Asymptotics in Nonlinear Evolution System with Dissipation and Ellipticity on Quadrant

RENJUN DUAN
Department of Mathematics, City University of Hong Kong
83 Tat Chee Avenue, Kowloon, Hong Kong, P.R. China

SHAOQIANG TANG
LTCS, Department of Mechanics and Engineering Science, Peking University
Beijing 100871, P.R. China

and

CHANGJIANG ZHU*
School of Mathematics and Statistics, Huazhong Normal University
Wuhan 430079, P.R. China

Abstract: In this paper, we consider an initial boundary value problem for some nonlinear evolution system with dissipation and ellipticity. We establish the global existence and furthermore obtain the $L^p (p \geq 2)$ decay rates of solutions corresponding to diffusion waves. The analysis is based on the energy method and pointwise estimates.

Key Words: Nonlinear evolution system, decay rate, energy method, pointwise estimates, a priori estimates.

1. Introduction

In this paper, we consider the following nonlinear evolution system with dissipation and ellipticity on the quadrant $\mathbb{R}_+ \times \mathbb{R}_+$:

\[
\begin{align*}
\psi_t &= -(1 - \alpha)\psi - \theta_x + \alpha \psi_{xx}, \\
\theta_t &= -(1 - \alpha)\theta + \nu \psi_x + 2 \psi \theta_x + \alpha \theta_{xx},
\end{align*}
\]

with initial data

\[(\psi, \theta)(x, 0) = (\psi_0, \theta_0)(x), \quad x \geq 0, \quad (1.2)\]

and the zero Dirichlet boundary condition

\[(\psi, \theta)(0, t) = (0, 0), \quad t \geq 0. \quad (1.3)\]

Here it is assumed that $(\psi_0, \theta_0)(0) = (0, 0)$, and $\lim_{x \to \infty}(\psi_0, \theta_0)(x) = (\psi_+, \theta_+)$. 

*Corresponding author. Email: cjzhu@mail.ccnu.edu.cn
The nonlinear interaction between dissipation and ellipticity plays a vital role in a number of physical systems, such as dynamic phase transitions, superposed fluids, Rayleigh-Benard problem, Taylor-Couette instability and fluid flow down an inclined plane, etc. [5, 8, 9, 14, 17]. These systems are far from well understood. A simplified system was thus proposed by Hsieh et al [7]:

\[
\begin{cases}
    \psi_t = -\left(\sigma - \alpha\right)\psi - \sigma \theta_x + \alpha \psi_{xx}, \\
    \theta_t = -(1 - \beta) \theta + \nu \psi_x + 2 \psi \theta_x + \beta \theta_{xx},
\end{cases}
\tag{1.4}
\]

where \(\alpha, \beta, \sigma\) and \(\nu\) are positive constants such that \(\alpha < \sigma\) and \(\beta < 1\). System (1.4) is reduced to (1.1) when \(\sigma = 1, \alpha = \beta\). Studies on this system are expected to yield insight into afore-mentioned physical systems with the similar mechanism.

To explain the complexity in system (1.4) by a rough argument, we ignore the damping and diffusion terms temporarily to obtain the linear system:

\[
\begin{pmatrix}
    \psi_t \\
    \theta_t
\end{pmatrix} =
\begin{pmatrix}
    0 & -\sigma \\
    \nu & 2\psi
\end{pmatrix}
\begin{pmatrix}
    \psi_x \\
    \theta_x
\end{pmatrix}.
\tag{1.5}
\]

It is elliptic for \(|\psi| < \sqrt{\sigma \nu}\), and hyperbolic for \(|\psi| > \sqrt{\sigma \nu}\). Around the zero equilibrium, the small disturbance triggers the growth in \(|\psi|\) because of the inherent instability of ellipticity. However, at a position where \(|\psi| > \sqrt{\sigma \nu}\), system (1.5) becomes hyperbolic. Correspondingly in (1.4), dissipation terms tend to draw the system back to the equilibrium. For suitable coefficients, a “switching back and forth” phenomenon is expected due to the interplay among ellipticity, hyperbolicity and dissipation. But this phenomenon won’t occur in the analysis of this paper since we will only consider small solutions so that the linear system (1.5) is always elliptic. Numerical investigations have also revealed exciting behaviors of the system (1.4) [6, 7]. Depending on the parameters, the system may admit spikes, periodic and quasiperiodic solutions, even chaos.

Next let’s recall some theoretical results on system (1.4). Based on the Fourier analysis and the energy method, the Cauchy problem of (1.4) with \(\sigma = 1\) and \(\alpha = \beta\) was studied in [16], where the global existence, nonlinear stability and optimal decay rates of solutions were obtained for coefficients with \(\nu < 4\alpha(1 - \alpha)\). Boundedness of initial data is assumed in the sense of

\[
(\psi_0, \theta_0)(x) \in L^2 \cap W^{1,\infty}(\mathbb{R}, \mathbb{R}^2),
\tag{1.6}
\]

which implies that \(\psi_0\) and \(\theta_0\) vanish at infinity. For initial data with different end states:

\[
\lim_{x \to \pm \infty} (\psi_0, \theta_0)(x) = (\psi_\pm, \theta_\pm), \quad (\psi_+, \theta_+) \neq (\psi_-, \theta_-),
\]

the global existence and asymptotics to the Cauchy problem were obtained by constructing a correct function and using the energy method [21]. More precisely, it was proved that the small perturbation make solutions decay exponentially around diffusion waves in the following system obtained by the Darcy’s law [3, 13],

\[
\begin{cases}
    \bar{\psi}_t = -(1 - \alpha) \bar{\psi} - \bar{\theta}_x + \alpha \bar{\psi}_{xx}, \\
    -(1 - \alpha) \bar{\theta} + \nu \bar{\psi}_x = 0.
\end{cases}
\tag{1.7}
\]

Similar results were obtained in [2] for the convergence toward another kind of diffusion waves defined by

\[
\begin{cases}
    \tilde{\psi}_t = -(1 - \alpha) \tilde{\psi} + \alpha \tilde{\psi}_{xx}, \\
    \tilde{\theta}_t = -(1 - \alpha) \tilde{\theta} + \alpha \tilde{\theta}_{xx}.
\end{cases}
\tag{1.8}
\]
In this paper, we shall investigate the global existence and asymptotics of the initial boundary value problem (1.1)-(1.3). Generally speaking, the presence of the boundary layer requires some special treatment in making estimates [4, 10, 11, 12, 18, 19]. Here we will consider the zero Dirichlet boundary condition instead of the general Dirichlet boundary condition to avoid the difficulties that arise due to a boundary layer. Precisely, for the zero Dirichlet boundary condition, we use the Green function on the half-space to obtain the representation of solutions to the derivation from diffusion waves (1.8) and furthermore apply the energy method to get some a priori estimates. The global existence of solutions follows from the standard argument that the local existence and uniform a priori estimates yield the global existence. This paper is the first step to deal with the general Dirichlet boundary condition.

The rest of this paper is arranged as follows. In Section 2, we study linear diffusion waves, and make decay estimates of solutions and their derivatives. In Sections 3, the global existence is established through the local existence and a priori estimates. In Section 4, we get the $L^2$ decay rate of the deviation from the diffusion wave through another a priori estimate. Furthermore, we obtain $L^p (p \geq 2)$ decay rates by pointwise estimates. Some concluding remarks are made in Section 5.

