The Boltzmann Equation with Forcing: Global Existence, Uniform Stability and Optimal Decay Rates

Renjun Duan

Department of Mathematics, City University of Hong Kong

This is a joint work with Professors Seiji Ukai, Tong Yang and Huijiang Zhao

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Outline

• Introduction
  ▶ Boltzmann equation
  ▶ Our aim
  ▶ Applications and problems

• Well-posedness of the Cauchy problem near Maxwellians
  ▶ Results: Existence and stability in $H^N_{x,\xi}$
  ▶ Macro-micro decomposition
  ▶ Refined energy estimates on macro component

• Convergence rate
  ▶ Results: Optimal rate in $H^N_{x,\xi}$
  ▶ High-order estimates and spectral analysis
  ▶ Nonhomogeneous damping transport equation
1. Introduction

1.1 Consider the Boltzmann equation:

\[ \partial_t f + \xi \cdot \nabla_x f + \nabla_x \varphi(x) \cdot \nabla_\xi f = Q(f,f), \quad (BE) \]

where

- \( t \geq 0 \) (time), \( x \in \mathbb{R}^n \) (position), \( \xi \in \mathbb{R}^n \) (velocity), \( n \geq 3 \);
- \( f = f(t,x,\xi) \geq 0 \) (number density), unknown;
- \( \varphi(x) \) (potential of stationary force), bounded & given;
- \( Q \) is a bilinear collision operator (hard sphere model):

\[
Q(f,g) = \frac{1}{2} \int_{\mathbb{R}^n \times S^{n-1}} (f'g_* + f_*g' - fg_* - f*g)|(\xi - \xi_*) \cdot \omega|d\omega d\xi_*,
\]

\[ f = f(t,x,\xi), \quad f' = f(t,x,\xi'), \quad f_* = f(t,x,\xi_*), \quad f'_* = f(t,x,\xi'_*), \]

likewise for \( g \),

\[ \xi' = \xi - [(\xi - \xi_*) \cdot \omega]\omega, \quad \xi_* = \xi_* + [(\xi - \xi_*) \cdot \omega]\omega, \quad \omega \in S^{n-1}. \]
1.2 (BE) has a stationary solution:

\[ e^{\phi(x)}M, \]

where

\[ M = \frac{1}{(2\pi)^{n/2}} \exp \left(-|\xi|^2/2 \right), \]

on the basis of the observations:

- Conservation of energy

\[
|\xi'|^2 + |\xi'_*|^2 = |\xi|^2 + |\xi_*|^2
\]

\[ \Rightarrow \]

\[ Q(e^{\phi(x)}M, e^{\phi(x)}M) = e^{2\phi(x)}Q(M, M) = 0, \]

- 

\[
\{ \partial_t + \xi \cdot \nabla_x + \nabla_x\phi(x) \cdot \nabla_\xi \} \exp \left(\phi(x) - |\xi|^2/2 \right) \equiv 0.
\]
1.3 Aim (D., PhD thesis, ’08):

- **Stability of stationary solution** $e^\phi M$:

\[
\left\{ \begin{array}{l}
\| f(0) - e^\phi M \|_X \ll 1 \\
|\phi| \ll 1
\end{array} \right\} \Rightarrow \sup_t \| f(t) - e^\phi M \|_X \leq C \| f(0) - e^\phi M \|_X,
\]

\[
X = H^N(\mathbb{R}^n_x \times \mathbb{R}^n_\xi; M^{-1/2} dx d\xi) : \text{ no time derivatives.}
\]

- **Uniform-in-time stability for two solutions**:

\[
\left\{ \begin{array}{l}
\| f(0) - e^\phi M \|_X \ll 1 \\
\| g(0) - e^\phi M \|_X \ll 1 \\
|\phi| \ll 1
\end{array} \right\} \Rightarrow \sup_t \| f(t) - g(t) \|_X \leq C \| f(0) - g(0) \|_X.
\]

- **Optimal convergence rate in** $X$:

\[
\text{Additional conditions on } f(0) - e^\phi M \text{ & } \phi \Rightarrow \| f(t) - e^\phi M \|_X \leq C_{f_0,\phi}(1 + t)^{-\frac{n}{4}}.
\]
1.4 Energy method

- Liu-Yu (CMP,’02), Liu-Yang-Yu (PD,’02),...
- Guo (IUMJ,’02),...

- Stability:

  Macro-micro decomposition (micro eqn + macro eqn) + macroscopic conservation laws

  $\Rightarrow$

  Energy inequality: $\frac{d}{dt}E(u(t)) + D(u(t)) \leq 0$

- Optimal convergence rate:

  High-order energy estimates + Spectral analysis

  $\Rightarrow$

  Optimal decay rate
1.5 Applications:

- **General force (non-potential)** [DUYZ, CMP, ’08]:

  \[ E(t, x), \quad E(t, x) + \xi \times B(t, x). \]

  Assumptions: \( E, B \) are small and decay in time \( \text{or} \ n \geq 5 \).

- **General intermolecular interaction law:**

  \[ |\xi - \xi_*| \gamma b(\theta), \quad 0 \leq \gamma \leq 1. \]

- **Time-periodic solution** [DUYZ, CMP, ’08]:

  time-periodic force: \( E(t+T, x) = E(t, x) \).

  Assumptions: \( E \) is small and \( n \geq 5 \).

- **Physical models such as:**
  one-species Vlasov-Poisson-Boltzmann system [DY, ’08];
  one-species Vlasov-Maxwell-Boltzmann system [in progress].
1.6 Back to (BE):

\[ \partial_t f + \xi \cdot \nabla_x f + \nabla_x \phi(x) \cdot \nabla_{\xi} f = Q(f, f). \]

Problems on the stability of \( e^{\phi} M \) in other cases:

(i) \( \phi(x) \) can be large:

This situation was studied in the case of the compressible Navier-Stokes equations.

