MATH 2010 Chapter 2

2.1 Linear Objects in $\mathbb{R}^n$

In this section, we will study linear objects in $\mathbb{R}^n$. Typical examples are 1-dimensional lines and 2-dimensional planes. We will also look at their higher dimensional analog.

2.1.1 Line

Consider the line L passing through $A = (1, 0)$ and $B = (0, 2)$ in $\mathbb{R}^2$.

Two standard ways to represent L is

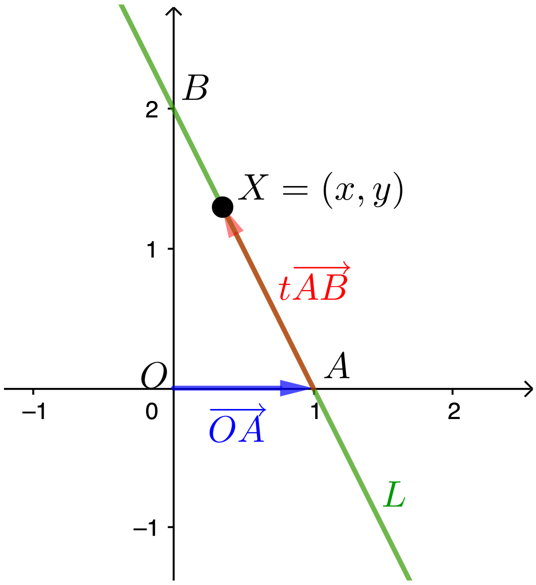
- **Equation form**

$$2x + y = 2$$

- **Parametric form**

$$\begin{aligned}(x, y) &= \overrightarrow{OA} + t\overrightarrow{AB} \\ &= (1, 0) + t(-1, 2) \\ &= (1 - t, 2t)\end{aligned}$$

  Varying $t \in \mathbb{R}$ gives all the points X on L.



- **Symmetric form** The parametric equation above implies

$$\begin{cases} x = 1 - t \\ y = 2t \end{cases} \Rightarrow \begin{cases} t = \dfrac{x-1}{-1} \\ t = \dfrac{y}{2} \end{cases}$$

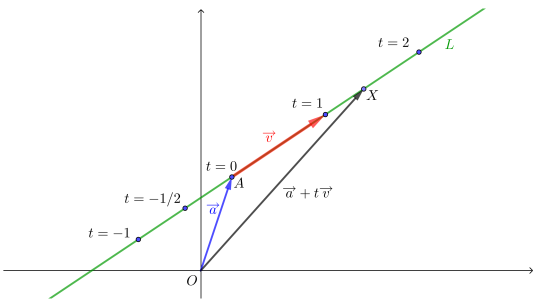
By eliminating t from the parametric form, we obtain another way to represent L. It is called the **symmetric form** :

$$\frac{x-1}{-1} = \frac{y-0}{2}$$

2.1.2 Parametric form of a line in $\mathbb{R}^n$

Let L be a line in $\mathbb{R}^n$. Let A be a point on it with $\vec{a} = \overrightarrow{OA}$ and $\vec{v}$ is a vector representing a direction of L. Then A parametric form of L is given by

$$\vec{x} = \vec{a} + t\vec{v}, \quad t \in \mathbb{R} \text{ is called a parameter}$$



L is said to be parametrized by $t \in \mathbb{R}$

Example 2.1. A line L passes through $A = (1, 2, 3)$ and $B = (-1, 3, 5)$. To find a parametric form of L, we can take

$$\vec{a} = (1, 2, 3) \quad \text{and} \quad \vec{v} = \overrightarrow{AB} = (-1-1, 3-2, 5-3) = (-2, 1, 2)$$

Hence, a parametric form is given by

$$(x, y, z) = (1, 2, 3) + t(-2, 1, 2)$$

Remark.

1. Parametric form is not unique. For instance,

   $$(x, y, z) = (-1, 3, 5) + t(2, -1, -2) \quad \text{and} \quad (x, y, z) = (-1, 3, 5) + t(-4, 2, 4)$$

   are two other parametrizations of L.