Notations: Throughout this paper, we denote any positive constants by $C$, without making ambiguity. The Lebesgue space on $\mathbb{R}_+$, $L^p = L^p(\mathbb{R}_+) (1 \leq p \leq \infty)$ is endowed with the norm $\| f \|_{L^p} = \left( \int_{\mathbb{R}_+} |f(x)|^p \, dx \right)^{\frac{1}{p}}$, $1 \leq p < \infty$, or $\| f \|_{L^\infty} = \sup_{x \in \mathbb{R}_+} |f(x)|$. When $p = 2$, we write $\| f \|_{L^2(\mathbb{R}_+)} = \| f \|$. The $l$-th order Hilbert space $H^l(\mathbb{R}_+)$ is endowed with the norm $\| f \|_{H^l(\mathbb{R}_+)} = \| f \|_l = \left( \sum_{i=0}^l \| \partial_x^i f \|^2 \right)^{\frac{1}{2}}$. For simplicity, $\| f(t) \|_{L^p}$ and $\| f(t) \|_l$ are denoted by $\| f(t) \|_{L^p}$ and $\| f(t) \|_l$, respectively.

2. Diffusion Waves

In this section, we construct diffusion waves corresponding to the nonlinear system (1.1). What’s more, the decay rates of diffusion waves are obtained with the help of their explicit representations.

As in [3, 13], we expect that solutions of (1.1) converge toward those of the following linear system

\[
\begin{aligned}
\tilde{\psi}_t &= -(1 - \alpha) \tilde{\psi} + \alpha \tilde{\psi}_{xx}, \\
\tilde{\theta}_t &= -(1 - \alpha) \tilde{\theta} + \alpha \tilde{\theta}_{xx}.
\end{aligned}
\]  

(2.1)

By setting $\tilde{\psi}(x,t) = \tilde{\phi}(x,t)e^{-(1-\alpha)t}$, we can derive a heat equation from (2.1) \[ 1 \]

\[
\tilde{\phi}_t = \alpha \tilde{\phi}_{xx}.
\]  

(2.2)

A self-similarity solution $\tilde{\phi}(x,t)$ takes the following form

\[
\tilde{\phi}(x,t) = p(\xi) = p \left( \frac{x}{\sqrt{1 + t}} \right), \quad 0 < \xi < \infty,
\]  

(2.3)
Lemma 2.1. The heat kernel function has the following property.

\[
\begin{align*}
-\frac{1}{2}\xi p'(\xi) &= \alpha p''(\xi), \\
p(0) &= 0, \quad p(\infty) = \psi_+.
\end{align*}
\]

We get by the direct calculation that

\[
\bar{\phi}(x, t) = p(\xi) = \psi_+ - \frac{2\psi_+}{\sqrt{4\pi\alpha(1 + t)}} \int_x^\infty \exp \left(-\frac{y^2}{4\alpha(1 + t)}\right) dy.
\]

Diffusion waves defined by (2.1) then read

\[
\begin{align*}
\bar{\psi}(x, t) &= \psi_+ e^{-(1 - \alpha)t} \left(1 - 2 \int_x^\infty G(y, t + 1) dy\right), \\
\bar{\theta}(x, t) &= \theta_+ e^{-(1 - \alpha)t} \left(1 - 2 \int_x^\infty G(y, t + 1) dy\right),
\end{align*}
\]

where \(G(x, t) = \frac{1}{\sqrt{4\pi\alpha t}} \exp\left\{-\frac{x^2}{4\alpha t}\right\}\) is the heat kernel function.

It is easy to show that

\[
\begin{align*}
(\bar{\psi}, \bar{\theta})(x, t) &\to (\psi_+ e^{-(1 - \alpha)t}, \theta_+ e^{-(1 - \alpha)t}), \quad x \to \infty, \\
(\bar{\psi}, \bar{\theta})(0, t) &= (0, 0), \quad t \geq 0.
\end{align*}
\]

On the other hand, as in [3, 13, 20], system (1.1) implies that

\[
\psi(x, t) \to \psi_+ e^{-(1 - \alpha)t}, \quad x \to \infty,
\]

and

\[
\theta(x, t) \to \theta_+ e^{-(1 - \alpha)t}, \quad x \to \infty.
\]

Therefore we have

\[
\begin{align*}
(\psi - \bar{\psi}, \theta - \bar{\theta})(x, t) &\to (0, 0), \quad x \to \infty, \\
(\psi - \bar{\psi}, \theta - \bar{\theta})(0, t) &= (0, 0), \quad t \geq 0.
\end{align*}
\]

Now we consider the asymptotics of \(\bar{\psi}(x, t), \bar{\theta}(x, t)\) and their derivatives in \(L^p(\mathbb{R}_+).\) First, the heat kernel function has the following property.

**Lemma 2.1.** When \(1 \leq p \leq +\infty, 0 \leq l, \ k < +\infty,\) it holds that

\[
\|\partial_l^k \partial_x^2 G(t)\|_{L^p} \leq C t^{-\frac{3}{2} (1 - \frac{l}{p}) - \frac{l}{p}}.
\]

By standard calculations, it is readily shown that diffusion waves \((\bar{\psi}, \bar{\theta})\) have the following properties.

**Lemma 2.2.** For functions \((\bar{\psi}, \bar{\theta})(x, t)\) defined by (2.5), it holds that

- (i) \(\|\partial_l^k \bar{\psi}(t)\|_{L^\infty} \leq C e^{-(1 - \alpha)t}, \quad \|\partial_l^k \bar{\theta}(t)\|_{L^\infty} \leq C e^{-(1 - \alpha)t}, \quad l = 0, 1, 2, \cdots;\)

- (ii) for any \(p\) with \(1 \leq p \leq +\infty,\)

\[
\begin{align*}
\|\partial_l^k \partial_x^2 \bar{\psi}(t)\|_{L^p} &\leq C |\psi_+| e^{-(1 - \alpha)t} (1 + t)^{\frac{1}{2p} - \frac{k}{2}}, \quad k = 1, 2, \cdots, \quad l = 0, 1, 2, \cdots, \\
\|\partial_l^k \partial_x^2 \bar{\theta}(t)\|_{L^p} &\leq C |\theta_+| e^{-(1 - \alpha)t} (1 + t)^{\frac{1}{2p} - \frac{k}{2}}, \quad k = 1, 2, \cdots, \quad l = 0, 1, 2, \cdots.
\end{align*}
\]
3. Global Existence

3.1. Reformulation and main results

Deviation from diffusion waves

\[
\begin{align*}
    u(x,t) &= \psi(x,t) - \bar{\psi}(x,t), \\
    v(x,t) &= \theta(x,t) - \bar{\theta}(x,t),
\end{align*}
\]

is governed by

\[
\begin{align*}
    u_t &= -(1 - \alpha)u - v_x + \alpha u_{xx} - \bar{\theta}_x, \\
    v_t &= -(1 - \alpha)v + \nu u_x + 2\psi u + 2\bar{\psi}_x u + F(x,t), \quad x > 0, \ t > 0,
\end{align*}
\]

with initial data

\[
(u, v)(x, 0) = (u_0, v_0)(x), \quad x \geq 0,
\]

and the boundary condition

\[
(u, v)(0, t) = (0, 0), \quad t \geq 0.
\]

Here we have used the following notations:

\[
\begin{align*}
    u_0(x) &= \psi_0(x) - \bar{\psi}(x, 0) \to 0, \quad \text{as} \ x \to \infty, \\
    v_0(x) &= \theta_0(x) - \bar{\theta}(x, 0) \to 0, \quad \text{as} \ x \to \infty, \\
    F(x, t) &= \nu \bar{\psi}_x + 2\bar{\psi}_x.
\end{align*}
\]

We seek solutions to the problem (3.2)-(3.4) in the set of functions

\[
X(0, T) = \{(u, v) | u, v \in L^\infty(0, T; H^1(\mathbb{R}^+)) \cap L^2(0, T; H^2(\mathbb{R}^+))\},
\]

where 0 < T ≤ ∞. Now we state our main result as follows.