▶ Matsumura-Padula (’92):

interior domain, smooth solutions,

\[ \phi \in H^4. \]

▶ Matsumura-Yamagata (’01):

the whole space \( \mathbb{R}^3 \), weak solutions,

\[ |\phi(x)| \leq \frac{C}{1 + |x|}, |\nabla_x \phi(x)| \leq \frac{C}{(1 + |x|)^2}, \ldots \]

\( C \) need not be small.
(ii) \( e^\phi M \) is connected to vacuum at infinity:

\[
e^\phi(x) M \to 0 \ (\text{or } \phi(x) \to -\infty) \text{ as } |x| \to \infty.
\]

Here, \( \phi \) is a confining potential.

**Remark:** Related results in this situation:

- **Kinetic Fokker-Planck equation:**

\[
\partial_t f + \xi \cdot \nabla_x f - \nabla_x \phi(x) \cdot \nabla_x f = \Delta_\xi f + \nabla_\xi \cdot (\xi f),
\]

Helffer-Nier, Hérau, Hérau-Nier, Desvillettes, Villani,...:

\[
\phi \in C^1(\mathbb{R}^3), \quad \inf \phi > -\infty.
\]

- **Linearized Boltzmann equation:**

\[
\partial_t u + \xi \cdot \nabla_\xi u - \nabla_x \phi(x) \cdot \nabla_\xi u = e^{-\phi}L u,
\]

Tabata (TTSP,'94):

\[
\phi = \phi(|x|), \quad \phi''(r) \geq C_1 > 0, \quad \phi'(r) \leq C_2 r + C_3, ...
\]
2. Well-posedness of the Cauchy problem

2.1 To expose the main idea, suppose

\[ n = 3. \]

Set the perturbation \( u \) by

\[ f = e^{\phi(x)} M + \sqrt{M} u. \]

The Cauchy problem for \((\text{BE})\) is reformulated as

\[
\begin{align*}
\partial_t u + \xi \cdot \nabla_x u + \nabla_x \phi(x) \cdot \nabla_x u - \frac{1}{2} \xi \cdot \nabla_x \phi(x) u &= e^{\phi(x)} Lu + \Gamma(u, u), \\
u(0, x, \xi) &= u_0(x, \xi).
\end{align*}
\]

\((CP)_1\)

Here

\[
Lu = M^{-1/2} \left[ Q(M, M^{1/2} u) + Q(M^{1/2} u, M) \right],
\]

\[
\Gamma(u, u) = M^{-1/2} Q(M^{1/2} u, M^{1/2} u).
\]
2.2 Recall some standard facts on $L$:

(a) \((Lu)(\xi) = -\nu(\xi)u(\xi) + (Ku)(\xi)\), where
\begin{itemize}
  \item \(\nu_0(1 + |\xi|) \leq \nu(\xi) \leq \nu_0^{-1}(1 + |\xi|), \nu_0 > 0;\)
  \item \(K\) is a self-adjoint compact operator on \(L^2(\mathbb{R}^3_\xi)\);
  \item \(\text{Ker}L = \text{span} \{ M^{1/2}; \xi_i M^{1/2}, i = 1, 2, 3; |\xi|^2 M^{1/2} \} := \mathcal{N};\)
\end{itemize}

(b) \(L\) is self-adjoint on \(L^2(\mathbb{R}^3_\xi)\) with the domain
\[
D(L) = \{ u \in L^2(\mathbb{R}^3_\xi) | \nu(\xi)u \in L^2(\mathbb{R}^3_\xi) \},
\]
and \(-L\) is locally coercive: \(\exists \lambda > 0 \text{ s.t.}\)
\[
-\int_{\mathbb{R}^3} uLu\,d\xi \geq \lambda \int_{\mathbb{R}^3} \nu(\xi) \left( \{I - P\}u \right)^2 \,d\xi, \forall u \in D(L)
\]
\[
= \lambda \| \{I - P\}u \|_{\nu}^2,
\]
where \(P\) is the projector from \(L^2(\mathbb{R}^3_\xi)\) to \(\mathcal{N}\).
2.3 Define the energy functional

\[
[[u(t)]]^2 = \sum_{|\alpha| + |\beta| \leq N} \| \partial_x^\alpha \partial_\xi^\beta u(t) \|^2 = \| u(t) \|^2_{H^N_{x,\xi}},
\]

and the dissipation rate

\[
[[u(t)]]^2_\nu = \| \{ I - P \} u(t) \|^2_\nu + \sum_{0 < |\alpha| \leq N} \| \partial_x^\alpha u(t) \|^2_\nu
\]

\[+ \sum_{|\alpha| + |\beta| \leq N, \ |\beta| > 0} \| \partial_x^\alpha \partial_\xi^\beta \{ I - P \} u(t) \|^2_\nu.\]

2.4 Assumptions on the potential \( \phi(x) \):

\( \text{(AP):} \)

\[
\phi \in L^\infty_x, \\
\delta_\phi := \| (1 + |x|)^2 \nabla_x \phi \|_{L^\infty_x} + \sum_{2 \leq |\alpha| \leq N} \| (1 + |x|) \partial_x^\alpha \phi \|_{L^\infty_x} \ll 1.
\]
Theorem I (Well-posedness)  Let \( f_0(x, \xi) = M + \sqrt{M}u_0(x, \xi) \geq 0 \).  
\( \exists \delta_0 > 0, \lambda_0 > 0 \) and \( C_0 > 0 \) s.t. if  
\[ \|u(0)\| + \delta \phi \leq \delta_0, \]

then \( \exists! u(t, x, \xi) \) to \((CP)_0\) s.t. \( f(t, x, \xi) = M + \sqrt{M}u(t, x, \xi) \geq 0 \), and  
\[ \|u(t)\|^2 + \lambda_0 \int_0^t \|u(s)\|^2 \nu ds \leq C_0\|u(0)\|^2, \forall t \geq 0. \]