2. By eliminating t from the parametric equation, we get a symmetric form of L

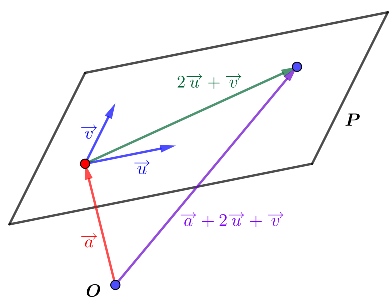
   $$\frac{x-1}{-2} = \frac{y-2}{1} = \frac{z-3}{2}$$

2.2 Planes in $\mathbb{R}^3$

A plane P in $\mathbb{R}^3$ can be uniquely determined by different sets of data, for example,

- 3 non-collinear points on P ; or
- A point on P and 2 linearly independent directions (not same or opposite) ; or
- A point on P and a normal vector

We will study how to represent a plane in equation or parametric form. Suppose P is a plane in $\mathbb{R}^3$, A is a point on it, $\vec{u}$ and $\vec{v}$ are two linearly independent directions of it. Let $\vec{a} = \overrightarrow{OA}$. Then the position vector of any point on P is given by the sum of $\vec{a}$ and a linear combination of $\vec{u}$ and $\vec{v}$.

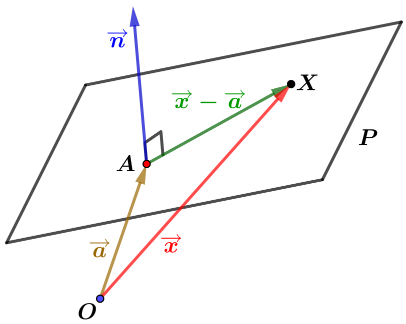


Hence, a parametric form of P can be given by

$$\vec{x} = \vec{a} + s\vec{u} + t\vec{v}$$

Here $s, t \in \mathbb{R}$ are parameters. By varying $s, t \in \mathbb{R}$, we obtain all the points on P. In another situation, suppose $\vec{a} = (a_1, a_2, a_3)$ is a point on P and $\vec{n} = (n_1, n_2, n_3)$ is a **normal vector** of P (that is, a vector perpendicular to the plane P). Let $\vec{x} = (x, y, z) \in \mathbb{R}^3$. Then:

$$\begin{aligned} \vec{x} \text{ is on } P \quad &\Longleftrightarrow \quad \vec{x} - \vec{a} \perp \vec{n} \\ &\Longleftrightarrow \quad (\vec{x} - \vec{a}) \cdot \vec{n} = \vec{0} \\ &\Longleftrightarrow \quad \vec{x} \cdot \vec{n} = \vec{a} \cdot \vec{n} \end{aligned}$$



The plane P can be described by the equation

$$n_1 x + n_2 y + n_3 z = a_1 n_1 + a_2 n_2 + a_3 n_3$$

Remark. If $(a, b, c) \neq \vec{0}$, the equation

$$ax + by + cz = d$$

describes a plane in $\mathbb{R}^3$ with normal vector (a, b, c).

Normal Vector
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Normal Vector as Cross Product
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Example 2.2. Suppose P is a plane passing through

$$A = (0, 0, 1), B = (0, 2, 0), C = (-1, 1, 0)$$

Represent P using parametric and equation form.

Solution. For parametric form,

$$\begin{aligned}\overrightarrow{AB} &= (0,2,0)-(0,0,1) = (0,2,-1)\\ \overrightarrow{AC} &= (-1,1,0)-(0,0,1) = (-1,1,-1)\end{aligned}$$

Hence

$$(x,y,z) = (0,0,1) + s(0,2,-1) + t(-1,1,-1)$$

To represent P by an equation, we take

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -1 \\ -1 & 1 & -1 \end{vmatrix} = (-1,1,2) \perp P$$

Then for any point (x,y,z) on P,

$$\begin{aligned}[(x,y,z)-(0,0,1)]\cdot(-1,1,2) &= 0\\ (-1)x + (1)y + 2(z-1) &= 0\\ -x+y+2z &= 2\end{aligned}$$

Example 2.3. Let two planes in $\mathbb{R}^3$ be given:

$$a_1x + b_1y + c_1z = d_1,$$

$$a_2x + b_2y + c_2z = d_2,$$

where $\vec{n}_i := \langle a_i, b_i, c_i\rangle \neq \vec{0}$ $(i = 1, 2)$.

