RESIDUE AT ∞

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consider the laurent expansion of $f(z)$ as $z \to \infty$.

Assume that

$$f(z) \sim \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \dots \quad z \to \infty \quad |z| > R.$$

THEN

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = \lim_{R\to\infty} \int_{C_R} \left(\frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \dots \right) dz = \lim_{R\to\infty} \int_0^{2\pi} (i a_{-1})\,d\varphi = 2\pi i\, a_{-1}.$$

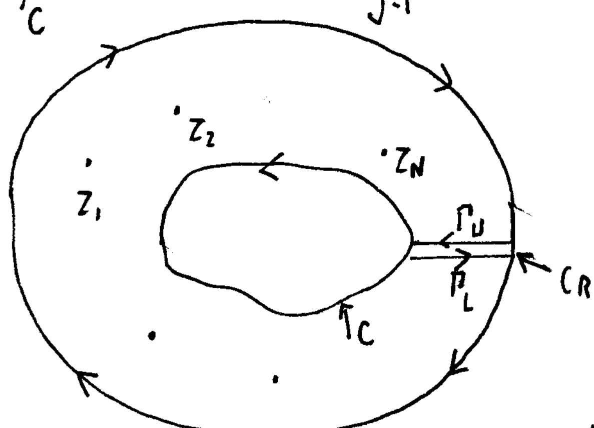
HERE C_R: $|z| = R$ $\quad 0 < \varphi < 2\pi$.

WE DEFINE $$\mathrm{RES}\,[f(\infty)] = -\frac{1}{2\pi i} \lim_{R\to\infty} \int_{C_R} f(z)\,dz = -a_{-1}.$$

EXAMPLE let C be counterclockwise - curve. let $f(z)$ be analytic on C AND analytic in exterior domain except at finitely many isolated poles at $z = z_i$, $i = 1, .., N$. Assume $f(z) = O(1/|z|)$ as $|z| \to \infty$.

THEN $$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{N} \mathrm{RES}\,[f; z_j] + 2\pi i\, \mathrm{RES}\,[f; \infty].$$

PROOF



$$\lim_{R\to\infty} \left(\int_C + \int_{\Gamma_L} + \int_{\Gamma_R} + \int_{C_R} \right) = 2\pi i \sum_{j=1}^{N} \mathrm{RES}\,[f; z_j]$$

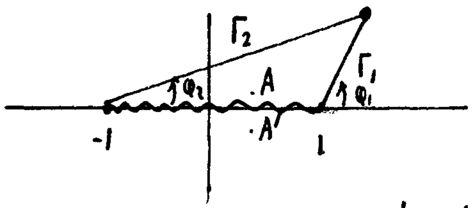
BUT $\int_{\Gamma_L} = -\int_{\Gamma_R}$. ALSO $$\int_C = +2\pi i \sum_{j=1}^{N} \mathrm{RES}\,[f; z_j] - \lim_{R\to\infty} \int_{C_R} f(z)\,dz$$

NOW $f(z) \sim a_{-1}/z$ as $z \to \infty$ $\rightarrow$ $$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{N} \mathrm{RES}\,[f; z_j] - 2\pi i\, a_{-1}$$

$-a_{-1} = \mathrm{RES}\,[f(\infty)]$.

REMARK

(i) pick a single branch of $f(z) = (z^2-1)^{1/2}$ that is analytic at ∞. (single-valued at ∞). (i.e. for $|z| > R$).



NOW $$f(z) = (r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2} \qquad |z-1| = r_1 \quad |z+1| = r_2$$

NOW TAKE $\theta_1 \in (0, 2\pi)$, $\theta_2 \in (0, 2\pi)$

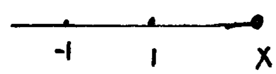
AT A: $z = x,\ y = 0^+$ $$f(z) = (1-x^2)^{1/2} e^{i(\pi/2)} = i(1-x^2)^{1/2}.$$ $\theta_1 = \pi$, $\theta_2 = 0$

AT A': $z = x,\ y = 0^-$ $$f(z) = (1-x^2)^{1/2} e^{i(\pi+2\pi)/2} = -i(1-x^2)^{1/2}$$ $\theta_1 = \pi$, $\theta_2 =$