Theorem II (Uniform stability)  Let  
\[ f_0(x, \xi) = M + \sqrt{M}u_0(x, \xi) \geq 0, \ q_0(x, \xi) = M + \sqrt{M}v_0(x, \xi) \geq 0. \]
\( \exists \delta_1 \in (0, \delta_0), \lambda_1 > 0 \) and \( C_1 > 0 \) s.t. if  
\[ \max\{\|u(0)\|, \|v(0)\|\} + \delta \phi \leq \delta_1, \]

then the solutions \( u(t, x, \xi), v(t, x, \xi) \) obtained in Theorem I satisfy  
\[ \|u(t) - v(t)\|^2 + \lambda_1 \int_0^t \|u(s) - v(s)\|^2 \nu ds \leq C_1\|u(0) - v(0)\|^2, \forall t \geq 0. \]
2.5 Related results:

- **Spaces based on the spectral analysis** ($\phi \equiv 0$):
  - **Ukai, Nishida-Imai**: $L^\infty_{\beta_1}(\mathbb{R}_\xi^3; H^k(\mathbb{R}_x^3))$, $\beta_1 > 5/2$, $k \geq 2$,
    where $L^\infty_{\beta_1}(\mathbb{R}_\xi^3) \equiv \{ u | (1 + |\xi|)^{\beta_1} u \in L^\infty(\mathbb{R}_\xi^3) \}$.
  - **Shizuta (Torus case)**: $L^\infty_{\beta_1}(\mathbb{R}_\xi^3; C^k(T_x^3))$, $\beta_1 > 5/2$, $k = 0,1,\cdots$.
  - **Ukai-Yang**: $L^2(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3) \cap L^\infty_{\beta_2}(\mathbb{R}_\xi^3; L^\infty(\mathbb{R}_x^3))$, $\beta_2 > 3/2$

**Remark** Notice that

$$L^\infty_{\beta_1}(\mathbb{R}_\xi^3) \subset L^\infty_{\beta_2}(\mathbb{R}_\xi^3) \subset L^2(\mathbb{R}_\xi^3),$$

where $\beta_1$ and $\beta_2$ are sufficiently close to $5/2$ and $3/2$, respectively.

- **Spaces based on the energy method**:
  - **Liu-Yu, Liu-Yang-Yu, Yang-Zhao, D. (JDE, '08)** (Refined energy method, no time derivative),
  - **Guo, Strain,**

$$H_{t,x,\xi}^{N(n_1,n_2,n_3)}(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3),$$

$N = N(n_1,n_2,n_3) \equiv n_1 + n_2 + n_3 \geq 4.$
Spaces based on the method of Green's function ($\phi \equiv 0$), Liu-Yu:

$$L^\infty(\mathbb{R}_x^3, w(t, x)dx; L^\infty_{\beta_3}(\mathbb{R}_\xi^3)), \quad \beta_3 \geq 3,$$

where the pointwise weight function,

$$w(t, x) = e^{-C(|x|+\beta_3 t)} + \frac{e^{-\frac{(|x| - \sqrt{5/3} t)^2}{Ct}}}{(1 + t)^2} + \{\text{acoustic cone}\},$$

exposes the wave structure of convergence. Notice that the initial perturbation $u_0$ decays exponentially in $x$.

**Remark** The energy method is an effective one in the presence of the external force.
2.6 Key points of the proof.

(a) Macro-micro decomposition: For fixed \((t, x)\),
\[
\begin{cases}
    u(t, x, \xi) = u_1 + u_2, \\
    u_1 \equiv Pu \in \mathcal{N}, \\
    u_2 \equiv \{I - P\}u \in \mathcal{N}^\perp.
\end{cases}
\]

(b) Our goal is to obtain the dissipation rate \([\|u(t)\|_\nu]_\nu^2\), which is equivalent with
\[
\sum_{|\alpha| + |\beta| \leq N} \| \partial_x^\alpha \partial_\xi^\beta \{I - P\}u(t) \|_\nu^2 + \sum_{0 < |\alpha| \leq N} \| \partial_x^\alpha Pu(t) \|_\nu^2.
\]

Remark

- (I)\(\rightleftarrows\)the local coercivity of \(-L\);
- (II)\(\rightleftarrows\)macro equations + local macro balance laws.
(c) Expand $u_1 = Pu$ as

$$u_1 = \left\{ a(t, x) + \sum_{i=1}^{3} b_i(t, x)\xi_i + c(t, x)|\xi|^2 \right\} M^{1/2}.$$ 

One can determine the evolution of $u_1$ and $(a, b, c)$ in terms of $u_2$:

- Macroscopic equation on $u_1$:

$$\partial_t u_1 + \xi \cdot \nabla_x u_1 + \nabla_x \phi \cdot \nabla \xi u_1 - \frac{1}{2} \xi \cdot \nabla_x \phi u_1 = r + \ell + n$$

with

$$r = -\partial_t u_2,$$
$$\ell = -\xi \cdot \nabla_x u_2 - \nabla_x \phi \cdot \nabla \xi u_2 + \frac{1}{2} \xi \cdot \nabla_x \phi u_2 + e^\phi Lu_2,$$
$$n = \Gamma(u, u).$$
• Macroscopic equations on coefficients \((a, b, c)\) of \(u_1\):

