1}{2}$ and

$$A\geq\left(E_2^4+E_4^4+H_1^4+1\right)^{\frac{6}{3-4\alpha}}=:A_4,$$

we infer from (5.15) and (5.17) that

$$E_{2,2}^2(t)\leq C\left(\|(u_{\rm in})_{\neq}\|^2_{H^2}+E_4^2(t)+1\right).$$

□

6. Estimates for L^2-norm of the density

First, we derive a lower bound for n that decreases exponentially.

Lemma 6.1. *For all $t\in[0,T]$, there holds*

$$\left\|\frac{1}{n_0(t)}\right\|_{L^\infty(\mathbb{T}^2)}\leq\left\|\frac{1}{n(t)}\right\|_{L^\infty(\mathbb{T}^3)}\leq\delta^{-1}{\rm e}^{\frac{\overline{n}}{A}t},\tag{6.1}$$

where $\delta>0$ is a constant.

Proof. Substituting the minimum point $(x_{\min}(t), y_{\min}(t), z_{\min}(t))$ into $(1.5)_1$ and using $(1.5)_2$, we obtain

$$\begin{aligned}&(\partial_t n)(x_{\min}, y_{\min}, z_{\min})\\ =&\frac{1}{A}(\triangle n)(x_{\min}, y_{\min}, z_{\min}) - y(\partial_x n)(x_{\min}, y_{\min}, z_{\min}) - \frac{1}{A}(u\cdot\nabla n)(x_{\min}, y_{\min}, z_{\min})\\ &-\frac{1}{A}(\nabla n\cdot\nabla c)(x_{\min}, y_{\min}, z_{\min}) - \frac{1}{A}(n\triangle c)(x_{\min}, y_{\min}, z_{\min})\\ \geq& -\frac{1}{A}(n\triangle c)(x_{\min}, y_{\min}, z_{\min}) \geq -\frac{1}{A}\overline{n}n(x_{\min}, y_{\min}, z_{\min}),\end{aligned}$$

which follows that

$$\frac{d}{dt}n_{\min}(t) \geq -\frac{1}{A}\overline{n}n_{\min}(t). \tag{6.2}$$

Due to $n_{\text{in}} > 0$ in $(x, y, z) \in \mathbb{T}^3$, there must be a $\delta > 0$, such that $n_{\text{in}} > \delta > 0$. Therefore, we get by (6.2) that

$$n_{\min}(t) \geq n_{\text{in}}\mathrm{e}^{\int_0^t -\frac{1}{A}\overline{n}dt} \geq \delta\mathrm{e}^{-\frac{\overline{n}}{A}t}.$$

The proof is complete. □

Motivated by [1], we next consider the following 2D free energy of n_0 on $\mathbb{T}^2$:

$$\mathcal{L}[n_0] = \int_{\mathbb{T}^2}\left[n_0\log n_0 - \frac{1}{2}(n_0 - \overline{n})c_0\right]dydz.$$

Lemma 6.2. *Under the conditions of Theorem 1.1 and the assumptions (1.6) and (2.5), there exists a positive constant A_5 independent of t and A, such that if $A \geq A_4$, there holds*

$$\mathcal{L}[n_0(t)] \leq \mathcal{L}[(n_{\text{in}})_0] + C, \quad \text{for} \quad t \in [0, T]. \tag{6.3}$$

Proof. Using (1.5), direct calculation shows that

$$\begin{aligned}\frac{d}{dt}\mathcal{L}[n_0] =& -\frac{1}{A}\int_{\mathbb{T}^2}n_0|\nabla\log n_0 - \nabla c_0|^2dydz - \frac{1}{A}\int_{\mathbb{T}^2}n_0u_0\cdot\nabla c_0dydz\\ &+\frac{1}{A}\int_{\mathbb{T}^2}\left[(n_{\neq}\nabla c_{\neq})_0 + (n_{\neq}u_{\neq})_0\right]\cdot(\nabla\log n_0 - \nabla c_0)\,dydz\\ =:& -\frac{1}{A}\int_{\mathbb{T}^2}n_0|\nabla\log n_0 - \nabla c_0|^2dydz + J_1 + J_2.\end{aligned} \tag{6.4}$$

For $j \in \{2, 3\}$, by (4.19), we decompose $u_{j,0}(t, y, z) = \overline{u}_{j,0}(t) + \widetilde{u}_{j,0}(t, y, z)$, therefore

$$\begin{aligned}J_1 =& -\frac{1}{A}\int_{\mathbb{T}^2}n_0u_{j,0}\partial_jc_0dydz\\ =& -\frac{1}{A}\int_{\mathbb{T}^2}n_0\overline{u}_{j,0}\partial_jc_0dydz - \frac{1}{A}\int_{\mathbb{T}^2}n_0\widetilde{u}_{j,0}\partial_jc_0dydz =: J_{11} + J_{12}.\end{aligned} \tag{6.5}$$

For J_{11}, using $(1.5)_2$ and integration by parts, we have

$$J_{11} = \frac{1}{A}\overline{u}_{j,0}(t)\int_{\mathbb{T}^2}\left(\partial_y^2 + \partial_z^2\right)c_0\partial_jc_0dydz - \frac{1}{A}\overline{u}_{j,0}(t)\overline{n}(t)\int_{\mathbb{T}^2}\partial_jc_0dydz = 0.$$

For J_{12}, due to $\|n_0\|_{L^3}\le \|n_0\|_{L^1}^{\frac13}\|n_0\|_{L^\infty}^{\frac23}$, by Lemma A.8 and Hölder's inequality, there holds

$$\begin{aligned}J_{12}\le&\frac{|\mathbb{T}|}{A}\|n_0\|_{L^\infty}\|\nabla c_0\|_{L^\infty}\|\widetilde{u}_{j,0}\|_{L^2}\le \frac{C}{A}\|n_0\|_{L^\infty}\|n_0-\overline{n}\|_{L^3}\|\widetilde{u}_{j,0}\|_{L^2}\\ \le&\frac{C}{A}\|n_0\|_{L^\infty}\|n_0\|_{L^1}^{\frac13}\|n_0\|_{L^\infty}^{\frac23}\|\widetilde{u}_{j,0}\|_{L^2}\le \frac{C}{A}m^{\frac13}\|n\|_{L^\infty}^{\frac53}\|\widetilde{u}_{j,0}\|_{L^2}.\end{aligned}$$

Combining J_{11} and J_{12} via (6.5), we obtain that

$$J_1\le \frac{C}{A}m^{\frac13}\|n\|_{L^\infty}^{\frac53}\|\widetilde{u}_{j,0}\|_{L^2}.$$

For J_2, using Lemma A.9, (6.1), Hölder's and Young's inequalities, one obtains

$$\begin{aligned}J_2\le&\frac{1}{2A}\int_{\mathbb{T}^2}\frac{|(n_{\neq}\nabla c_{\neq})_0+(n_{\neq}u_{\neq})_0|^2}{n_0}dydz+\frac{1}{2A}\int_{\mathbb{T}^2}n_0\left|\nabla\log n_0-\nabla c_0\right|^2dydz\\ \le&\frac{1}{2A}\left\|\frac{1}{n_0}\right\|_{L^\infty}\|n_{\neq}\|_{L^\infty}^2\left(\|\nabla c_{\neq}\|_{L^2}^2+\|u_{\neq}\|_{L^2}^2\right)+\frac{1}{2A}\int_{\mathbb{T}^2}n_0\left|\nabla\log n_0-\nabla c_0\right|^2dydz\\ \le&\frac{1}{2A\delta}\mathrm{e}^{\frac{\overline{n}}{A}t}\|n_{\neq}\|_{L^\infty}^2\left(\|n_{\neq}\|_{L^2}^2+\|u_{\neq}\|_{L^2}^2\right)+\frac{1}{2A}\int_{\mathbb{T}^2}n_0\left|\nabla\log n_0-\nabla c_0\right|^2dydz.\end{aligned}$$

Combining J_1 and J_2, we get by (6.4) that

$$\begin{aligned}\frac{d}{dt}\mathcal{L}[n_0]\le&-\frac{1}{2A}\int_{\mathbb{T}^2}n_0\left|\nabla\log n_0-\nabla c_0\right|^2dydz+\frac{C}{A}m^{\frac13}\|n\|_{L^\infty}^{\frac53}\|\widetilde{u}_{j,0}\|_{L^2}\\ &+\frac{1}{2A\delta}\mathrm{e}^{\frac{\overline{n}}{A}t}\|n_{\neq}\|_{L^\infty}^2\left(\|n_{\neq}\|_{L^2}^2+\|u_{\neq}\|_{L^2}^2\right),\end{aligned}$$

which follows that

$$\begin{aligned}&\mathcal{L}[n_0]-\mathcal{L}[(n_{\mathrm{in}})_0]\\ \le&-\frac{1}{2A}\int_0^t\int_{\mathbb{T}^2}n_0\left|\nabla\log n_0-\nabla c_0\right|^2dydzds+\frac{C}{A}m^{\frac13}\|n\|_{L^\infty L^\infty}^{\frac53}\int_0^t\|\widetilde{u}_{j,0}\|_{L^2}ds\\ &+\frac{1}{2A\delta}\|n_{\neq}\|_{L^\infty L^\infty}^2\int_0^t\mathrm{e}^{\frac{\overline{n}}{A}s}\left(\|n_{\neq}\|_{L^2}^2+\|u_{\neq}\|_{L^2}^2\right)ds.\end{aligned}\tag{6.6}$$

Using Lemma 4.3, we have

$$\frac{\int_0^t\|\widetilde{u}_{j,0}\|_{L^2}ds}{A}\le\frac{C\epsilon\int_0^t\mathrm{e}^{-\frac{s}{2A}}ds}{A}\le C\epsilon.\tag{6.7}$$

Moreover, by Lemma A.1, Lemma 3.1 and assumptions (2.5), one deduces

$$\mathrm{e}^{2aA^{-\frac13}t}\|u_{\neq}\|_{L^2}^2\le C\mathrm{e}^{2aA^{-\frac13}t}\left(\|\partial_x\omega_{2,\neq}\|_{L^2}^2+\|\triangle u_{2,\neq}\|_{L^2}^2\right)\le CE_2^2,$$

and $\mathrm{e}^{2aA^{-\frac13}t}\|n_{\neq}\|_{L^2}^2\le C\|\partial_x^2n_{\neq}\|_{X_b}^2\le CE_2^2$. They imply that

$$\|u_{\neq}\|_{L^2}^2\le CE_2^2\mathrm{e}^{-2aA^{-\frac13}t},\quad \|n_{\neq}\|_{L^2}^2\le CE_2^2\mathrm{e}^{-2aA^{-\frac13}t}.$$

Therefore, when A is sufficiently large satisfying $A \geq \left(\frac{\overline{n}}{a}\right)^{\frac{3}{2}}$, we get

$$\int_0^t \mathrm{e}^{\frac{\overline{n}}{A}s}\left(\|n_{\neq}\|_{L^2}^2+\|u_{\neq}\|_{L^2}^2\right)ds \leq CE_2^2\int_0^{\infty}\mathrm{e}^{\frac{\overline{n}}{A}s-2aA^{-\frac{1}{3}}s}ds \leq CE_2^2\int_0^{\infty}\mathrm{e}^{-aA^{-\frac{1}{3}}s}ds=\frac{CE_2^2A^{\frac{1}{3}}}{a}.$$

Combining it with (2.5) and (6.7), (6.6) yields

$$\mathcal{L}[n_0]-\mathcal{L}[(n_{\rm in})_0] \leq -\frac{1}{2A}\int_0^t\int_{\mathbb{T}^2} n_0|\nabla\log n_0-\nabla c_0|^2dydzds+C\epsilon m^{\frac{1}{3}}E_3^{\frac{5}{3}}+\frac{CE_2^2E_3^2}{A^{\frac{2}{3}}a\delta}.$$

Hence, when

$$A \geq \max\left\{A_2, A_3, A_4, \left(\frac{E_2^2E_3^2}{a\delta}\right)^{\frac{3}{2}}, \left(\frac{\overline{n}}{a}\right)^{\frac{3}{2}}, \left(\frac{1}{4a}\right)^{\frac{3}{2}}\right\}=: A_5,$$

as long as ϵ is enough small satisfying

$$\epsilon m^{\frac{1}{3}}E_3^{\frac{5}{3}} \leq C, \tag{6.8}$$

we obtain that

$$\mathcal{L}[n_0]-\mathcal{L}[(n_{\rm in})_0] \leq C.$$

□

Next, as in [1] or [17], we use (6.3) and (A.6) to get a bound on $\|n_0\log^+ n_0\|_{L^1}$.

Lemma 6.3. *Under the assumptions of Lemma 6.2 and $m=\|(n_{\rm in})_0\|_{L^1}<8\pi$, there exists a constant $C_{L\log L}(n_{\rm in})$ such that*

$$\int_{\mathbb{T}^2} n_0\log^+ n_0dydz \leq C_{L\log L}(n_{\rm in}). \tag{6.9}$$

Proof. Let $Y=(y,z)\in\mathbb{T}^2$ be fixed. Define the cut-off function $\varphi(\tau)\in C^{\infty}$ such that

$$\begin{aligned} {\rm supp}(\varphi)&=B(Y,1/4),\\ \varphi(\tau)&=1,\quad \forall\tau\in B(Y,1/8),\\ {\rm supp}\left(\nabla\varphi(\tau)\right)&\subset\overline{B}(Y,1/4)\backslash B(Y,1/8). \end{aligned} \tag{6.10}$$

By periodically extending $n_0(\tau)$ and $c_0(\tau)$ to $\mathbb{R}^2$, we can rewrite the equation $-\triangle c_0=n_0-\overline{n}$ that holds on $\mathbb{T}^2$ as the following equation that holds on $\mathbb{R}^2$:

$$\begin{aligned} -\triangle_{\tau}\left(\varphi(\tau)c_0(\tau)\right)=&-2\nabla_{\tau}\varphi(\tau)\cdot\nabla_{\tau}c_0(\tau)-c_0(\tau)\triangle_{\tau}\varphi(\tau)-\varphi(\tau)\triangle_{\tau}c_0(\tau)\\ =&-2\nabla_{\tau}\varphi(\tau)\cdot\nabla_{\tau}c_0(\tau)-c_0(\tau)\triangle_{\tau}\varphi(\tau)+\left(n_0(\tau)-\overline{n}\right)\varphi(\tau). \end{aligned} \tag{6.11}$$

Since $\varphi(Y) = 1$ as $Y \in B(Y, 1/8)$ and $\text{supp}(\varphi) = B(Y, 1/4)$, using (6.11) and the fundamental solution of the Laplacian on $\mathbb{R}^2$, we get

$$\begin{aligned}
&c_0(Y) = c_0(Y)\varphi(Y) \\
=& -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log(|Y-\tau|)\left[(n_0(\tau)-\overline{n})\varphi(\tau) - 2\nabla_\tau\varphi(\tau)\cdot\nabla_\tau c_0(\tau) - c_0(\tau)\triangle_\tau\varphi(\tau)\right] d\tau \\
=& -\frac{1}{2\pi}\int_{|Y-\tau|\le\frac{1}{4}} \log(|Y-\tau|)(n_0(\tau)-\overline{n})\varphi(\tau)d\tau - \frac{1}{\pi}\int_{|Y-\tau|\le\frac{1}{4}} \nabla_\tau\cdot\left[\log(|Y-\tau|)\nabla_\tau\varphi(\tau)\right]c_0(\tau)d\tau \\
&+\frac{1}{2\pi}\int_{|Y-\tau|\le\frac{1}{4}} \log(|Y-\tau|)\triangle_\tau\varphi(\tau)c_0(\tau)d\tau.
\end{aligned} \tag{6.12}$$

Due to the support of φ, we can identify the above with an analogous integral on $\mathbb{T}^2$ with $|Y-\tau|$ replaced by $d(Y,\tau)$. Multiplying (6.12) by $-\frac{1}{2}(n_0(Y)-\overline{n})$ and integrating with respect to Y over $\mathbb{T}^2$, we have

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{T}^2}(n_0(Y)-\overline{n})\,c_0(Y)dY \\
=&\frac{1}{4\pi}\int\int_{\mathbb{T}^2\times\mathbb{T}^2, d(Y,\tau)\le\frac{1}{4}} \log d(Y,\tau)\,(n_0(Y)-\overline{n})\,(n_0(\tau)-\overline{n})\,\varphi(\tau)d\tau dY \\
&+\frac{1}{2\pi}\int\int_{\mathbb{T}^2\times\mathbb{T}^2, \frac{1}{8}\le d(Y,\tau)\le\frac{1}{4}} (n_0(Y)-\overline{n})\,\nabla_\tau\cdot\left(\log d(Y,\tau)\nabla_\tau\varphi(\tau)\right)c_0(\tau)d\tau dY \\
&-\frac{1}{4\pi}\int\int_{\mathbb{T}^2\times\mathbb{T}^2, \frac{1}{8}\le d(Y,\tau)\le\frac{1}{4}} (n_0(Y)-\overline{n})\log d(Y,\tau)\triangle_\tau\varphi(\tau)c_0(\tau)d\tau dY.
\end{aligned} \tag{6.13}$$

By using $(6.10)_2$, one deduces that

$$\begin{aligned}
&\frac{1}{4\pi}\int\int_{d(Y,\tau)\le\frac{1}{4}} \log d(Y,\tau)\,(n_0(Y)-\overline{n})\,(n_0(\tau)-\overline{n})\,\varphi(\tau)d\tau dY \\
=&\frac{1}{4\pi}\int\int_{\mathbb{T}^2\times\mathbb{T}^2} \log d(Y,\tau)n_0(Y)n_0(\tau)d\tau dY - \frac{1}{4\pi}\int\int_{d(Y,\tau)>\frac{1}{8}} \log d(Y,\tau)n_0(Y)n_0(\tau)d\tau dY \\
&-\frac{1}{2\pi}\overline{n}\int\int_{d(Y,\tau)\le\frac{1}{8}} \log d(Y,\tau)n_0(Y)d\tau dY + \frac{1}{4\pi}(\overline{n})^2\int\int_{d(Y,\tau)\le\frac{1}{8}} \log d(Y,\tau)d\tau dY \\
&+\frac{1}{4\pi}\int\int_{\frac{1}{8}\le d(Y,\tau)\le\frac{1}{4}} \log d(Y,\tau)\,(n_0(Y)-\overline{n})\,(n_0(\tau)-\overline{n})\,\varphi(\tau)d\tau dY.
\end{aligned}$$

Combining it with (6.13), we have

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{T}^2}(n_0(Y)-\overline{n})\,c_0(Y)dY\\
=&\frac{1}{4\pi}\int\int_{\mathbb{T}^2\times\mathbb{T}^2}\log d(Y,\tau)n_0(Y)n_0(\tau)d\tau dY-\frac{1}{4\pi}\int\int_{d(Y,\tau)>\frac{1}{8}}\log d(Y,\tau)n_0(Y)n_0(\tau)d\tau dY\\
&-\frac{1}{2\pi}\overline{n}\int\int_{d(Y,\tau)\le\frac{1}{8}}\log d(Y,\tau)n_0(Y)d\tau dY+\frac{1}{4\pi}(\overline{n})^2\int\int_{d(Y,\tau)\le\frac{1}{8}}\log d(Y,\tau)d\tau dY\\
&+\frac{1}{4\pi}\int\int_{\frac{1}{8}\le d(Y,\tau)\le\frac{1}{4}}\log d(Y,\tau)\,(n_0(Y)-\overline{n})\,(n_0(\tau)-\overline{n})\,\varphi(\tau)d\tau dY\\
&+\frac{1}{2\pi}\int\int_{\frac{1}{8}\le d(Y,\tau)\le\frac{1}{4}}(n_0(Y)-\overline{n})\,\nabla_\tau\cdot(\log d(Y,\tau)\nabla_\tau\varphi(\tau))\,c_0(\tau)d\tau dY\\
&-\frac{1}{4\pi}\int\int_{\frac{1}{8}\le d(Y,\tau)\le\frac{1}{4}}(n_0(Y)-\overline{n})\log d(Y,\tau)\triangle_\tau\varphi(\tau)c_0(\tau)d\tau dY=:I_{1,1}+I_{1,2}+\cdots+I_{1,7}.
\end{aligned}$$

