MATH 2010 Chapter 9

9.1 Application of Chain Rule

9.1.1 Level Sets

Let

$$f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}, c \in \mathbb{R}$$

Recall that the level set of f corresponding to $c \in \mathbb{R}$ is:

$$L_c = f^{-1}(c) = \{x \in \Omega : f(x) = c\}$$

Example 9.1. Some level sets of $f(x, y) = x^2 + y^2$:

$$f^{-1}(1) = \{x^2 + y^2 = 1\}$$

$$f^{-1}(4) = \{x^2 + y^2 = 4\}$$

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Theorem 9.2. *Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}, \Omega$ is open,*
Let $c \in \mathbb{R}, S = f^{-1}(c)$ and $a \in S$.
Suppose f is differentiable at a, and $\nabla f(a) \neq 0$. Then, $\nabla f(a) \perp S$ at a.

Example 9.3.

$$f(x, y) = x^2 + y^2 \quad \nabla f = (2x, 2y)$$

Let $S = f^{-1}(25)$, then $(4, 3) \in S$
$\nabla f(4, 3) = (8, 6)$
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Example 9.4. $f(x, y) = x^2 - y^2 \quad \nabla f(x, y) = (2x, -2y)$
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Example 9.5.

$$S : x^2 + 4y^2 + 9z^2 = 22 \quad \text{(Ellipsoid)}$$

Find equation of tangent plane of S at $(3, 1, 1)$

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Solution. Let $f(x, y, z) = x^2 + 4y^2 + 9z^2, S = f^{-1}(22)$

Also $f(3, 1, 1) = 22$, so $(3, 1, 1) \in S$

$\nabla f = (2x, 8y, 18z)$

$\nabla f(3, 1, 1) = (6, 8, 18) \perp S$ at $(3, 1, 1)$

$\therefore$ $(6, 8, 18)$ is a normal vector for the tangent plane. Equation of the tangent plane:

$$\begin{aligned}
[(x, y, z) - (3, 1, 1)] \cdot (6, 8, 18) = 0 \\
6(x - 3) + 8(y - 1) + 18(z - 1) = 0 \\
3x + 4y + 9z = 22
\end{aligned}$$

Proof of Example 9.5. Let $r(t)$ be a curve on S , $r(0) = a$.

Then $r(t)$ on $S = f^{-1}(c)$

$$\Rightarrow f(r(t)) = c \text{ is a constant}$$

By the chain rule,

$$\nabla f(r(t)) \cdot r'(t) = \frac{df}{dt} = 0$$

Put $t = 0$, then $\nabla f(a) \cdot r'(0) = 0$

$\therefore$ $\nabla f(a) \perp$ any curve on S at a .

$\therefore$ $\nabla f(a) \perp S$ at a . $\square$

9.1.2 Implicit Differentiation

Consider the curve:

$$C : \quad x^2 + y^2 = 1$$

Find $\frac{dy}{dx}$ at $(\frac{3}{5}, -\frac{4}{5})$. Locally near $(\frac{3}{5}, -\frac{4}{5})$, we have:

$$y^2 = 1 - x^2, y < 0 \Rightarrow y = -\sqrt{1 - x^2}$$

$\therefore$ y is a function of x near $(\frac{3}{5}, -\frac{4}{5})$.

To find $\frac{dy}{dx}$ at $(\frac{3}{5}, -\frac{4}{5})$,

Method 1: Compute:

$$\frac{d}{dx}\left(-\sqrt{1 - x^2}\right)$$

Method 2: Implicit Differentiation (chain rule)

$$x^2 + y^2 = 1 \left(\begin{array}{c} \text{Regard } x \text{ as a variable} \\ y \text{ as a function of } x \end{array} \right)$$

Differentiating both sides:

$$\frac{d}{dx} : 2x + 2y\frac{dy}{dx} = 0$$

Evaluating at $(x, y) = (\frac{3}{5}, -\frac{4}{5})$, we have:

$$2(\frac{3}{5}) + 2(-\frac{4}{5})\frac{dy}{dx} = 0$$

$\therefore \left.\frac{dy}{dx}\right|_{(\frac{3}{5},-\frac{4}{5})} = \frac{3}{4}$

Example 9.6. Consider

$$S : x^3 + z^2 + ye^{xz} + z\cos y = 0 \quad \circledast$$

Given that z can be regarded as a function $z = z(x, y)$ of independent variables x, y locally near $(0, 0, 0)$.

Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(0, 0, 0)$.

Remark. It is not easy to express z in terms of x, y .

Solution. Take $\frac{\partial}{\partial x}$ to $\circledast$,

$$3x^2 + 2z\frac{\partial z}{\partial x} + ye^{xz}(z + x\frac{\partial z}{\partial x}) + \frac{\partial z}{\partial x}\cos y = 0$$

Substitute $(x, y, z) = (0, 0, 0)$,

$$0 + 0 + 0 + \frac{\partial z}{\partial x}(1) = 0 \Rightarrow \boxed{\frac{\partial z}{\partial x}(0, 0) = 0}$$

Similarly, take $\frac{\partial}{\partial y}$ to $\circledast$

$$0 + 2z\frac{\partial z}{\partial y} + e^{xz} + ye^{xz}(x\frac{\partial z}{\partial y}) + \frac{\partial z}{\partial y}\cos y - z\sin y = 0$$

Substitute $(x, y, z) = (0, 0, 0)$, then

$$0 + 0 + 1 + 0 + \frac{\partial z}{\partial y}(1) - 0 = 0 \Rightarrow \boxed{\frac{\partial z}{\partial y}(0, 0) = -1}$$

Remark. From computations above, we have:

$$\frac{\partial z}{\partial x} = -\frac{3x^2 + yze^{xz}}{2z + xye^{xz} + \cos y}$$

$$\frac{\partial z}{\partial y} = \frac{z\sin y - e^{xz}}{2z + xye^{xz} + \cos y}$$

whenever the denominator is non-zero.

9.2 Finding Extrema (Maximum or Minimum)

Definition 9.7. Let:

$$f : A \subseteq \mathbb{R}^n \to \mathbb{R}, \quad a \in A.$$

1. The function f is said to have **global (absolute) maximum** at a if:

$$f(x) \leq f(a)$$

   for all $x \in A$.

2. The function f is said to have **local (relative) maximum** at a if:

$$f(x) \leq f(a)$$

   for all $x \in A$ near a, (i.e. There exists $\epsilon > 0$ such that $f(x) \leq f(a)$ for all $x \in A \cap B_\varepsilon(a)$.)

3. **Global (absolute) minimum** and **local (relative) minimum** are defined similarly.

Remark. Any global extremum (max/min) is also a local extremum.

A function does not necessarily have a global maximum/minimum.

Example 9.8. Let $f(x) = e^x$: on $\mathbb{R}$

$$\lim_{x \to -\infty} f(x) = 0.$$

$$\lim_{x \to \infty} f(x) = \infty.$$

But $f(x) > 0$ for all $x \in \mathbb{R}$. Hence, f has neither global maximum nor global minimum.

Example 9.9. Let $f(x) = x$ on $(-1, 1]$ (Domain is not closed).
Then f attains its global maximum at $x = 1$, but it has no global minimum.

Example 9.10. Let:

$$f : [-1, 1] \to \mathbb{R}$$

$$f(x) = \begin{cases} 1 - x & \text{if} \quad x \in (0, 1] \\ 0 & \text{if} \quad x = 0 \\ -1 - x & \text{if} \quad x \in [-1, 0) \end{cases}$$

The function f has neither global maximum nor global minimum.
(f is not continuous)

Question: When must a function have global extremum?

Theorem 9.11 (Extreme Value Theorem EVT). *Let A be closed and bounded subset of $\mathbb{R}^n$. Let $f : A \to \mathbb{R}$ be a continuous function.*

Then f has a global maximum and a global minimum.