**Theorem 3.1.** Let \((u_0, v_0)(x) \in H^1(\mathbb{R}^+, \mathbb{R}^2), \ 0 < \alpha < 1\) and \(0 < \nu < 4\alpha(1 - \alpha)\). Assume that both \(\delta = |\psi_+| + |\theta_+|\) and \(\delta_0 = \|u_0\|_1^2 + \|v_0\|_1^2\) are sufficiently small. Then there exist unique global solutions \((u, v)(x, t) \in X(0, \infty)\) to the initial boundary value problem (3.2)-(3.4), which satisfy

\[
\|u(t)\|_1^2 + \|v(t)\|_1^2 + \int_0^t (\|u(\tau)\|_2^2 + \|v(\tau)\|_2^2) d\tau \leq C(\delta + \delta_0)^2, \quad \text{for any} \ t \geq 0
\]

and

\[
\sup_{x \in \mathbb{R}^+} |(u, v)(x, t)| \to 0, \quad \text{as} \ t \to \infty.
\]

Furthermore, for any \(2 \leq p \leq \infty\) and \(t \geq \tau > 0\), it holds that

\[
\sum_{k=0}^1 \left( \|\partial_x^k u(t)\|_{L^p} + \|\partial_x^k v(t)\|_{L^p} \right) \leq C(\tau)e^{-\frac{l}{2}t},
\]

where the positive constant \(l\) is defined by (4.2) in Section 4 and \(C(\tau)\) is a positive constant depending only on \(\tau\).
3.2. Local existence

To order to get the explicit representation of solutions to system (3.2), let’s first consider the initial boundary value problem

\[
\begin{cases}
  u_t = -(1 - \alpha)u + \alpha u_{xx}, \\
  \bar{v}_t = -(1 - \alpha)\bar{v} + \alpha \bar{v}_{xx}, & x > 0, \ t > 0, \\
  (\bar{u}, \bar{v})(0, t) = (0, 0), & t \geq 0, \\
  (\bar{u}, \bar{v})(x, 0) = (\bar{u}_0, \bar{v}_0)(x), & x \geq 0.
\end{cases}
\]

(3.10)

By introducing the following transformation

\[
\bar{u}(x, t) = w(x, t)e^{-(1-\alpha)t}, \quad \bar{v}(x, t) = z(x, t)e^{-(1-\alpha)t},
\]

it is easy to see that system (3.10) has solutions

\[
\begin{cases}
  \bar{u}(x, t) = e^{-(1-\alpha)t} \int_0^\infty \left( G(x, t) - G(x+y, t) \right) \bar{u}_0(y) dy, \\
  = \frac{e^{-(1-\alpha)t}}{\sqrt{4\pi\alpha t}} \int_0^\infty \left( e^{-\frac{(x-y)^2}{4\alpha t}} - e^{-\frac{(x+y)^2}{4\alpha t}} \right) \bar{u}_0(y) dy, \\
  \bar{v}(x, t) = e^{-(1-\alpha)t} \int_0^\infty \left( G(x, t) - G(x+y, t) \right) \bar{v}_0(y) dy, \\
  = \frac{e^{-(1-\alpha)t}}{\sqrt{4\pi\alpha t}} \int_0^\infty \left( e^{-\frac{(x-y)^2}{4\alpha t}} - e^{-\frac{(x+y)^2}{4\alpha t}} \right) \bar{v}_0(y) dy.
\end{cases}
\]

(3.11)

Thus we define an integral operator \( K(t) \) by

\[
(K(t) * f)(x) = e^{-(1-\alpha)t} \int_0^{+\infty} \left( G(x, t) - G(x-y, t) \right) f(y) dy
\]

\[
= e^{-(1-\alpha)t} \left( G(\cdot, t) * f \right)(x) - e^{-(1-\alpha)t} \left( G(\cdot, t) * f \right)(-x),
\]

(3.12)

for any \( x \geq 0 \) and any function \( f(x) \). Hence \( K(t) * f \) is the difference between two convolutions. From Lemma 2.1, we have the following estimates about the integral operator.

**Lemma 3.2.** When \( 1 \leq p \leq +\infty \), \( 0 \leq l \), \( k < +\infty \), we have

\[
\|\partial^l_x \partial^k_y K(t)\|_{L^p} \leq C e^{-(1-\alpha)t} t^{-\frac{l}{2}(1-\frac{1}{p}) - \frac{k}{2}}.
\]

(3.13)

In view of the Duhamel’s principle, we can rewrite (3.2)-(3.4) in the following integral form

\[
\begin{cases}
  u(x, t) = (K(t) * u_0)(x) + \int_0^t K(t-s) * (-v_x - \bar{\theta}_x)(\cdot, s) ds, \\
  v(x, t) = (K(t) * v_0)(x) + \int_0^t K(t-s) * (\nu u_x + 2uv_x + 2\bar{\theta}_x u + F)(\cdot, s) ds.
\end{cases}
\]

(3.14)

Hence we can construct the approximation sequences of solutions and obtain the local existence by implementing the standard arguments with the Brower fixed point principle, cf. [1].
Lemma 3.3 (Local existence). If \((u_0, v_0)(x) \in H^1(\mathbb{R}_+, \mathbb{R}^2)\), then there exists \(t_0 > 0\) depending only on \(\| (u_0, v_0)(x) \|_{H^1(\mathbb{R}_+, \mathbb{R}^2)}\), such that the initial boundary value problem (3.2)-(3.4) admits unique smooth solutions \((u, v)(x, t) \in X(0, t_0)\) satisfying

\[
\|(u, v)(\cdot, t)\|_{H^1(\mathbb{R}_+, \mathbb{R}^2)} \leq 2\|(u_0, v_0)(x)\|_{H^1(\mathbb{R}_+, \mathbb{R}^2)}, \quad 0 \leq t \leq t_0.
\]  

\((3.15)\)

3.3. Global existence

In order to get the global existence, we must show that solutions obtained in Lemma 3.3 are bounded from above by a positive constant that depends only on the initial data.

Throughout this subsection, we suppose that \((u, v)(x, t) \in X(0, T)\) are solutions to the problem (3.2)-(3.4). Next let’s devote ourselves to making estimates of \((u, v)\) under the following \textit{a priori} assumption

\[
N(T) = \sup_{0 \leq t \leq T} \left\{ \sum_{k=0}^{1} \left( \| \partial_x^k u(t) \|^2 + \| \partial_x^k v(t) \|^2 \right) \right\} \leq \delta_1^2, \tag{3.16}
\]

where \(0 < \delta_1 \ll 1\).

From the Sobolev inequality \(\| f \|_{L^\infty} \leq \| f \|_{L^2}^{\frac{1}{2}} \| f_x \|_{L^2}^{\frac{1}{2}}\), we have

\[
\|(u, v)(x, t)\|_{L^\infty([0, T] \times \mathbb{R}_+, \mathbb{R}^2)} \leq \delta_1. \tag{3.17}
\]

Moreover, if \(\nu < 4\alpha(1 - \alpha)\) as in [16], we can find \(\varepsilon \in (0, 2), c_0 > 0\) such that

\[
\begin{cases} 
2c_0 \alpha - \frac{1}{(1 - \alpha)\varepsilon} > 0, \\
2(1 - \alpha) - \frac{c_0 \nu^2}{\alpha \varepsilon} > 0.
\end{cases} \tag{3.18}
\]

In fact, we set \(k = \frac{\nu}{4\alpha(1 - \alpha)}\), then \(k \in (0, 1)\). Choosing

\[
\begin{cases} 
\varepsilon = k + 1, \\
c_0 = \frac{1}{\alpha(1 - \alpha)} \left( \frac{1}{2(k + 1)} + \frac{k + 1}{8k^2} \right),
\end{cases} \tag{3.19}
\]

one can easily check that \(\varepsilon\) and \(c_0\) satisfy (3.18).