First of all, direct calculations show that

$$I_{1,2}+I_{1,3}+I_{1,4}+I_{1,5}\ge -Cm^2.$$

Moreover, in the region $\frac{1}{8}\le|Y-\tau|\le\frac{1}{4}$, note that

$$\begin{aligned}
&|\nabla_\tau\cdot[\log(|Y-\tau|)\nabla_\tau\varphi(\tau)]|\le 8|\nabla_\tau\varphi(\tau)|+\log 8|\triangle_\tau\varphi(\tau)|\le C,\\
&|\log(|Y-\tau|)\triangle_\tau\varphi(\tau)|\le\log 8|\triangle_\tau\varphi(\tau)|\le C,
\end{aligned}$$

we have

$$|I_{1,6}|+|I_{1,7}|\le C\int_{\mathbb{T}^2}n_0(Y)dY\int_{\mathbb{T}^2}c_0(\tau)d\tau+C\overline{n}\int_{\mathbb{T}^2}c_0(\tau)d\tau\le Cm\|c_0\|_{L^1(\mathbb{T}^2)}.$$

Denoting $K(\tau)=-\frac{1}{2\pi}\log|\tau|$ to be the fundamental solution of the Laplacian on $\mathbb{T}^2$, as $-\triangle c_0=n_0-\overline{n}$, by Young's inequality, we obtain

$$\|c_0\|_{L^1(\mathbb{T}^2)}=\|K*(n_0-\overline{n})\|_{L^1(\mathbb{T}^2)}\le\|K\|_{L^1(\mathbb{T}^2)}\|n_0-\overline{n}\|_{L^1(\mathbb{T}^2)}\le Cm.$$

This follows that

$$I_{1,6}+I_{1,7}\ge -Cm^2.$$

Combining the estimates of $I_{1,2}-I_{1,7}$, we conclude that

$$-\frac{1}{2}\int_{\mathbb{T}^2}(n_0-\overline{n})c_0dY\ge\frac{1}{4\pi}\int\int_{\mathbb{T}^2\times\mathbb{T}^2}\log d(Y,\tau)n_0(Y)n_0(\tau)d\tau dY-Cm^2.$$

From which along with (6.3), we arrive at

$$\begin{aligned}
\mathcal{L}[(n_{\rm in})_0] &\geq \int_{\mathbb{T}^2} n_0 \log n_0 dY - \int_{\mathbb{T}^2} \frac{1}{2}(n_0-\overline{n})c_0 dY - C \\
&\geq \int_{\mathbb{T}^2} n_0 \log n_0 dY + \frac{1}{4\pi}\int_{\mathbb{T}^2\times\mathbb{T}^2} \log d(Y,\tau) n_0(Y) n_0(\tau) d\tau dY - Cm^2 - C \\
&= \left(1-\frac{m}{8\pi}\right)\int_{\mathbb{T}^2} n_0 \log n_0 dY \\
&\quad + \frac{m}{8\pi}\left(\int_{\mathbb{T}^2} n_0 \log n_0 dY + \frac{2}{m}\int\int_{\mathbb{T}^2\times\mathbb{T}^2} \log d(Y,\tau) n_0(Y) n_0(\tau) d\tau dY\right) - Cm^2 - C.
\end{aligned}$$

Applying (A.6) to it, one obtains

$$\mathcal{L}[(n_{\rm in})_0] \geq \left(1-\frac{m}{8\pi}\right)\int_{\mathbb{T}^2} n_0 \log n_0 dY - C(m) - Cm^2,$$

which implies that

$$\int_{\mathbb{T}^2} n_0 \log n_0 dY \leq \frac{\mathcal{L}[(n_{\rm in})_0] + C(m) + Cm^2}{1-\frac{m}{8\pi}} \leq C_{L\log L}(n_{\rm in}) < \infty$$

under the condition of $m < 8\pi$. Due to

$$\log^- n_0 = -\min\{0, \log n_0\} = \begin{cases} -\log n_0, & 0 < n_0 < 1, \\ 0, & n_0 \geq 1, \end{cases}$$

there holds

$$\int_{\mathbb{T}^2} n_0 \log^+ n_0 dY = \begin{cases} 0, & 0 < n_0 < 1, \\ \int_{\mathbb{T}^2} n_0 \log n_0 dY, & n_0 \geq 1. \end{cases}$$

This shows that

$$\int_{\mathbb{T}^2} n_0 \log^+ n_0 dY \leq C_{L\log L}(n_{\rm in}) < \infty.$$

The proof is complete. □

The following lemma gives a uniform in time L^2 bound of n_0.

Lemma 6.4. *Under the assumptions of Lemma 6.2, there holds*

$$\|n_0\|_{L^2} \leq C\left(\|(n_{\rm in})_0\|_{L^2} + m + 1\right) =: H_1. \tag{6.14}$$

Proof. Let $Q > \max\{1, \overline{n}\}$ be a constant, to be chosen later. Noting that $(n_0-Q)_+ = \max\{0, n_0 - Q\}$, and using (6.9), we have

$$\begin{aligned}
&\int_{\mathbb{T}^2} (n_0-Q)_+ dydz = \int_{n_0>Q} (n_0-Q) dydz \leq \int_{n_0>Q} n_0 dydz \\
&= \int_{n_0>Q} \frac{1}{\log^+ n_0} n_0 \log^+ n_0 dydz \leq \frac{1}{\log Q}\int_{n_0>Q} n_0 \log^+ n_0 dydz \leq \frac{C_{L\log L}}{\log Q}.
\end{aligned} \tag{6.15}$$

Recall that n_0 satisfies

$$\partial_t n_0 = \frac{1}{A}\triangle n_0 - \frac{1}{A}\nabla\cdot(n_0\nabla c_0) - \frac{1}{A}\nabla\cdot(n_{\neq}\nabla c_{\neq})_0 - \frac{1}{A}(u_0\cdot\nabla n_0) - \frac{1}{A}(u_{\neq}\cdot\nabla n_{\neq})_0. \tag{6.16}$$

As $(n_0-Q)_+(n_0-Q)_-=0$, multiplying (6.16) by $(n_0-Q)_+$ and integrating with respect to (y,z) over $\mathbb{T}^2$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(n_0-Q)_+\|^2_{L^2(\mathbb{T}^2)}=&\frac{1}{A}\Big(\int_{\mathbb{T}^2}(n_0-Q)_+\triangle n_0dydz-\int_{\mathbb{T}^2}(n_0-Q)_+\nabla\cdot(n_0\nabla c_0)dydz\\&-\int_{\mathbb{T}^2}(n_0-Q)_+\nabla\cdot(n_{\neq}\nabla c_{\neq})_0dydz-\int_{\mathbb{T}^2}(n_0-Q)_+(u_0\cdot\nabla n_0)dydz\\&-\int_{\mathbb{T}^2}(n_0-Q)_+(u_{\neq}\cdot\nabla n_{\neq})_0dydz\Big)=:J_1+\cdots+J_5.\end{aligned}\tag{6.17}$$

For J_1, integration by parts shows that

$$J_1=-\frac{1}{A}\int_{\mathbb{T}^2}|\nabla((n_0-Q)_+)|^2dydz.$$

For J_2, using (1.5) and integration by parts, we have

$$\begin{aligned}J_2=&\frac{1}{2A}\int_{\mathbb{T}^2}\left((n_0-Q)_+\right)^2(n_0-\overline{n})dydz+\frac{Q}{A}\int_{\mathbb{T}^2}(n_0-Q)_+(n_0-\overline{n})dydz\\=&\frac{1}{2A}\int_{\mathbb{T}^2}\left((n_0-Q)_+\right)^3dydz+\frac{3Q-\overline{n}}{2A}\int_{\mathbb{T}^2}\left((n_0-Q)_+\right)^2dydz+\frac{Q^2-Q\overline{n}}{A}\int_{\mathbb{T}^2}(n_0-Q)_+dydz\\\leq&\frac{1}{2A}\int_{\mathbb{T}^2}\left((n_0-Q)_+\right)^3dydz+\frac{3Q}{2A}\int_{\mathbb{T}^2}\left((n_0-Q)_+\right)^2dydz+\frac{Q^2m}{A}.\end{aligned}$$

For J_3 and J_5, by integration by parts, one deduces

$$J_3+J_5\leq\frac{1}{16A}\int_{\mathbb{T}^2}|\nabla\left((n_0-Q)_+\right)|^2dydz+\frac{C(\|(n_{\neq}\nabla c_{\neq})_0\|^2_{L^2}+\|(n_{\neq}u_{\neq})_0\|^2_{L^2})}{A}.$$

For J_4, using $\nabla\cdot u_0=0$, for $j\in\{2,3\}$, we have

$$\begin{aligned}J_4=&-\frac{1}{A}\int_{\mathbb{T}^2}(n_0-Q)_+u_{j,0}\partial_jn_0dydz=-\frac{1}{A}\int_{\mathbb{T}^2}(n_0-Q)_+u_{j,0}\partial_j(n_0-Q)_+dydz\\=&-\frac{1}{2A}\int_{\mathbb{T}^2}u_{j,0}\partial_j((n_0-Q)_+)^2dydz=0.\end{aligned}$$

Collecting J_1-J_5, we get by (6.17) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(n_0-Q)_+\|^2_{L^2}\leq&-\frac{7\|\nabla(n_0-Q)_+\|^2_{L^2}}{8A}+\frac{\|(n_0-Q)_+\|^3_{L^3}}{2A}+\frac{3Q\|(n_0-Q)_+\|^2_{L^2}}{2A}\\&+\frac{Q^2m}{A}+\frac{C}{A}\left(\|(n_{\neq}\nabla c_{\neq})_0\|^2_{L^2}+\|(u_{\neq}n_{\neq})_0\|^2_{L^2}\right).\end{aligned}\tag{6.18}$$

Using Lemma A.7 and (6.15), one obtains

$$\begin{aligned}\|(n_0-Q)_+\|^3_{L^3(\mathbb{T}^2)}\leq&C\|(n_0-Q)_+\|_{L^1(\mathbb{T}^2)}\|\nabla\left((n_0-Q)_+\right)\|^2_{L^2(\mathbb{T}^2)}\\\leq&\frac{C_{L\log L}}{\log Q}\|\nabla\left((n_0-Q)_+\right)\|^2_{L^2(\mathbb{T}^2)}.\end{aligned}$$

Thus, we choose Q depending only on $C_{L\log L}$ such that

$$-\frac{7}{8A}\|\nabla\left((n_0-Q)_+\right)\|^2_{L^2(\mathbb{T}^2)}+\frac{1}{2A}\|(n_0-Q)_+\|^3_{L^3(\mathbb{T}^2)}\leq-\frac{1}{2A}\|\nabla\left((n_0-Q)_+\right)\|^2_{L^2(\mathbb{T}^2)}.\tag{6.19}$$

Similarly, it follows from Lemma A.7 and (6.15) that

$$-\|\nabla\left((n_0-Q)_+\right)\|_{L^2(\mathbb{T}^2)}^2\leq-\frac{\|(n_0-Q)_+\|_{L^2(\mathbb{T}^2)}^4}{C\|(n_0-Q)_+\|_{L^1(\mathbb{T}^2)}^2}\leq-\frac{1}{Cm^2}\|(n_0-Q)_+\|_{L^2(\mathbb{T}^2)}^4. \tag{6.20}$$

Substituting (6.19) and (6.20) into (6.18), we get

$$\begin{aligned}\frac{d}{dt}\|(n_0-Q)_+\|_{L^2(\mathbb{T}^2)}^2\leq&-\frac{1}{ACm^2}\|(n_0-Q)_+\|_{L^2(\mathbb{T}^2)}^4+\frac{3Q}{A}\|(n_0-Q)_+\|_{L^2(\mathbb{T}^2)}^2\\&+\frac{2Q^2m}{A}+\frac{C}{A}\left(\|(n_{\neq}\nabla c_{\neq})_0\|_{L^2(\mathbb{T}^2)}^2+\|(u_{\neq}n_{\neq})_0\|_{L^2(\mathbb{T}^2)}^2\right).\end{aligned} \tag{6.21}$$

We denote $G(t)$ by

$$G(t):=\frac{C}{A}\int_0^t\left(\|(n_{\neq}\nabla c_{\neq})_0\|_{L^2(\mathbb{T}^2)}^2+\|(u_{\neq}n_{\neq})_0\|_{L^2(\mathbb{T}^2)}^2\right)ds,\quad\text{for}\quad t\geq0.$$

Then using Lemma A.1, Lemma A.9, Lemma 3.1 and assumption (2.5), direct calculations indicate that

$$\begin{aligned}G(t)\leq&\frac{C}{A}\left(\|n_{\neq}\|_{L^\infty L^\infty}^2\|\nabla c_{\neq}\|_{L^2L^2}^2+\|n_{\neq}\|_{L^\infty L^\infty}^2\|u_{\neq}\|_{L^2L^2}^2\right)\\\leq&\frac{C}{A^{\frac{2}{3}}}\|n\|_{L^\infty L^\infty}^2\left(\|\partial_x n_{\neq}\|_{X_a}^2+\|\partial_x\omega_{2,\neq}\|_{X_a}^2+\|\triangle u_{2,\neq}\|_{X_a}^2\right)\leq\frac{CE_2^2E_3^2}{A^{\frac{2}{3}}}\leq C\end{aligned} \tag{6.22}$$

provided with $A\geq A_5$. Moreover, using Young's inequality, we rewrite (6.21) as

$$\begin{aligned}\frac{d}{dt}\left(\|(n_0-Q)_+\|_{L^2}^2-G(t)\right)\leq&-\frac{\left[\|(n_0-Q)_+\|_{L^2}^2-G(t)-(9C^2m^4Q^2+4Cm^3Q^2)^{\frac{1}{2}}\right]}{2ACm^2}\\&\times\left[\|(n_0-Q)_+\|_{L^2}^2+\left(9C^2m^4Q^2+4Cm^3Q^2\right)^{\frac{1}{2}}\right],\end{aligned}$$

which implies that

$$\|(n_0-Q)_+\|_{L^2}^2-G(t)\leq\|(n_{\rm in})_0\|_{L^2}^2+2\left(9C^2m^4Q^2+4Cm^3Q^2\right)^{\frac{1}{2}}.$$

Combining it with (6.22), one deduces

$$\|(n_0-Q)_+\|_{L^2}\leq C\left(\|(n_{\rm in})_0\|_{L^2}+m+1\right). \tag{6.23}$$

By decomposing $n_0=(n_0-Q)_++\min\{n_0,Q\}$ and using (6.23), we get

$$\begin{aligned}\|n_0\|_{L^2}\leq&\|(n_0-Q)_+\|_{L^2}+\|\min\{n_0,Q\}\|_{L^2}\\\leq&\|(n_0-Q)_+\|_{L^2}+Q^{\frac{1}{2}}m^{\frac{1}{2}}\leq C\left(\|(n_{\rm in})_0\|_{L^2}+m+1\right),\end{aligned}$$

which gives the result.

The proof is complete. □

7. Estimates for L^∞-norm of the density $E_3(t)$: Proof of Proposition 2.2

Proof of Proposition 2.2. For $p=2^j$ with $j\geq 1$, multiplying $(1.5)_1$ by $2pn^{2p-1}$, and integrating by parts the resulting equation over $\mathbb{T}^3$, one deduces

$$\begin{aligned}&\frac{d}{dt}\|n^p\|_{L^2}^2+\frac{2(2p-1)}{Ap}\|\nabla n^p\|_{L^2}^2=\frac{2(2p-1)}{A}\int_{\mathbb{T}^3}n^p\nabla c\cdot\nabla n^p dxdydz\\ \leq&\frac{2(2p-1)}{A}\|n^p\nabla c\|_{L^2}\|\nabla n^p\|_{L^2}\leq\frac{2p-1}{Ap}\|\nabla n^p\|_{L^2}^2+\frac{(2p-1)p}{A}\|n^p\nabla c\|_{L^2}^2.\end{aligned}$$

Using Hölder's inequality and Gagliardo-Nirenberg inequality, we get

$$\|n^p\nabla c\|_{L^2}^2\leq\|n^p\|_{L^4}^2\|\nabla c\|_{L^4}^2\leq C\|n^p\|_{L^2}^{\frac12}\|\nabla n^p\|_{L^2}^{\frac32}\|\nabla c\|_{L^4}^2+C\|n^p\|_{L^2}^2\|\nabla c\|_{L^4}^2,$$

which follows that

$$\begin{aligned}&\frac{d}{dt}\|n^p\|_{L^2}^2+\frac{2(2p-1)}{Ap}\|\nabla n^p\|_{L^2}^2\\ \leq&\frac{2p-1}{Ap}\|\nabla n^p\|_{L^2}^2+\frac{C(2p-1)p}{A}\|n^p\|_{L^2}^{\frac12}\|\nabla n^p\|_{L^2}^{\frac32}\|\nabla c\|_{L^4}^2+\frac{C(2p-1)p}{A}\|n^p\|_{L^2}^2\|\nabla c\|_{L^4}^2\\ \leq&\frac{5(2p-1)}{4Ap}\|\nabla n^p\|_{L^2}^2+\frac{C(2p-1)p^7}{A}\|n^p\|_{L^2}^2\|\nabla c\|_{L^4}^8+\frac{C(2p-1)p}{A}\|n^p\|_{L^2}^2\|\nabla c\|_{L^4}^2.\end{aligned}$$

Consequently, there holds

$$\frac{d}{dt}\|n^p\|_{L^2}^2+\frac{1}{2A}\|\nabla n^p\|_{L^2}^2\leq\frac{Cp^8}{A}\|n^p\|_{L^2}^2\left(1+\|\nabla c\|_{L^4}^8\right).\tag{7.1}$$

Using Gagliardo-Nirenberg inequality

$$\|n^p\|_{L^2}\leq C\left(\|n^p\|_{L^1}^{\frac25}\|\nabla n^p\|_{L^2}^{\frac35}+\|n^p\|_{L^1}\right),$$

we infer from (7.1) that

$$\frac{d}{dt}\|n^p\|_{L^2}^2\leq-\frac{\|n^p\|_{L^2}^{\frac{10}{3}}}{2AC\|n^p\|_{L^1}^{\frac43}}+\frac{Cp^8}{A}\|n^p\|_{L^2}^2\left(1+\|\nabla c\|_{L^\infty L^4}^8\right).$$

Applying Lemma A.8, Lemma A.9, Lemma 6.4 and the assumption (1.6), there holds

$$\begin{aligned}\|\nabla c\|_{L^\infty L^4}\leq&\|\nabla c_{\neq}\|_{L^\infty L^4}+\|\nabla c_0\|_{L^\infty L^4}\\ \leq&C\left(\|\partial_x^2 n_{\neq}\|_{L^\infty L^2}+\|n_0\|_{L^\infty L^2}\right)\leq C\left(E_2+H_1\right).\end{aligned}$$

Therefore

$$\frac{d}{dt}\|n^p\|_{L^2}^2\leq-\frac{\|n^p\|_{L^2}^{\frac{10}{3}}}{2CA\|n^p\|_{L^1}^{\frac43}}+\frac{Cp^8}{A}\|n^p\|_{L^2}^2(1+E_2^8+H_1^8),$$

which indicates that

$$\sup_{t\geq0}\|n^p\|_{L^2}^2\leq\max\left\{8C^3(1+E_2^8+H_1^8)^{\frac32}p^{12}\sup_{t\geq0}\|n^p\|_{L^1}^2,2\|n_{\rm in}^p\|_{L^2}^2\right\}.\tag{7.2}$$

Next, the Moser-Alikakos iteration is used to determine E_3. Recall $p = 2^j$ with $j \geq 1$, and we rewrite (7.2) into

$$\sup_{t\geq 0}\int_{\mathbb{T}^3}|n(t)|^{2^{j+1}}dxdydz \leq \max\Big\{C_1p^{12}\Big(\sup_{t\geq 0}\int_{\mathbb{T}^3}|n(t)|^{2^j}dxdydz\Big)^2, 2\int_{\mathbb{T}^3}|n_{\rm in}|^{2^{j+1}}dxdydz\Big\}, \quad (7.3)$$

where $C_1 = 8C^3(1+E_2^8+H_1^8)^{\frac{3}{2}}$. From Lemma 6.4, we note that

$$\|n_0\|_{L^\infty L^2} \leq H_1.$$

Hence

$$\sup_{t\geq 0}\|n(t)\|_{L^2} \leq |\mathbb{T}|\|n_0\|_{L^\infty L^2} + \|n_{\neq}\|_{L^\infty L^2} \leq |\mathbb{T}|H_1 + E_2.$$

By interpolation inequality, for $0<\theta<1$, we have

$$\|n_{\rm in}\|_{L^{2^j}} \leq \|n_{\rm in}\|_{L^2}^{\theta}\|n_{\rm in}\|_{L^\infty}^{1-\theta} \leq \|n_{\rm in}\|_{L^2} + \|n_{\rm in}\|_{L^\infty} \leq |\mathbb{T}|H_1 + E_2 + \|n_{\rm in}\|_{L^\infty}$$

for $j \geq 1$. This yields that

$$2\int_{\mathbb{T}^3}|n_{\rm in}|^{2^{j+1}}dxdydz \leq 2\left(|\mathbb{T}|H_1 + E_2 + \|n_{\rm in}\|_{L^\infty}\right)^{2^{j+1}} \leq K^{2^{j+1}},$$

where $K = 2(|\mathbb{T}|H_1 + E_2 + \|n_{\rm in}\|_{L^\infty})$.