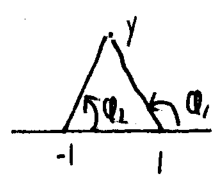
NOW FOR LARGE values OF z we have
$$f(z) = \left[z^2(1 - 1/z^2)\right]^{1/2}$$
$$f(z) = \pm z\,(1 - 1/z^2)^{1/2}.$$

$$(*) \qquad f(z) = \pm z\left(1 - 1/2z^2 + \dots\right) = \pm z \mp 1/2z + \dots$$

NOW WHICH SIGN is consistent with branch chosen. $\theta_1 = \theta_2 = 0$



- let z be real and positive with $z > 1$. THEN
$$f(z) = (x^2-1)^{1/2} e^{i0} \sim x \quad \text{as } x \to \infty.$$
FROM $(*)$ ALONG $z = x$ $\quad f(z) = \pm x \mp 1/2x + \dots$
need + sign.



- let $z = iy$ WITH $y \to \infty$. $\quad f(z) = (y^2-1)^{1/2} e^{i(\pi/2+\pi/2)/2}$
$$f(z) \sim i(y^2-1)^{1/2} \sim iy \quad \text{as } y \to \infty. \text{ HENCE + sign.}$$

EXAMPLE CHOOSE A BRANCH OF

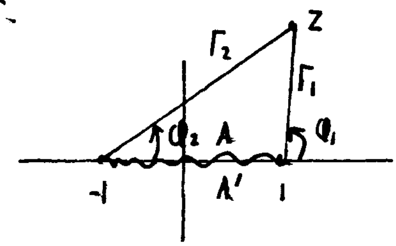
$$f(z) = (1-z^2)^{1/2}$$

which is analytic outside $|z| = R$.

WE WANT THE CUT TO LIE ALONG REAL AXIS FROM (-1) TO (1).

$$f(z) = \left(-1\left(z^2-1\right)\right)^{1/2} = (-1)^{1/2}\left(z^2-1\right)^{1/2}.$$

CHOOSE $(-1)^{1/2} = i$.



HENCE $f(z) = i\left(z^2-1\right)^{1/2}$

$$f(z) = i\left(r_1 r_2\right)^{1/2} e^{i(\varphi_1+\varphi_2)/2} \qquad 0 < \varphi_1 < 2\pi,\ 0 < \varphi_2 < 2\pi$$

AT POINT A: $z = x$, $y = 0^+$ $\quad \varphi_1 = \pi,\ \varphi_2 = 0.$

$$f(z) = i\left(1-x^2\right)^{1/2} e^{i\pi/2} = -\left(1-x^2\right)^{1/2}$$

AT POINT A$'$: $z = x$, $y = 0^-$ $\quad \varphi_1 = \pi,\ \varphi_2 = 2\pi$

$$f(z) = i\left(1-x^2\right)^{1/2} e^{3\pi i/2} = \left(1-x^2\right)^{1/2}.$$

NOW AS $z \to \infty$ WE HAVE

$$f(z) = \pm iz\left(1 - 1/2z^2 + \cdots\right)$$

$$f(z) = \pm iz \mp i/2z + \ldots, \qquad |z| \to \infty.$$

WHICH SIGN?

TAKE $z = x$ $\quad$ x REAL $\quad x \to \infty.$ $\quad \varphi_1 = 0,\ \varphi_2 = 0$

$$f(z) = i\left(x^2-1\right)^{1/2} \longrightarrow f(z) = \pm ix \mp i/2x \ldots \qquad + \text{ SIGN}$$

TAKE $z = iy$ $\quad$ y REAL $\quad y \to \infty$ $\quad \varphi_1 = \pi/2,\ \varphi_2 = \pi/2$

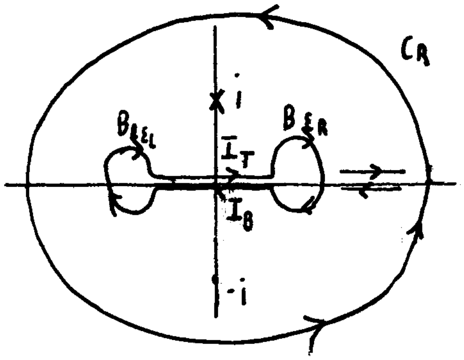
$$f(z) = i\left(y^2-1\right)^{1/2} e^{i\pi/2} = -\left(y^2-1\right)^{1/2} \sim -y$$

$$f(z) = \pm i(iy) = \mp y \ldots \qquad \text{TAKE + ROOT.}$$

$$f(z) \sim iz - i/2z \qquad \text{AS } |z| \to \infty.$$

EXAMPLE $I = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\, dx$ $\qquad f(z) = \frac{(1-z^2)^{1/2}}{1+z^2}$.

