Math 2010 Chapter 6

6.1 Differentiability

Definition 6.1. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and $f : \Omega \longrightarrow \mathbb{R}$ a real-valued function on Ω. Let r be a non-negative integer.

The function f is said to be a C^r **function** if all partial derivatives of f up to order r exist and are continuous on Ω.

The function f is said to be a C^∞ **function** if it is C^r for any $r \geq 0$.

Example 6.2.

- f is C^0 is if it continuous.
- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is C^2 if:

$$f,\ f_x,\ f_y,\ f_{xx},\ f_{xy},\ f_{yx},\ f_{yy}$$

are all defined and continuous everywhere.

Polynomials, rational functions, exponentials, logarithms, trigonometric functions, and their sum/difference/product quotient/compositions are all C^∞ functions on any open set where all partial derivatives of all orders are defined.

For example:

$$f(x, y) = e^{x^2 - y} \sin \frac{x}{y}$$

is C^∞ on:

$$\Omega = \big\{(x, y) \in \mathbb{R}^2 : y \neq 0\big\}$$

Theorem 6.3 (Generalization of Clairaut's Theorem)**.** *Let r be a non-negative integer. If a function f is C^r on an open set $\Omega \subseteq \mathbb{R}^n$, then the order of differentiation does not matter for all partial derivatives of order up to r.*

Example 6.4. If $f(x, y, z)$ is C^3, then:

$$f_{xz} = f_{zx}, \quad f_{xyz} = f_{yzx} = f_{zyx}, \quad f_{xxy} = f_{xyx} = f_{yxx}$$

6.1.1 Differentiability for Functions in One Variable

Recall the following definition of differentiability in one-variable calculus: A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable at a if:

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. In geometric terms, this means that the graph $y = f(x)$ of f "resembles" the line $y = L(x)$ with slope $f'(a)$ which passes through $(x, f(a))$:

$$f(x) \approx L(x) := f(a) + f'(a)(x - a)$$

for x "near" a.

The degree 1 polynomial $L(x)$ is called the **linear approximation** (or **linearization**) of f at $x = a$. The error of the approximation is simply the difference:

$$\varepsilon(x) = f(x) - L(x) = f(x) - f(a) - f'(a)\underbrace{(x - a)}_{\Delta x}.$$

Observe that:

$$\frac{\varepsilon(x)}{x - a} = \frac{f(x) - f(a)}{x - a} - f'(a).$$

Hence,

$$\lim_{x \to a} \frac{\varepsilon(x)}{x - a} = f'(a) - f'(a) = 0,$$

which is equivalent to:

$$\lim_{x \to a} \frac{\varepsilon(x)}{|x - a|} = 0.$$

This motivates an equivalent formulation of differentiability for functions in one variable, namely:

A real-valued function f is differentiable if there exists a line $y = L(x)$ such that the "error of approximation" $\varepsilon(x) := f(x) - L(x)$ satisfies the condition:

$$\lim_{x \to a} \frac{\varepsilon(x)}{|x - a|} = 0.$$

The benefit of such a formulation is that it readily extends to a definition of differentiability for functions in multiple variables.

Consider a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is two variables. A possible formulation for the differentiability of f at (a, b) is as follows:

There exists a plane $L(x, y) = f(a, b) + C(x - a) + D(y - b)$ which well approximates $f(x, y)$ near $(x, y) = (a, b)$, in the sense that:

$$\lim_{(x,y) \to (a,b)} \frac{f(x, y) - L(x, y)}{\|(x - a, y - b)\|} = 0.$$

Notice that if the limit above exists, then the limit along every path towards (a, b) must be the same, in particular, fixing $y = b$ and letting $x \to a^+$:

$$\begin{aligned} 0 &= \lim_{(x,b)\to(a^+,b)} \frac{f(x,b) - L(x,y)}{\|(x-a, b-b)\|} \\ &= \lim_{x\to a^+} \frac{f(x,b) - L(x,b)}{|x-a|} \\ &= \lim_{x\to a^+} \frac{f(x,b) - L(x,b)}{x-a} \\ &= \lim_{x\to a^+} \frac{f(x,b) - f(a,b) - C(x-a)}{x-a} \\ &= \lim_{x\to a^+} \frac{f(x,b) - f(a,b)}{x-a} - C \end{aligned}$$