Now, we rewrite (7.3) as

$$\sup_{t\geq 0}\int_{\mathbb{T}^3}|n(t)|^{2^{j+1}}dxdydz \leq \max\left\{C_1 4096^j\left(\sup_{t\geq 0}\int_{\mathbb{T}^3}|n(t)|^{2^j}dxdydz\right)^2, K^{2^{j+1}}\right\}.$$

For $j=k$, we get

$$\sup_{t\geq 0}\int_{\mathbb{T}^3}|n(t)|^{2^{k+1}}dxdydz \leq C_1^{a_k}4096^{b_k}K^{2^{k+1}},$$

where $a_k = 1 + 2a_{k-1}$ and $b_k = k + 2b_{k-1}$.

Generally, one can obtain the following formulas

$$a_k = 2^k - 1, \text{ and } b_k = 2^{k+1} - k - 2.$$

Therefore, one deduces

$$\sup_{t\geq 0}\left(\int_{\mathbb{T}^3}|n(t)|^{2^{k+1}}dxdydz\right)^{\frac{1}{2^{k+1}}} \leq C_1^{\frac{2^k-1}{2^{k+1}}}4096^{\frac{2^{k+1}-k-2}{2^{k+1}}}K.$$

Letting $k\to\infty$, there holds

$$\sup_{t\geq 0}\|n(t)\|_{L^\infty} \leq C(1+E_2^8+H_1^8)^{\frac{3}{4}}(|\mathbb{T}|H_1 + E_2 + ||n_{\rm in}||_{L^\infty}) =: E_3. \quad (7.4)$$

□

8. Energy estimates for $E_4(t)$: Proof of Proposition 2.3

To estimate $\|\partial_x^2 u_{2,\neq}\|_{X_b}$ and $\|\partial_x^2 u_{3,\neq}\|_{X_b}$, it is important to introduce the new quantity W defined by

$$W = u_{2,\neq} + \kappa u_{3,\neq},$$

where

$$V = y + \frac{\mathbf{U}_2}{A}, \quad \text{and } \kappa = \frac{\partial_z V}{\partial_y V}. \tag{8.1}$$

The similar quality was first proposed by Wei-Zhang in [36] and further applied in [6] and [10].

For $j \in \{2,3\}$, there holds $(u\cdot\nabla u_j)_{\neq} = u_0\cdot\nabla u_{j,\neq} + u_{\neq}\cdot\nabla u_{j,0} + (u_{\neq}\cdot\nabla u_{j,\neq})_{\neq}$. Then we infer from (1.3) that

$$\begin{cases} \mathcal{L}_V u_{2,\neq} + \frac{\mathbf{U}_1\partial_x u_{2,\neq}}{A} + \frac{g_{2,1}+g_{2,2}+G_{2,3}}{A} + \frac{\partial_y(P^{N_1}_{\neq}+P^{N_2}_{\neq})}{A} = 0, \\ \mathcal{L}_V u_{3,\neq} + \frac{\mathbf{U}_1\partial_x u_{3,\neq}}{A} + \frac{g_{3,1}+g_{3,2}+G_{3,3}}{A} + \frac{\partial_z(P^{N_1}_{\neq}+P^{N_2}_{\neq})}{A} = 0, \end{cases} \tag{8.2}$$

where $\mathcal{L}_V$ can be found in (5.10) and

$$g_{j,1} = u_{2,0}\partial_y u_{j,\neq} + u_{3,0}\partial_z u_{j,\neq}, \quad g_{j,2} = u_{\neq}\cdot\nabla u_{j,0}, \quad G_{j,3} = (u_{\neq}\cdot\nabla u_{j,\neq})_{\neq}. \tag{8.3}$$

Due to div $u = 0$, we have

$$\begin{aligned} \text{div } (u\cdot\nabla u)_{\neq} &= \partial_x(u\cdot\nabla u_1)_{\neq} + \partial_y(u\cdot\nabla u_2)_{\neq} + \partial_z(u\cdot\nabla u_3)_{\neq} \\ &= \text{div } (u_{\neq}\cdot\nabla u_{\neq})_{\neq} + 2(\partial_y u_{1,0}\partial_x u_{2,\neq} + \partial_z u_{1,0}\partial_x u_{3,\neq}) + 2\partial_y g_{2,2} + 2\partial_z g_{3,2}, \end{aligned}$$

which along with $\partial_y V = 1 + \frac{\partial_y \mathbf{U}_2}{A}$ implies that

$$\begin{aligned} \frac{P^{N_1}_{\neq} + P^{N_2}_{\neq}}{A} =& -2\triangle^{-1}\left(\partial_x u_{2,\neq} + \frac{\text{div } (u\cdot\nabla u)_{\neq}}{2A} - \frac{\partial_x n_{\neq}}{2A}\right) \\ =& -2\triangle^{-1}\Bigg(\left(1 + \frac{\partial_y\mathbf{U}_2}{A}\right)\partial_x u_{2,\neq} + \frac{\partial_z\mathbf{U}_2}{A}\partial_x u_{3,\neq} + \frac{\text{div } (u_{\neq}\cdot\nabla u_{\neq})_{\neq}}{2A} \\ &+ \frac{\partial_y\mathbf{U}_1\partial_x u_{2,\neq} + \partial_z\mathbf{U}_1\partial_x u_{3,\neq}}{A} + \frac{\partial_y g_{2,2} + \partial_z g_{3,2}}{A} - \frac{\partial_x n_{\neq}}{2A}\Bigg) \\ =& -2\triangle^{-1}\left(\partial_y V\partial_x W + \frac{\partial_y g_{2,2} + \partial_z g_{3,2}}{A} + \frac{P_{1,1} + P_{1,2} + P_{1,3}}{A}\right), \end{aligned}$$

where

$$P_{1,1} = \frac{\text{div } (u_{\neq}\cdot\nabla u_{\neq})_{\neq}}{2}, \quad P_{1,2} = \partial_y\mathbf{U}_1\partial_x u_{2,\neq} + \partial_z\mathbf{U}_1\partial_x u_{3,\neq} \quad P_{1,3} = -\frac{\partial_x n_{\neq}}{2}. \tag{8.4}$$

Using the above decomposition, we rewrite (8.2) into

$$\begin{cases} \mathcal{L}_V u_{2,\neq} - 2\partial_y\triangle^{-1}(\partial_y V\partial_x W) + \frac{\mathbf{U}_1\partial_x u_{2,\neq}}{A} + \frac{g_{2,1}+g_{2,2}+G_{2,3}}{A} = \frac{2\partial_y\triangle^{-1}(\partial_y g_{2,2}+\partial_z g_{3,2})}{A} \\ \qquad + \frac{2\partial_y\triangle^{-1}(P_{1,1}+P_{1,2}+P_{1,3})}{A}, \\ \mathcal{L}_V u_{3,\neq} - 2\partial_z\triangle^{-1}(\partial_y V\partial_x W) + \frac{\mathbf{U}_1\partial_x u_{3,\neq}}{A} + \frac{g_{3,1}+g_{3,2}+G_{3,3}}{A} = \frac{2\partial_z\triangle^{-1}(\partial_y g_{2,2}+\partial_z g_{3,2})}{A} \\ \qquad + \frac{2\partial_z\triangle^{-1}(P_{1,1}+P_{1,2}+P_{1,3})}{A}. \end{cases} \tag{8.5}$$

Therefore $W = u_{2,\neq} + \kappa u_{3,\neq}$ satisfies

$$\widetilde{\mathcal{L}_V} W + \frac{\mathbf{U}_1 \partial_x W}{A} + \frac{G^{(1)} + G^{(2)}}{A} = \left(\partial_t \kappa - \frac{\triangle \kappa}{A}\right) u_{3,\neq} - \frac{2\nabla\kappa\cdot\nabla u_{3,\neq}}{A}, \tag{8.6}$$

where

$$\widetilde{\mathcal{L}_V} W = \mathcal{L}_V W - 2(\partial_y + \kappa\partial_z)\triangle^{-1}(\partial_y V \partial_x W)$$

and

$$G^{(1)} = G_{2,3} + \kappa G_{3,3} - 2(\partial_y + \kappa\partial_z)\triangle^{-1}(P_{1,1} + P_{1,2} + P_{1,3}),$$
$$G^{(2)} = g_{2,1} + g_{2,2} + \kappa(g_{3,1} + g_{3,2}) - 2(\partial_y + \kappa\partial_z)\triangle^{-1}(\partial_y g_{2,2} + \partial_z g_{3,2}).$$

In addition, $\triangle W$ satisfies

$$\mathcal{L}_V \triangle W = \triangle\left(-\frac{2\nabla\kappa\cdot\nabla u_{3,\neq}}{A}\right) + \text{good terms}.$$

To remove the singular term $\triangle\left(-\frac{2\nabla\kappa\cdot\nabla u_{3,\neq}}{A}\right)$, motivated by the quasi-linear method in [36], we introduce the following decomposition

$$\nabla\kappa\cdot\nabla u_{3,\neq} = \rho_1 \nabla V\cdot\nabla u_{3,\neq} + \rho_2(\partial_z - \kappa\partial_y)u_{3,\neq}, \tag{8.7}$$

where

$$\rho_1 = \frac{\partial_y\kappa + \kappa\partial_z\kappa}{\partial_y V(1+\kappa^2)}, \quad \rho_2 = \frac{\partial_z\kappa - \kappa\partial_y\kappa}{1+\kappa^2}. \tag{8.8}$$

As $(\partial_z - \kappa\partial_y)$ has a good commutative relation with $\mathcal{L}_V$, it is a good derivative. Thus the second term in (8.7) is good. To handle the first term in (8.7) and obtain a sharp threshold of velocity, we need to make a further decomposition for W as follows

$$W = W^{(1)} + \frac{1}{A}W^{(2)},$$

where $W^{(1)}$ and $W^{(2)}$ solve

$$\begin{cases} \mathcal{L}_V \triangle W^{(1)} = \text{good terms}, \\ \mathcal{L}_V W^{(2)} = -\rho_1 \nabla V\cdot\nabla u_{3,\neq}, \\ W^{(1)}(0) = W(0),\ W^{(2)}(0) = 0. \end{cases}$$

Furthermore, an additional quantity $W^{(3)}$ is also needed satisfying

$$\mathcal{L}_V W^{(3)} = -\nabla V\cdot\nabla u_{3,\neq}, \quad W^{(3)}(0) = 0.$$

We denote

$$\triangle u_{3,\neq} = \left(\triangle u_{3,\neq} - 2\partial_x W^{(3)}\right) + 2\partial_x W^{(3)},$$

which satisfies

$$\begin{aligned} \mathcal{L}_V\left(\triangle u_{3,\neq} - 2\partial_x W^{(3)}\right) =& 2\partial_z\left(\partial_y V\partial_x W\right) - \frac{\triangle(\mathbf{U}_1\partial_x u_{3,\neq})}{A} - \frac{\triangle(g_{3,1} + g_{3,2} + G_{3,3})}{A} \\ &+ \frac{2\partial_z\left(\partial_y g_{2,2} + \partial_z g_{3,2}\right)}{A} + \frac{2\partial_z\left(P_{1,1} + P_{1,2} + P_{1,3}\right)}{A} - \triangle V\partial_x u_{3,\neq} \end{aligned} \tag{8.9}$$

and

$$\mathcal{L}_V W^{(3)} = -\nabla V\cdot\nabla u_{3,\neq}. \tag{8.10}$$

In this way, by space-time estimates, we will prove that

$$E_4^2(t) \leq E_{5,1}^2(t) + E_{5,2}^2(t) \leq C\left(\|u_{\rm in}\|_{H^2}^2 + 1\right),$$

where $E_{5,1}^2(t)$ and $E_{5,2}^2(t)$ are the auxiliary norms defined by

$$\begin{aligned} E_{5,1}(t) &= A^{-\frac{2}{3}}\|\triangle u_{3,\neq}\|_{X_b}, \\ E_{5,2}(t) &= \sum_{j=2}^{3}\left(\|\partial_x^2 u_{j,\neq}\|_{X_b} + \|\partial_x(\partial_z - \kappa\partial_y)u_{j,\neq}\|_{X_b}\right) + \|\partial_x \nabla W\|_{X_b}. \end{aligned}$$

Lemma 8.1. *Under the result of Lemma 4.5, it holds that*

$$\begin{gathered} \|\kappa\|_{H^1} \leq CA^{-1}\|\triangle \mathbf{U}_2\|_{L^2} \leq C\epsilon, \quad \|\kappa\|_{H^3} \leq CA^{-1}\|\triangle \mathbf{U}_2\|_{H^2} \leq C\epsilon, \\ \|\partial_t\kappa\|_{H^1} \leq CA^{-1}\|\partial_t \mathbf{U}_2\|_{H^2} \leq CA^{-1}\epsilon, \quad \|\partial_t\rho_1\|_{L^2} \leq CA^{-1}\|\partial_t \mathbf{U}_2\|_{H^2} \leq CA^{-1}\epsilon, \\ \|\rho_1\|_{H^2} + \|\rho_2\|_{H^2} \leq CA^{-1}\|\triangle \mathbf{U}_2\|_{H^2} \leq C\epsilon, \end{gathered}$$

where the definitions of κ, ρ_1 and ρ_2 are given by (8.1) and (8.8).

The following lemma gives the estimates of $W^{(2)}$ and $W^{(3)}$ associated with good derivatives.

Lemma 8.2. *Under the conditions of Theorem 1.1 and the assumptions (2.5), it holds that*

$$\begin{aligned} &\text{(i)} \quad \|\partial_x^2 W^{(3)}\|_{X_b}^2 + \|\partial_x(\partial_z - \kappa\partial_y)W^{(3)}\|_{X_b}^2 \leq CA^{\frac{4}{3}}E_{5,2}^2(t), \\ &\text{(ii)} \quad \|\partial_x^2 W^{(2)}\|_{X_b}^2 \leq C\epsilon^2 A^{\frac{4}{3}}E_{5,2}^2(t), \quad \|\partial_x \nabla W^{(2)}\|_{X_b}^2 \leq C\epsilon^2 A^2 E_{5,2}^2(t), \\ &\text{(iii)} \quad \|\partial_x(W^{(2)} - \rho_1 W^{(3)})\|_{X_b}^2 \leq CA^{\frac{2}{3}}\epsilon^2 E_{5,2}^2(t). \end{aligned}$$

Proof. **Estimate (i).** Applying Proposition A.3 to (8.10), we obtain

$$\begin{aligned} \|\partial_x^2 W^{(3)}\|_{X_b}^2 + \|\partial_x(\partial_z - \kappa\partial_y)W^{(3)}\|_{X_b}^2 &\leq CA^{\frac{1}{3}}\big(\|{\rm e}^{bA^{-\frac{1}{3}}t}\partial_x^2(\nabla V\cdot\nabla u_{3,\neq})\|_{L^2L^2}^2 \\ &+ \|{\rm e}^{bA^{-\frac{1}{3}}t}\partial_x(\partial_z - \kappa\partial_y)(\nabla V\cdot\nabla u_{3,\neq})\|_{L^2L^2}^2\big) =: CA^{\frac{1}{3}}\left(I_1 + I_2\right). \end{aligned} \tag{8.11}$$

Recalling that $V = y + \frac{\mathbf{U}_2(t,y,z)}{A}$, by Lemma 4.5 and Lemma 8.1, we have

$$\|\nabla V\|_{L^\infty} \leq 1 + A^{-1}\|\nabla \mathbf{U}_2\|_{L^\infty} \leq C\left(1 + A^{-1}\|\triangle \mathbf{U}_2\|_{H^2}\right) \leq C, \tag{8.12}$$

$$\|(\partial_z - \kappa\partial_y)\nabla V\|_{L^\infty} \leq CA^{-1}\left(1 + \|\kappa\|_{H^3}\right)\|\triangle \mathbf{U}_2\|_{H^2} \leq C. \tag{8.13}$$

Then for I_1, using (8.12), we have

$$I_1 \leq \|\nabla V\|_{L^\infty L^\infty}^2 \|{\rm e}^{bA^{-\frac{1}{3}}t}\nabla\partial_x^2 u_{3,\neq}\|_{L^2L^2}^2 \leq CA\|\partial_x^2 u_{3,\neq}\|_{X_b}^2 \leq CAE_{5,2}^2(t).$$

For I_2, using (8.12) and (8.13), we deduce that

$$\|\partial_x(\partial_z - \kappa\partial_y)\left(\nabla V\cdot\nabla u_{3,\neq}\right)\|_{L^2} \leq C\left(\|\nabla\partial_x(\partial_z - \kappa\partial_y)u_{3,\neq}\|_{L^2} + \|\nabla\partial_x^2 u_{3,\neq}\|_{L^2}\right),$$

which implies

$$I_2 \leq CA\left(\|\partial_x(\partial_z - \kappa\partial_y)u_{3,\neq}\|_{X_b}^2 + \|\partial_x^2 u_{3,\neq}\|_{X_b}^2\right) \leq CAE_{5,2}^2(t).$$

Combining the estimates of I_1 and I_2, (8.11) gives the result of (i).