CONSIDER CONTOUR AS SHOWN



NOW
$$\lim_{R\to\infty}\int_{C_R} + \lim_{\varepsilon\to 0}\int_{B_{\varepsilon L}} + \lim_{\varepsilon\to 0}\int_{B_{\varepsilon R}} + \int_{I_B} + \int_{I_T} = 2\pi i\Big(\text{RES}[\ ,i] + \text{RES}[\ ,-i]\Big)$$

BRANCH CHOSEN

$$(1-z^2)^{1/2} = i(z^2-1)^{1/2}$$
$$(z^2-1)^{1/2} = i(r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2}$$
$$0 < \theta_1 < 2\pi, \quad 0 < \theta_2 < 2\pi.$$

RECALL **ON TOP**: $z = x,\ y = 0^+,\ \theta_1 = \pi,\ \theta_2 = 0 \qquad dz = dx$

$$f(z) = \frac{i(1-x^2)^{1/2} e^{i\pi/2}}{1+x^2} = -\frac{(1-x^2)^{1/2}}{1+x^2}$$

ON BOTTOM: $z = x,\ y = 0^-\quad \theta_1 = \pi,\ \theta_2 = 2\pi \qquad dz = dx$

$$f(z) = \frac{i(1-x^2)^{1/2}}{1+x^2} e^{3\pi i/2} = \frac{(1-x^2)^{1/2}}{1+x^2}.$$

NOW

$$\int_{I_T} = -\int_{-1}^{1} \frac{(1-x^2)^{1/2}}{1+x^2} \qquad \int_{I_B} = \int_{1}^{-1} \frac{(1-x^2)^{1/2}}{1+x^2}\, dx.$$

AND
$$\int_{I_B} + \int_{I_T} = 2\pi i\Big[\text{RES}[\ ,i] + \text{RES}[\ ,-i]\Big] - \lim_{R\to\infty}\int_{C_R}.$$

NOW AS SHOWN EARLIER $f(z) \sim \frac{iz - i/2z + \cdots}{z^2} \sim i/z$ AS $z \to \infty$.

HENCE

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = \lim_{R\to\infty} \int_{C_R} \frac{i}{z}\,dz = i\, i \int_0^{2\pi} d\varphi = -2\pi.$$

$$z = Re^{i\varphi}$$

$$\longrightarrow \quad -2\int_{-1}^{1} \frac{(1-x^2)^{1/2}}{1+x^2}\,dx = 2\pi + 2\pi i\left(\mathrm{RES}(\quad, i) + \mathrm{RES}(\quad, -i)\right).$$

RES$(\quad, i)$: $z = i$ RES: $\frac{(1-z^2)^{1/2}}{2i} = \frac{i(z^2-1)^{1/2}}{2i} = \frac{i\,2^{1/2}}{2i} e^{i(\varphi_1+\varphi_2)/2} = -\frac{2^{1/2}}{2i}$

$\varphi_1 = 3\pi/4 \quad \varphi_2 = \pi/4 \quad \frac{\varphi_1+\varphi_2}{2} = \pi/2.$

RES$(\quad, -i)$ $z = -i$ $\frac{(1-z^2)^{1/2}}{-2i} = \frac{i\,2^{1/2}}{-2i} e^{i(5\pi/4 + 7\pi/4)/2} = \frac{2^{1/2}}{-2i}.$

$\varphi_1 = 5\pi/4, \quad \varphi_1 = 7\pi/4$

THUS

$$\mathrm{RES}\left[\frac{(1-z^2)^{1/2}}{1+z^2};\ i\right] = \frac{-2^{1/2}}{2i}$$

$$\mathrm{RES}\left[\frac{(1-z^2)^{1/2}}{1+z^2};\ -i\right] = -\frac{2^{1/2}}{2i}.$$

$$\longrightarrow \quad -2\int_{-1}^{1} \frac{(1-x^2)^{1/2}}{1+x^2}\,dx = 2\pi + 2\pi i\left(-\sqrt{2}/i\right) = 2\pi - 2\sqrt{2}\,\pi$$