\[
\begin{align*}
\partial_t a + b \cdot \nabla_x \phi &= -\partial_t \tilde{r}^{(0)}(0) + \ell^{(0)}(0) + n^{(0)}(0) \equiv \gamma^{(0)}, \\
\partial_i b_i + \partial_i a - (a \partial_i \phi - 2c \partial_i \phi) &= -\partial_t \tilde{r}^{(1)}_i(1) + \ell^{(1)}_i(1) + n^{(1)}_i(1) \equiv \gamma^{(1)}_i, \\
\partial_t c + \partial_i b_i - b_i \partial_i \phi &= -\partial_t \tilde{r}^{(2)}_i(2) + \ell^{(2)}_i(2) + n^{(2)}_i(2) \equiv \gamma^{(2)}_i, \\
\partial_i b_j + \partial_j b_i - (b_j \partial_i \phi + b_i \partial_j \phi) &= -\partial_t \tilde{r}^{(2)}_{ij}(2) + \ell^{(2)}_{ij}(2) + n^{(2)}_{ij}(2) \equiv \gamma^{(2)}_{ij}, \quad i \neq j, \\
\partial_i c - c \partial_i \phi &= -\partial_t \tilde{r}^{(3)}_i(3) + \ell^{(3)}_i(3) + n^{(3)}_i(3) \equiv \gamma^{(3)}_i.
\end{align*}
\]

**Remark** An important observation from Guo is that \(b = (b_1, b_2, b_3)\) satisfies an elliptic-type equation:

\[
-\Delta_x b_j - \partial_j \partial_j b_j = \sum_{i \neq j} \partial_j (b_i \partial_i \phi) + \sum_{i \neq j} \partial_j \gamma^{(2)}_i \\
- \sum_{i \neq j} \partial_i (b_j \partial_i \phi + b_i \partial_j \phi) - \sum_{i \neq j} \partial_i \gamma^{(2)}_{ij} \\
- 2 \partial_j (b_j \partial_j \phi) - 2 \partial_j \gamma^{(2)}_j.
\]
• Macroscopic balance laws on coefficients \((a, b, c)\) of \(u_1\):

\[
\partial_t a - \frac{1}{2} \nabla_x \cdot \left\langle |\xi|^2 \xi \sqrt{M}, u_2 \right\rangle = -\frac{1}{2} b \cdot \nabla_x \phi, \\
\partial_t b_i + \partial_i(a + 5c) + \nabla_x \cdot \left\langle \xi \xi_i \sqrt{M}, u_2 \right\rangle = (a + 3c) \partial_i \phi, \\
\partial_t c + \frac{1}{3} \nabla_x \cdot b + \frac{1}{6} \nabla_x \cdot \left\langle |\xi|^2 \xi \sqrt{M}, u_2 \right\rangle = \frac{1}{6} b \cdot \nabla_x \phi,
\]

where it is noticed that one also has the conservation of mass

\[
\partial_t(a + 3c) + \nabla_x \cdot b = 0.
\]

**Remark** The time derivatives \(\partial_t(a, b, c)\) can be replaced by the spatial derivatives and the nonlinear product terms. For the product terms, the Hardy inequality

\[
\int_{\mathbb{R}^3} \frac{|g(x)|^2}{|x|^2} dx \leq 2 \int_{\mathbb{R}^3} |\nabla_x g(x)|^2 dx, \quad \forall \ g \in H^1(\mathbb{R}^3),
\]

is used to gain the spatial derivatives.
• (BE) can be exactly written as the linearized viscous compressible Navier-Stokes equations with remaining terms only related to 13 moments of the micro part $u_2$ and product terms between $(a, b, c)$ and $\nabla_x \phi$:

\[
\begin{align*}
\partial_t (a + 3c) + \nabla_x \cdot b &= 0, \\
\partial_t b + \nabla_x (a + 3c) + 2\nabla_x c - \Delta_x b - \frac{1}{3} \nabla_x \nabla_x \cdot b &= R^b, \\
\partial_t c + \frac{1}{3} \nabla_x \cdot b - \Delta_x c &= R^c,
\end{align*}
\]

where $R^b = (R^b_1, R^b_2, R^b_3)$ and $R^c$ are defined by

\[
R^b_j = -\nabla_x \cdot \langle \xi \xi_j \sqrt{M}, u_2 \rangle - \frac{1}{3} \partial_j \nabla_x \cdot \langle |\xi|^2 \xi \sqrt{M}, u_2 \rangle \\
- \sum_{i \neq j} \partial_i \gamma^{(2)}_{ij} - 2 \partial_j \gamma^{(2)}_j + \{\text{product terms}\},
\]

\[
R^c = -\frac{1}{6} \nabla_x \cdot \langle |\xi|^2 \xi \sqrt{M}, u_2 \rangle - \sum_i \partial_i \gamma^{(3)}_i + \{\text{product terms}\}.
\]
2.7 Proof of Theorem I:

- Local existence: ...
- \textit{A priori estimates:}

\textbf{Part I: To obtain the microscopic dissipation}

(i) \textit{Estimates on zero-order:}

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \lambda \|u_2\|^2_\nu \leq C[u(t)][u(t)]^2_\nu + C\delta_\phi \|\nabla_x u_1\|^2.
\]