Estimate (ii). Notice that

$$\mathcal{L}_V\partial_x^2 W^{(2)} = \partial_x^2\mathcal{L}_V W^{(2)} = -\rho_1\nabla V\cdot\nabla\partial_x^2 u_{3,\neq}$$

and $\partial_x^2 W^{(2)}(0)=0$. Then applying Proposition A.1, there holds

$$\|\partial_x^2 W^{(2)}\|_{X_b}^2 \le CA^{\frac13}\|\mathrm{e}^{bA^{-\frac13}t}\rho_1\nabla V\cdot\nabla\partial_x^2 u_{3,\neq}\|_{L^2L^2}^2.$$

Using Lemma 8.1 and (8.12), we obtain

$$\|\rho_1\nabla V\cdot\nabla\partial_x^2 u_{3,\neq}\|_{L^2}\le C\|\rho_1\|_{H^2}\|\nabla V\|_{L^\infty}\|\nabla\partial_x^2 u_{3,\neq}\|_{L^2}\le C\epsilon\|\nabla\partial_x^2 u_{3,\neq}\|_{L^2}.$$

This indicates that

$$\|\partial_x^2 W^{(2)}\|_{X_b}^2 \le C\epsilon^2 A^{\frac13}\|\mathrm{e}^{bA^{-\frac13}t}\nabla\partial_x^2 u_{3,\neq}\|_{L^2L^2}^2\le C\epsilon^2 A^{\frac43}\|\partial_x^2 u_{3,\neq}\|_{X_b}^2\le C\epsilon^2 A^{\frac43}E_{5,2}^2(t). \tag{8.14}$$

For $j\in\{1,2,3\}$, we get

$$\mathcal{L}_V\partial_x\partial_j W^{(2)}=\partial_x\partial_j\mathcal{L}_V W^{(2)}-\partial_j V\partial_x^2 W^{(2)}=-\partial_j\left(\rho_1\nabla V\cdot\nabla\partial_x u_{3,\neq}\right)-\partial_j V\partial_x^2 W^{(2)}$$

satisfying $\partial_x\partial_j W^{(2)}(0)=0$. Applying Proposition A.1 to it, one deduces

$$\|\partial_x\partial_j W^{(2)}\|_{X_b}^2\le CA^{\frac13}\|\mathrm{e}^{bA^{-\frac13}t}\partial_j V\partial_x^2 W^{(2)}\|_{L^2L^2}^2+CA\|\mathrm{e}^{bA^{-\frac13}t}\rho_1\nabla V\cdot\nabla\partial_x u_{3,\neq}\|_{L^2L^2}^2. \tag{8.15}$$

Due to (8.12) and Lemma 8.1, there holds

$$\begin{aligned}&\|\partial_j V\partial_x^2 W^{(2)}\|_{L^2}\le\|\partial_j V\|_{L^\infty}\|\partial_x^2 W^{(2)}\|_{L^2}\le C\|\partial_x^2 W^{(2)}\|_{L^2},\\&\|\rho_1\nabla V\cdot\nabla\partial_x u_{3,\neq}\|_{L^2}\le C\|\rho_1\|_{H^2}\|\nabla V\|_{L^\infty}\|\nabla\partial_x u_{3,\neq}\|_{L^2}\le C\epsilon\|\nabla\partial_x^2 u_{3,\neq}\|_{L^2},\end{aligned}$$

which implies that

$$\begin{aligned}&\|\mathrm{e}^{bA^{-\frac13}t}\partial_j V\partial_x^2 W^{(2)}\|_{L^2L^2}^2\le C\|\mathrm{e}^{bA^{-\frac13}t}\partial_x^2 W^{(2)}\|_{L^2L^2}^2\le CA^{\frac13}\|\partial_x^2 W^{(2)}\|_{X_b}^2,\\&\|\mathrm{e}^{bA^{-\frac13}t}\rho_1\nabla V\cdot\nabla\partial_x u_{3,\neq}\|_{L^2L^2}^2\le C\epsilon^2\|\mathrm{e}^{bA^{-\frac13}t}\nabla\partial_x^2 u_{3,\neq}\|_{L^2L^2}^2\le C\epsilon^2 A\|\partial_x^2 u_{3,\neq}\|_{X_b}^2.\end{aligned}$$

Substituting the above estimations into (8.15) and using (8.14), we arrive at

$$\|\partial_x\nabla W^{(2)}\|_{X_b}^2\le CA^{\frac23}\|\partial_x^2 W^{(2)}\|_{X_b}^2+C\epsilon^2A^2\|\partial_x^2 u_{3,\neq}\|_{X_b}^2\le C\epsilon^2A^2E_{5,2}^2(t).$$

Estimate (iii). Due to

$$\begin{aligned}\mathcal{L}_V(\rho_1 f)-\rho_1\mathcal{L}_V f=&(\partial_t\rho_1-A^{-1}\triangle\rho_1)f-2A^{-1}\nabla\rho_1\cdot\nabla f\\=&(\partial_t\rho_1+A^{-1}\triangle\rho_1)f-2A^{-1}\nabla\cdot(f\nabla\rho_1)\end{aligned}$$

and $\mathcal{L}_V W^{(2)}=\rho_1\mathcal{L}_V W^{(3)}$, there holds

$$\begin{aligned}\mathcal{L}_V\partial_x\left(W^{(2)}-\rho_1W^{(3)}\right)=&\partial_x\mathcal{L}_V\left(W^{(2)}-\rho_1W^{(3)}\right)=\partial_x\left(\rho_1\mathcal{L}_V W^{(3)}-\mathcal{L}_V(\rho_1W^{(3)})\right)\\=&-\partial_x\left[(\partial_t\rho_1+A^{-1}\triangle\rho_1)W^{(3)}-2A^{-1}\nabla\cdot(W^{(3)}\nabla\rho_1)\right]\\=&-\triangle Q+2A^{-1}\nabla\cdot\left(\partial_xW^{(3)}\nabla\rho_1\right),\end{aligned}$$

where $\triangle Q=\left(\partial_t\rho_1+A^{-1}\triangle\rho_1\right)\partial_xW^{(3)}$.

Applying Proposition A.1, we get

$$\|\partial_x(W^{(2)}-\rho_1W^{(3)})\|_{X_b}^2\le C\left(A\|\mathrm{e}^{bA^{-\frac13}t}\nabla Q\|_{L^2L^2}^2+A^{-1}\|\mathrm{e}^{bA^{-\frac13}t}\partial_xW^{(3)}\nabla\rho_1\|_{L^2L^2}^2\right). \tag{8.16}$$

It follows from Lemma 8.1 that $\|\nabla\rho_1\|_{H^1}\le\|\rho_1\|_{H^2}\le C\epsilon$ and

$$\|\partial_t\rho_1+A^{-1}\triangle\rho_1\|_{L^2}\le\|\partial_t\rho_1\|_{L^2}+A^{-1}\|\rho_1\|_{H^2}\le CA^{-1}\epsilon.$$

Combining them with Lemma A.5, one obtains

$$\begin{aligned}\|\nabla Q\|_{L^2} =&\|\nabla\triangle^{-1}\triangle Q\|_{L^2} = \|\nabla\triangle^{-1}[\left(\partial_t\rho_1 + A^{-1}\triangle\rho_1\right)\partial_x W^{(3)}]\|_{L^2}\\ \leq& C\|\partial_t\rho_1 + A^{-1}\triangle\rho_1\|_{L^2}\left(\|\partial_x W^{(3)}\|_{L^2} + \|\partial_x\left(\partial_z-\kappa\partial_y\right)W^{(3)}\|_{L^2}\right)\\ \leq& CA^{-1}\epsilon\left(\|\partial_x^2 W^{(3)}\|_{L^2} + \|\partial_x(\partial_z-\kappa\partial_y)W^{(3)}\|_{L^2}\right)\end{aligned} \tag{8.17}$$

and

$$\begin{aligned}\|\partial_x W^{(3)}\nabla\rho_1\|_{L^2} \leq& C\|\nabla\rho_1\|_{H^1}\left(\|\partial_x W^{(3)}\|_{L^2} + \|\partial_x(\partial_z-\kappa\partial_y)W^{(3)}\|_{L^2}\right)\\ \leq& C\epsilon\left(\|\partial_x^2 W^{(3)}\|_{L^2} + \|\partial_x(\partial_z-\kappa\partial_y)W^{(3)}\|_{L^2}\right).\end{aligned} \tag{8.18}$$

Substituting (8.17) and (8.18) into (8.16), and using the result of (i), we have

$$\begin{aligned}\|\partial_x(W^{(2)}-\rho_1 W^{(3)})\|_{X_b}^2 \leq& CA^{-1}\epsilon^2\left(\|e^{bA^{-\frac13}t}\partial_x^2 W^{(3)}\|_{L^2L^2}^2 + \|e^{bA^{-\frac13}t}\partial_x(\partial_z-\kappa\partial_y)W^{(3)}\|_{L^2L^2}^2\right)\\ \leq& CA^{-1}\epsilon^2 A^{\frac13}\left(\|\partial_x^2 W^{(3)}\|_{X_b}^2 + \|\partial_x(\partial_z-\kappa\partial_y)W^{(3)}\|_{X_b}^2\right) \leq CA^{\frac23}\epsilon^2 E_{5,2}^2(t).\end{aligned}$$

□

Lemma 8.3. *Under the assumptions of Theorem 1.1 and (2.5), it holds that*

$$\begin{aligned}&\|e^{bA^{-\frac13}t}\nabla(\mathbf{U}_1\partial_x W)\|_{L^2L^2}^2 \leq C\left(\|(u_{1,\text{in}})_0\|_{H^1}^2 + H_1^2\right)\left(A^{\frac12}\|\triangle W^{(1)}\|_{X_b}^2 + \epsilon^2 A^{\frac23}E_{5,2}^2(t)\right),\\ &\|e^{bA^{-\frac13}t}\nabla(\mathbf{U}_1\partial_x u_{3,\neq})\|_{L^2L^2}^2 \leq C\left(\|(u_{1,\text{in}})_0\|_{H^1}^2 + H_1^2\right)AE_{5,2}^2(t),\\ &\|e^{bA^{-\frac13}t}\mathbf{U}_1\partial_x^2(u_{2,\neq},u_{3,\neq})\|_{L^2L^2}^2 \leq C\left(\|(u_{1,\text{in}})_0\|_{H^1}^2 + H_1^2\right)A^{\frac13(2\alpha+1)}E_{5,2}^2(t),\\ &\|e^{bA^{-\frac13}t}P_{1,2}\|_{L^2L^2}^2 \leq C\left(\|(u_{1,\text{in}})_0\|_{H^1}^2 + H_1^2\right)A^{\frac23}E_{5,2}^2(t),\end{aligned} \tag{8.19}$$

where $\alpha\in(\frac12,\frac34)$ and the definition of H_1 is the same as in Lemma 6.4.

Proof. **Estimate** $(8.19)_1$. Recalling that $W = W^{(1)} + \frac{1}{A}W^{(2)}$, we have

$$\|e^{bA^{-\frac13}t}\nabla(\mathbf{U}_1\partial_x W)\|_{L^2L^2}^2 \leq \|e^{bA^{-\frac13}t}\nabla(\mathbf{U}_1\partial_x W^{(1)})\|_{L^2L^2}^2 + \frac{1}{A^2}\|e^{bA^{-\frac13}t}\nabla(\mathbf{U}_1\partial_x W^{(2)})\|_{L^2L^2}^2.$$

Using Lemma 4.4, Lemma 6.4, Lemma A.2, Lemma A.3 and

$$\|e^{bA^{-\frac13}t}\partial_x\nabla f\|_{L^2L^2} = \|e^{bA^{-\frac13}t}\nabla\triangle^{-1}\partial_x(\triangle f)\|_{L^2L^2} \leq \|\triangle f\|_{X_b},$$

one obtains

$$\begin{aligned}&\|e^{bA^{-\frac13}t}\nabla(\mathbf{U}_1\partial_x W^{(1)})\|_{L^2L^2}^2\\ \leq&\|\mathbf{U}_1\|_{L^\infty_{t,z}L^2_y}^2\|e^{bA^{-\frac13}t}\nabla\partial_x W^{(1)}\|_{L^2_{t,x,z}L^\infty_y}^2 + \|\nabla\mathbf{U}_1\|_{L^\infty_t L^2_{y,z}}^2\|e^{bA^{-\frac13}t}\partial_x W^{(1)}\|_{L^2_{t,x}L^\infty_{y,z}}^2\\ \leq& C\|\mathbf{U}_1\|_{L^\infty H^1}^2\Big(\|e^{bA^{-\frac13}t}\partial_x\nabla W^{(1)}\|_{L^2L^2}\|e^{bA^{-\frac13}t}\partial_x\triangle W^{(1)}\|_{L^2L^2}\\ &+\|e^{bA^{-\frac13}t}\partial_x\nabla W^{(1)}\|_{L^2L^2}^{\frac32-\alpha}\|e^{bA^{-\frac13}t}\partial_x\triangle W^{(1)}\|_{L^2L^2}^{\alpha-\frac12}\Big)\\ \leq& C\left(\|(u_{1,\text{in}})_0\|_{H^1}^2 + H_1^2\right)A^{\frac12}\|\triangle W^{(1)}\|_{X_b}^2,\end{aligned} \tag{8.20}$$

where $\alpha\in(\frac{1}{2},\frac{3}{4})$. Similarly, by Lemma 4.4, Lemma 6.4, Lemma A.2, Lemma A.3 and (ii) of Lemma 8.2, there holds

$$\begin{aligned}
&\|e^{bA^{-\frac{1}{3}}t}\nabla(\mathbf{U}_1\partial_x W^{(2)})\|_{L^2L^2}^2\\
\leq&C\|\mathbf{U}_1\|_{L^\infty H^1}^2\big(\|e^{bA^{-\frac{1}{3}}t}\partial_x\nabla W^{(2)}\|_{L^2L^2}\|e^{bA^{-\frac{1}{3}}t}\partial_x\triangle W^{(2)}\|_{L^2L^2}\\
&+\|e^{bA^{-\frac{1}{3}}t}\partial_x\nabla W^{(2)}\|_{L^2L^2}^{3-2\alpha}\|e^{bA^{-\frac{1}{3}}t}\partial_x\triangle W^{(2)}\|_{L^2L^2}^{2\alpha-1}\big)\\
\leq&C(\|(u_{1,\rm in})_0\|_{H^1}^2+H_1^2)A^{\frac{2}{3}}\|\partial_x\nabla W^{(2)}\|_{X_b}^2\leq C(\|(u_{1,\rm in})_0\|_{H^1}^2+H_1^2)\epsilon^2A^{\frac{8}{3}}E_{5,2}^2(t).
\end{aligned}\tag{8.21}$$

Collecting (8.20) and (8.21), we get the first result.

Estimate $(8.19)_2$**.** By Lemma 4.4, Lemma 6.4, Lemma A.2 and Lemma A.4, we arrive

$$\begin{aligned}
&\|e^{bA^{-\frac{1}{3}}t}\nabla(\mathbf{U}_1\partial_x u_{3,\neq})\|_{L^2L^2}^2\\
\leq&\|\nabla\mathbf{U}_1\|_{L^\infty L^2}^2\|e^{bA^{-\frac{1}{3}}t}\partial_x u_{3,\neq}\|_{L^2_{t,x}L^\infty_{y,z}}^2+\|\mathbf{U}_1\|_{L^\infty_{t,y}L^2_z}^2\|e^{bA^{-\frac{1}{3}}t}\nabla\partial_x u_{3,\neq}\|_{L^2_{t,x,y}L^\infty_z}^2\\
\leq&C\|\mathbf{U}_1\|_{L^\infty H^1}^2\left(\|e^{bA^{-\frac{1}{2}}t}\nabla(\partial_x,\partial_z-\kappa\partial_y)\partial_x u_{3,\neq}\|_{L^2L^2}^2+\|e^{bA^{-\frac{1}{3}}t}(\partial_x,\partial_z-\kappa\partial_y)\nabla\partial_x u_{3,\neq}\|_{L^2L^2}^2\right)\\
\leq&C\|\mathbf{U}_1\|_{L^\infty H^1}^2\left(\|e^{bA^{-\frac{1}{3}}t}\nabla(\partial_x,\partial_z-\kappa\partial_y)\partial_x u_{3,\neq}\|_{L^2L^2}^2+\|\kappa\|_{H^3}^2\|e^{bA^{-\frac{1}{3}}t}\partial_x\partial_y u_{3,\neq}\|_{L^2L^2}^2\right)\\
\leq&C\left(\|(u_{1,\rm in})_0\|_{H^1}^2+H_1^2\right)AE_{5,2}^2(t),
\end{aligned}$$

where we use Lemma 8.1 and $(\partial_z-\kappa\partial_y)\nabla\partial_x u_{3,\neq}=\nabla(\partial_z-\kappa\partial_y)\partial_x u_{3,\neq}+\nabla\kappa\partial_y\partial_x u_{3,\neq}$.

Estimate $(8.19)_3$. Due to Lemma 4.4, Lemma 6.4, Lemma A.2 and Lemma A.3, for $j\in\{2,3\}$ and $\alpha\in(\frac{1}{2},\frac{3}{4})$, there holds

$$\begin{aligned}
\|e^{bA^{-\frac{1}{3}}t}\mathbf{U}_1\partial_x^2 u_{j,\neq}\|_{L^2L^2}^2\leq&\|\mathbf{U}_1\|_{L^\infty_tL^\infty_yL^2_z}^2\|e^{bA^{-\frac{1}{3}}t}\partial_x^2 u_{j,\neq}\|_{L^2_tL^\infty_zL^2_{x,y}}^2\\
\leq&C\|\mathbf{U}_1\|_{L^\infty H^1}^2\|e^{bA^{-\frac{1}{3}}t}\partial_x^2 u_{j,\neq}\|_{L^2L^2}^{2-2\alpha}\|e^{bA^{-\frac{1}{3}}t}\nabla\partial_x^2 u_{j,\neq}\|_{L^2L^2}^{2\alpha}\\
\leq&C\left(\|(u_{1,\rm in})_0\|_{H^1}^2+H_1^2\right)A^{\frac{1}{3}(2\alpha+1)}E_{5,2}^2(t).
\end{aligned}$$

Estimate $(8.19)_4$. According to $P_{1,2}=\partial_y\mathbf{U}_1\partial_x u_{2,\neq}+\partial_z\mathbf{U}_1\partial_x u_{3,\neq}$ in (8.4), using Lemma 4.4, Lemma 6.4 and Lemma A.4, for $j\in\{2,3\}$, we get

$$\begin{aligned}
&\|e^{bA^{-\frac{1}{3}}t}P_{1,2}\|_{L^2L^2}^2\leq\|e^{bA^{-\frac{1}{3}}t}\partial_j\mathbf{U}_1\partial_x u_{j,\neq}\|_{L^2L^2}^2\leq\|\partial_j\mathbf{U}_1\|_{L^\infty_tL^2_{y,z}}^2\|e^{bA^{-\frac{1}{3}}t}\partial_x u_{j,\neq}\|_{L^2_tL^\infty_{y,z}L^2_x}^2\\
\leq&C\|\mathbf{U}_1\|_{L^\infty H^1}^2\|e^{bA^{-\frac{1}{3}}t}(\partial_x,\partial_z-\kappa\partial_y)\partial_x u_{j,\neq}\|_{L^2L^2}\|e^{bA^{-\frac{1}{3}}t}\nabla(\partial_x,\partial_z-\kappa\partial_y)\partial_x u_{j,\neq}\|_{L^2L^2}\\
\leq&C\left(\|(u_{1,\rm in})_0\|_{H^1}^2+H_1^2\right)A^{\frac{2}{3}}E_{5,2}^2(t).
\end{aligned}$$

□

Lemma 8.4. *Under the assumptions of Theorem 1.1 and (2.5), it holds that*

$$\begin{aligned}
&\|\mathrm{e}^{bA^{-\frac13}t}\nabla g_{3,1}\|^2_{L^2L^2}\leq CA^{\frac53}\epsilon^2\left(E^2_{5,1}(t)+E^2_{5,2}(t)\right),\\
&\|\mathrm{e}^{bA^{-\frac13}t}\nabla(g_{2,1}+\kappa g_{3,1})\|^2_{L^2L^2}+\|\mathrm{e}^{bA^{-\frac13}t}\partial_x g_{3,1}\|^2_{L^2L^2}\\
&+\|\mathrm{e}^{bA^{-\frac13}t}\nabla g_{2,2}\|^2_{L^2L^2}+\|\mathrm{e}^{bA^{-\frac13}t}\nabla g_{3,2}\|^2_{L^2L^2}\leq C\epsilon^2AE^2_{5,2}(t).
\end{aligned}\tag{8.22}$$

Proof. **Estimate** $(8.22)_1$. By Lemma 4.1, Lemma A.2 and $\partial_z u_{3,0}=-\partial_y u_{2,0}$, we get

$$\|\nabla(u_{2,0},u_{3,0})\|^2_{L^2L^\infty}\leq CA\epsilon^2,\tag{8.23}$$

which along with (3.19) and

$$\|\nabla u_{3,\neq}\|^2_{L^2}\leq\|u_{3,\neq}\|_{L^2}\|\triangle u_{3,\neq}\|_{L^2}\leq A^{\frac23}\|\partial_x^2u_{3,\neq}\|^2_{L^2}+A^{-\frac23}\|\triangle u_{3,\neq}\|^2_{L^2}$$

shows that

$$\begin{aligned}
&\|\mathrm{e}^{bA^{-\frac13}t}\nabla g_{3,1}\|^2_{L^2L^2}=\|\mathrm{e}^{bA^{-\frac13}t}\nabla\left(u_{2,0}\partial_y u_{3,\neq}+u_{3,0}\partial_z u_{3,\neq}\right)\|^2_{L^2L^2}\\
\leq&\|\nabla(u_{2,0},u_{3,0})\|^2_{L^2L^\infty}\|\mathrm{e}^{bA^{-\frac13}t}\nabla u_{3,\neq}\|^2_{L^\infty L^2}+\|(u_{2,0},u_{3,0})\|^2_{L^\infty L^\infty}\|\mathrm{e}^{bA^{-\frac13}t}\triangle u_{3,\neq}\|^2_{L^2L^2}\\
\leq&CA\epsilon^2\|\nabla u_{3,\neq}\|^2_{X_b}\leq CA^{\frac53}\epsilon^2\big(\|\partial_x^2u_{3,\neq}\|^2_{X_b}+A^{-\frac43}\|\triangle u_{3,\neq}\|^2_{X_b}\big)\leq CA^{\frac53}\epsilon^2\left(E^2_{5,2}(t)+E^2_{5,1}(t)\right).
\end{aligned}$$