Lemma 3.4. Let the assumptions in Theorem 3.1 hold. If \(\delta\) and \(\delta_1\) are sufficiently small, then for any \(0 \leq t \leq T\), it holds that

\[
\int_0^\infty (u^2 + v^2)dx + \int_0^t \int_0^\infty (u^2 + v^2)dxd\tau + \int_0^t \int_0^\infty (u_x^2 + v_x^2)dxd\tau \leq C_1(\delta + \delta_0), \tag{3.20}
\]

where \(C_1\) is a positive constant depending only on \(\alpha\) and \(\nu\).
Proof. Multiplying the first equation of (3.2) by $2u$, the second equation by $2c_0v$, and integrating over $(0, \infty) \times (0, t)$, by the Cauchy-Schwarz inequality we arrive at

$$
\int_0^\infty (u^2 + c_0v^2) dx + 2(1 - \alpha) \int_0^t \int_0^\infty (u^2 + c_0v^2) dxd\tau + 2\alpha \int_0^t \int_0^\infty (u_x^2 + c_0v_x^2) dxd\tau
= \|u_0\|^2 + c_0\|v_0\|^2 - 2 \int_0^t \int_0^\infty uv_x dxd\tau + 2\nu c_0 \int_0^t \int_0^\infty u\theta_x dxd\tau - 2 \int_0^t \int_0^\infty u\theta_x dxd\tau
+ 4c_0 \int_0^t \int_0^\infty uv_x dxd\tau - 2c_0 \int_0^t \int_0^\infty \psi_x^2 dxd\tau + 4c_0 \int_0^t \int_0^\infty \theta_x u\theta dxd\tau
+ 2c_0 \int_0^t \int_0^\infty vF(x, \tau) dxd\tau
\leq (1 + c_0)\delta_0 + \varepsilon(1 - \alpha) \int_0^t \int_0^\infty u^2 dxd\tau + \frac{1}{\varepsilon(1 - \alpha)} \int_0^t \int_0^\infty v_x^2 dxd\tau + \varepsilon\alpha \int_0^t \int_0^\infty u_x^2 dxd\tau
+ \frac{c_0^2\nu^2}{\varepsilon\alpha} \int_0^t \int_0^\infty v^2 dxd\tau + \delta(1 - \alpha) \int_0^t \int_0^\infty u^2 dxd\tau + \frac{1}{\delta(1 - \alpha)} \int_0^t \int_0^\infty \theta_x^2 dxd\tau
+ 2c_0\|u\|_{L^\infty([0, T] \times \mathbb{R}^+)} \int_0^t \int_0^\infty (v^2 + v_x^2) dxd\tau + 2c_0\|\psi_x\|_{L^\infty([0, T] \times \mathbb{R}^+)} \int_0^t \int_0^\infty v^2 dxd\tau
+ 2c_0\|\theta_x\|_{L^\infty([0, T] \times \mathbb{R}^+)} \int_0^t \int_0^\infty (u^2 + v^2) dxd\tau
+ c_0\delta \int_0^t \int_0^\infty \theta_x^2 dxd\tau + \frac{c_0}{\delta} \int_0^t \int_0^\infty F^2(x, \tau) dxd\tau.
$$

(3.21)

Using Lemma 2.2 and (3.17), we derive from the above inequality

$$
\int_0^\infty (u^2 + c_0v^2) dx + \{(2 - \varepsilon)(1 - \alpha) - C\delta\} \int_0^t \int_0^\infty u^2 dxd\tau + (2 - \varepsilon)\alpha \int_0^t \int_0^\infty u_x^2 dxd\tau
+ \left\{2(1 - \alpha) - \frac{c_0\nu^2}{\varepsilon\alpha} - C(\delta_1 + \delta)\right\} \int_0^t \int_0^\infty c_0v^2 dxd\tau
+ \left\{2c_0\alpha - \frac{1}{\varepsilon(1 - \alpha)} - C\delta_1\right\} \int_0^t \int_0^\infty v_x^2 dxd\tau
\leq C(\delta + \delta_0) + \frac{1}{\delta(1 - \alpha)} \int_0^t \int_0^\infty \theta_x^2 dxd\tau + \frac{c_0}{\delta} \int_0^t \int_0^\infty F^2(x, \tau) dxd\tau.
$$

(3.22)

For the last two terms on the right hand side of (3.22), Lemma 2.2 yields

$$
\frac{1}{\delta(1 - \alpha)} \int_0^t \int_0^\infty \theta_x^2 dxd\tau \leq \frac{1}{\delta(1 - \alpha)} C\delta^2 \int_0^t e^{-2(1-\alpha)\tau} d\tau \leq C\delta,
$$

(3.23)

and

$$
\frac{c_0}{\delta} \int_0^t \int_0^\infty F^2(x, \tau) dxd\tau \leq \frac{C}{\delta} \int_0^t \int_0^\infty (\psi_x^2 + \psi^2\theta_x^2) dxd\tau \leq C\delta.
$$

(3.24)
So we end up with
\[
\int_0^\infty (u^2 + c_0 v^2)dx + \{(2 - \varepsilon)(1 - \alpha) - C\delta\} \int_0^t \int_0^\infty u^2 dx d\tau + (2 - \varepsilon)\alpha \int_0^t \int_0^\infty u_x^2 dx d\tau \\
\quad + \left\{2(1 - \alpha) - \frac{c_0 v^2}{\varepsilon\alpha} - C(\delta + \delta_1)\right\} \int_0^t \int_0^\infty c_0 v^2 dx d\tau \\
\quad + \left\{2c_0\alpha - \frac{1}{\varepsilon(1 - \alpha)} - C\delta_1\right\} \int_0^t \int_0^\infty v_x^2 dx d\tau
\leq C(\delta + \delta_0).
\]

(3.25)

This ends the proof of Lemma 3.4 with the help of (3.18).

**Lemma 3.5.** Let the assumptions in Theorem 3.1 hold. If \(\delta\) and \(\delta_1\) are sufficiently small, then for any \(0 \leq t \leq T\), it holds that
\[
\int_0^\infty (u_x^2 + v_x^2)dx + \int_0^t \int_0^\infty (u_{xx}^2 + v_{xx}^2)dx d\tau \leq C_2(\delta + \delta_0)^{\frac{1}{2}},
\]
where \(C_2\) is a positive constant depending only on \(\alpha\) and \(\nu\).