(ii) \textit{Estimates on pure spatial derivatives:}

\[
\frac{1}{2} \frac{d}{dt} \sum_{1 \leq |\alpha| \leq N} \|\partial_\alpha^x u\|^2 + \lambda \sum_{1 \leq |\alpha| \leq N} \|\partial_\alpha^x u_2\|^2_\nu \leq C[u(t)][u(t)]^2_\nu + C\delta_\phi \sum_{1 \leq |\alpha| \leq N} \|\partial_\alpha^x u_1\|^2 + C\delta_\phi \sum_{1 \leq |\alpha| \leq N-1} \|\partial_\alpha^x \nabla_\xi u_2\|^2.
\]
(iii) Estimates on mixed derivatives:

\[
\frac{1}{2} \frac{d}{dt} \sum_{|\alpha|+|\beta| \leq N} \| \partial_x^\alpha \partial_\xi^\beta u_2 \|^2 + \lambda \sum_{|\alpha|+|\beta| \leq N} \| \partial_x^\alpha \partial_\xi^\beta u_2 \|^2_
u \\
\leq C[[u(t)][[u(t)]]^2 + C \sum_{|\alpha| \leq N-k+1} \| \partial_x^\alpha u_2 \|^2_
u \\
+ C \chi \{k \geq 2\} \sum_{1 \leq |\beta| \leq k-1} \| \partial_x^\alpha \partial_\xi^\beta u_2 \|^2_
u + C \sum_{|\alpha| \leq N-k} \| \partial_x^\alpha \nabla_x (a, b, c) \|^2,
\]

where the integer \(1 \leq k \leq N\) and \(\chi \{k \geq 2\}\) denotes the characteristic function of the set \(\{k \geq 2\}\).

**Remark** The above estimate is based on the equation:

\[
\partial_t u_2 + \xi \cdot \nabla_x u_2 + \nabla_x \phi \cdot \nabla_\xi u_2 - 1/2 \xi \cdot \nabla_x \phi u_2 + e^\phi \nu(\xi) u_2 \\
= e^\phi Ku_2 + \Gamma(u, u) - \partial_t u_1 - \xi \cdot \nabla_x u_1 - \nabla_x \phi \cdot \nabla_\xi u_1 + 1/2 \xi \cdot \nabla_x \phi u_1.
\]
Proper linear combinations of (i), (ii), (iii) \( k \Rightarrow \)

\[
\frac{1}{2} \frac{d}{dt} \left( \sum_{|\alpha| \leq N} \| \partial_x^\alpha u \|^2 + \sum_{|\alpha| + |\beta| \leq N, \ |\beta| \geq 1} \| \partial_x^\alpha \partial_\xi^\beta u_2 \|^2 \right) \\
+ \lambda \sum_{|\alpha| + |\beta| \leq N} \| \partial_x^\alpha \partial_\xi^\beta u_2 \|^2_{\nu} \\
\leq C [u(t)] [u(t)]^2 + C \sum_{|\alpha| \leq N-1} \| \partial_x^\alpha \nabla_x (a, b, c) \|^2.
\]
Part II: To obtain the macroscopic dissipation

\[ 2 \frac{d}{dt} \mathcal{I}(u(t)) + \sum_{|\alpha| \leq N-1} \| \nabla_x \partial_\alpha^x (a, b, c) \|^2 \leq C \left\{ \sum_{|\alpha| \leq N} \| \partial_\alpha^x u_2 \|^2 + [[u(t)]]^2 [[u(t)]]^2 \right\}, \]

where \( \mathcal{I}(u(t)) \) is called the interactive energy functional:

\[ \mathcal{I}(u(t)) = \sum_{|\alpha| \leq N-1} \sum_{i=1}^3 \left[ \mathcal{I}^a_{\alpha,i}(u(t)) + \mathcal{I}^b_{\alpha,i}(u(t)) + \mathcal{I}^c_{\alpha,i}(u(t)) + \mathcal{I}^{ab}_{\alpha,i}(u(t)) \right], \]

\[ \mathcal{I}^a_{\alpha,i}(u(t)) = \left\langle \partial_\alpha^x \tilde{r}_i^{(1)}, \partial_i \partial_\alpha^x a \right\rangle, \]
\[ \mathcal{I}^b_{\alpha,i}(u(t)) = - \sum_{j \neq i} \left\langle \partial_\alpha^x \tilde{r}_j^{(2)}, \partial_i \partial_\alpha^x b_i \right\rangle + \sum_{j \neq i} \left\langle \partial_\alpha^x \tilde{r}_{ji}^{(2)}, \partial_j \partial_\alpha^x b_i \right\rangle + 2 \left\langle \partial_\alpha^x \tilde{r}_i^{(2)}, \partial_i \partial_\alpha^x b_i \right\rangle, \]
\[ \mathcal{I}^c_{\alpha,i}(u(t)) = \left\langle \partial_\alpha^x \tilde{r}_i^{(3)}, \partial_i \partial_\alpha^x c \right\rangle, \]
\[ \mathcal{I}^{ab}_{\alpha,i}(u(t)) = \left\langle \partial_i \partial_\alpha^x a, \partial_\alpha^x b_i \right\rangle. \]
**Idea for Part II:**

\[-\Delta_x b_j = -\partial_j \partial_t \tilde{r}^{(2)} + \cdots \]

\[
\Rightarrow
\]

\[
\|\nabla_x \partial_x^\alpha b_j\|^2 = -\frac{d}{dt} \langle \partial_j \partial_x^\alpha \tilde{r}^{(2)}, \partial_x^\alpha b_j \rangle + \langle \partial_j \partial_x^\alpha \tilde{r}^{(2)}, \partial_x^\alpha \partial_t b_j \rangle + \cdots
\]

\[
= -\frac{d}{dt} \left( \langle \partial_x^\alpha \tilde{r}^{(2)}, -\partial_j \partial_x^\alpha b_j \rangle \right) + \langle \partial_j \partial_x^\alpha \tilde{r}^{(2)}, \partial_x^\alpha \partial_t b_j \rangle + \cdots.
\]