Estimate $(8.22)_2$. As $g_{2,1}+\kappa g_{3,1}=\left(u_{2,0}\partial_y+u_{3,0}\partial_z\right)W-u_{3,\neq}(u_{2,0}\partial_y+u_{3,0}\partial_z)\kappa$, by (3.19) and (8.23), we get

$$\|\mathrm{e}^{bA^{-\frac13}t}\nabla(g_{2,1}+\kappa g_{3,1})\|^2_{L^2L^2}\leq CA\epsilon^2\left(\|\mathrm{e}^{bA^{-\frac13}t}\nabla W\|^2_{Y_0}+\|\mathrm{e}^{bA^{-\frac13}t}u_{3,\neq}\nabla\kappa\|^2_{Y_0}\right).$$

Direct calculations show that

$$\|\mathrm{e}^{bA^{-\frac13}t}\nabla W\|^2_{Y_0}\leq\|\mathrm{e}^{bA^{-\frac13}t}\partial_x\nabla W\|^2_{Y_0}\leq\|\partial_x\nabla W\|^2_{X_b}\leq E^2_{5,2}(t).$$

On the other hand, using Lemma 8.1, one obtains

$$\begin{aligned}
&\|u_{3,\neq}\nabla\kappa\|_{L^2}\leq\|u_{3,\neq}\|_{L^2}\|\nabla\kappa\|_{L^\infty}\leq C\|\partial_x^2u_{3,\neq}\|_{L^2},\\
&\|\nabla(u_{3,\neq}\nabla\kappa)\|_{L^2}\leq\|u_{3,\neq}\nabla\kappa\|_{H^1}\leq C\|u_{3,\neq}\|_{H^1}\|\nabla\kappa\|_{H^2}\leq C\|\nabla\partial_x^2u_{3,\neq}\|_{L^2},
\end{aligned}$$

which indicates that

$$\begin{aligned}
\|\mathrm{e}^{bA^{-\frac13}t}u_{3,\neq}\nabla\kappa\|^2_{Y_0}\leq&C\left(\|\mathrm{e}^{bA^{-\frac13}t}\partial_x^2u_{3,\neq}\|^2_{L^\infty L^2}+\frac1A\|\mathrm{e}^{bA^{-\frac13}t}\nabla\partial_x^2u_{3,\neq}\|^2_{L^2L^2}\right)\\
\leq&C\|\partial_x^2u_{3,\neq}\|^2_{X_b}\leq CE^2_{5,2}(t).
\end{aligned}$$

This gives

$$\|\mathrm{e}^{bA^{-\frac13}t}\nabla(g_{2,1}+\kappa g_{3,1})\|^2_{L^2L^2}\leq C\epsilon^2AE^2_{5,2}(t).\tag{8.24}$$

Recall that $\partial_x g_{3,1}=(u_{2,0}\partial_y+u_{3,0}\partial_z)\partial_x u_{3,\neq}$. By (3.19), we have

$$\|\mathrm{e}^{bA^{-\frac13}t}\partial_x g_{3,1}\|^2_{L^2L^2}\leq\|(u_{2,0},u_{3,0})\|^2_{L^\infty L^\infty}\|\mathrm{e}^{bA^{-\frac13}t}\nabla\partial_x u_{3,\neq}\|^2_{L^2L^2}\leq C\epsilon^2AE^2_{5,2}(t).\tag{8.25}$$

For $g_{2,2}=u_{\neq}\cdot\nabla u_{2,0}$ and $j\in\{2,3\}$, it follows from Lemma A.5 and Lemma 4.1 that

$$\begin{aligned}\|\nabla g_{2,2}\|_{L^2}=&\|\nabla\left(u_{j,\neq}\partial_j u_{2,0}\right)\|_{L^2}\leq C\|\partial_j u_{2,0}\|_{H^1}\left(\|u_{j,\neq}\|_{H^1}+\|(\partial_z-\kappa\partial_y)u_{j,\neq}\|_{H^1}\right)\\ \leq& C\|\nabla u_{2,0}\|_{H^1}\left(\|\nabla\partial_x^2 u_{j,\neq}\|_{L^2}+\|\nabla\partial_x(\partial_z-\kappa\partial_y)u_{j,\neq}\|_{L^2}\right)\end{aligned}$$

and

$$\begin{aligned}\|e^{bA^{-\frac13}t}\nabla g_{2,2}\|^2_{L^2L^2}\leq& C\epsilon^2\left(\|e^{bA^{-\frac13}t}\nabla\partial_x^2 u_{j,\neq}\|^2_{L^2L^2}+\|e^{bA^{-\frac13}t}\nabla\partial_x(\partial_z-\kappa\partial_y)u_{j,\neq}\|^2_{L^2L^2}\right)\\ \leq& C\epsilon^2A\left(\|\partial_x^2 u_{j,\neq}\|^2_{X_b}+\|\partial_x(\partial_z-\kappa\partial_y)u_{j,\neq}\|^2_{X_b}\right)\leq C\epsilon^2AE^2_{5,2}(t).\end{aligned}\tag{8.26}$$

For $g_{3,2}=u_{\neq}\cdot\nabla u_{3,0}$, we rewrite it into

$$g_{3,2}=\left(u_{2,\neq}\partial_y+u_{3,\neq}\partial_z\right)u_{3,0}=W\partial_y u_{3,0}+u_{3,\neq}\left(\partial_z u_{3,0}-\kappa\partial_y u_{3,0}\right),$$

which implies that

$$\|e^{bA^{-\frac13}t}\nabla g_{3,2}\|^2_{L^2L^2}\leq\|e^{bA^{-\frac13}t}\nabla(W\partial_y u_{3,0})\|^2_{L^2L^2}+\|e^{bA^{-\frac13}t}\nabla\left(u_{3,\neq}(\partial_z u_{3,0}-\kappa\partial_y u_{3,0})\right)\|^2_{L^2L^2}.\tag{8.27}$$

By Lemma A.2, Lemma A.3, Lemma 4.1 and $\partial_z u_{3,0}=-\partial_y u_{2,0}$, we get

$$\begin{aligned}&\|e^{bA^{-\frac13}t}\nabla(W\partial_y u_{3,0})\|^2_{L^2L^2}\\ \leq&\|e^{bA^{-\frac13}t}\nabla W\|^2_{L^\infty L^2}\|\partial_y u_{3,0}\|^2_{L^2L^\infty}+\|e^{bA^{-\frac13}t}W\|^2_{L^\infty_{t,y}L^2_{x,z}}\|\nabla\partial_y u_{3,0}\|^2_{L^2_{t,y}L^\infty_z}\\ \leq& C\left(\|u_{3,0}\|^2_{L^2H^2}+\|u_{2,0}\|^2_{L^2H^3}\right)\|e^{bA^{-\frac13}t}\nabla\partial_x W\|^2_{L^\infty L^2}\leq C\epsilon^2AE^2_{5,2}(t).\end{aligned}\tag{8.28}$$

As $\mathbf{U}_2(0)=0$, using Lemma 4.5, there holds

$$\|\triangle\mathbf{U}_2\|_{L^2}\leq\|\mathbf{U}_2(t)\|_{H^2}\leq\int_0^t\|\partial_t\mathbf{U}_2(s)\|_{H^2}ds\leq C\epsilon t.$$

On the other hand, $\|\triangle\mathbf{U}_2(t)\|_{H^2}\leq CA\epsilon$. Therefore

$$\|\triangle\mathbf{U}_2\|_{L^2}\leq C\epsilon A\min\{A^{-1}t,1\},$$

which implies that

$$\|\kappa\|_{H^1}\leq CA^{-1}\|\triangle\mathbf{U}_2\|_{L^2}\leq C\epsilon\min\{A^{-1}t,1\}.\tag{8.29}$$

From this, along with Lemma 4.1 and Lemma 8.1, one obtains

$$\begin{aligned}\|\kappa\nabla u_{3,0}\|_{H^1}\leq& C\|\kappa\|_{H^2}\left(\|\nabla u_{3,0}\|_{L^2}+\|\triangle u_{3,0}\|_{L^2}\right)\\ \leq& C\|\kappa\|^{\frac12}_{H^1}\|\kappa\|^{\frac12}_{H^3}\left(\|\nabla u_{3,0}\|_{L^2}+\|\triangle u_{3,0}\|_{L^2}\right)\\ \leq& C\epsilon\min\{A^{-1}t,1\}^{\frac12}\left(\|\nabla u_{3,0}\|_{L^2}+\|\triangle u_{3,0}\|_{L^2}\right)\\ \leq& C\epsilon\left(\|\nabla u_{3,0}\|_{L^2}+\|\min\left(A^{-\frac23}+A^{-1}t,1\right)^{\frac12}\triangle u_{3,0}\|_{L^2}\right)\leq C\epsilon^2.\end{aligned}\tag{8.30}$$

Due to Lemma 4.1, Lemma A.5, (8.30) and $\partial_z u_{3,0} = -\partial_y u_{2,0}$, we arrive

$$\begin{aligned}
&\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\nabla\left(u_{3,\neq}(\partial_z u_{3,0} - \kappa\partial_y u_{3,0})\right)\|^2_{L^2L^2}\\
\leq& C\left(\|\partial_z u_{3,0}\|^2_{L^\infty H^1} + \|\kappa\nabla u_{3,0}\|^2_{L^\infty H^1}\right)\left(\|\mathrm{e}^{bA^{-\frac{1}{3}}t}u_{3,\neq}\|^2_{L^2H^1} + \|\mathrm{e}^{bA^{-\frac{1}{3}}t}(\partial_z - \kappa\partial_y)u_{3,\neq}\|^2_{L^2H^1}\right)\\
\leq& C\epsilon^2\left(\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\nabla\partial_x^2 u_{3,\neq}\|^2_{L^2L^2} + \|\mathrm{e}^{bA^{-\frac{1}{3}}t}\nabla(\partial_z - \kappa\partial_y)u_{3,\neq}\|^2_{L^2L^2}\right) \leq C\epsilon^2 A E^2_{5,2}(t).
\end{aligned} \tag{8.31}$$

Combining (8.28) with (8.31), we get from (8.27) that

$$\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\nabla g_{3,2}\|^2_{L^2L^2} \leq C\epsilon^2 A E^2_{5,2}(t). \tag{8.32}$$

Then $(8.22)_2$ follows from (8.24), (8.25), (8.26) and (8.32).

□

Lemma 8.5. *Under the assumptions of Theorem 1.1 and (2.5), it holds that*

$$\begin{aligned}
\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\nabla G^{(1)}\|^2_{L^2L^2} \leq& CA^{\frac{1}{6}+\alpha}E_2^4 + CA^{\frac{1}{2}+\frac{2}{3}\alpha}E_2^4 + CA^{\frac{\alpha}{3}+\frac{1}{2}}E_2^2E_5^2\\
&+ C\left(\|(u_{1,\mathrm{in}})_0\|^2_{H^1} + H_1^2\right)A^{\frac{2}{3}}E^2_{5,2}(t) + CA^{\frac{1}{3}}E_2^2,\\
\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\nabla G^{(2)}\|^2_{L^2L^2} \leq& C\epsilon^2 A E^2_{5,2}(t),
\end{aligned}$$

where $\alpha \in (\frac{1}{2}, \frac{3}{4})$.

Proof. Due to (8.3) and Lemma 3.2, there holds

$$\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\nabla G_{2,3}\|^2_{L^2L^2} \leq \|\mathrm{e}^{bA^{-\frac{1}{3}}t}\nabla(u_{\neq}\cdot\nabla u_{2,\neq})_{\neq}\|^2_{L^2L^2} \leq CA^{\frac{1}{2}+\frac{2}{3}\alpha}E_2^4.$$

For $G_{3,3} = (u_{\neq}\cdot\nabla u_{3,\neq})_{\neq}$ in (8.3), using Lemma 3.2 and Lemma 8.1, we have

$$\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\nabla\kappa G_{3,3}\|^2_{L^2L^2} \leq C\|\kappa\|^2_{L^\infty H^3}\|\mathrm{e}^{bA^{-\frac{1}{3}}t}u_{\neq}\cdot\nabla u_{3,\neq}\|^2_{L^2L^2} \leq CA^{\frac{2}{3}}E_2^4. \tag{8.33}$$

On the other hand, by Lemma 8.1 and (8.29), one deduces

$$\begin{aligned}
\|\kappa\nabla G_{3,3}\|_{L^2} \leq& C\|\kappa\|_{H^2}\|\nabla(u_{\neq}\cdot\nabla u_{3,\neq})\|_{L^2} \leq C\|\kappa\|^{\frac{1}{2}}_{H^1}\|\kappa\|^{\frac{1}{2}}_{H^3}\|\nabla(u_{\neq}\cdot\nabla u_{3,\neq})\|_{L^2}\\
\leq& C(A^{-1}t)^{\frac{1}{2}}\|\nabla(u_{\neq}\cdot\nabla u_{3,\neq})\|_{L^2},
\end{aligned}$$

which along with Lemma 3.2 and $(A^{-\frac{1}{3}}t)^{\frac{1}{2}} \leq C\mathrm{e}^{(2a-b)A^{-\frac{1}{3}}t}$ implies that

$$\begin{aligned}
\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\kappa\nabla G_{3,3}\|^2_{L^2L^2} \leq& CA^{-\frac{2}{3}}\|(A^{-\frac{1}{3}}t)^{\frac{1}{2}}\mathrm{e}^{bA^{-\frac{1}{3}}t}\nabla(u_{\neq}\cdot\nabla u_{3,\neq})\|^2_{L^2L^2}\\
\leq& CA^{-\frac{2}{3}}\|\mathrm{e}^{2aA^{-\frac{1}{3}}t}\nabla(u_{\neq}\cdot\nabla u_{3,\neq})\|^2_{L^2L^2} \leq CA^{\frac{\alpha}{3}+\frac{1}{2}}E_2^2E_5^2 + CA^{\frac{2}{3}}E_2^4.
\end{aligned} \tag{8.34}$$

It follows from (8.33) and (8.34) that

$$\|\mathrm{e}^{bA^{-\frac{1}{3}}t}\nabla(\kappa G_{3,3})\|^2_{L^2L^2} \leq CA^{\frac{2}{3}}E_2^4 + CA^{\frac{\alpha}{3}+\frac{1}{2}}E_2^2E_5^2.$$

Using Lemma 3.2, we note that

$$\begin{aligned}\|e^{bA^{-\frac13}t}P_{1,1}\|^2_{L^2L^2}\leq & C\big(\|e^{bA^{-\frac13}t}\partial_x(u_{\neq}\cdot\nabla u_{1,\neq})\|^2_{L^2L^2}+\|e^{bA^{-\frac13}t}\partial_y(u_{\neq}\cdot\nabla u_{2,\neq})\|^2_{L^2L^2}\\ &+\|e^{bA^{-\frac13}t}\partial_z(u_{\neq}\cdot\nabla u_{3,\neq})\|^2_{L^2L^2}\big)\leq CA^{\frac16+\alpha}E_2^4+CA^{\frac12+\frac23\alpha}E_2^4.\end{aligned}$$

From this, along with (8.4), $(8.19)_4$ and Lemma 8.1, one obtains

$$\begin{aligned}&\|e^{bA^{-\frac13}t}\nabla\left((\partial_y+\kappa\partial_z)\triangle^{-1}(P_{1,1}+P_{1,2}+P_{1,3})\right)\|^2_{L^2L^2}\\ \leq & C\left(\|e^{bA^{-\frac13}t}P_{1,1}\|^2_{L^2L^2}+\|e^{bA^{-\frac13}t}P_{1,2}\|^2_{L^2L^2}+\|e^{bA^{-\frac13}t}P_{1,3}\|^2_{L^2L^2}\right)\\ \leq & CA^{\frac16+\alpha}E_2^4+CA^{\frac12+\frac23\alpha}E_2^4+C(\|(u_{1,\mathrm{in}})_0\|^2_{H^1}+H_1^2)A^{\frac23}E^2_{5,2}(t)+CA^{\frac13}E_2^2.\end{aligned}\tag{8.35}$$

Based on the above estimates, we conclude that

$$\begin{aligned}\|e^{bA^{-\frac13}t}\nabla G^{(1)}\|^2_{L^2L^2}\leq & CA^{\frac16+\alpha}E_2^4+CA^{\frac12+\frac23\alpha}E_2^4+CA^{\frac{\alpha}{3}+\frac12}E_2^2E_5^2\\ &+C\left(\|(u_{1,\mathrm{in}})_0\|^2_{H^1}+H_1^2\right)A^{\frac23}E^2_{5,2}(t)+CA^{\frac13}E_2^2.\end{aligned}\tag{8.36}$$

For $G^{(2)}=g_{2,1}+g_{2,2}+\kappa(g_{3,1}+g_{3,2})-2(\partial_y+\kappa\partial_z)\triangle^{-1}(\partial_y g_{2,2}+\partial_z g_{3,2})$, by Lemma 8.1 and Lemma 8.4, there holds

$$\begin{aligned}\|e^{bA^{-\frac13}t}\nabla G^{(2)}\|^2_{L^2L^2}\leq & C\big(\|e^{bA^{-\frac13}t}\nabla(g_{2,1}+\kappa g_{3,1})\|^2_{L^2L^2}+\|e^{bA^{-\frac13}t}\nabla g_{2,2}\|^2_{L^2L^2}\\ &+\|e^{bA^{-\frac13}t}\nabla g_{3,2}\|^2_{L^2L^2}\big)\leq C\epsilon^2AE^2_{5,2}(t).\end{aligned}\tag{8.37}$$

Combining (8.36) with (8.37), we finish the proof.

□

Lemma 8.6. *Under the assumptions of Theorem 1.1 and (2.5), there exists a positive constant A_6 independent of t and A, such that if $A\geq A_6$, then there holds*

$$E_4(t)+E_{5,1}(t)\leq C\left(\|(u_{\mathrm{in}})_{\neq}\|_{H^2}+1\right)=:E_4+E_5.$$

Proof. **Step I. Estimate** $\|\triangle u_{3,\neq}-2\partial_x W^{(3)}\|_{X_b}$. Applying Proposition A.1 to (8.9), we obtain

$$\begin{aligned}&\|\triangle u_{3,\neq}-2\partial_xW^{(3)}\|^2_{X_b}\\ \leq & C\big(\|(\triangle u_{3,\mathrm{in}})_{\neq}\|^2_{L^2}+A^{\frac13}\|e^{bA^{-\frac13}t}\triangle V\partial_x u_{3,\neq}\|^2_{L^2L^2}\\ &+A^{\frac13}\|e^{bA^{-\frac13}t}\partial_z(\partial_yV\partial_xW)\|^2_{L^2L^2}+\frac1A\|e^{bA^{-\frac13}t}\nabla(\mathbf{U}_1\partial_xu_{3,\neq})\|^2_{L^2L^2}\\ &+\frac1A\|e^{bA^{-\frac13}t}\nabla(g_{2,2}+g_{3,1}+g_{3,2}+G_{3,3})\|^2_{L^2L^2}+\frac1A\|e^{bA^{-\frac13}t}(P_{1,1}+P_{1,2}+P_{1,3})\|^2_{L^2L^2}\big),\end{aligned}\tag{8.38}$$

where we use $\partial_xW^{(3)}(0)=0$. Due to $\|\triangle V\|_{L^\infty}\leq CA^{-1}\|\triangle\mathbf{U}_2\|_{H^2}\leq C$, there holds

$$\begin{aligned}\|e^{bA^{-\frac13}t}\triangle V\partial_xu_{3,\neq}\|^2_{L^2L^2}\leq & \|\triangle V\|^2_{L^\infty L^\infty}\|e^{bA^{-\frac13}t}\partial_xu_{3,\neq}\|^2_{L^2L^2}\\ \leq & C\|e^{bA^{-\frac13}t}\partial_x^2u_{3,\neq}\|^2_{L^2L^2}\leq CA^{\frac13}\|\partial_x^2u_{3,\neq}\|^2_{X_b}\leq CA^{\frac13}E^2_{5,2}(t).\end{aligned}\tag{8.39}$$

Moreover, for $\|\partial_y V\|_{H^3}\leq C\left(1+A^{-1}\|\triangle \mathbf{U}_2\|_{H^2}\right)\leq C$, one deduces

$$\begin{aligned}&\|\partial_z(\partial_y V\partial_x W)\|_{L^2}\leq \|\partial_y V\|_{H^3}\|\partial_x W\|_{H^1}\leq C\|\nabla\partial_x W\|_{L^2},\\&\|\mathrm{e}^{bA^{-\frac13}t}\partial_z(\partial_y V\partial_x W)\|^2_{L^2L^2}\leq C\|\mathrm{e}^{bA^{-\frac13}t}\nabla\partial_x W\|^2_{L^2L^2}\leq CA^{\frac13}E^2_{5,2}(t).\end{aligned}\tag{8.40}$$

Recall that $G_{3,3}=(u_{\neq}\cdot\nabla u_{3,\neq})_{\neq}$ in (8.3), by Lemma 3.2, and we get

$$\|\mathrm{e}^{bA^{-\frac13}t}\nabla G_{3,3}\|^2_{L^2L^2}\leq C\left(A^{\frac{\alpha}{3}+\frac76}E_2^2E_5^2+A^{\frac43}E_2^4\right),\tag{8.41}$$

where $\alpha\in(\frac12,\frac34)$.