Suppose $\vec{n}_1$ and $\vec{n}_2$ are not parallel to each other. Then, the two planes are non-parallel, and the intersection of the two planes is a line parallel to the vector $\vec{v} = \vec{n}_1 \times \vec{n}_2$. Note that the vector $\vec{v}$ is nonzero, since $\vec{n}_1$ and $\vec{n}_2$ are by assumption non-parallel.

Theorem 2.4. *Given a plane in $\mathbb{R}^3$ corresponding to:*

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0,$$

The (minimal) distance between a point $P \in \mathbb{R}^3$ and the plane is:

$$d = \left|\mathrm{Proj}_{\vec{n}}\overrightarrow{P_0P}\right| = \left|\overrightarrow{P_0P}\cdot\frac{\vec{n}}{|\vec{n}|}\right|,$$

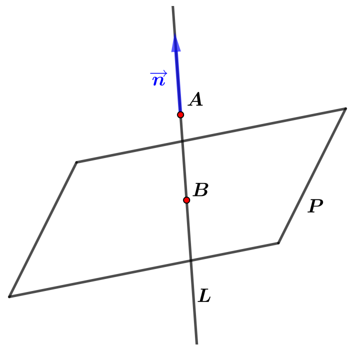
where $P_0 = (x_0, y_0, z_0)$ and $\vec{n} = \langle a, b, c\rangle$.

Example 2.5. Find the distance between $A = (2, 1, 1)$ and the plane P: $-x + 2y - z = -4$.

Solution. From the equation of P, $\vec{n} = (-1, 2, -1) \perp P$. Consider the line L defined by

$$\overrightarrow{X}(t) = \overrightarrow{A} + t\vec{n} = (2, 1, 1) + t(-1, 2, -1)$$

Let $B = L \cap P$ be the intersection of L and P.



Then B is the point of P closest to A. To find B, put:

$$\overrightarrow{X}(t) = (2 - t, 1 + 2t, 1 - t)$$

into the equation of P. Then:

$$-(2 - t) + 2(1 + 2t) - (1 - t) = -4 \Rightarrow 6t - 1 = -4 \Rightarrow 6t = -3 \Rightarrow t = -\frac{1}{2}$$

We have $B = \overrightarrow{X}(-\frac{1}{2}) = (\frac{5}{2}, 0, \frac{3}{2})$. The distance between A and P is

$$= \|\overrightarrow{AB}\| = \sqrt{\left(\frac{5}{2} - 2\right)^2 + (0 - 1)^2 + \left(\frac{3}{2} - 1\right)^2} = \frac{\sqrt{6}}{2}$$

Exercise 2.6. Find the distance between the lines

$$L_1(s) = (-4, 9, -4) + s(4, -3, 0)$$
$$L_2(t) = (5, 2, 10) + t(4, 3, 2)$$

Hint: Find A on L_1, B on L_2 such that $\overrightarrow{AB} \perp L_1, L_2$

2.2.1 Line in $\mathbb{R}^3$ by equations

Can we describe the a straight line in $\mathbb{R}^3$ by an equation? Note that each non-trivial linear equation in x, y, z can only represent a plane. At least two such equations are needed to describe a line. For instance,

Example 2.7. Consider the y-axis in $\mathbb{R}^3$. A point (x, y, z) is on the y-axis if and only if both the x and z coordinates are zero. Hence, y-axis can be described using the equations

$$\begin{cases} x = 0 \\ z = 0 \end{cases}$$

Geometrically, each of the equations $x = 0$ and $z = 0$ represents a plane in $\mathbb{R}^3$. The y-axis is the intersection of the two planes.

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Given a linear object, for example, a line or a plane, we can describe it using either parametric form or a system of linear equations. It is easy to convert between the two using linear algebra.