Remark.

1. A closed and bounded subset of $\mathbb{R}^n$ is said to be **compact**.
2. The theorem provides a sufficient, but not necessary, condition for the existence of global extrema.

Example 9.12. Let:

$$f : A = [0, 4] \to \mathbb{R},$$

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Observe that A is closed and bounded, and f is continuous.

Hence, by Extreme Value Theorem (EVT), the function f has a global maximum and a global minimum on A.

Recall that in one-variable calculus: local extrema can only occur at:

1. Critical points (i.e. points a in the interior of the domain where $f'(a) = 0$ or DNE.)
2. The boundary points of the domain.

Definition 9.13. Let:

$$f : A \to \mathbb{R}, \quad a \in \text{Int}(A).$$

Then, a is called a **critical point** of f if either of the following conditions holds:

1. $\nabla f(a)$ DNE (i.e. $\frac{\partial f}{\partial x_i}(a)$ DNE for some i)
2. $\nabla f(a) = \vec{0}$ (i.e. $\frac{\partial f}{\partial x_i}(a) = 0$ for all i)

Theorem 9.14. *Suppose $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ attains a local extremum at $a \in \text{Int}(A)$, then a is a critical point of f.*

Proof of Theorem 9.14. Suppose f has a local extremum at $a \in \text{Int}(A)$.

If $\nabla f(a)$ DNE, then a is a critical point.

If $\nabla f(a)$ exists, then all $\frac{\partial f}{\partial x_i}(a)$ exist.

For any $i = 1, \cdots, n$, let:

$$g_i(t) = f(a + te_i)$$

Note that $a \in \text{Int}(A)$ implies that $g_i(t)$ is defined for t near 0.

By assumption, $g_i'(0) = \frac{\partial f}{\partial x_i}(a)$ exists.
Hence, f has a local extremum at a.
This implies that g_i has a local extremum at 0.
This in turn implies that $g_i'(0) = 0$ since by assumption $g_i'(0)$ exists.
We conclude that:

$$\frac{\partial f}{\partial x_i}(a) = 0 \quad (\text{for all } i = 1, 2, \cdots, n).$$

Hence, $\nabla f(a) = \vec{0}$. So, a is a critical point. □

9.2.1 Finding Extrema on a Bounded Region

Strategy for finding extrema:
Given:

$$f : A \subseteq \mathbb{R}^n \to \mathbb{R}.$$

To find the extrema of f:

1. Find critical points of f in $\mathrm{Int}(A)$.
2. Consider the restriction of f to the boundary ∂A of A.
   Find maximum/minimum of f on ∂A
3. Comparing values of f at points found in 1. and 2.

Example 9.15. Find global maximum/minimum of

$$f(x, y) = x^2 + 2y^2 - x + 3$$

on the region:

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

Remark. The region A is is closed and bounded.
Moreover, since f is polynomial, it is continuous.
So, by Extreme Value Theorem (EVT), the function f has global maximum and minimum on A.

Solution. We follow the strategy above:
Step 1 Consider the critical points of f in $\mathrm{Int}(A)$:
First, notice that $\nabla f = (2x - 1, 4y)$ exists everywhere. Moreover:

$$\begin{aligned}\nabla f = \vec{0} \;&\Leftrightarrow\; \begin{cases} 2x - 1 = 0 \\ 4y = 0 \end{cases} \\ &\Leftrightarrow\; (x, y) = \left(\frac{1}{2}, 0\right)\end{aligned}$$

Also, $(\frac{1}{2}, 0)$ lies in $\mathrm{Int}(A) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

So, we conclude that f has only one critical point $\left(\frac{1}{2}, 0\right)$ in $\mathrm{Int}(A)$, with:

$$f\left(\frac{1}{2}, 0\right) = \left(\frac{1}{2}\right)^2 + 0 - \frac{1}{2} + 3 = \frac{11}{4}$$