Collecting (8.39)-(8.41), and using (8.35), Lemma 8.3 and Lemma 8.4, we get from (8.38) that

$$\|\triangle u_{3,\neq}-2\partial_x W^{(3)}\|^2_{X_b}\leq C\left(\|(u_{\rm in})_{\neq}\|^2_{H^2}+A^{\frac23}E^2_{5,2}(t)+A^{\frac{\alpha}{3}+\frac16}E_2^2E_5^2+A^{\frac13}E_2^4+\epsilon^2A^{\frac23}E^2_{5,1}(t)+1\right)\tag{8.42}$$

provided with $A\geq\max\{A_5,E_2^{\frac{24}{5-6\alpha}},E_2^{\frac{24}{3-4\alpha}},(\|(u_{1,\rm in})_0\|^2_{H^1}+H_1^2)^{\frac32}\}=:A_{6,1}$.

Step II. Estimate $\|\triangle W^{(1)}\|_{X_b}$. By (8.6), there holds

$$\widetilde{\mathcal{L}_V}W+\frac{\mathbf{U}_1\partial_x W}{A}+\frac{G^{(1)}+G^{(2)}}{A}=\left(\partial_t\kappa-\frac{\triangle\kappa}{A}\right)u_{3,\neq}-\frac{2\nabla\kappa\cdot\nabla u_{3,\neq}}{A}.$$

Using the following decomposition

$$\nabla\kappa\cdot\nabla u_{3,\neq}=\rho_1\nabla V\cdot\nabla u_{3,\neq}+\rho_2(\partial_z-\kappa\partial_y)u_{3,\neq}$$

and $W=W^{(1)}+\frac1A W^{(2)}$, we get

$$\begin{aligned}&\widetilde{\mathcal{L}_V}W^{(1)}+\frac{\mathbf{U}_1\partial_x W}{A}+\frac{G^{(1)}+G^{(2)}}{A}-\left(\partial_t\kappa-\frac{\triangle\kappa}{A}\right)u_{3,\neq}\\=&-\frac2A\left(\rho_1\nabla V\cdot\nabla u_{3,\neq}+\rho_2(\partial_z-\kappa\partial_y)u_{3,\neq}\right)-\frac1A\widetilde{\mathcal{L}_V}W^{(2)}=-\frac2A\rho_2(\partial_z-\kappa\partial_y)u_{3,\neq}-\frac1A J_{11},\end{aligned}\tag{8.43}$$

where

$$J_{11}=\widetilde{\mathcal{L}_V}W^{(2)}+2\rho_1\nabla V\cdot\nabla u_{3,\neq}.\tag{8.44}$$

Applying Proposition A.4 to (8.43), one deduces

$$\begin{aligned}\|\triangle W^{(1)}\|^2_{X_b}\leq& C\big(\|\triangle W^{(1)}(0)\|^2_{L^2}+A^{-1}\|\mathrm{e}^{bA^{-\frac13}t}\nabla\left(\rho_2(\partial_z-\kappa\partial_y)u_{3,\neq}\right)\|^2_{L^2L^2}\\&+A^{-1}\|\mathrm{e}^{bA^{-\frac13}t}\nabla J_{11}\|^2_{L^2L^2}+A\|\mathrm{e}^{bA^{-\frac13}t}\nabla((\partial_t\kappa-A^{-1}\triangle\kappa)u_{3,\neq})\|^2_{L^2L^2}\\&+A^{-1}\|\mathrm{e}^{bA^{-\frac13}t}\nabla(\mathbf{U}_1\partial_x W)\|^2_{L^2L^2}+A^{-1}\|\mathrm{e}^{bA^{-\frac13}t}\nabla(G^{(1)}+G^{(2)})\|^2_{L^2L^2}\big).\end{aligned}\tag{8.45}$$

It follows from Lemma 8.1 that $\|\rho_2\|_{H^2}\leq C\epsilon$ and

$$\|\partial_t\kappa-A^{-1}\triangle\kappa\|_{H^1}\leq\|\partial_t\kappa\|_{H^1}+CA^{-1}\|\kappa\|_{H^3}\leq CA^{-1}\epsilon.\tag{8.46}$$

Hence we have

$$\|\nabla\left(\rho_2(\partial_z-\kappa\partial_y)u_{3,\neq}\right)\|_{L^2}\leq C\|\rho_2\|_{H^2}\|\nabla(\partial_z-\kappa\partial_y)u_{3,\neq}\|_{L^2}\leq C\epsilon\|\nabla\partial_x(\partial_z-\kappa\partial_y)u_{3,\neq}\|_{L^2}.\tag{8.47}$$

Besides, using (8.46) and Lemma A.5, there holds

$$\|\nabla\left((\partial_t\kappa-A^{-1}\triangle\kappa)u_{3,\neq}\right)\|_{L^2}\leq CA^{-1}\epsilon\left(\|\nabla\partial_x^2u_{3,\neq}\|_{L^2}+\|\nabla\partial_x(\partial_z-\kappa\partial_y)u_{3,\neq}\|_{L^2}\right).\tag{8.48}$$

For J_{11} in (8.44), we rewrite it as follows:

$$
\begin{aligned}
J_{11} =& \mathcal{L}_V W^{(2)} - 2(\partial_y + \kappa\partial_z)\triangle^{-1}(\partial_y V \partial_x W^{(2)}) + 2\rho_1 \nabla V \cdot \nabla u_{3,\neq} \\
=& -2(\partial_y + \kappa\partial_z)\triangle^{-1}(\partial_y V \partial_x W^{(2)}) + \rho_1 \nabla V \cdot \nabla u_{3,\neq} \\
=& -(\partial_y + \kappa\partial_z)\triangle^{-1}(\partial_y V J_{12}) - (\partial_y + \kappa\partial_z)\triangle^{-1}(\partial_y V \rho_1 \triangle u_{3,\neq}) + \rho_1 \nabla V \cdot \nabla u_{3,\neq},
\end{aligned}
\tag{8.49}
$$

where $J_{12} = 2\partial_x W^{(2)} - \rho_1 \triangle u_{3,\neq} = 2\partial_x(W^{(2)} - \rho_1 W^{(3)}) - \rho_1 \left(\triangle u_{3,\neq} - 2\partial_x W^{(3)}\right)$. Using Lemma 8.1 and (8.12), we arrive

$$
J_{12} \le C\left(\|\partial_x(W^{(2)} - \rho_1 W^{(3)})\|_{L^2} + \epsilon\|\triangle u_{3,\neq} - 2\partial_x W^{(3)}\|_{L^2}\right)
$$

and

$$
\begin{aligned}
&\|(\partial_y + \kappa\partial_z)\triangle^{-1}(\partial_y V J_{12})\|_{H^1} \le C\left(\|\partial_y \triangle^{-1}(\partial_y V J_{12})\|_{H^1} + \|\kappa\|_{H^3}\|\partial_z \triangle^{-1}(\partial_y V J_{12})\|_{H^1}\right) \\
&\le C\|J_{12}\|_{L^2} \le C\left(\|\partial_x(W^{(2)} - \rho_1 W^{(3)})\|_{L^2} + \epsilon\|\triangle u_{3,\neq} - 2\partial_x W^{(3)}\|_{L^2}\right).
\end{aligned}
\tag{8.50}
$$

Thanks to $\nabla V \cdot \nabla u_{3,\neq} = \partial_y V(\partial_y + \kappa\partial_z)u_{3,\neq}$, there holds

$$
\begin{aligned}
&-(\partial_y + \kappa\partial_z)\triangle^{-1}(\partial_y V \rho_1 \triangle u_{3,\neq}) + \rho_1 \nabla V \cdot \nabla u_{3,\neq} \\
=& -\left[(\partial_y + \kappa\partial_z)\triangle^{-1}, \partial_y V \rho_1\right] \triangle u_{3,\neq} \\
=& -\left[\partial_y \triangle^{-1}, \partial_y V \rho_1\right] \triangle u_{3,\neq} - \kappa\left[\partial_z \triangle^{-1}, \partial_y V \rho_1\right] \triangle u_{3,\neq}.
\end{aligned}
$$

From this, along with (8.12), Lemma 8.1 and Lemma A.5, we get

$$
\begin{aligned}
&\|-(\partial_y + \kappa\partial_z)\triangle^{-1}\left(\partial_y V \rho_1 \triangle u_{3,\neq}\right) + \rho_1 \nabla V \cdot \nabla u_{3,\neq}\|_{H^1} \\
\le& \|\left[\partial_y \triangle^{-1}, \partial_y V \rho_1\right] \triangle u_{3,\neq}\|_{H^1} + \|\kappa\|_{L^\infty}\|\left[\partial_z \triangle^{-1}, \partial_y V \rho_1\right] \triangle u_{3,\neq}\|_{H^1} \\
\le& C\epsilon\left(\|\nabla \partial_x^2 u_{3,\neq}\|_{L^2} + \|\partial_x(\partial_z - \kappa\partial_y)\nabla u_{3,\neq}\|_{L^2}\right).
\end{aligned}
\tag{8.51}
$$

Then we conclude from (8.49), (8.50) and (8.51) that

$$
\begin{aligned}
\|\nabla J_{11}\|_{L^2} \le& C\left(\|\partial_x(W^{(2)} - \rho_1 W^{(3)})\|_{L^2} + \epsilon\|\triangle u_{3,\neq} - 2\partial_x W^{(3)}\|_{L^2}\right) \\
&+ C\epsilon\left(\|\nabla \partial_x^2 u_{3,\neq}\|_{L^2} + \|\nabla \partial_x(\partial_z - \kappa\partial_y)u_{3,\neq}\|_{L^2}\right).
\end{aligned}
\tag{8.52}
$$

Collecting (8.42), (8.47), (8.48), (8.52) and (iii) of Lemma 8.2, when $A \ge A_{6,1}$, one obtains

$$
\begin{aligned}
&A^{-1}\|e^{bA^{-\frac{1}{3}}t}\nabla J_{11}\|_{L^2L^2}^2 + A^{-1}\|e^{bA^{-\frac{1}{3}}t}\nabla\left(\rho_2(\partial_z - \kappa\partial_y)u_{3,\neq}\right)\|_{L^2L^2}^2 \\
&+ A\|e^{bA^{-\frac{1}{3}}t}\nabla\left((\partial_t\kappa - \nu\triangle\kappa)u_{3,\neq}\right)\|_{L^2L^2}^2 \\
\le& CA^{-\frac{2}{3}}\left(\|\partial_x\left(W^{(2)} - \rho_1 W^{(3)}\right)\|_{X_b}^2 + \epsilon^2\|\triangle u_{3,\neq} - 2\partial_x W^{(3)}\|_{X_b}^2\right) \\
&+ C\epsilon^2\left(\|\partial_x^2 u_{3,\neq}\|_{X_b}^2 + \|\partial_x(\partial_z - \kappa\partial_y)u_{3,\neq}\|_{X_b}^2\right) \\
\le& C\epsilon^2\left(\|(u_{\rm in})_{\neq}\|_{H^2}^2 + E_{5,1}^2(t) + E_{5,2}^2(t) + 1\right),
\end{aligned}
$$

Combining above with Lemma 8.3 and Lemma 8.5, when

$$
A \ge C\max\{\epsilon^{-6}(\|(u_{1,\rm in})_0\|_{H^1}^2 + H_1^2)^3, (\epsilon^{-1}E_2)^{\frac{12}{3-2\alpha}}, (E_2E_5)^{\frac{12}{3-2\alpha}}, A_{6,1}\} =: A_{6,2},
$$

we get from (8.45) that

$$
\|\triangle W^{(1)}\|_{X_b}^2 \le C\left(\|(u_{\rm in})_{\neq}\|_{H^2}^2 + \epsilon^2 E_{5,1}^2(t) + \epsilon^2 E_{5,2}^2(t) + 1\right) + CA^{-\frac{1}{2}}\left(\|(u_{1,\rm in})_0\|_{H^1}^2 + H_1^2\right)\|\triangle W^{(1)}\|_{X_b}^2.
$$

This implies that

$$\|\triangle W^{(1)}\|_{X_b}^2 \leq C\left(\|(u_{\rm in})_{\neq}\|_{H^2}^2+\epsilon^2E_{5,1}^2(t)+\epsilon^2E_{5,2}^2(t)+1\right). \tag{8.53}$$

Step III. Estimate $\|\partial_x^2u_{j,\neq}\|_{X_b}+\|\partial_x(\partial_z-\kappa\partial_y)u_{j,\neq}\|_{X_b}$. As $W=W^{(1)}+\frac{1}{A}W^{(2)}$, for $j\in\{2,3\}$, we rewrite (8.5) into

$$\begin{aligned}\mathcal{L}_Vu_{j,\neq}=&2\partial_j\triangle^{-1}(\partial_yV\partial_xW^{(1)})+\frac{2}{A}\partial_j\triangle^{-1}(\partial_yV\partial_xW^{(2)})-\frac{\mathbf{U}_1\partial_xu_{j,\neq}}{A}-\frac{g_{j,1}+g_{j,2}+G_{j,3}}{A}\\&+\frac{2\partial_j\triangle^{-1}(\partial_yg_{2,2}+\partial_zg_{3,2})}{A}+\frac{2\partial_j\triangle^{-1}(P_{1,1}+P_{1,2}+P_{1,3})}{A}.\end{aligned}$$

Applying Proposition A.3 to it, there holds

$$\begin{aligned}&\|\partial_xu_{j,\neq}\|_{X_b}^2+\|\partial_x(\partial_z-\kappa\partial_y)u_{j,\neq}\|_{X_b}^2\\ \leq& C\big(\|(u_{\rm in})_{\neq}\|_{H^2}^2+\|{\rm e}^{bA^{-\frac13}t}\partial_j(\partial_yV\partial_xW^{(1)})\|_{L^2L^2}^2+A^{-\frac53}\|{\rm e}^{bA^{-\frac13}t}\partial_yV\partial_x^2W^{(2)}\|_{L^2L^2}^2\\&+A^{-1}\|{\rm e}^{bA^{-\frac13}t}\mathbf{U}_1\partial_x^2u_{j,\neq}\|_{L^2L^2}^2+A^{-1}\|{\rm e}^{bA^{-\frac13}t}\partial_x(g_{j,1}+G_{j,3})\|_{L^2L^2}^2\\&+A^{-1}\|{\rm e}^{bA^{-\frac13}t}\nabla(g_{2,2}+g_{3,2})\|_{L^2L^2}^2+A^{-1}\|{\rm e}^{bA^{-\frac13}t}(P_{1,1}+P_{1,2}+P_{1,3})\|_{L^2L^2}^2\big),\end{aligned} \tag{8.54}$$

where we use $\|\partial_x^2f\|_{L^2}+\|\partial_x(\partial_z-\kappa\partial_y)f\|_{L^2}\leq C(1+\|\kappa\|_{L^\infty})\|\partial_x\nabla f\|_{L^2}\leq C\|\partial_x\nabla f\|_{L^2}$.

By Lemma 4.5, we arrive

$$\begin{aligned}\|{\rm e}^{bA^{-\frac13}t}\partial_j(\partial_yV\partial_xW^{(1)})\|_{L^2L^2}^2\leq& C\left(\|\nabla V\|_{L^\infty L^\infty}^2+\|\nabla^2V\|_{L^\infty L^\infty}^2\right)\|{\rm e}^{bA^{-\frac13}t}\nabla\partial_xW^{(1)}\|_{L^2L^2}^2\\ \leq& C\left(1+\frac{\|\triangle\mathbf{U}_2\|_{L^\infty H^2}^2}{A}\right)\|\triangle W^{(1)}\|_{X_b}^2\leq C\|\triangle W^{(1)}\|_{X_b}^2\end{aligned}$$

and

$$A^{-\frac53}\|{\rm e}^{bA^{-\frac13}t}\partial_yV\partial_x^2W^{(2)}\|_{L^2L^2}^2\leq A^{-\frac53}\|\partial_yV\|_{L^\infty L^\infty}^2\|{\rm e}^{bA^{-\frac13}t}\partial_x^2W^{(2)}\|_{L^2L^2}^2\leq CA^{-\frac43}\|\partial_x^2W^{(2)}\|_{X_b}^2.$$

Due to Lemma 8.1 and Lemma 8.4, one obtains

$$\|{\rm e}^{bA^{-\frac13}t}\partial_xg_{j,1}\|_{L^2L^2}^2\leq C\left(\|{\rm e}^{bA^{-\frac13}t}\nabla(g_{2,1}+\kappa g_{3,1})\|_{L^2L^2}^2+\|{\rm e}^{bA^{-\frac13}t}\partial_xg_{3,1}\|_{L^2L^2}^2\right)\leq C\epsilon^2AE_{5,2}^2(t).$$

Moreover, it follows from Lemma 3.2 that

$$\|{\rm e}^{bA^{-\frac13}t}\partial_xG_{j,3}\|_{L^2L^2}^2\leq C\|{\rm e}^{2aA^{-\frac13}t}\partial_x(u_{\neq}\cdot\nabla u_{\neq})\|_{L^2L^2}^2\leq CA^{\frac16+\alpha}E_2^4.$$

By combining the above estimates and using Lemma 8.3, Lemma 8.4 and (8.35), when

$$A\geq C\max\{\epsilon^{\frac{-3}{1-\alpha}}\left(\|(u_{1,\rm in})_0\|_{H^1}^2+H_1^2\right)^{\frac{3}{2(1-\alpha)}},A_{6,2}\}=:A_6.$$

we get from (8.54) that

$$\begin{aligned}&\|\partial_xu_{j,\neq}\|_{X_b}^2+\|\partial_x(\partial_z-\kappa\partial_y)u_{j,\neq}\|_{X_b}^2\\ \leq& C\left(\|(u_{\rm in})_{\neq}\|_{H^2}^2+\|\triangle W^{(1)}\|_{X_b}^2+A^{-\frac43}\|\partial_x^2W^{(2)}\|_{X_b}^2+\epsilon^2E_{5,2}^2(t)+1\right).\end{aligned} \tag{8.55}$$

Step IV. Estimate $E_{5,1}^2(t)+E_{5,2}^2(t)$ and $E_4^2(t)$. For $W=W^{(1)}+\frac{1}{A}W^{(2)}$, there holds

$$\|\partial_x\nabla W\|_{X_b}\leq\frac{1}{A}\|\partial_x\nabla W^{(2)}\|_{X_b}+\|\partial_x\nabla W^{(1)}\|_{X_b}\leq\frac{1}{A}\|\partial_x\nabla W^{(2)}\|_{X_b}+\|\triangle W^{(1)}\|_{X_b},$$

which along with (8.55) gives

$$E_{5,2}^2(t)\leq C\big(\|(u_{\rm in})_{\neq}\|_{H^2}^2+\|\triangle W^{(1)}\|_{X_b}^2+A^{-\frac{4}{3}}\|\partial_x^2W^{(2)}\|_{X_b}^2+\epsilon^2E_{5,2}^2(t)+A^{-2}\|\partial_x\nabla W^{(2)}\|_{X_b}^2+1\big).$$

By (8.53) and (ii) of Lemma 8.2, the above inequality indicates that

$$E_{5,2}^2(t)\leq C\left(\|(u_{\rm in})_{\neq}\|_{H^2}^2+\epsilon^2E_{5,1}^2(t)+1\right)+C\epsilon^2E_{5,2}^2(t).$$

Taking ϵ small enough satisfying $C\epsilon^2\leq\frac{1}{2}$, we have

$$E_{5,2}^2(t)\leq C\left(\|(u_{\rm in})_{\neq}\|_{H^2}^2+\epsilon^2E_{5,1}^2(t)+1\right). \tag{8.56}$$

Using (8.42) and (i) of Lemma 8.2, one deduces

$$\begin{aligned}\|\triangle u_{3,\neq}\|_{X_b}^2\leq&C\left(\|\triangle u_{3,\neq}-2\partial_xW^{(3)}\|_{X_b}^2+\|\partial_xW^{(3)}\|_{X_b}^2\right)\\ \leq&C\left(\|(u_{\rm in})_{\neq}\|_{H^2}^2+A^{\frac{4}{3}}E_{5,2}^2(t)+A^{\frac{\alpha}{3}+\frac{1}{6}}E_2^2E_5^2+A^{\frac{1}{3}}E_2^4+\epsilon^2A^{\frac{2}{3}}E_{5,1}^2(t)+1\right).\end{aligned} \tag{8.57}$$

Combining (8.56) and (8.57), when $A\geq A_6$, there holds

$$E_{5,1}^2(t)+E_{5,2}^2(t)=A^{-\frac{4}{3}}\|\triangle u_{3,\neq}\|_{X_b}^2+E_{5,2}^2(t)\leq C\left(\|(u_{\rm in})_{\neq}\|_{H^2}^2+1\right)+C\epsilon^2E_{5,1}^2(t).$$

Choosing ϵ small enough satisfying $C\epsilon^2\leq\frac{1}{2}$, we conclude that

$$E_4^2(t)\leq C(E_{5,1}^2(t)+E_{5,2}^2(t))\leq C\left(\|(u_{\rm in})_{\neq}\|_{H^2}^2+1\right).$$

The proof is complete. □

Corollary 8.1. *Under the assumptions of Theorem 1.1 and (2.5), according to Lemma 5.1, Lemma 5.2 and Lemma 8.6, there exists a positive constant $A\geq\max\{A_3,A_4,A_6\}=:A_7$, such that if $A\geq A_7$, there holds*

$$E_2(t)\leq C\left(\|(\partial_x^2n_{\rm in})_{\neq}\|_{L^2}+\|(u_{\rm in})_{\neq}\|_{H^2}+1\right)=:E_2.$$

Appendix A. Some useful estimates and lemmas in the proof

A.1. **Several useful lemmas.** We first prove an embedding inequality for non-zero modulus functions.