**Proof.** First, we multiply the first equation of (3.2) by \((-2u_{xx})\), the second equation by \((-2c_0v_{xx})\) respectively, and take integration over \((0, \infty) \times (0, t)\). Then similar to (3.21), we have
\[
\int_0^\infty (u_x^2 + c_0 v_x^2)dx + 2(1 - \alpha) \int_0^t \int_0^\infty (u_x^2 + c_0 v_x^2)dx d\tau + 2\alpha \int_0^t \int_0^\infty (u_{xx}^2 + c_0 v_{xx}^2)dx d\tau \\
\leq (1 + c_0)\delta_0 + \frac{\varepsilon\alpha}{2} \int_0^t \int_0^\infty u_{xx}^2 dx d\tau + \frac{2}{\varepsilon\alpha} \int_0^t \int_0^\infty v_{xx}^2 dx d\tau + \frac{1}{2\varepsilon(1 - \alpha)} \int_0^t \int_0^\infty v_{xx}^2 dx d\tau \\
+ 2\nu^2 c_0^2 (1 - \alpha) \int_0^t \int_0^\infty u_{xx}^2 dx d\tau + \frac{\varepsilon\alpha}{2} \int_0^t \int_0^\infty u_{xx}^2 dx d\tau + \frac{2}{\varepsilon\alpha} \int_0^t \int_0^\infty v_{xx}^2 dx d\tau + \frac{1}{2\varepsilon(1 - \alpha)} \int_0^t \int_0^\infty v_{xx}^2 dx d\tau \\
+ (\delta + \delta_0)^{\frac{1}{2}} \int_0^t \int_0^\infty v_{xx}^2 dx d\tau + C(\delta + \delta_0)^{-\frac{1}{2}} \int_0^t \int_0^\infty v_{xx}^2 dx d\tau \\
+ 2c_0 \left\|\bar\theta_x\right\|_{L^\infty([0, T] \times \mathbb{R}^+)} \int_0^t \int_0^\infty (u^2 + v_{xx}^2)dx d\tau \\
+ c_0\delta \int_0^t \int_0^\infty v_{xx}^2 dx d\tau + \frac{c_0}{\delta} \int_0^t \int_0^\infty F^2(x, \tau) dx d\tau.
\]

(3.27)

From (3.17), (3.23), (3.24) and Lemma 2.2, we have
\[
\int_0^\infty (u_x^2 + c_0 v_x^2)dx + 2(1 - \alpha) \int_0^t \int_0^\infty (u_x^2 + c_0 v_x^2)dx d\tau + 2\alpha \int_0^t \int_0^\infty (u_{xx}^2 + c_0 v_{xx}^2)dx d\tau \\
\leq C(\delta + \delta_0) + \varepsilon\alpha \int_0^t \int_0^\infty u_{xx}^2 dx d\tau + \left\{\frac{1}{\varepsilon(1 - \alpha)} + C(\delta + \delta_0)^{\frac{1}{2}}\right\} \int_0^t \int_0^\infty v_{xx}^2 dx d\tau \\
+ C \int_0^t \int_0^\infty (u^2 + u_x^2 + v_x^2)dx d\tau + C(\delta + \delta_0)^{-\frac{1}{2}} \int_0^t \int_0^\infty v_{xx}^2 dx d\tau.
\]

(3.28)
With the help of Lemma 3.4, (3.28) gives
\begin{align*}
\int_0^{\infty} (u_x^2 + c_0 v_x^2) dx + (2 - \varepsilon) \alpha \int_0^t \int_0^{\infty} u_{xx}^2 dx d\tau \\
+ \left\{ 2c_0 \alpha - \frac{1}{\varepsilon (1 - \alpha)} - C(\delta + \delta_0)^{\frac{1}{2}} \right\} \int_0^t \int_0^{\infty} v_{xx}^2 dx d\tau \leq C(\delta + \delta_0)^{\frac{1}{2}}. \tag{3.29}
\end{align*}
Thus the proof of Lemma 3.5 is completed by (3.18) and (3.29) provided that \( \delta \) and \( \delta_1 \) are sufficiently small.

Now we are in a position to complete the proof of the global existence of solutions to the problem (3.2)-(3.4). By the standard argument, the local existence and uniform a priori estimates yield the global existence. In fact, let \( \delta \) and \( \delta_1 \) sufficiently small such that we can find \( C_1 > 0 \) and \( C_2 > 0 \) independent of any \( T > 0 \) to make Lemmas 3.4 and 3.5 hold. Furthermore, let \( \delta \) and \( \delta_0 \) sufficiently small such that
\[ \max \left\{ 2\delta_0^2, C_1(\delta + \delta_0) + C_2(\delta + \delta_0)^{\frac{1}{2}} \right\} < \delta_1^2. \]
Define
\[ T^* = \sup \left\{ T : \sum_{k=0}^1 \left( \| \partial_x^k u(t) \|^2 + \| \partial_x^k v(t) \|^2 \right) \leq \delta_1^2 \text{ for any } 0 \leq t \leq T \right\}. \]
Thus it follows from Lemma 3.3 that \( 0 < T^* \leq \infty \). If \( T^* < \infty \), then by Lemmas 3.4 and 3.5, we have that for any \( 0 \leq t \leq T^* \),
\[ \sum_{k=0}^1 \left( \| \partial_x^k u(t) \|^2 + \| \partial_x^k v(t) \|^2 \right) \leq C_1(\delta + \delta_0) + C_2(\delta + \delta_0)^{\frac{1}{2}} < \delta_1^2, \]
which is a contradiction with the definition of \( T^* \). So there exist global solutions \((u, v)\) to the problem (3.2)-(3.4) such that
\[ \sum_{k=0}^1 \left( \| \partial_x^k u(t) \|^2 + \| \partial_x^k v(t) \|^2 \right) \leq \delta_1^2, \quad 0 \leq t < \infty. \]
Again by Lemmas 3.4-3.5, we have (3.7).

Finally, we prove that (3.8) is true. For this purpose, we introduce the following lemma proved in [13].

**Lemma 3.6.** If \( g(t) \geq 0, \ g(t) \in L^1(0, \infty) \) and \( g'(t) \in L^1(0, \infty) \), then \( g(t) \to 0 \) as \( t \to \infty \).

Taking \( g(t) = \|u_x(t)\|^2 \) in Lemma 3.6, we can conclude from (3.7) that \( g(t) \in L^1(0, \infty) \). Denote the inner product by \( \langle \cdot, \cdot \rangle \) in \( L^2(\mathbb{R}_+) \). By using the definition of \( L^2 \)-inner product and integrating by parts, we have that \( g'(t) = 2 \langle u, u_{xx} \rangle = -2 \langle u, u_{xt} \rangle \). It is easy to verify from (3.7) that
\[ -\langle u_t, u_{xx} \rangle = -\langle -u_t, u_{xx} \rangle = \langle (1 - \alpha)u + v_x - \alpha u_{xx} + \theta_x, u_{xx} \rangle \in L^1(0, \infty). \]
Hence, \( g'(t) \in L^1(0, \infty) \), and
\[ g(t) = \|u_x(t)\|^2 \to 0 \text{ as } t \to \infty. \tag{3.30} \]
Applying the Sobolev inequality, we have from (3.30) and (3.7) that
\[ \sup_{x \in \mathbb{R}^+} |u(x, t)| \leq \|u(t)\|^\frac{1}{2} \|u_x(t)\|^\frac{1}{2} \xrightarrow{t \to \infty} 0. \]  
(3.31)

The same argument can apply to \(v(x, t)\), i.e.
\[ \sup_{x \in \mathbb{R}^+} |v(x, t)| \leq \|v(t)\|^\frac{1}{2} \|v_x(t)\|^\frac{1}{2} \xrightarrow{t \to \infty} 0. \]  
(3.32)

This ends the proof of (3.8).