Example 2.8. Let L be the line represented by the system

$$\begin{cases} x + y + 6z = 6 \\ x - y - 2z = -2 \end{cases}$$

By Gaussian elimination,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}.$$

The solution describes L in parametric form.

Conversely, from the above parametric form

$$\begin{cases} x &= 2 - 2t \\ y &= 4 - 4t \\ z &= t \end{cases}$$

The first two equations imply $2x - y = 0$ while the last two imply $y = 4 - 4z$.

We obtain another set of linear equation representing L:

$$\begin{cases} 2x - y = 0 \\ y + 4z = 4 \end{cases}$$

2.2.2 Intersection of Planes

Example 2.9. Consider a system of three non-trivial equations of the form $ax + by + cz = d$. Each of them represents a plane in $\mathbb{R}^3$. What can be their intersections?

- Case 1: Unique solution
- Case 2: Infinitely many solutions
- Case 3: No solution

- All three planes are parallel to each other.

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- Only two planes are parallel to each other.

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- The intersection of each pair of planes is a line and three such lines are parallel to each other.

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- Their intersection is a line.

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- Their intersection is a point, e.g. the xy-plane, yz-plane and zx-plane intersect at $(0, 0, 0)$.

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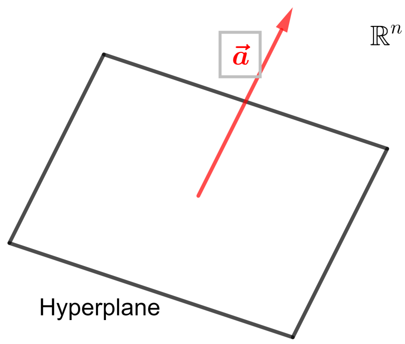
2.2.3 General linear objects in $\mathbb{R}^n$

Similar to lines in $\mathbb{R}^3$, we need a system of equations to describe a 2-dimensional plane in $\mathbb{R}^n$ when $n \geq 4$. Generally in $\mathbb{R}^n$, an equation of the form

$$\vec{a} \cdot \vec{x} = a_1x_1 + a_2x_2 + \cdots + a_nx_n = c, \;\; \vec{a} \neq \vec{0}$$

describes a hyperplane (dimension $= n - 1$) with normal vector $\vec{a}$:



A k-dimensional "plane" P (called k-plane) in $\mathbb{R}^n$ can be described in parametric form or by equation(s).

1. **Parametric form**

$$\vec{x} = \vec{q} + \sum_{i=1}^{k} t_i \vec{v}_i$$

where

- $\vec{q} \in P$
- $\vec{v}_1, \cdots, \vec{v}_k$ are k linearly independent vectors parallel to P
- $t_1, \cdots, t_k$ are parameters

2. $n - k$ **non-redundant equations**

$$\sum_{j=1}^{n} a_{ij} x_j = c_i \text{ for } i = 1, 2, \cdots, n-k$$

Here non-redundant means that the $(n-k) \times n$ coefficient matrix $A = (a_{ij})$ has rank $n - k$. The solution of the system of $n - k$ equations corresponds to the intersection of the $n - k$ hyperplanes.

2.3 Curves in $\mathbb{R}^n$

Definition 2.10. Let $I \subseteq \mathbb{R}$ be an interval.
A **curve** in $\mathbb{R}^n$ is a continuous function:

$$\vec{x} : I \longrightarrow \mathbb{R}^n$$

That is, $\vec{x}$ is defined as:

$$\vec{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)), \quad t \in \mathbb{R}$$

where x_i is a continuous real-valued function on I for each i.

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Example 2.11. Let $\vec{v} : [-1, 1) \longrightarrow \mathbb{R}^2$ be defined by $\vec{v}(t) = (t^2, t)$. Then $y^2 = t^2 = x$ and the curve lies on the parabola $x = y^2$.

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Example 2.12. Let $\vec{p}, \vec{q} \in \mathbb{R}^3, \vec{q} \neq \vec{0}$. Define $\vec{x} : \mathbb{R} \longrightarrow \mathbb{R}^3$ by $\vec{x}(t) = \vec{p} + t\vec{q}$. Then $\vec{x}(t)$ is a straight line.