- (I) is bounded by the temporal energy;
- (II) is estimated by the Cauchy-Schwarz inequality and the balance law for \( b_j \):

\[
\partial_t b_j = -\partial_j (a + 5c) - \nabla_x \cdot \langle \xi_j \sqrt{M}, u_2 \rangle + (a + 3c) \partial_j \phi.
\]
Further linear combination of Part I and Part II ⇒ 

\[
\frac{d}{dt} E_M(u(t)) + \lambda D(u(t)) \leq C \sqrt{E_M(u(t))} D(u(t)),
\]

where the energy functional is in the form

\[
E_M(u(t)) \sim \sum_{|\alpha| \leq N} \| \partial_x^\alpha u \|^2 + \sum_{|\alpha| + |\beta| \leq N, |\beta| \geq 1} \| \partial_x^\alpha \partial_\xi^\beta u_2 \|^2 \\
+ \frac{M}{2} \sum_{|\alpha| \leq N} \| \partial_x^\alpha u \|^2 + 2I(u(t)) \\
\sim [u(t)]^2,
\]

and the dissipation rate is in the form

\[
D(u(t)) \sim \sum_{|\alpha| + |\beta| \leq N} \| \partial_x^\alpha \partial_\xi^\beta u_2 \|^2 \nu + \sum_{|\alpha| \leq N-1} \| \nabla_x \partial_x^\alpha (a, b, c) \|^2 \\
\sim [u(t)]^2 \nu.
\]
2.8 Proof of Theorem II:

Set

\[ w(t, x, \xi) = u(t, x, \xi) - v(t, x, \xi). \]

Then \( w \) satisfies

\[
\partial_t w + \xi \cdot \nabla_x w + \nabla_x \phi(x) \cdot \nabla_\xi w - \frac{1}{2} \xi \cdot \nabla_x \phi(x) w = e^{\phi(x)} Lw + \Gamma(w, u) + \Gamma(v, w).
\]

Similar proof yields the Lyapunov-type inequality

\[
\frac{d}{dt} \mathcal{E}_M(w(t)) + C \mathcal{D}(w(t)) \leq C \{ \mathcal{D}(u(t)) + \mathcal{D}(v(t)) \} \mathcal{E}_M(w(t)).
\]

By using the time integrability

\[
\int_0^\infty \{ \mathcal{D}(u(s)) + \mathcal{D}(v(s)) \} ds < \infty
\]

and the Gronwall’s inequality, the uniform-in-time stability estimate holds true.
2.9 Generalized to the general collision kernel:

\[ |\xi - \xi^*|^{\gamma} b(\theta), \quad 0 \leq \gamma \leq 1. \]

**Problem:** one of the source terms

\[-1/2 \xi \cdot \nabla_x \phi(x) u\]

can not be controlled in terms of the dissipation \(Lu\) since

\[-\langle Lu, u \rangle \geq \lambda \int (1 + |\xi|)^{\gamma}(u_2)^2 dx d\xi.\]

**Idea:** use another kind of perturbation

\[f = e^\phi M + \sqrt{e^\phi Mu}.\]

Then the reformulated equation reads

\[\partial_t u + \xi \cdot \nabla_x u + \nabla_x \phi(x) \cdot \nabla \xi u = e^\phi(x) Lu + e^\phi(x)/2 \Gamma(u, u).\]
2.10 Application: Vlasov-Poisson-Boltzmann system (DY, recent work)

\[ \partial_t f + \xi \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_\xi f = Q(f, f), \]

\[ \Delta_x \Phi = \int_{\mathbb{R}^3} f(t, x, \xi) d\xi - \bar{\rho}, \]

where \( \bar{\rho} > 0 \) is (or near) a positive constant.

**Energy functional:**

\[ [[u(t)]]^2 \equiv \sum_{|\alpha| + |\beta| \leq N} \| \partial^\alpha x \partial^\beta_\xi u \|^2 + \sum_{|\alpha| \leq N} \| \partial^\alpha x \nabla_x \Phi \|^2 \]

**Dissipation rate:**

\[ [[u(t)]]^2_\nu \equiv \sum_{|\alpha| + |\beta| \leq N, |\beta| > 0} \| \partial^\alpha x \partial^\beta_\xi \{ I - P \} u \|^2_\nu + \sum_{0 < |\alpha| \leq N} \| \partial^\alpha x u \|^2_\nu + \sum_{|\alpha| \leq N - 1} \| \partial^\alpha x \nabla_x \Phi \|^2 + \sum_{|\alpha| \leq N - 1} \| \partial^\alpha x b \|^2. \]
A prior estimate:

\[
\frac{d}{dt} E(u(t)) + \lambda D(u(t)) \leq \sqrt{E(u(t))}D(u(t)),
\]

\[
E(u(t)) \sim [u(t)]^2, \quad D(u(t)) \sim [u(t)]^2_\nu.
\]

**Difficulties:**

- No time-derivatives;
- Dissipations include:

\[
\| \nabla_x \Phi \|^2 \quad \text{(Poisson equation?)}
\]

\[
\| b \|^2 \quad \text{(Elliptic equation?)}
\]

**Remark** \( \| \nabla_x \Phi \| \) or \( \| b \| \) is necessary to be included since the source term contains

\[
\iint \xi \cdot \nabla_x \Phi u_1^2 dx d\xi = \int \nabla_x \Phi \cdot b(a + 5c) dx.
\]
3. Convergence rate

3.1 The case of $\phi \equiv 0$: the solution semigroup $\{e^{Bt}\}_{t \geq 0}$, where

$$B = -\xi \cdot \nabla_x + L,$$

decays with an algebraic rate:

$$\|\nabla_x m e^{Bt} g\|_{L^2_{x,\xi}} \leq C(q, m)(1 + t)^{-\sigma_{q,m}} (\|g\|_{Z_q} + \|\nabla_x^m g\|_{L^2_{x,\xi}}),$$

$$m \geq 0, \quad q \in [1, 2], \quad Z_q = L^2_{\xi}(L^q_x),$$

$$\sigma_{q,m} = \frac{3}{2} \left( \frac{1}{q} - \frac{1}{2} \right) + \frac{m}{2}.$$