4. Decay Rate

In this section, we will prove (3.9) in Theorem 3.1. First let’s consider \(L^2\) decay rates of solutions to the problem (3.2)-(3.4) under the following \textit{a priori} assumption
\[ \sup_{0 \leq t \leq T} \left\{ e^{lt} \sum_{k=0}^{1} \left( \|\partial_x^k u(t)\|^2 + \|\partial_x^k v(t)\|^2 \right) \right\} \leq 1, \]  
(4.1)

with
\[ l = \min \left\{ (2 - \varepsilon)(1 - \alpha), \ 2(1 - \alpha) - \frac{c_0}{\varepsilon \alpha} \right\}, \]  
(4.2)

where \(\varepsilon\) and \(c_0\) are defined by (3.18). Our aim is to prove uniform estimates
\[ e^{lt} \sum_{k=0}^{1} \left( \|\partial_x^k u(t)\|^2 + \|\partial_x^k v(t)\|^2 \right) \leq \frac{1}{2}, \quad 0 \leq t \leq T, \]  
provided that \(\delta\) and \(\delta_0\) are sufficiently small. Thus by the same contradiction argument as in the last section, we obtain the \(L^2\) decay rates of solutions \((u, v)\). Hence we omit its details for brevity and only prove the above uniform estimates.

By the Sobolev inequality, we have from (4.1) that
\[ \|u, v\|_{L^\infty} \leq e^{-\frac{1}{2}t}. \]  
(4.3)

The following Gronwall’s inequality will be also used later on.

\textbf{Lemma 4.1 (Gronwall’s inequality).} Let \(\eta(\cdot)\) be a nonnegative continuous function on \([0, \infty)\), which satisfies the differential inequality
\[ \eta'(t) + \lambda \eta(t) \leq \omega(t), \quad 0 \leq t < \infty, \]  
where \(\lambda\) is a positive constant and \(\omega(t)\) is a nonnegative continuous function on \([0, \infty)\). Then
\[ \eta(t) \leq \left( \eta(0) + \int_0^t e^{\lambda \tau} \omega(\tau) d\tau \right) e^{-\lambda t}, \quad 0 \leq t < \infty. \]  
(4.4)

Now we can state the main result about \(L^2\) decay rates of solutions.
Theorem 4.2. Suppose that \((u, v)(x, t)\) are solutions to the problem (3.2)-(3.4) under the assumptions imposed in Theorem 3.1. If \(\delta\) and \(\delta_0\) are sufficiently small, then we have that for any \(0 \leq t \leq T\),
\[
\| \partial_x^k u(t) \|^2 + \| \partial_x^k v(t) \|^2 \leq C_3(\delta + \delta_0)^{\frac{1}{2}} e^{-lt}, \quad k = 0, 1,
\]  
(4.5)
where \(C_3\) is a positive constant depending only on \(\alpha\) and \(\nu\) and \(l\) is defined by (4.2).

Proof. The proof consists of two steps.

Step 1. Taking (3.2) \times 2u + (3.2) \times 2c_0v and integrating over \(x \in \mathbb{R}_+\), we reach at
\[
\begin{align*}
&\frac{d}{dt} \int_0^\infty (u^2 + c_0v^2)dx + 2(1 - \alpha) \int_0^\infty (u^2 + c_0v^2)dx + 2\alpha \int_0^\infty (u_x^2 + c_0v_x^2)dx \\
&= -2 \int_0^\infty uv_x dx + 2c_0 \int_0^\infty v_x dx - 2 \int_0^\infty u\theta_x dx - 2c_0 \int_0^\infty (u_x + \psi_x)v^2 dx \\
&+ 4c_0 \int_0^\infty \theta_x uv x dx + 2c_0 \int_0^\infty vF(x, t) dx \\
&\leq \varepsilon(1 - \alpha) \int_0^\infty u^2 dx + \frac{1}{\varepsilon(1 - \alpha)} \int_0^\infty v^2 dx + \varepsilon\alpha \int_0^\infty u_x^2 dx + \frac{c_0^2\nu^2}{\varepsilon\alpha} \int_0^\infty v^2 dx - 2 \int_0^\infty u\theta_x dx \\
&+ 2c_0\|\theta_x\|_{L^\infty} \int_0^\infty u^2 dx + 2c_0(\|\psi_x\|_{L^\infty} + \|\theta_x\|_{L^\infty}) \int_0^\infty v^2 dx \\
&- 2c_0 \int_0^\infty u_x v^2 dx + 2c_0 \int_0^\infty vF(x, t) dx. \tag{4.6}
\end{align*}
\]

By Lemma 2.2, rearrangement of the terms in the inequality gives
\[
\frac{d}{dt} \int_0^\infty (u^2 + c_0v^2)dx + (2 - \varepsilon)(1 - \alpha) \int_0^\infty u^2 dx + \left\{2(1 - \alpha) - \frac{c_0^2\nu^2}{\varepsilon\alpha}\right\} \int_0^\infty c_0v^2 dx \\
+ (2 - \varepsilon)\alpha \int_0^\infty u_x^2 dx + \left\{2c_0\alpha - \frac{1}{\varepsilon(1 - \alpha)}\right\} \int_0^\infty v_x^2 dx \\
\leq C\delta e^{-(1-\alpha)t} \int_0^\infty (u^2 + c_0v^2)dx - 2 \int_0^\infty u\theta_x dx - 2c_0 \int_0^\infty u_x v^2 dx + 2c_0 \int_0^\infty vF(x, t) dx. \tag{4.7}
\]

Next, we estimate terms in the right hand side of (4.7).

In fact, we have from the assumption (4.1) and Lemma 2.2
\[
-2 \int_0^\infty u\theta_x dx \leq \delta e^{\frac{1}{2}(1-\alpha)t} \int_0^\infty u^2 dx + \frac{1}{\delta} e^{-(\frac{1}{2} - \alpha)t} \int_0^\infty \theta_x^2 dx \\
\leq C\delta e^{-(\frac{1}{2} + (1-\alpha)t)}. \tag{4.8}
\]

Moreover, we get from (3.7) by Sobolev inequality
\[
\|(u, v)\|_{L^\infty} \leq C(\delta + \delta_0)^{\frac{1}{2}}. \tag{4.9}
\]

We derive by the Cauchy-Schwarz inequality from (4.1), (4.3) and (4.9)
\[
-2c_0 \int_0^\infty u_x v^2 dx \leq c_0 \left( \int_0^\infty u_x^2 v dx + \int_0^\infty |v|^3 dx \right) \\
\leq c_0 \|v\|_{L^\infty} \|v\|_{L^\infty} \int_0^\infty (u_x^2 + v^2) dx \\
\leq C(\delta + \delta_0)^{\frac{1}{2}} e^{-\frac{\nu}{4}t}. \tag{4.10}
\]
In addition, again from the assumption (4.1) and Lemma 2.2, we deduce
\[ 2c_0 \int_0^\infty vF(x,t)dx \leq \delta e^{\left(\frac{1}{2}-(1-\alpha)\right)t} \int_0^\infty v^2dx + \frac{c_0^2}{\delta} e^{-\left(\frac{1}{2}-(1-\alpha)\right)t} \int_0^\infty F^2(x,t)dx \]
\[ \leq \delta e^{\left(\frac{1}{2}-(1-\alpha)\right)t} e^{-\alpha t} + \frac{C}{\delta} e^{-\left(\frac{1}{2}-(1-\alpha)\right)t} \int_0^\infty (\psi^2 + \theta^2)dx \]
\[ \leq C\delta e^{-\left(\frac{1}{2}+(1-\alpha)\right)t}. \] (4.11)

Thus, we get from (4.7), (4.8), (4.10) and (4.11) that
\[ \frac{d}{dt} \int_0^\infty (u^2 + c_0v^2)dx + (2 - \varepsilon)(1 - \alpha) \int_0^\infty u^2dx + \left\{ 2(1 - \alpha) - \frac{c_0\nu^2}{\varepsilon\alpha} \right\} \int_0^\infty c_0v^2dx \]
\[ \leq C(\delta + \delta_0)^{\frac{1}{2}} e^{-\frac{\varepsilon t}{2}} + C\delta e^{-\left(\frac{1}{2}+(1-\alpha)\right)t}. \] (4.12)

Recalling the definition of \( l \) in (4.2), we have
\[ \frac{d}{dt} \int_0^\infty (u^2 + c_0v^2)dx + l \int_0^\infty (u^2 + c_0v^2)dx \leq C(\delta + \delta_0)^{\frac{1}{2}} e^{-\frac{\varepsilon t}{2}} + C\delta e^{-\left(\frac{1}{2}+(1-\alpha)\right)t}. \] (4.13)

Noticing that \( \frac{l}{2} \leq 1 - \alpha \), we obtain from Lemma 4.1
\[ \int_0^\infty (u^2 + c_0v^2)dx \leq C(\delta + \delta_0)^{\frac{1}{2}} e^{-\alpha t}, \] (4.14)
which implies (4.5) for \( k = 0 \).