Step 2 Consider f on $\partial A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Parametrize ∂A as follows:

$$(x, y) = (\cos\theta, \sin\theta), \theta \in [0, 2\pi]$$

$$\begin{aligned} f(\cos\theta, \sin\theta) &= \cos^2\theta + 2\sin^2\theta - \cos\theta + 3 \\ &= \cos^2\theta + 2(1 - \cos^2\theta) - \cos\theta + 3 \\ &= -\cos^2\theta - \cos\theta + 5 \\ &= -(\cos\theta + \frac{1}{2})^2 + \frac{1}{4} + 5 \\ &= \frac{21}{4} - (\cos\theta + \frac{1}{2})^2 \end{aligned}$$

Maximum value of f on ∂A is $\frac{21}{4}$. It is attained when:

$$x = \cos\theta = -\frac{1}{2} \quad \text{i.e. } (x, y) = (-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$$

Minimum value of f on ∂A is 3. It is attained when:

$$x = \cos\theta = 1, \quad \text{i.e. } (x, y) = (1, 0)$$

Step 3 Reviewing the values of f at the points obtained in Steps 1 and 2, we have:

$$\begin{aligned} f\left(\frac{1}{2}, 0\right) &= \frac{11}{4} \quad \text{(minimum)} \\ f\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) &= \frac{21}{4} \quad \text{(maximum)} \\ f(1, 0) &= 3 \end{aligned}$$

Hence, the maximum value of f is $\frac{21}{4}$. It is attained at $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

The minimum value of f is $\frac{11}{4}$. It is attained at $\left(\frac{1}{2}, 0\right)$.

Example 9.16. Find the global extrema of

$$f(x, y) = \sqrt{x^2 + y^4} - y$$

on $R = \{(x, y) \in \mathbb{R}^2 : -1 \le x, y \le 1\}$

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Solution. R is the square $[-1,1] \times [-1,1] \subseteq \mathbb{R}^2$

R is closed and bounded.

Also, f is continuous.

By Extreme Value Theorem (EVT), the function f has a global maximum and a global minimum.

First, observe that: $\operatorname{Int}(R) = \{(x,y) \in \mathbb{R}^2, -1 < x, y < 1\}$

Exercise : Show that $\frac{\partial f}{\partial x}(0,0)$ DNE. (Hint: $(f(x,0)) = |x|$).)

For$(x,y) \neq (0,0)$, the gradient ∇f exists, with:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{x}{\sqrt{x^2+y^4}}, \frac{2y^3}{\sqrt{x^2+y^4}} - 1\right)$$

Hence:

$$\nabla f = (0,0) \Leftrightarrow \begin{cases} \frac{x}{\sqrt{x^2+y^4}} = 0 \\ \frac{2y^3}{\sqrt{x^2+y^4}} - 1 = 0 \end{cases}$$

Hence, $\nabla f(x,y) = \vec{0}$ if and only if: $x = 0$, and

$$\frac{2y^3}{y^2} - 1 = 0,$$

which holds if and only if $y = \frac{1}{2}$.

Therefore, f has two critical points in $\operatorname{Int}(R)$:

$$\underbrace{(0,0)}_{\nabla f \text{ DNE}}, \quad \underbrace{\left(0, \frac{1}{2}\right)}_{\nabla f = \vec{0}}$$

Note that:

$$f(0,0) = 0, \quad f\left(0, \frac{1}{2}\right) = -\frac{1}{4}$$

Consider f on ∂R:

$$f(x,y) = \sqrt{x^2+y^4} - y$$

$$\partial R = \{(x,y) : |x| = 1, -1 \leq y \leq 1\} \cup \{(x,y) : |y| = 1, -1 \leq x \leq 1\}$$

Consider different parts of ∂R:

1. $y = 1, -1 \leq x \leq 1$

$$f(x,1) = \sqrt{x^2+1} - 1 \Rightarrow 0 \leq f \leq \sqrt{2} - 1$$

2. $y = -1, -1 \leq x \leq 1$

$$f(x,-1) = \sqrt{x^2+1} + 1 \Rightarrow 2 \leq f \leq \sqrt{2} + 1$$

3. $|x| = 1, -1 \leq y \leq 1$

$$f(x,y) = \sqrt{1+y^4} - y.$$

If $-1 \leq y \leq 1$, then $1 \leq \sqrt{1+y^4} \leq \sqrt{2}$, and $-1 \leq -y \leq 1$. Hence:

$$0 = 1 - 1 \leq \sqrt{1+y^4} - y \leq \sqrt{2} + 1.$$

Restricted to $C = \{(x,y) \mid |x| = 1, -1 \leq y \leq 1\}$, the maximum value of $f(x,y)$ is therefore $f(\pm 1, -1) = \sqrt{2} + 1$.

Since we already know that $f(0,1) = 0$, which is less than all possible values of f restricted to C. The exact minimum of f on C is of little interest to us.

Hence, on ∂R, the function f has a minimum value of 0 at $(0,1)$, and a maximum value of $\sqrt{2}+1$ at $(\pm 1, -1)$.

Comparing values of f at points obtained in Steps 1 and 2:

$$\begin{aligned} f(0,0) &= 0 \\ f(0,\frac{1}{2}) &= -\frac{1}{4} \quad \text{(minimum)} \\ f(0,1) &= 0 \\ f(\pm 1,-1) &= \sqrt{2}+1 \quad \text{(maximum)} \\ f(\pm 1,1) &= \sqrt{2}-1, \end{aligned}$$

we conclude that the maximum value of f is $\sqrt{2}+1$, attained at $(\pm 1, -1)$, and the minimum value is $-\frac{1}{4}$, attained at $(0, \frac{1}{2})$.

9.2.2 Finding Extrema on an Unbounded Region

Example 9.17. Find the global extrema of

$$f(x,y) = x^2 + y^2 - 4x + 6y + 7$$

on $\mathbb{R}^2$.

Remark. $\mathbb{R}^2$ is not bounded. So f might not have global extrema. Observe that:

$$\lim_{\underbrace{(x,y)\to\infty}} f(x,y) = +\infty$$

"(x,y) are far away from origin."

Hence,

1. f has no global maximum on $\mathbb{R}^2$
2. Strategy for finding global minimum

   Find a closed and bounded region A such that f is "large enough" outside R. Then, the minimum of f on A is equal to the minimum of f of $\mathbb{R}$.

$$\text{min on } R = \text{ min on } \mathbb{R}^2$$

Solution. Find the critical points of f.

$$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (2x-4, 2y+6)$$

is defined everywhere on $\mathbb{R}^2$.

$$\begin{aligned} \nabla f = (0,0) &\Leftrightarrow \begin{cases} 2x-4=0 \\ 2y+6=0 \end{cases} \\ &\Leftrightarrow (x,y) = (2,-3) \end{aligned}$$

Hence, the function f has only one critical point $(2,-3)$, with $f(2,-3) = -6$.
We want to show that f has a global minimum at $(2,-3)$:
For $(x,y) \in \mathbb{R}^2$, let $r = \sqrt{x^2+y^2}$

$$\begin{aligned} \text{Then } f(x,y) &= x^2+y^2-4x+6y+7 \\ &\geq r^2-4r-6r+7 \\ &= r(r-10)+7. \end{aligned}$$

This is because:

$$r = \sqrt{x^2+y^2} \geq |x|, |y|,$$

which implies:

$$\begin{cases} x \leq r \Rightarrow -4x \geq -4r \\ -y \leq r \Rightarrow 6y \geq -6r \end{cases}$$

Hence, if $\sqrt{x^2+y^2} = r \geq 10$, then $f(x,y) \geq 7 > f(2,-3)$.

Let $A = \overline{B_{10}(0,0)}$. Let $f|_R$ denote the restriction of f on R.

By Extreme Value Theorem (EVT) , the function $f|_R$ has global a minimum.