Lemma A.1. *Let f be a function such that $f_{\neq}\in H^1(\mathbb{T}^3)$, there holds*

$$\|f_{\neq}\|_{L^2(\mathbb{T}^3)}\leq C\|\partial_xf_{\neq}\|_{L^2(\mathbb{T}^3)}\leq C\|\nabla f_{\neq}\|_{L^2(\mathbb{T}^3)}.$$

Proof. It follows from Poincaré inequality immediately and we omit it. □

The following lemma can be used to estimate the L^∞ norm for the zero mode.

Lemma A.2 (Lemma 3.1 in [10]). *For a given function $f(x,y,z)$ and $f_0=\frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} f(t,x,y,z)dx$, we have*

$$\begin{aligned}
&\|f_0\|_{L^\infty}\leq C\left(\|\partial_y f_0\|_{L^2}^{\frac12}\|f_0\|_{L^2}^{\frac12}+\|\partial_z\nabla f_0\|_{L^2}^{\frac12}\|\partial_z f_0\|_{L^2}^{\alpha-\frac12}\|f_0\|_{L^2}^{1-\alpha}+\|f_0\|_{L^2}\right),\\
&\|f_0\|_{L^\infty}\leq C\left(\|\partial_y f_0\|_{L^2}^{\frac12}\|f_0\|_{L^2}^{\frac12}+\|\partial_z\nabla f_0\|_{L^2}^{\alpha-\frac12}\|\partial_z f_0\|_{L^2}^{\frac12}\|\partial_y f_0\|_{L^2}^{1-\alpha}+\|f_0\|_{L^2}\right),\\
&\|f_0\|_{L^\infty_z L^2_y}\leq C\left(\|f_0\|_{L^2}+\|\partial_z f_0\|_{L^2}^{\alpha}\|f_0\|_{L^2}^{1-\alpha}\right),\\
&\|f_0\|_{L^\infty_y L^2_z}\leq C(\|\partial_y f_0\|_{L^2}^{\frac12}\|f_0\|_{L^2}^{\frac12}+\|f_0\|_{L^2}),
\end{aligned}\tag{A.1}$$

where α is a constant with $\alpha\in\left(\frac12,\frac34\right)$.

The following lemma can be used to estimate the L^∞ norm for the non-zero mode. We only prove $(A.2)_{1,2}$, and the remaining results are similar to Lemma 3.2 in [10]. The proof is omitted.

Lemma A.3. *For a given function $g=g(x,y,z)$ and $\alpha\in(\frac12,\frac34)$, if $g_0=\frac{1}{|\mathbb{T}|}\int_{\mathbb{T}} g(t,x,y,z)dx=0$, then we have*

$$\begin{aligned}
&\|g\|_{L^\infty}\leq C\big(\|\partial_x\nabla g\|_{L^2}^{\frac12}\|\partial_x\partial_z g\|_{L^2}^{1-\alpha}\|\partial_z^2 g\|_{L^2}^{\alpha-\frac12}+\|\partial_x\nabla g\|_{L^2}^{\frac12}\|\partial_x g\|_{L^2}^{\frac12}\big),\\
&\|g\|_{L^\infty}\leq C\big(\|\partial_z\nabla g\|_{L^2}^{\frac12}\|\partial_x\partial_z g\|_{L^2}^{1-\alpha}\|\partial_x^2 g\|_{L^2}^{\alpha-\frac12}+\|\partial_x\nabla g\|_{L^2}^{\frac12}\|\partial_x g\|_{L^2}^{\frac12}\big),\\
&\|g\|_{L^\infty_{y,z}L^2_x}\leq C\min\big\{\|\partial_y g\|_{L^2}^{\frac12}\|g\|_{L^2}^{\frac12}+\|\partial_z\nabla g\|_{L^2}^{\frac12}\|\partial_z g\|_{L^2}^{\alpha-\frac12}\|g\|_{L^2}^{1-\alpha}+\|g\|_{L^2},\\
&\qquad\qquad\|\partial_y g\|_{L^2}^{\frac12}\|g\|_{L^2}^{\frac12}+\|\partial_z g\|_{L^2}^{\frac12}\|\partial_z\nabla g\|_{L^2}^{\alpha-\frac12}\|\partial_y g\|_{L^2}^{1-\alpha}+\|g\|_{L^2}\big\},\\
&\|g\|_{L^\infty_{x,y}L^2_z}\leq C\big(\|\partial_x g\|_{L^2}^{\frac12}\|\partial_x\partial_y g\|_{L^2}^{\alpha-\frac12}\|\partial_y g\|_{L^2}^{1-\alpha}+\|\partial_x g\|_{L^2}\big),\\
&\|g\|_{L^\infty_{x,z}L^2_y}\leq C\big(\|\partial_x g\|_{L^2}^{\alpha}\|g\|_{L^2}^{1-\alpha}+\|\partial_x\partial_z g\|_{L^2}^{\alpha}\|g\|_{L^2}^{1-\alpha}\big),\\
&\|g\|_{L^\infty_x L^2_{y,z}}\leq C\|\partial_x g\|_{L^2}^{\alpha}\|g\|_{L^2}^{1-\alpha},\\
&\|g\|_{L^\infty_z L^2_{y,x}}\leq C(\|g\|_{L^2}+\|\partial_z g\|_{L^2}^{\alpha}\|g\|_{L^2}^{1-\alpha}),\\
&\|g\|_{L^\infty_y L^2_{x,z}}\leq C\big(\|\partial_y g\|_{L^2}^{\frac12}\|g\|_{L^2}^{\frac12}+\|g\|_{L^2}\big),\\
&\|g\|_{L^\infty_{x,z}L^2_y}\leq C\big(\|\partial_x g\|_{L^2}^{\alpha}\|g\|_{L^2}^{1-\alpha}+\|\partial_x\partial_z g\|_{L^2}^{\frac12}\|\partial_x g\|_{L^2}^{\alpha-\frac12}\|\partial_z g\|_{L^2}^{\alpha-\frac12}\|g\|_{L^2}^{\frac32-2\alpha}\big).
\end{aligned}\tag{A.2}$$

Proof. Due to $g_0=0$, we denote $g(x,y,z)$ by $g=\sum_{k_1,k_3\in\mathbb{Z},k_1\neq0}\widehat{g}_{k_1,k_3}(y)\mathrm{e}^{i(k_1x+k_3z)}$, then

$$\|g\|_{L^\infty}\leq\sum_{k_1\neq0,k_3\in\mathbb{Z}}\|\widehat{g}_{k_1,k_3}\|_{L^\infty_y}\leq C\sum_{k_1\neq0,k_3\in\mathbb{Z}}(\|\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\frac12}\|\partial_y\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\frac12}+\|\widehat{g}_{k_1,k_3}\|_{L^2_y}).\tag{A.3}$$

First, there holds

$$\begin{aligned}
&\sum_{k_1\neq 0,k_3\in\mathbb{Z}}\|\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\frac12}\|\partial_y\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\frac12}\\
=&\sum_{k_1\neq 0,k_3\in\mathbb{Z}}\frac{\|k_1\partial_y\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\frac12}\|k_3\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\frac12}|k_1k_3|^{\alpha-\frac12}+\|k_1\partial_y\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\frac12}\|\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\frac12}|k_1|^{\alpha-\frac12}}{|k_1|^\alpha(1+|k_3|^\alpha)}\\
\leq&\sum_{k_1\neq 0,k_3\in\mathbb{Z}}\frac{\|k_1\partial_y\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\frac12}\|k_1k_3\widehat{g}_{k_1,k_3}\|_{L^2_y}^{1-\alpha}\|k_3^2\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\alpha-\frac12}+\|k_1\partial_y\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\frac12}\|\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\frac12}|k_1|^{\alpha-\frac12}}{|k_1|^\alpha(1+|k_3|^\alpha)}.
\end{aligned}$$

Using Hölder's inequality, we get

$$\sum_{k_1\neq 0,k_3\in\mathbb{Z}}\|\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\frac12}\|\partial_y\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\frac12}\leq C\big(\|\partial_x\partial_y g\|_{L^2}^{\frac12}\|\partial_x\partial_z g\|_{L^2}^{1-\alpha}\|\partial_z^2 g\|_{L^2}^{\alpha-\frac12}+\|\partial_x\partial_y g\|_{L^2}^{\frac12}\|\partial_x g\|_{L^2}^{\frac12}\big). \tag{A.4}$$

Thus, one deduces

$$\begin{aligned}
\sum_{k_1\neq 0,k_3\in\mathbb{Z}}\|\widehat{g}_{k_1,k_3}\|_{L^2_y}\leq&\sum\frac{\|k_1\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\frac12}\|k_1k_3\widehat{g}_{k_1,k_3}\|_{L^2_y}^{1-\alpha}\|k_3^2\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\alpha-\frac12}+\|k_1\widehat{g}_{k_1,k_3}\|_{L^2_y}^{\alpha}\|\widehat{g}_{k_1,k_3}\|_{L^2_y}^{1-\alpha}}{|k_1|^\alpha(1+|k_3|^\alpha)}\\
\leq& C\big(\|\partial_x g\|_{L^2}^{\frac12}\|\partial_x\partial_z g\|_{L^2}^{1-\alpha}\|\partial_z^2 g\|_{L^2}^{\alpha-\frac12}+\|\partial_x g\|_{L^2}\big).
\end{aligned}\tag{A.5}$$

Combining (A.3), (A.4) and (A.5), we finish the proof of $(A.1)_1$.

Similarly, we can prove $(A.1)_2$. ☐

Lemma A.4. *For a given function $g=g(x,y,z)$, if $g_0=\frac{1}{|\mathbb{T}|}\int_{\mathbb{T}}g(t,x,y,z)dx=0$, there holds*

$$\begin{aligned}
\|g\|_{L^\infty_{y,z}L^2_x}&\leq C\|(\partial_x,\partial_z-\kappa\partial_y)g\|_{L^2}^{\frac12}\|\nabla(\partial_x,\partial_z-\kappa\partial_y)g\|_{L^2}^{\frac12},\\
\|g\|_{L^\infty_z L^2_{y,x}}&\leq C\|(\partial_x,\partial_z-\kappa\partial_y)g\|_{L^2}.
\end{aligned}$$

Proof. Let $G(X,Y,Z)$ such that $G(x,V(t,y,z),z)=g(x,y,z)$. Note that $(\partial_z-\kappa\partial_y)g(x,y,z)=\partial_Z G(x,V(t,y,z),z)$ and $\|g\|_{L^2}\leq\|\partial_x g\|_{L^2}$. Then the results follow from $(A.2)_3$ and $(A.2)_7$. ☐

Lemma A.5 (**Lemma 5.5** in [36])**.** *Under the conditions of Lemma 8.1, if $\partial_x f_1=0, P_0 f_2=0$, it holds that*

$$\begin{aligned}
\|f_1f_2\|_{L^2}&\leq C\|f_1\|_{H^1}\left(\|f_2\|_{L^2}+\|(\partial_z-\kappa\partial_y)f_2\|_{L^2}\right),\\
\|\nabla\triangle^{-1}(f_1f_2)\|_{L^2}&\leq C\|f_1\|_{L^2}\left(\|f_2\|_{L^2}+\|(\partial_z-\kappa\partial_y)f_2\|_{L^2}\right),\\
\|\nabla(f_1f_2)\|_{L^2}&\leq C\|f_1\|_{H^1}\left(\|f_2\|_{H^1}+\|(\partial_z-\kappa\partial_y)f_2\|_{H^1}\right),
\end{aligned}$$

and for $j\in\{2,3\}$,

$$\|[\partial_j\triangle^{-1},f_1]\triangle f_2\|_{H^1}\leq C\|f_1\|_{H^2}\left(\|\nabla f_2\|_{L^2}+\|(\partial_z-\kappa\partial_y)\nabla f_2\|_{L^2}\right).$$

Next, we introduce a logarithmic Hardy-Littlewood-Sobolev inequality that plays an important role in estimation $\|n_0\|_{L^2}$ and can be found in [29].

Lemma A.6. *Let $\mathcal{M}$ be a 2D Riemannian compact manifold. For all $m > 0$, there exists a constant $C(m)$ such that for all nonnegative functions $f \in L^1(\mathcal{M})$ such that $f \log f \in L^1$, if $\int_{\mathcal{M}} f dx = m$, then*

$$\int_{\mathcal{M}} f \log f dx + \frac{2}{m} \int_{\mathcal{M}} \int_{\mathcal{M}} f(x) f(y) \log d(x,y) \geq -C(m), \tag{A.6}$$

where $d(x,y)$ is the distance on the Riemannian manifold.

The following Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{T}^n$ is frequently used in the proof.

Lemma A.7 (Lemma 9.2 in [24])**.** *Suppose $f \in C^\infty(\mathbb{T}^n), n \geq 2$, and the set where f vanishes is nonempty. Assume that $q, r > 0, \infty > q > r$, and $\frac{1}{n} - \frac{1}{2} + \frac{1}{r} > 0$. Then*

$$\|f\|_{L^q} \leq C(n,q) \|\nabla f\|_{L^2}^{\theta} \|f\|_{L^r}^{1-\theta}, \quad \theta = \frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{n} - \frac{1}{2} + \frac{1}{r}}.$$

For a fixed n, the constant $C(n,q)$ is bounded uniformly when q varies in any compact set in $(0, \infty)$.

A.2. **Elliptic estimates.** The following elliptic estimates are necessary.

Lemma A.8. *Let c_0 and n_0 be the zero mode of c and n, respectively, satisfying*

$$-\triangle c_0 + \bar{n} = n_0,$$

then there hold

$$\begin{aligned} &\|\triangle c_0(t)\|_{L^2} \leq \|n_0(t)\|_{L^2}, \\ &\|\nabla c_0(t)\|_{L^\infty} \leq C \|n_0(t) - \bar{n}\|_{L^3} \\ &\|\nabla c_0(t)\|_{L^4} \leq C \|n_0(t)\|_{L^2}, \end{aligned} \tag{A.7}$$

for any $t \geq 0$.

Proof. The basic energy estimates yield

$$\|\triangle c_0(t)\|_{L^2}^2 + |\bar{n}|^2 |\mathbb{T}|^2 = \|n_0(t)\|_{L^2}^2,$$

which implies $(A.7)_1$. Using Gagliardo-Nirenberg inequality and Hölder's inequality, we have

$$\begin{aligned} \|\nabla c_0(t)\|_{L^\infty} \leq & C \left(\|\nabla c_0(t)\|_{L^2}^{\frac{1}{4}} \|\triangle c_0(t)\|_{L^3}^{\frac{3}{4}} + \|\nabla c_0(t)\|_{L^2} \right) \\ \leq & C \|\triangle c_0(t)\|_{L^3} \leq C \|n_0(t) - \bar{n}\|_{L^3}, \end{aligned}$$

which gives $(A.7)_2$.

Moreover, it follows from Lemma A.7 and $(A.7)_1$ that

$$\|\nabla c_0(t)\|_{L^4} \leq C \|\nabla c_0(t)\|_{L^2}^{\frac{1}{2}} \|\triangle c_0(t)\|_{L^2}^{\frac{1}{2}} \leq C \|\triangle c_0(t)\|_{L^2} \leq C \|n_0(t)\|_{L^2},$$

which gives $(A.7)_3$. □

Lemma A.9. *There holds*

$$\|\partial_x^j \triangle c_{\neq}(t)\|_{L^2} \le \|\partial_x^j n_{\neq}(t)\|_{L^2},$$
$$\|\partial_x^j \nabla c_{\neq}(t)\|_{L^4} \le C\|\partial_x^j n_{\neq}(t)\|_{L^2},$$

for any $t \ge 0$ *and* $j \ge 0$.

Proof. By integration by parts, we have

$$\|\triangle c_{\neq}(t)\|_{L^2}^2 = \|n_{\neq}(t)\|_{L^2}^2.$$

Using Gagliardo-Nirenberg inequality and Lemma A.1, we obtain

$$\|\nabla c_{\neq}(t)\|_{L^4} \le C\left(\|c_{\neq}(t)\|_{L^2}^{\frac{1}{8}}\|\triangle c_{\neq}(t)\|_{L^2}^{\frac{7}{8}} + \|c_{\neq}(t)\|_{L^2}\right) \le C\|n_{\neq}(t)\|_{L^2}.$$

□

A.3. **Space-time estimates.** First, we need to prove the following space-time estimate, and this result is also an improvement on previous time-space estimates [9, 36], which allows us to estimate the non-zero mode $n_{\neq}$ in the periodic domain $\mathbb{T}^3$.

Proposition A.1. *Assume that* f *satisfies*

$$\partial_t f - \frac{1}{A}\triangle f + \left(y + \frac{\mathbf{U}_2(t,y,z)}{A}\right)\partial_x f = \partial_x f_1 + f_2 + \nabla\cdot f_3$$

for $t \in [0,T]$, *where* f, f_1, f_2 *and* f_3 *are given functions and* $P_0 f = P_0 f_1 = P_0 f_2 = P_0 f_3 = 0$. *As long as*

$$\frac{\|\triangle \mathbf{U}_2\|_{L^\infty H^2}}{A} + \|\partial_t \mathbf{U}_2\|_{L^\infty L^\infty} < c, \tag{A.8}$$

for some small c *independent of* A *and* t, *then there holds*

$$\|f_{\neq}\|_{X_a}^2 \le C\left(\|(f_{\rm in})_{\neq}\|_{L^2}^2 + \|e^{aA^{-\frac{1}{3}}t}\nabla f_{1,\neq}\|_{L^2L^2}^2 + A^{\frac{1}{3}}\|e^{aA^{-\frac{1}{3}}t}f_{2,\neq}\|_{L^2L^2}^2 + A\|e^{aA^{-\frac{1}{3}}t}f_{3,\neq}\|_{L^2L^2}^2\right),$$

where "a" is a positive constant.

Proof. Our proof was inspired the coordinate transform method of [36]. First, we state the following result, which is classical.