**Remark** The above rate was obtained by the spectral analysis due to Ukai (’74) and Nishida-Imai (’76). Recently, an extra decay was obtained by Ukai-Yang (AA-’06) if g is purely microscopic.
Theorem III (Optimal convergence rates) Let all conditions in Theorem I hold. Further assume that $\|u_0\|_{Z_1}$ is bounded and

$$\| |x|\phi \|_{L^\infty_x} + \| |x|\nabla_x\phi \|_{L^2_x}$$

is small enough. Then the solution $u$ obtained in Theorem I satisfies

$$[[u(t)]] \leq C (1 + t)^{-\frac{3}{4}} ([u_0]) + \|u_0\|_{Z_1}, \ \forall \ t \geq 0.$$ 

Remarks

(a) By optimal, it means that the decay rate is the same as one of $e^{Bt}$ when $\phi = 0$, at the level of zero order ($\sigma_{1,0} = 3/4$).

(b) The proof is based on

- the energy estimates of higher order,
- the decay-in-time estimates on $e^{Bt}$ (Ukai-Yang),
- and the analysis on the damping transport operator

$$\partial_t + \xi \cdot \nabla_x + \nu(\xi).$$
3.2 Hypocoercivity:

\[
\text{degenerate dissipative operator} + \text{conservative operator} \downarrow \text{full dissipation and convergence (Villani, etc.)}
\]

Models: Boltzmann equation, Fokker-Planck equation, Classical Landau equation, BGK model, etc.
3.3 Some known results on the convergence rates:

- **Without forces:**
  - Exponential convergence rate in bounded domain and torus: Ukai ('74), Giraud ('75), Shizuta-Asano ('77),...
  - Algebraic convergence rate in unbounded domain: Ukai ('76), Nishida-Imai ('76), Ukai-Asano ('83),...
  - Almost exponential convergence rate: Strain-Guo ('05), Desvillettes-Villani ('05)
  - Optimal convergence rate (extra decay): Ukai-Yang ('06)

- **With forces:**
  - Convergence rate in $L^\infty$ framework: Asano ('02),...
  - Convergence rate in $L^2$ framework: Ukai-Yang-Zhao ('05),...
  - Torus case: Mouhout-Neumann ('06),...
3.4 Sketch of proof of Theorem III.

Step 1. Energy estimates of higher order:

\[
\frac{d}{dt} \mathcal{E}_{h.o.}(u(t)) + \lambda \|[u(t)]\|_\nu^2 \leq C \|\nabla_x (a, b, c)\|^2,
\]

where

\[
\frac{1}{C} \|[u(t)]\|_0^2 \leq \mathcal{E}_{h.o.}(u(t)) \leq C \|[u(t)]\|_0^2,
\]

\[
\|[u(t)]\|_0^2 \equiv \|\{I - P\}u(t)\|^2 + \sum_{0 < |\alpha| \leq N} \|\partial_x^\alpha u(t)\|^2
\]

\[
+ \sum_{|\alpha| + |\beta| \leq N, \ |\beta| > 0} \|\partial_x^\alpha \partial_\xi^\beta \{I - P\}u(t)\|^2.
\]

Notice that

\[
\|[u(t)]\|_0 \leq C \|[u(t)]\|_\nu.
\]

Then

\[
\frac{d}{dt} \mathcal{E}_{h.o.}(u(t)) + \lambda \mathcal{E}_{h.o.}(u(t)) \leq C \|\nabla_x (a, b, c)\|^2.
\]
Step 2. Decay-in-time estimates from the spectral analysis on $\nabla_x u$:

$$u(t) = e^{Bt}u_0 + \int_0^t e^{B(t-s)}S[u](s)ds,$$

where

$$S[u] = -\nabla_x \phi \cdot \nabla_x u + \frac{1}{2} \nabla_x \phi \cdot \xi u + (e^\phi - 1) Lu + \Gamma(u, u).$$

Decomposition of $e^{Bt}$ (Ukai-Yang):

$$e^{Bt} = E_0(t) + E_1(t) + E_2(t),$$

where

$$E_0(t)u \equiv e^{-\nu(\xi)t}u(x - \xi t, \xi),$$

$$\|\nabla_x^m E_1(t)\nu u\|_{Z_2} \leq C(1 + t)^{-\sigma_q m} \|u\|_{Z_q},$$

$$\|\nabla_x^m E_1(t)\{I - P\}\nu u\|_{Z_2} \leq C(1 + t)^{-\sigma_q m+1} \|u\|_{Z_q} : \text{extra decay},$$

$$\|\nabla_x^m E_2(t)\nu u\|_{Z_2} \leq Ce^{-\lambda t} \|\nabla_x^m u\|_{Z_2}.$$
Remark: A technical lemma will be used to deal with the velocity increasing in the source term $S$. The trouble comes from the transport part in the semigroup $e^{Bt}$. For this, define $\Psi[h](t, x, \xi)$ as the solution to

$$\partial_t u + \xi \cdot u + \nu(\xi)u = \nu(\xi)h(t, x, \xi), \quad u|_{t=0} = 0,$$

where $\nu_0 > 0$ is such that

$$\nu_0 (1 + |\xi|) \leq \nu(\xi) \leq \frac{1}{\nu_0} (1 + |\xi|).$$

Claim: $\forall \lambda \in (0, \nu_0), \exists C$ s.t.

$$\int_0^t e^{-\lambda(t-s)} \| \Psi[h](s) \|^2_{L^2_{x,\xi}} \, ds \leq C \int_0^t e^{-\lambda(t-s)} \| h(s) \|^2_{L^2_{x,\xi}} \, ds.$$