In $\mathrm{Int}(R)$, the point $(2,-3)$ is the only critical point, with:

$$f(2,-3) = -6$$

On ∂A, we have $f(x,y) \geq 7 > f(2,-3)$. Hence, $f|_R$ has a global minimum at $(2,-3)$.

For $(x,y) \notin R$, we have $f(x,y) \geq 7 > f(2,-3)$. Hence, f has no global maximum, but it has a global minimum value of -6 at $(2,-3)$.

Remark. 1. It is in fact easier to solve this problem using elementary algebra:

Since

$$\begin{aligned} f(x,y) &= x^2+y^2-4x+6y+7 \\ &= (x-2)^2+(y+3)^2-6, \end{aligned}$$

it is quite clear what the global minimum of f is.

2. A function can have neither global maximum nor global minimum.

For example, let $g(x,y) = x^2-y^2-4x+6y+7$

Along the line $x=0$, we have $g(0,y) = -y^2+6y+7$. So:

$$\lim_{y\to\pm\infty} g(0,y) = -\infty \Rightarrow \text{ no global minimum.}$$

Along $y=0$, we have $g(x,0) = x^2-4x+7$. So:

$$\lim_{x\to\pm\infty} g(x,0) = \infty \Rightarrow \text{ no global maximum.}$$

Another example of extrema on unbounded region.

Example 9.18. Make a box (without top) with volume $= 16$

Cost:

Base \$2/unit area

Side \$0.5/unit area

Question :

How to minimize cost?

Solution. Want to minimize

$$\begin{aligned} C(x,y) &= 2xy + (\frac{16}{xy}x + \frac{16}{xy}y)(2)(0.5) \\ &= 2xy + \frac{16}{x} + \frac{16}{y} \end{aligned}$$

on the domain $\Omega = \{(x,y) \in \mathbb{R}^2 : x,y > 0\}$

- Ω is neither closed nor bounded. Hence, Extreme Value Theorem (EVT) cannot be applied directly.
- C is large if x or y is small or large.

Strategy: Find a rectangle R such that the values of $C|_{\mathbb{R}^2\setminus R}$ are all greater than the minimum of $C|_R$.

Step 1

Find critical points

$$\nabla C = (2y - \frac{16}{x^2}, 2x - \frac{16}{y^2}) \text{ exists everywhere}$$

$$\nabla C = \vec{0} \Leftrightarrow \begin{cases} 2y - \frac{16}{x^2} = 0 \\ 2x - \frac{16}{y^2} = 0 \end{cases}$$

Hence, $y = \frac{8}{x^2}, x = \frac{8}{y^2} = \frac{8}{\frac{64}{x^4}} = \frac{x^4}{8}, x > 0 \Rightarrow x^3 = 8, x = 2, y = 2$ Hence, Only one critical point $(2, 2), C(2, 2) = 24$.

Step 2

Choose R s.t. $C > 24$ on ∂R and outside R.

$$C(x, y) = 2xy + \frac{16}{x} + \frac{16}{y}$$

One possible choice: $R = [0.1, 1000] \times [0.1, 1000]$

- If $x \leq 0.1$ or $y \leq 0.1$,

$$\text{then } C > \frac{16}{x} + \frac{16}{y} > \frac{16}{0.1} = 160 > 24$$

- If $(x \geq 0.1, y \geq 1000)$ or $(y \geq 0.1, x \geq 1000)$,

$$\text{then } C > 2(0.1)(1000) = 200 > 24$$

Step 3

Analysis

- R is closed and bounded, C is continuous.

  By Extreme Value Theorem (EVT), $C|_R$ has minimum.
- C has only one critical point $(2, 2) \in \Omega$ $(2, 2) \in \mathrm{Int}(R), C(2, 2) = 24$ $C > 24$ on $\partial R \Rightarrow C|_R$ has minimum value 24 at $(2, 2)$
- $C > 24$ outside R

Hence, C has the minimum value of 24 at $(2, 2)$ on Ω.