This implies that:

$$\lim_{x\to a^+} \frac{f(x,b) - f(a,b)}{x-a} = C$$

and likewise:

$$\lim_{x\to a^-} \frac{f(x,b) - f(a,b)}{x-a} = C$$

Hence, for the plane L to even have a *chance* to well approximate f near (a, b), the partial derivative:

$$f_x(a,b) = \lim_{x\to a} \frac{f(x,b) - f(a,b)}{x-a}$$

must exist, and the coefficient C must be equal to $f_x(a, b)$. Similarly, $f_y(a, b)$ must exist and be equal to D.

The only candidate for a plane which well approximates f near (a, b) is therefore:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b),$$

provided that $f_x(a, b)$ and$f_y(a, b)$ both exist.

Note that this is a *necessary* but *not sufficient* condition for f to be well approximated by a plane near (a, b).

6.1.2 Differentiability for Function in Multiple Variables

Definition 6.5. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $\vec{a} = (a_1, a_2, \ldots, a_n) \in \Omega$. A function $f : \Omega \longrightarrow \mathbb{R}$ is said to be **differentiable** at $\vec{a}$ if:

- Each first order partial derivative $f_{x_i}(\vec{a})$ exists, for $i = 1, 2, \ldots, n$.

- For:
$$L(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} f_{x_i}(\vec{a})(x_i - a_i)$$
(i.e. L is the linear approximation of f at $\vec{a}$), and:
$$\varepsilon(\vec{x}) = f(\vec{x}) - L(\vec{x})$$
(i.e. "error" of the approximation), we have:
$$\lim_{\vec{x}\to\vec{a}} \frac{\varepsilon(\vec{x})}{\|\vec{x}-\vec{a}\|} = 0.$$

Remark. Observe that:

- $L(\vec{x})$ is a polynomial of degree ≥ 1.
- $L(\vec{a}) = f(\vec{a})$.
- $\dfrac{\partial L}{\partial x_i}(\vec{a}) = \dfrac{\partial f}{\partial x_i}(\vec{a})$, for $i = 1, 2, \ldots, n$.
- The graph $y = L(\vec{x})$ is the n-dimensional "tangent plane" to $y = f(\vec{x})$ at $(\vec{a}, f(\vec{a}))$, in exact analogy to the fact that $y = f(a) + f'(a)(x-a)$ is the tangent line to the graph $y = f(x)$ of a differentiable function f at $(a, f(a))$ in one-variable calculus.

Example 6.6. Let $f(x, y) = x^2 y$.

1. Show that f is differentiable at $(1, 2)$.
2. Approximate $f(1.1, 1.9)$ using linearization
3. Find tangent plane of $z = f(x, y)$ at $(1, 2, f(1, 2)) = (1, 2, 2)$.

Solution. 1. Since:
$$\frac{\partial f}{\partial x} = 2xy, \qquad \frac{\partial f}{\partial y} = x^2,$$
$$\frac{\partial f}{\partial x}(1,2) = 4, \qquad \frac{\partial f}{\partial y}(1,2) = 1,$$
the linearization of f at $(1, 2)$ is:
$$\begin{aligned} L(x,y) &= f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2) \\ &= 2 + 4(x-1) + (y-2) \end{aligned}$$

with error term:

$$\begin{aligned}\varepsilon(x,y) &= f(x,y) - L(x,y) \\ &= x^2y - 2 - 4(x-1) - (y-2)\end{aligned}$$

To show that f is differentiable at $(1,2)$, we compute the limit:

$$\lim_{(x,y)\to(1,2)} \frac{\varepsilon(x,y)}{\|(x,y)-(1,2)\|}$$

$$= \lim_{(x,y)\to(1,2)} \frac{x^2y - 2 - 4(x-1) - (y-2)}{\sqrt{(x-1)^2 + (y-2)^2}}$$

Let $h = x - 1, k = y - 2.$

$$= \lim_{(h,k)\to(0,0)} \frac{(1+h)^2(2+k) - 2 - 4h - k}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{h^2k + 2hk + 2h^2}{\sqrt{h^2+k^2}}$$

Let $h = r\cos\theta, k = r\sin\theta$

$$= \lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta + 2r^2\cos\theta\sin\theta + 2r^2\cos^2\theta}{r}$$

$$= \lim_{r\to 0} r^2\cos^2\theta\sin\theta + 2r\cos\theta\sin\theta + 2r\cos^2\theta$$

$= 0$ by Sandwich theorem

Hence, f is differentiable at $(1,2)$.