Lemma A.10. *Let* $g : \mathbb{T}^3 \to \mathbb{R}^3$ *be a* C^1 *map such that* $\|\nabla g\|_{L^\infty} < \frac{1}{2}$. *Then it holds that*

(i) $\|f\circ(\mathrm{Id}+g)\|_{L^p} \sim \|f\|_{L^p}, \|\nabla(f\circ(\mathrm{Id}+\mathrm{g}))\|_{L^p} \sim \|\nabla f\|_{L^p}$ *for every* $1 \le p \le +\infty$ *and* $f \in W^{1,p}$. *Here* $A \sim B$ *means* $C^{-1}A \le B \le CA$ *for some absolute constant* C.
(ii) *There exists a unique* C^1 *solution* h *to* $h(x,y,z) = g((x,y,x) + h(x,y,z))$ *satisfying* $\|\nabla h\|_{L^\infty} \le C\|\nabla g\|_{L^\infty}$.

Then, we define the map $(\mathrm{Id}+g) : \mathbb{T}^3 \to \mathbb{R}^3$ by

$$(x,y,z) \mapsto (X,Y,Z) = (x, V(t,y,z), z), \quad V = y + \frac{\mathbf{U}_2(t,y,z)}{A}.$$

Denote

$$\begin{aligned}&F(t,x,V(t,y,z),z)=f(t,x,y,z),\\&F_j(t,x,V(t,y,z),z)=f_j(t,x,y,z),\ \text{for } j=1,2,\\&F_3(t,x,V(t,y,z),z)=\text{div }f_3(t,x,y,z).\end{aligned}\tag{A.9}$$

According to (A.8), by choosing c small enough, there holds

$$\frac{1}{A}\|\nabla\mathbf{U}_2\|_{L^\infty}\leq\frac{C}{A}\|\nabla\mathbf{U}_2\|_{H^2}\leq\frac{C}{A}\|\triangle\mathbf{U}_2\|_{H^2}<\frac{1}{2},$$

which along with Lemma A.10 implies that $F(t,X,Y,Z)\in C^1$ is well-defined.

Let $w_t(t,Y,Z),w_y(t,Y,Z)$ and $w_z(t,Y,Z)$ be such that

$$\begin{aligned}w_t(t,V(t,y,z),z)&=\frac{\partial_t\mathbf{U}_2(t,y,z)}{A}=\partial_tV(t,y,z),\\w_y(t,V(t,y,z),z)&=\frac{\partial_y\mathbf{U}_2(t,y,z)}{A}=\partial_yV(t,y,z)-1,\\w_z(t,V(t,y,z),z)&=\frac{\partial_z\mathbf{U}_2(t,y,z)}{A}=\partial_zV(t,y,z).\end{aligned}\tag{A.10}$$

Direct calculations indicate that

$$\begin{aligned}&\partial_tf=(\partial_t+w_t\partial_Y)\,F,\quad\partial_xf=\partial_XF,\\&\partial_yf=(1+w_y)\,\partial_YF,\quad\partial_zf=(\partial_Z+w_z\partial_Y)\,F.\end{aligned}\tag{A.11}$$

Therefore, we obtain that $V\partial_xf=Y\partial_XF$, and

$$\begin{aligned}\triangle f=&\left(\partial_x^2+\partial_y^2+\partial_z^2\right)f\\=&\partial_X^2F+\partial_y\left[(1+w_y)\,\partial_YF\right]+\partial_z\left[(\partial_Z+w_z\partial_Y)\,F\right]\\=&\left(\partial_X^2+\partial_Y^2+\partial_Z^2\right)F+\left[(1+w_y)^2+w_z^2-1\right]\partial_Y^2F+2w_z\partial_Z\partial_YF+(w_{yy}+w_{zz})\,\partial_YF\\=:&\triangle F+G\partial_Y^2F+2w_z\partial_Z\partial_YF+H\partial_YF,\end{aligned}\tag{A.12}$$

where

$$G=(1+w_y)^2+w_z^2-1,\ \ H(t,V(t,y,z),z)=\frac{\triangle\mathbf{U}_2(t,y,z)}{A}.$$

Using (A.11)-(A.12), we write the operator $\mathcal{L}_V$ into

$$\begin{aligned}\mathcal{L}_Vf=&\partial_tf-\frac{1}{A}\triangle f+V\partial_xf\\=&(\partial_t+w_t\partial_Y)\,F-\frac{1}{A}\left(\triangle F+G\partial_Y^2F+2w_z\partial_Z\partial_YF+H\partial_YF\right)+Y\partial_XF\\=&\mathcal{L}F-\frac{1}{A}\left(G\partial_Y^2F+2w_z\partial_Z\partial_YF\right)+\left(w_t-\frac{H}{A}\right)\partial_YF,\end{aligned}\tag{A.13}$$

where $\mathcal{L} = \partial_t - \frac{1}{A}\triangle + Y\partial_X$. Due to (A.10) and (A.11), there hold

$$\begin{aligned} \partial_Y F =& \frac{\partial_y f}{\partial_y V}, \quad \partial_Z F = \partial_z f - \frac{\partial_z V \partial_y f}{\partial_y V}, \\ G(t, V(t,y,z), z) =& \left(1 + \frac{\partial_y \mathbf{U}_2}{A}\right)^2 + \left(\frac{\partial_z \mathbf{U}_2}{A}\right)^2 - 1 \\ =& (\partial_y V)^2 + (\partial_z V)^2 - 1, \end{aligned}$$

which imply that

$$\begin{aligned} \partial_Y G + 2\partial_Z w_z =& \frac{\partial_y\left((\partial_y V)^2 + (\partial_z V)^2 - 1\right)}{\partial_y V} + 2\left[\partial_z - \left(\frac{\partial_z V}{\partial_y V}\right)\partial_y\right]\partial_z V \\ =& 2\partial_y^2 V + \frac{2\partial_z V \partial_y \partial_z V}{\partial_y V} + 2\partial_z^2 V - 2\frac{\partial_z V}{\partial_y V}\partial_y\partial_z V \\ =& 2\triangle V = \frac{2}{A}\triangle \mathbf{U}_2 = 2H. \end{aligned} \tag{A.14}$$

Combining (A.13) and (A.14), one deduces that

$$\mathcal{L}_V f = \mathcal{L}F - \frac{1}{A}\partial_Y(G\partial_Y F) - \frac{2}{A}\partial_Z\left(w_z \partial_Y F\right) + \left(w_t + \frac{H}{A}\right)\partial_Y F. \tag{A.15}$$

From (A.9), we get $\mathcal{L}_V f = \partial_X F_1 + F_2 + F_3$. Then (A.15) yields

$$\mathcal{L}F = \partial_X F_1 + F_2 + F_3 + \frac{1}{A}\partial_Y(G\partial_Y F) + \frac{2}{A}\partial_Z\left(w_z\partial_Y F\right) - \left(w_t + \frac{H}{A}\right)\partial_Y F.$$

This implies that

$$\begin{aligned} \|F\|_{X_a}^2 \leq& C\bigg(\|F(0)\|_{L^2}^2 + \|\mathrm{e}^{aA^{-\frac{1}{3}}t}\nabla F_1\|_{L^2L^2}^2 + A^{\frac{1}{3}}\|\mathrm{e}^{aA^{-\frac{1}{3}}t}F_2\|_{L^2L^2}^2 + A\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\nabla\triangle^{-1}F_3\|_{L^2L^2}^2 \\ &+ \frac{1}{A}\left\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\left(G, 2w_z\right)\partial_Y F\right\|_{L^2L^2}^2 + A^{\frac{1}{3}}\left\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\left(w_t + \frac{H}{A}\right)\partial_Y F\right\|_{L^2L^2}^2\bigg) \\ :=& C\bigg(\|F(0)\|_{L^2}^2 + \|\mathrm{e}^{aA^{-\frac{1}{3}}t}\nabla F_1\|_{L^2L^2}^2 + A^{\frac{1}{3}}\|\mathrm{e}^{aA^{-\frac{1}{3}}t}F_2\|_{L^2L^2}^2 \\ &+ A\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\nabla\triangle^{-1}F_3\|_{L^2L^2}^2 + K_1 + K_2\bigg). \end{aligned} \tag{A.16}$$

For K_1, recall that $w_z = \frac{\partial_z \mathbf{U}_2}{A}$ and

$$G = (\partial_y V)^2 + (\partial_z V)^2 - 1 = \left(\frac{\partial_y \mathbf{U}_2}{A} + 1\right)^2 + \left(\frac{\partial_z \mathbf{U}_2}{A}\right)^2 - 1 = \frac{|\nabla \mathbf{U}_2|^2}{A^2} + \frac{2\partial_y \mathbf{U}_2}{A}.$$

Then using (A.8), one deduces

$$\begin{aligned} K_1 \leq& \left(\|G\|_{L^\infty L^\infty}^2 + \|2w_z\|_{L^\infty L^\infty}^2\right)\frac{\|\mathrm{e}^{aA^{-\frac{1}{3}}t}\partial_Y F\|_{L^2L^2}^2}{A} \\ \leq& C(c^4 + c^2)\|F\|_{X_a}^2, \end{aligned} \tag{A.17}$$

where we use the Poincaré inequality $\|\nabla \mathbf{U}_2\|_{L^2} \leq C\|\triangle \mathbf{U}_2\|_{L^2}$.

For K_2, as $w_t = \frac{\partial_t \mathbf{U}_2}{A}$ and $H = \frac{\triangle \mathbf{U}_2}{A}$, using (A.8), we have

$$K_2 \leq A^{\frac{1}{3}} \left(\|w_t\|^2_{L^\infty L^\infty} + \left\| \frac{H}{A} \right\|^2_{L^\infty L^\infty} \right) \|\mathrm{e}^{aA^{-\frac{1}{3}}t} \partial_Y F\|^2_{L^2L^2} \leq \frac{Cc^2}{A^{\frac{2}{3}}} \|F\|^2_{X_a}. \tag{A.18}$$

Combining (A.17) with (A.18), then choosing c small enough, we get by (A.16) that

$$\begin{aligned}\|F\|^2_{X_a} \leq & C\Big(\|F(0)\|^2_{L^2} + \|\mathrm{e}^{aA^{-\frac{1}{3}}t} \nabla F_1\|^2_{L^2L^2} + A^{\frac{1}{3}} \|\mathrm{e}^{aA^{-\frac{1}{3}}t} F_2\|^2_{L^2L^2} \\ & + A \|\mathrm{e}^{aA^{-\frac{1}{3}}t} \nabla \triangle^{-1} F_3\|^2_{L^2L^2} \Big).\end{aligned} \tag{A.19}$$

It follows from Lemma A.10 that

$$\|\mathrm{e}^{aA^{-\frac{1}{3}}t} \nabla F\|^2_{L^2L^2} \leq C \|\mathrm{e}^{aA^{-\frac{1}{3}}t} \nabla f\|^2_{L^2L^2}, \ \ \|\mathrm{e}^{aA^{-\frac{1}{3}}t} F\|^2_{L^2L^2} \leq C \|\mathrm{e}^{aA^{-\frac{1}{3}}t} f\|^2_{L^2L^2} \tag{A.20}$$

and

$$\|\mathrm{e}^{aA^{-\frac{1}{3}}t} \nabla F_1\|^2_{L^2L^2} \leq C \|\mathrm{e}^{aA^{-\frac{1}{3}}t} \nabla f_1\|^2_{L^2L^2}, \ \ \|\mathrm{e}^{aA^{-\frac{1}{3}}t} F_2\|^2_{L^2L^2} \leq C \|\mathrm{e}^{aA^{-\frac{1}{3}}t} f_2\|^2_{L^2L^2}. \tag{A.21}$$

Next we claim

$$\|\nabla \triangle^{-1} F\|_{L^2} \sim \|\nabla \triangle^{-1} f\|_{L^2}. \tag{A.22}$$

As $V = y + \frac{\mathbf{U}_2(t,y,z)}{A}$ and (A.8), we get

$$\|\nabla V\|_{L^\infty} + \|\nabla^2 V\|_{L^\infty} \leq 1 + \frac{\|\nabla \mathbf{U}_2\|_{L^\infty}}{A} + \frac{\|\nabla^2 \mathbf{U}_2\|_{L^\infty}}{A} \leq 1 + C \frac{\|\triangle \mathbf{U}_2\|_{H^2}}{A} \leq C. \tag{A.23}$$

Denote

$$\begin{aligned}\check{F}_1 = \triangle^{-1} F, \quad \check{f}_1(t,x,y,z) = \check{F}_1(t,x,V(t,y,z),z), \\ \check{f}_2 = \triangle^{-1} f, \quad \check{f}_2(t,x,y,z) = \check{F}_2(t,x,V(t,y,z),z).\end{aligned}$$

Then using (A.20) and (A.23), one deduces

$$\begin{aligned}\|\nabla \triangle^{-1} F\|^2_{L^2} =& \|\nabla \check{F}_1\|^2_{L^2} = - < F, \check{F}_1 >= - < f, \partial_y V \check{f}_1 > \\ =& < \nabla \check{f}_2, \nabla(\partial_y V \check{f}_1) > \leq \|\nabla \check{f}_2\|_{L^2} \|\nabla(\partial_y V \check{f}_1)\|_{L^2} \\ \leq & C \|\nabla \check{f}_2\|_{L^2} \left(\|\nabla V\|_{L^\infty} \|\nabla \check{f}_1\|_{L^2} + \|\nabla^2 V\|_{L^\infty} \|\check{f}_1\|_{L^2} \right) \\ \leq & C \|\nabla \check{f}_2\|_{L^2} \|\nabla \check{f}_1\|_{L^2} \leq C \|\nabla \check{f}_2\|_{L^2} \|\nabla \check{f}_1\|_{L^2},\end{aligned} \tag{A.24}$$

which implies

$$\|\nabla \check{F}_1\|_{L^2} \leq C \|\nabla \check{f}_2\|_{L^2}. \tag{A.25}$$

On the other hand, there holds

$$\begin{aligned}\|\nabla \triangle^{-1} f\|^2_{L^2} =& \|\nabla \check{f}_2\|^2_{L^2} = - < f, \check{f}_2 >= - < F, \check{F}_2 / \partial_y V > \\ =& < \nabla \check{F}_1, \nabla(\check{F}_2 / \partial_y V) > \leq \|\nabla \check{F}_1\|_{L^2} \|\nabla(\check{F}_2 / \partial_y V)\|_{L^2}.\end{aligned}$$

Notice that $\partial_y V = \frac{1}{A} \partial_y \mathbf{U}_2 + 1$ and $\frac{1}{A} \|\nabla \mathbf{U}_2\|_{L^\infty} < \frac{1}{2}$, we have $\frac{1}{2} < \partial_y V < \frac{3}{2}$. Along this with (A.23), we get $\|\nabla(1/\partial_y V)\|_{L^\infty} \leq C$, and

$$\begin{aligned}\|\nabla \triangle^{-1} f\|_{L^2} =& \|\nabla \check{f}_2\|^2_{L^2} \leq C \|\nabla \check{F}_1\|_{L^2} \left(\|\nabla \check{F}_2\|_{L^2} + \|\check{F}_2\|_{L^2} \right) \\ \leq & C \|\nabla \check{F}_1\|_{L^2} \|\nabla \check{F}_2\|_{L^2} \leq C \|\nabla \check{F}_1\|_{L^2} \|\nabla \check{f}_2\|_{L^2},\end{aligned} \tag{A.26}$$

which gives

$$\|\nabla \check{f}_2\|_{L^2} \leq C\|\nabla \check{F}_1\|_{L^2}. \tag{A.27}$$

Combining with (A.24)-(A.27), we conclude that

$$\|\nabla \triangle^{-1} F\|_{L^2} = \|\nabla \check{F}_1\|_{L^2} \sim \|\nabla \check{f}_2\|_{L^2} = \|\nabla \triangle^{-1} f\|_{L^2},$$

which indicates (A.22) holds. Similarly, we obtain

$$\|e^{aA^{-\frac{1}{3}}t}\nabla \triangle^{-1} F_3\|^2_{L^2L^2} \leq C\|e^{aA^{-\frac{1}{3}}t} f_3\|^2_{L^2L^2}. \tag{A.28}$$

From (A.19), (A.21) and (A.28), we complete the proof. ☐

To estimate the coupled terms $(\triangle u_{2,\neq}, \partial_x \omega_{2,\neq})$, we also need the following proposition.

Proposition A.2. *Let (h_1, h_2) satisfy*

$$\begin{cases} \mathcal{L}_V h_1 = \nabla \cdot g_1, \\ \mathcal{L}_V h_2 + \partial_x \partial_z \triangle^{-1} h_1 = \nabla \cdot g_2, \end{cases}$$

for $t \in [0, T]$. Assume that

$$\frac{\|\triangle \mathbf{U}_2\|_{L^\infty H^2}}{A} + \|\partial_t \mathbf{U}_2\|_{L^\infty L^\infty} \leq c \tag{A.29}$$

for some small constant c independent of A and T. If $P_0 h_1 = P_0 h_2 = P_0 g_1 = P_0 g_2 = 0$, then for $a \geq 0$, it holds that

$$\|h_{1,\neq}\|^2_{X_a} + \|h_{2,\neq}\|^2_{X_a} \leq C\left(\|(h_{1,\mathrm{in}})_{\neq}\|^2_{L^2} + \|(h_{2,\mathrm{in}})_{\neq}\|^2_{L^2} + A\|e^{aA^{-\frac{1}{3}}t} g_{1,\neq}\|^2_{L^2L^2} + A\|e^{aA^{-\frac{1}{3}}t} g_{2,\neq}\|^2_{L^2L^2}\right).$$

Proof. From Proposition A.2 in [10], we find that if (h_1, h_2) satisfy

$$\begin{cases} \partial_t h_1 - \frac{1}{A}\triangle h_1 + y\partial_x h_1 = \nabla \cdot g_1, \\ \\ \partial_t h_2 - \frac{1}{A}\triangle h_2 + y\partial_x h_2 + \partial_x \partial_z \triangle^{-1} h_2 = \nabla \cdot g_2, \end{cases}$$

for $t \in [0, T]$, then it holds that

$$\|h_{1,\neq}\|^2_{X_a} + \|h_{2,\neq}\|^2_{X_a} \leq C\left(\|(h_{1,\mathrm{in}})_{\neq}\|^2_{L^2} + \|(h_{2,\mathrm{in}})_{\neq}\|^2_{L^2} + A\|e^{aA^{-\frac{1}{3}}t} g_{1,\neq}\|^2_{L^2L^2} + A\|e^{aA^{-\frac{1}{3}}t} g_{2,\neq}\|^2_{L^2L^2}\right).$$

Next following the same route as in Proposition A.1, we complete the proof. ☐

Proposition A.3 (Proposition 4.7 in [36]**).** *Let f satisfy*

$$\mathcal{L}_V f = f_1 + f_2 + f_3$$

for $t \in [0, T]$. Moreover, $\mathbf{U}_2$ satisfies (A.29) and $P_0 f = P_0 f_1 = P_0 f_2 = P_0 f_3 = 0$, then for $a \geq 0$, it holds that

$$\begin{aligned} \|\partial_x^2 f_{\neq}\|^2_{X_a} + \|\partial_x(\partial_z - \kappa\partial_y) f_{\neq}\|^2_{X_a} \leq C\big(&\|(f_{\mathrm{in}})_{\neq}\|^2_{H^2} + \|e^{aA^{-\frac{1}{3}}t}\triangle f_{1,\neq}\|^2_{L^2L^2} + A^{\frac{1}{3}}\|e^{aA^{-\frac{1}{3}}t}\partial_x^2 f_{2,\neq}\|^2_{L^2L^2} \\ &+ A^{\frac{1}{3}}\|e^{aA^{-\frac{1}{3}}t}\partial_x(\partial_z - \kappa\partial_y) f_{2,\neq}\|^2_{L^2L^2} + A\|e^{aA^{-\frac{1}{3}}t}\partial_x f_{3,\neq}\|^2_{L^2L^2}\big). \end{aligned}$$

Proposition A.4 (**Proposition 4.9** in [36]). *Let f satisfy $\widetilde{\mathcal{L}_V} f = f_1$ for $t \in [0,T]$. Moreover, $\mathbf{U}_2$ satisfies (A.29) and $P_0 f = P_0 f_1 = 0$, then for $a \geq 0$, it holds that*

$$\|\triangle f_{\neq}\|_{X_a}^2 \leq C \left(\|(\triangle f_{\text{in}})_{\neq}\|_{L^2}^2 + A\|\mathrm{e}^{aA^{-\frac{1}{3}}t} \nabla f_{1,\neq}\|_{L^2L^2}^2 \right).$$

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