Step 2. Taking \((3.2)_1 \times (-2u_{xx}) + (3.2)_2 \times (-2c_0v_{xx})\) and integrating with respect to \( x \) over \( \mathbb{R}_+ \), we have
\[ \frac{d}{dt} \int_0^\infty (u^2_x + c_0v^2_x)dx + 2(1 - \alpha) \int_0^\infty (u^2_x + c_0v^2_x)dx + 2\alpha \int_0^\infty (u^2_{xx} + c_0v^2_{xx})dx \]
\[ \leq \varepsilon(1 - \alpha)c_0 \int_0^\infty u^2_x dx + \frac{1}{\varepsilon(1 - \alpha)c_0} \int_0^\infty u^2_{xx}dx + \frac{c_0\nu^2}{\varepsilon\alpha} \int_0^\infty u^2_{xx}dx + \varepsilon\alpha c_0 \int_0^\infty v^2_{xx}dx \]
\[ + 2 \int_0^\infty u_{xx} \check{\theta}_x dx - 4c_0 \int_0^\infty u_{vx}v_{xx}dx - 4c_0 \int_0^\infty \check{\psi}_{vx}v_{xx}dx - 4c_0 \int_0^\infty \check{\psi}_xu_{xx}dx \]
\[ - 2c_0 \int_0^\infty v_{xx}F(x,t)dx. \] (4.15)

From Lemma 2.2, it may be shown that
\[ 2 \int_0^\infty u_{xx} \check{\theta}_x dx \leq \delta e^{\left(\frac{1}{2}-(1-\alpha)\right)t} \int_0^\infty u^2_{xx}dx + \frac{1}{\delta} e^{-\left(\frac{1}{2}-(1-\alpha)\right)t} \int_0^\infty \theta^2_x dx \]
\[ \leq \delta \int_0^\infty u^2_{xx} + C\delta e^{-\left(\frac{1}{2}+(1-\alpha)\right)t}. \] (4.16)
From (4.3) and (4.9), we have

\[-4c_0 \int_0^\infty uv_xv_{xx}dx\]

\[\leq (\delta + \delta_0)^{\frac{1}{4}} \int_0^\infty v_{xx}^2dx + 4c_0^4(\delta + \delta_0)^{-\frac{1}{4}} \int_0^\infty \psi^2v_x^2dx\]

\[\leq (\delta + \delta_0)^{\frac{1}{4}} \int_0^\infty v_{xx}^2dx + C(\delta + \delta_0)^{-\frac{1}{4}}\|u\|_{L^\infty(I_\circ)}^\frac{3}{2}\|u\|_{L^\infty(I_\circ)}^{\frac{1}{2}} \int_0^\infty v_x^2dx\]

\[\leq (\delta + \delta_0)^{\frac{1}{4}} \int_0^\infty v_{xx}^2dx + C(\delta + \delta_0)^{\frac{1}{4}}e^{-\frac{\delta}{2}t}. \quad (4.17)\]

Again using (4.1) and Lemma 2.2, we get

\[\int_0^\infty \psi v_x v_{xx}dx \leq (\delta + \delta_0)^{\frac{1}{4}} \int_0^\infty v_{xx}^2dx + 4c_0^4(\delta + \delta_0)^{-\frac{1}{4}} \int_0^\infty \psi^2v_x^2dx\]

\[\leq (\delta + \delta_0)^{\frac{1}{4}} \int_0^\infty v_{xx}^2dx + C(\delta + \delta_0)^{\frac{1}{4}}e^{-2(1-\alpha)t}, \quad (4.18)\]

and

\[\int_0^\infty \bar{\theta}_xuv_xdx \leq \int_0^\infty v_{xx}^2dx + 4c_0^4 \delta^{-1} \int_0^\infty \bar{\theta}_x^2u^2dx\]

\[\leq \delta \int_0^\infty v_{xx}^2dx + C\delta e^{-2(1-\alpha)t}. \quad (4.19)\]

Moreover, we have

\[\int_0^\infty v_{xx}Fdtx \leq \int_0^\infty v_{xx}^2dx + 4c_0^4 \delta^{-1} \int_0^\infty F^2dx\]

\[\leq \delta \int_0^\infty v_{xx}^2dx + C\delta e^{-2(1-\alpha)t}. \quad (4.20)\]

Substituting (4.16)-(4.20) into (4.15) and noticing that

\[-\{2(1-\alpha) + l\} < -\left\{\frac{l}{2} + (1-\alpha)\right\}, \quad -2(1-\alpha) < -\left\{\frac{l}{2} + (1-\alpha)\right\},\]

we have

\[
\frac{d}{dt} \int_0^\infty (u_x^2 + c_0v_x^2)dx + \left\{2(1-\alpha) - \frac{c_0v_x^2}{e^\alpha}\right\} \int_0^\infty u_x^2dx + (2-\varepsilon)(1-\alpha) \int_0^\infty c_0v_x^2dx
\]

\[+ \left\{2\alpha - \frac{1}{\varepsilon(1-\alpha)c_0} - \delta\right\} \int_0^\infty u_x^2dx + \left\{(2-\varepsilon)\alpha c_0 - 2(\delta + \delta_0)^{\frac{1}{2}} - 2\delta\right\} \int_0^\infty v_x^2dx\]

\[\leq C\delta e^{-l^{\frac{1}{2}}(1-\alpha)t} + C(\delta + \delta_0)^{\frac{1}{4}}e^{-\frac{\delta}{2}t} + C(\delta + \delta_0)^{\frac{1}{4}}e^{-2(1-\alpha)t}. \quad (4.21)\]

Together with (4.2), (4.21) gives

\[
\frac{d}{dt} \int_0^\infty (u_x^2 + c_0v_x^2)dx + l \int_0^\infty (u_x^2 + c_0v_x^2)dx
\]

\[\leq C\delta e^{-l^{\frac{1}{2}}(1-\alpha)t} + C(\delta + \delta_0)^{\frac{1}{4}}e^{-\frac{\delta}{2}t} + C(\delta + \delta_0)^{\frac{1}{4}}e^{-2(1-\alpha)t}. \quad (4.22)\]

Thus we have from Lemma 4.1 that

\[\int_0^\infty (u_x^2 + c_0v_x^2)dx \leq C(\delta + \delta_0)^{\frac{1}{4}}e^{-lt}, \quad (4.23)\]

which implies (4.5) for $k = 1$. The proof of Theorem 4.2 is completed.
**Remark.** Immediately it follows from Theorem 4.2 that

\[
\sum_{k=0}^{1} \left( \| \partial_x^k u(t) \|^2 + \| \partial_x^k v(t) \|^2 \right) \leq e^{-lt}, \quad 0 \leq t < \infty.
\]

In fact the same contradiction argument as in the Section 3 can be applied. For brevity, we omit its proof.