2. Using the linearization L of f at $(1, 2)$ we have:

$$f(1.1, 1.9) \approx L(1.1, 1.9)$$

$$= 2 + 4(1.1 - 1) + (1.9 - 2)$$

$$= 2 + 0.4 + (-0.1)$$

$$= 2.3$$

   Compare: $f(1, 1, 1.9) = 2.299$.

3. The tangent plane to $z = f(x, y)$ at $(1, 1, 2)$ is:

$$z = L(x, y)$$

$$= 2 + 4(x - 1) + (y - 2)$$

$$z = -4 + 4x + y$$

Exercise 6.7. Is $f(x, y) = \sqrt{|xy|}$ differentiable at $(0, 0)$?

Solution.

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

Similarly:

$$\frac{\partial f}{\partial y}(0, 0) = 0$$

Hence,

$$L(x, y) = f(0, 0) + \frac{\partial f}{\partial x}(0, 0)(x - 0) + \frac{\partial f}{\partial y}(0, 0)(y - 0)$$

$$= 0 + 0 + 0 = 0.$$

So, $L(x, y)$ is the zero function.

The error term is:

$$\varepsilon(x, y) = f(x, y) - L(x, y) = \sqrt{|xy|}$$

So,

$$\lim_{(x,y)\to(0,0)} \frac{\varepsilon(x,y)}{\|(x,y)-(0,0)\|} = \lim_{(x,y\to 0,0)} \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}$$

$$= \lim_{r\to 0} \frac{\sqrt{|r^2\cos\theta\sin\theta|}}{r}$$

$$= \lim_{r\to 0} \sqrt{|\cos\theta\sin\theta|},$$

which varies with θ. Hence, the limit does not exist.
We conclude that f is not differentiable at $(0,0)$.

Theorem 6.8. *If a real-valued function f in multiple variables is differentiable at $\vec{a}$, then f is continuous at $\vec{a}$.*

Proof of Theorem 6.8. Let L be the linear approximation of f at $\vec{a}$, that is:

$$f(\vec{x}) = L(\vec{x}) + \varepsilon(\vec{x})$$

By definition of differentiability, we have:

$$\lim_{\vec{x}\to\vec{a}} \frac{\varepsilon(\vec{x})}{\|\vec{x}-\vec{a}\|} = 0$$

Hence,

$$\begin{aligned}
\lim_{\vec{x}\to\vec{a}} \varepsilon(\vec{x}) &= \lim_{\vec{x}\to\vec{a}} \frac{\varepsilon(\vec{x})}{\|\vec{x}-\vec{a}\|} \cdot \|\vec{x}-\vec{a}\| \\
&= \lim_{\vec{x}\to\vec{a}} \frac{\varepsilon(\vec{x})}{\|\vec{x}-\vec{a}\|} \cdot \lim_{\vec{x}\to\vec{a}} \|\vec{x}-\vec{a}\| \\
&= 0 \cdot 0 = 0.
\end{aligned}$$

It now follows that:

$$\begin{aligned}
\lim_{\vec{x}\to\vec{a}} f(\vec{x}) &= \lim_{\vec{x}\to\vec{a}} L(\vec{x}) + \lim_{\vec{x}\to\vec{a}} \varepsilon(\vec{x}) \\
&= \lim_{\vec{x}\to\vec{a}} f(\vec{a} + \underbrace{\lim_{\vec{x}\to\vec{a}} \left(\sum_{i=1}^{n} f_{x_i}(\vec{a})(x_i - a_i) \right)}_{=0} + 0 \\
&= f(\vec{a}).
\end{aligned}$$

$\square$