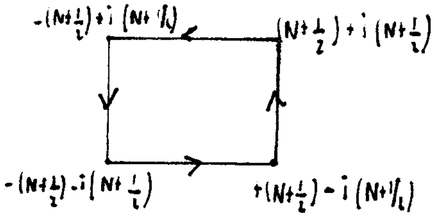
SUMMATION OF SERIES

CONSIDER $\displaystyle\int_{C_N} \pi \cot(\pi z)\, f(z)\, dz$ WHERE C_N IS SQUARE WITH VERTICES AT $\pm (N+1/2) \pm i(N+1/2)$



NOW THE INTEGRAND HAS RESIDUES AT SINGULARITIES OF $\cot(\pi z)$ AND SINGULARITIES OF $f(z)$.

$\cot \pi z = \infty \longrightarrow \sin(\pi z) = 0 \qquad z_K = \pm K \qquad K = 0, 1, 2, \ldots$ ARE SIMPLE POLES.

NOW ASSUME $f(z_K) \neq 0$ AND THAT f HAVE SINGULARITIES AT ζ_j FOR $j = 1, 2, \ldots, S$, WHERE $\cot(\pi \zeta_j) \neq 0$. THEN

$$\lim_{N\to\infty} \int_{C_N} \pi \cot(\pi z)\, f(z)\, dz = 2\pi i \sum_{K=-\infty}^{\infty} \text{RES}\,[\pi \cot(\pi z) f ; K] + 2\pi i \sum_{j=1}^{S} \text{RES}\,[\pi \cot(\pi z) f ; \zeta_j].$$

NOW $\text{RES}\,[f \pi \cot(\pi z) ; K] = f(K)\, \pi \dfrac{\cos(\pi K)}{\pi \cos(\pi K)} = f(K).$

$$\longrightarrow \lim_{N\to\infty} \int_{C_N} \pi \cot(\pi z)\, f(z)\, dz = 2\pi i \sum_{K=-\infty}^{\infty} f(K) + 2\pi i \sum_{j=1}^{S} \text{RES}\,[\pi \cot(\pi z) f ; \zeta_j]. \quad (*)$$

NOW IT CAN BE SHOWN THAT

$$|\cot \pi z| = \left| \frac{\cos \pi z}{\sin \pi z} \right| = \left| \frac{e^{\pi z} + e^{-\pi z}}{e^{\pi z} - e^{-\pi z}} \right| \leq \frac{|e^{\pi z}| + |e^{-\pi z}|}{|e^{\pi z}| - |e^{-\pi z}|}$$

NOW AS $z \to \infty i$ IN upper 1/2 PLANE $|\cot \pi z|$ IS bounded

$z \to -\infty i$ IN LOWER 1/2 PLANE $|\cot \pi z|$ IS bounded

$z \to \infty$ IN right 1/2 PLANE $|\cot \pi z|$ IS bounded

$z \to -\infty$ IN LEFT 1/2 PLANE $|\cot \pi z|$ IS bounded.

THUS $\displaystyle\left| \int_{C_N} f(z) \cot(\pi z)\, \pi\, dz \right| \leq K \left| \int_{C_N} f(z)\, dz \right| \to 0$ AS $N \to \infty$ PROVIDED $|f(z)|\,|z| \to 0$ AS $|z| \to \infty$

$\leq K \max_{C_N} |f|$ length (C_N)

SUBSTITUTE IN (*) TO GET $$0 = 2\pi i \sum_{K=-\infty}^{\infty} f(K) + 2\pi i \sum_{j=1}^{s} \text{RES}\left[\pi \cot(\pi z) f \,;\, \zeta_j\right].$$

$$(\dagger) \qquad \sum_{K=-\infty}^{\infty} f(K) = -\sum_{j=1}^{s} \text{RES}\left[f(z)\pi \cot(\pi z) \,;\, \zeta_j\right]$$

IN $(\dagger)$ we have to ensure that

- $|z f(z)| \to 0$ AS $|z| \to \infty$ is needed
- ζ_j is a pole of $f(z)$ AND f is analytic at all other points besides $\zeta_1, \dots, \zeta_s$.

<u>EXAMPLE</u> SHOW $$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a} \coth(\pi a) \qquad a>0.$$

let $f(z) = \dfrac{1}{z^2+a^2}$. IT HAS SIMPLE POLES AT $z = \pm ia$. THEN

$$\text{RES}\left[\pi f(z) \cot(\pi z);\, ia\right] = \frac{\pi \cot(\pi i a)}{2ia} = \frac{-\pi}{2a} \coth(\pi a)$$

$$\text{RES}\left[\pi f(z) \cot(\pi z);\, -ia\right] = \frac{\pi \cot(-\pi i a)}{-2ia} = \frac{-\pi}{2a} \coth(\pi a)$$

$$\cot(\pi i a) = -i \coth(\pi a).$$

THEREFORE $$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = -\left[\frac{-\pi}{2a}\coth(\pi a) - \frac{\pi}{2a}\coth(\pi a)\right] = \frac{\pi}{a}\coth(\pi a)$$

$$\longrightarrow \sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a)$$

NOTICE $$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} + \frac{1}{a^2} + \sum_{n=-\infty}^{-1} \frac{1}{n^2+a^2} = \sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2}$$

$$\longrightarrow 2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} + \frac{1}{a^2} = \frac{\pi}{a}\coth(\pi a) \longrightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a}\coth(\pi a) - \frac{1}{2a^2}.$$

let $a \to 0$. $\tanh z = z - z^3/3 + \dots \quad z \to 0 \qquad \coth z = \dfrac{1}{(z - z^3/3 + \dots)} = \dfrac{1}{z(1 - z^2/3 + \dots)} = \dfrac{1}{z}(1 + z^2/3 + \dots)$

$\coth(\pi a) = \dfrac{1}{\pi a}\left(1 + \pi^2 a^2/3 + \dots\right) \quad a \to 0.$ HENCE $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} \to \frac{\pi}{2a}\frac{1}{\pi a}\left(1 + \frac{\pi^2 a^2}{3} + \dots\right) - \frac{1}{2a^2} \quad a \to$

$$\longrightarrow \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6.$$