Sketch of proof for Claim: (a) Decompose

$$\| \Psi[h](s) \|^2_{L^2_{x,\xi}} = \sum_{R=0}^\infty \| \Psi[h](s) \|^2_{L^2(\mathbb{R}^3_x \times \Omega_\xi(R))},$$

where

$$\Omega_\xi(R) = \{ \xi \in \mathbb{R}^3; R \leq |\xi| < R + 1 \}.$$
(b) Use the pointwise estimates: \( \xi \in \Omega \_\xi (R) \),

\[
|\Psi[h](s, x, \xi)| = \left| \int_0^s e^{-\nu(\xi)(s-\theta)} \nu(\xi) h(\theta, x - (s - \theta)\xi, \xi) d\theta \right| \\
\leq \int_0^s e^{-\nu_0(1+R)(s-\theta)} \frac{1}{\nu_0} (2 + R) |h(\theta, x - (s - \theta)\xi, \xi)| d\theta.
\]

(c) Minkowski and Hölder inequalities and Fubini theorem yield

\[
\int_0^t e^{-\lambda(t-s)} \|\Psi[h](s)\|^2_{L^2_{x,\xi}} ds \\
\leq \sum_{R=0}^{\infty} \frac{(2 + R)^2}{\nu_0^4 (1 + R)^2} \int_0^t e^{-\lambda(t-\theta)} \|h(\theta)\|^2_{L^2(\mathbb{R}^3 \times \Omega \_\xi (R))} d\theta \\
\leq \frac{4}{\nu_0^4} \sum_{R=0}^{\infty} \int_0^t e^{-\lambda(t-\theta)} \|h(\theta)\|^2_{L^2(\mathbb{R}^3 \times \Omega \_\xi (R))} d\theta.
\]
Step 3. Time-decay estimates on higher order: From Steps 1 and 2, Gronwall + Hardy + Claim ⇒

\[ E_{h.o.}(u(t)) \leq e^{-\lambda t} E_{h.o.}(u(0)) + \int_0^t e^{-\lambda(t-s)} \| \nabla_x u(s) \|^2_{L^2_{x,\xi}} \, ds \]

\[ \leq C(1 + t)^{-\frac{5}{2}} [\delta_0^2 + K_0^2 + (\delta_0^2 + \delta_\phi^2) E_{h.o.}(t)], \quad \forall \, t \geq 0, \]

where

\[ E_{h.o.}^{\infty}(t) = \sup_{0 \leq s \leq t} (1 + s)^{\frac{5}{2}} E_{h.o.}(u(s)), \]

\[ \delta_0 = [|u_0|] = \|u_0\|_{H^N_{x,\xi}} , \quad K_0 = \|u_0\|_{Z_1}, \]

Then

\[ E_{h.o.}^{\infty}(t) \leq C(\delta_0^2 + K_0^2). \]
Step 4. Decay-in-time rate for zero order: Add $\|P u(t)\|_{L_{x,\xi}^2}^2$ to both sides of zero-order energy estimates to obtain

$$\frac{d}{dt} \mathcal{E}_{z.o.}(u(t)) + \lambda \mathcal{E}_{z.o.}(u(t)) \leq C \|u(t)\|_{L_{x,\xi}^2}^2,$$

where

$$\mathcal{E}_{z.o.}(u(t)) \sim \|u(t)\|^2 = \|u(t)\|^2_{H_{x,\xi}^N}.$$

Then,

$$\mathcal{E}_{z.o.}(u(t)) \leq e^{-\lambda t} \mathcal{E}_{z.o.}(u(0)) + C \int_0^t e^{-\lambda(t-s)} \|u(s)\|_{L_{x,\xi}^2}^2 \, ds.$$

The rest computation is similar as before:

- Use the mild form of $u(t)$ to iterate once;
- Use the decomposition of $e^{Bt}$ and the Claim to find the time-decay rate.

$\Rightarrow$

$$\mathcal{E}_{z.o.}(u(t)) \lesssim (1 + t)^{-\frac{3}{2}}.$$
3.5 Application 1:

The following more general linearized Boltzmann equation with linear and variant-coefficient sources can be considered in the same way:

\[
\partial_t u + \xi \cdot \nabla_x u - Lu = A_0 Ku + \sum_{|\alpha|+|\beta|\leq 1} A_{\alpha\beta} \partial_x^\alpha \partial_\xi^\beta u,
\]

\[
\equiv \{A_0 K + A_{00}\} u + A_{10} \cdot \nabla_x u + A_{01} \cdot \nabla_\xi u,
\]

where

\[
A_0 = A_0(t, x, \xi), \quad A_{\alpha\beta} = A_{\alpha\beta}(t, x, \xi)
\]

satisfies some conditions on smallness in \((t, x, \xi)\) and increase in \(\xi\). A physical force inducing the above equation is in the form:

\[
F(t, x, \xi) = E(t, x) + \xi \times B(t, x).
\]
3.5 Application 2:

\[ \partial_t f + \xi \cdot \nabla_{\xi} f + F(t, x) \cdot \nabla_{\xi} f = Q(f, f) \]

**P:** Force \( F(t, x) \) is time-periodic \( \Rightarrow \exists \) Time-periodic solution?

**A:** \( \uparrow \) Yes if \( n \geq 5 \), D.-Ukai-Yang-Zhao (CMP, ’08);

**Proof:**
(i) Optimal time-decay estimates on the linearized equation
(ii) Find the fixed point for certain nonlinear mapping \( \Psi \):

\[ \Psi[u](t) = \int_{-\infty}^{t} U(t, s) S_F[u](s) ds, \quad \forall t \in \mathbb{R}. \]

(Well-defined since \( U(t, s) \lesssim (1 + t - s)^{-\frac{n}{4}} \) and \( \frac{n}{4} > 1 \))

\( \uparrow \) Open for \( 1 \leq n \leq 4 \), in particular, \( n = 3 \) (Physical).
Thanks!