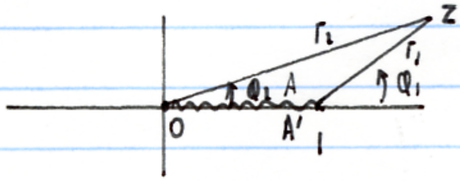
$$\int_{-1}^{1} \frac{(1-x^2)^{1/2}}{1+x^2}\,dx = (\sqrt{2}-1)\pi.$$

EXAMPLE $\quad I = \int_0^1 x^{\alpha-1}(1-x)^{-\alpha}\,dx \qquad 0<\alpha<1$

let $\quad f(z) = z^{\alpha-1}(1-z)^{-\alpha} = z^{\alpha-1}\left(-1\,(z-1)\right)^{-\alpha} \qquad (-1)^{-\alpha} = e^{-\alpha\pi i}$ choose without loss of generali[ty]

$$f(z) = e^{-\alpha\pi i} z^{\alpha-1}(z-1)^{-\alpha}.$$

NOW put a cut between 0 AND 1.



$$z^{\alpha-1}(z-1)^{-\alpha} = r_2^{\alpha-1} r_1^{-\alpha} e^{i[(\alpha-1)\theta_2 - \alpha\theta_1]}$$

IF $\quad \theta_1 \in (0,2\pi), \quad \theta_2 \in (0,2\pi)$

we get cut as shown

AT A $\qquad \theta_1 = \pi, \ \theta_2 = 0 \qquad z = x \quad$ WITH $\quad 0<x<1$

$$\rightarrow \quad z^{\alpha-1}(z-1)^{-\alpha} = x^{\alpha-1}(1-x)^{-\alpha} e^{-i\alpha\pi}$$

$$\rightarrow \quad f(z) = e^{-\alpha\pi i} z^{\alpha-1}(z-1)^{-\alpha} = x^{\alpha-1}(1-x)^{-\alpha} e^{-2i\alpha\pi}.$$

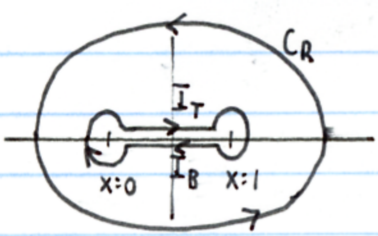
AT A' $\qquad \theta_1 = \pi, \ \theta_2 = 2\pi \qquad z = x \quad$ WITH $\quad 0<x<1$

$$\rightarrow \quad z^{\alpha-1}(z-1)^{-\alpha} = x^{\alpha-1}(1-x)^{-\alpha} e^{i[(\alpha-1)2\pi - \alpha\pi]} = x^{\alpha-1}(1-x)^{-\alpha} e^{i\pi}$$

$$\rightarrow \quad f(z) = e^{-\alpha\pi i} z^{\alpha-1}(z-1)^{-\alpha} = x^{\alpha-1}(1-x)^{-\alpha}.$$

NOW AS $\quad z\to\infty \qquad f(z) \sim e^{-\alpha\pi i} z^{\alpha-1}(z)^{-\alpha} \sim \dfrac{e^{-\alpha\pi i}}{z} + \dots \qquad |z|\to\infty.$

HENCE



$$\lim_{R\to\infty}\left(\int_{I_T} + \int_{I_B} + \int_{C_R}\right) = 0$$

no contribution from blobs near $x = 0, 1$.

$$\rightarrow \quad \underbrace{\int_0^1 x^{\alpha-1}(1-x)^{-\alpha} e^{-2i\alpha\pi}\,dx}_{\text{TOP}} + \underbrace{\int_1^0 x^{\alpha-1}(1-x)^{-\alpha}\,dx}_{\text{BOT}} = -2\pi i\, e^{-\alpha\pi i}$$

$$\left(-1 + e^{-2i\alpha\pi}\right) I = -2i\,\pi e^{-\alpha\pi i} \quad\rightarrow\quad \frac{\left(e^{\pi i\alpha} - e^{-\pi i\alpha}\right)}{2i} I = \pi$$

$$\rightarrow \quad \int_0^1 x^{\alpha-1}(1-x)^{-\alpha}\,dx = \pi/\sin(\pi\alpha)$$