Finally, we devote ourselves to the proof of \( L^p \) \((p \geq 2)\) decay rates (3.9) by pointwise estimates. For the case when \( k = 0 \), it follows from the Gagliardo-Nirenberg inequality that for any \( t \geq 0 \),

\[
\| u(t) \|_{L^p} \leq C \| u(t) \|_{L^\infty}^{\frac{p+2}{p+2}} \| \partial_x u(t) \|_{L^2}^{\frac{p-2}{p+2}} \leq Ce^{-\frac{1}{2}t}. \tag{4.24}
\]

Similarly we have that for any \( t \geq 0 \),

\[
\| v(t) \|_{L^p} \leq Ce^{-\frac{1}{2}t}. \tag{4.25}
\]

For the case when \( k = 1 \), we need the following inequality.

**Lemma 4.3 (Young’s inequality).** If \( f \in L^p, g \in L^r \), \( p \geq 1, r \geq 1 \) and \( \frac{1}{p} + \frac{1}{r} \geq 1 \), then \( h = f * g \in L^q \), where \( \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1 \). Furthermore, it holds that

\[
\| h \|_{L^q} \leq \| f \|_{L^p} \cdot \| g \|_{L^r}.
\]

From (3.14), we have that

\[
\begin{alignat}{2}
\partial_x u(x,t) &= (\partial_x K(t) * u_0)(x) \\
&\quad + \int_0^t \partial_x K(t-s) * (-v_x - \tilde{\theta}_x)(\cdot, s) \, ds,
\end{alignat}
\]

\[
\partial_x v(x,t) = (\partial_x K(t) * v_0)(x)
\]

\[
&\quad + \int_0^t \partial_x K(t-s) * (\nu u_x + 2w_x + 2\tilde{\psi} v_x + 2\tilde{\theta} u + F)(\cdot, s) \, ds. \tag{4.26}
\]

Then it holds that

\[
\| \partial_x u(t) \|_{L^p} \leq \| \partial_x K(t) * u_0 \|_{L^p} + \int_0^t \| \partial_x K(t-s) * v_x \|_{L^p} \, ds
\]

\[
+ \int_0^t \| \partial_x K(t-s) * \tilde{\theta}_x \|_{L^p} \, ds \tag{4.27}
\]

and

\[
\| \partial_x v(t) \|_{L^p} \leq \| \partial_x K(t) * v_0 \|_{L^p} + \int_0^t \| \partial_x K(t-s) * \nu u_x \|_{L^p} \, ds
\]

\[
+ \int_0^t \| \partial_x K(t-s) * (2w_x + 2\tilde{\psi} v_x + 2\tilde{\theta} u) \|_{L^p} \, ds
\]

\[
+ \int_0^t \| \partial_x K(t-s) * (\nu \tilde{\psi} x + 2\tilde{\psi} \tilde{\theta}_x) \|_{L^p} \, ds. \tag{4.28}
\]
Thus it follows from the Young’s inequality that

\[ \| \partial_x K(t) * u_0 \|_{L^p} \leq \| \partial_x K(t) \|_{L^{p+2}} \| u_0 \|_{L^2}. \]

Other terms can be similarly dealt with. Thus by using Lemma 2.2 and Theorem 4.2, we have from (4.27) and (4.28) that

\begin{align*}
\| \partial_x u(t) \|_{L^p} & \leq C e^{-(1-\alpha)t} t^{-\frac{3}{4} + \frac{1}{2p}} + C \int_0^t e^{-(1-\alpha)(t-s)} (t-s)^{-\frac{3}{4} + \frac{1}{2p}} e^{-\frac{1}{2}s} ds \\
& \quad + C \int_0^t e^{-(1-\alpha)(t-s)} (t-s)^{-\frac{3}{4} + \frac{1}{2p}} e^{-(1-\alpha)s(1+s)^{-\frac{1}{4}} ds} \tag{4.29}
\end{align*}

and

\begin{align*}
\| \partial_x v(t) \|_{L^p} & \leq C e^{-(1-\alpha)t} t^{-\frac{3}{4} + \frac{1}{2p}} + C \int_0^t e^{-(1-\alpha)(t-s)} (t-s)^{-\frac{3}{4} + \frac{1}{2p}} e^{-\frac{1}{2}s} ds \\
& \quad + C \int_0^t e^{-(1-\alpha)(t-s)} (t-s)^{-\frac{3}{4} + \frac{1}{2p}} e^{-(1-\alpha)s(1+s)^{-\frac{1}{4}} ds} \tag{4.30}
\end{align*}

Next we only prove that

\[ \int_0^t e^{-(1-\alpha)(t-s)} (t-s)^{-\frac{3}{4} + \frac{1}{2p}} e^{-\frac{1}{2}s} ds \leq C e^{-\frac{1}{2}t}. \]

Other integrals can be controlled in the same way. In fact, since \( 1 - \alpha > l/2 \), it holds that

\begin{align*}
\int_0^t e^{-(1-\alpha)(t-s)} (t-s)^{-\frac{3}{4} + \frac{1}{2p}} e^{-\frac{1}{2}s} ds & = e^{-\frac{1}{2}t} \int_0^t e^{-(1-\alpha-\frac{1}{2})x} x^{-\frac{3}{4} + \frac{1}{2p}} dx \\
& \leq C e^{-\frac{1}{2}t} \int_0^\infty x^{-\frac{3}{4} + \frac{1}{2p}} e^{-x} dx \\
& \leq C e^{-\frac{1}{2}t}.
\end{align*}

Thus it follows from (4.29) and (4.30) that for any \( t \geq \tau > 0 \),

\[ \| \partial_x u(t) \|_{L^p} \leq C e^{-(1-\alpha)t} t^{-\frac{3}{4} + \frac{1}{2p}} + C e^{-\frac{1}{2}t} + C e^{-(1-\alpha)t} t^{\frac{1}{4} + \frac{1}{2p}} \leq C(\tau) e^{-\frac{1}{2}t} \tag{4.31} \]

and

\[ \| \partial_x v(t) \|_{L^p} \leq C e^{-(1-\alpha)t} t^{-\frac{3}{4} + \frac{1}{2p}} + C e^{-\frac{1}{2}t} + C e^{-\min(1-\alpha,l)t} t^{\frac{1}{4} + \frac{1}{2p}} + C e^{-(1-\alpha)t} t^{\frac{1}{4} + \frac{1}{2p}} \]

\[ \leq C(\tau) e^{-\frac{1}{2}t}. \tag{4.32} \]

Hence combining (4.24), (4.25), (4.31) and (4.32) yields (3.9). The proof of Theorem 3.1 is completed.
5. Discussions

In this paper, we have established the global existence for the initial boundary value problem of system (1.1) on a quadrant, and the exponential decay rates of deviation from diffusion waves. This approach applies equally well to the general system (1.4) [15, 16]. As a matter of fact, as long as dissipation terms dominate, diffusion waves are stable in general. Furthermore, our approach also applies to a companion system of (1.4), called conservative form in [7]:

\[
\begin{align*}
\psi_t &= - (\sigma - \alpha) \psi - \sigma \theta_x + \alpha \psi_{xx}, \\
\theta_t &= -(1 - \beta) \theta + \nu \psi_x + (\psi \theta)_x + \beta \theta_{xx},
\end{align*}
\]

Though numerical results have demonstrated the drastic difference between system (5.1) and (1.4), in dissipation dominating regime of parameters, both system converges to diffusion waves in the same manner.

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References