Definition 2.13. A curve $\vec{x} : [a, b] \longrightarrow \mathbb{R}^n$ is said to be:

- **closed** if $\vec{x}(a) = \vec{x}(b)$.
- **simple** if $\vec{x}(t_1) \neq \vec{x}(t_2)$ for any $a \leq t_1 < t_2 \leq b$, except possibly at $t_1 = a, t_2 = b$.

Example 2.14.

$$\vec{x} : [1, \infty) \longrightarrow \mathbb{R}^2,$$
$$\vec{x}(t) = \left(\frac{1}{t}, \frac{1}{t^2}\right), \quad t \in \mathbb{R}.$$

Definition 2.15. Let $\vec{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$, where x_i are real-valued functions. The **derivative** of $\vec{x}$ at t is:

$$\vec{x}'(t) = \lim_{h \to 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h}.$$

For any a in the domain of $\vec{x}$, if $\vec{x}'(a)$ exists, then $\vec{x}'(a)$ is called the **tangent vector** of $\vec{x}$ at $t = a$.

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Theorem 2.16. *Let $\vec{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$. Then:*

- $$\lim_{t \to a} \vec{x}(t) = \left(\lim_{t \to a} x_1(t), \lim_{t \to a} x_2(t), \ldots, \lim_{t \to a} x_n(t)\right)$$
- *If $\vec{x}'(t)$ exists, then each x_i is differentiable at t, and:*

  $$\vec{x}'(t) = (x_1'(t), x_2'(t), \ldots, x_n'(t)).$$

In physics, if we let $\vec{x}(t)$ be the **displacement** (position) of a moving particle at time t. Then:

- $\vec{x}'(t)$ is the **velocity** of the particle at time t.
- $\vec{x}''(t) = (\vec{x}')'(t)$ is the **acceleration** of the particle at time t.

Example 2.17.

$$\vec{x}(t) = (\cos t, \sin t), 0 \leqslant t \leqslant 2\pi$$

$$\begin{aligned} \vec{v}(t) &= \vec{x}'(t) = (-\sin t, \cos t) \perp \vec{x}(t) \\ \vec{a}(t) &= \vec{x}''(t) = (-\cos t, -\sin t) = -\vec{x}(t) \end{aligned}$$

Also speed $= \|\vec{v}(t)\| = 1$

Example 2.18. Let $\vec{x} : [1, \infty) \to \mathbb{R}^2$ be defined by

$$\vec{x}(t) = \left(\frac{1}{t}, \frac{1}{t^2} \right).$$

Then:

$$\begin{aligned} \lim_{t \to \infty} \vec{x}(t) &= \left(\lim_{t \to \infty} \frac{1}{t}, \lim_{t \to \infty} \frac{1}{t^2} \right) \\ &= (0, 0) \end{aligned}$$

Theorem 2.19. *Let $\vec{x}(t), \vec{y}(t)$ be curves in $\mathbb{R}^n$, and $c \in \mathbb{R}$ a scalar constant. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a real-valued function.*

1. $(\vec{x} \pm \vec{y})'(t) = \vec{x}'(t) \pm \vec{y}'(t)$.
2. $(c\vec{x}(t))' = c\vec{x}'(t)$.
3. $(f(t)\vec{x}(t))' = f'(t)\vec{x}(t) + f(t)\vec{x}'(t)$.
4. $(\vec{x}(t) \cdot \vec{y}(t))' = \vec{x}'(t) \cdot \vec{y}(t) + \vec{x}(t) \cdot \vec{y}'(t)$.
5. *If* $n = 3$,
   $(\vec{x}(t) \times \vec{y}(t))' = \vec{x}'(t) \times \vec{y}(t) + \vec{x}(t) \times \vec{y}'(t)$.

2.4 Arclength

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Definition 2.20. Let $\vec{x} : [a, b] \longrightarrow \mathbb{R}^n$ be a curve such that $\vec{x}'$ exists and is continuous on (a, b).

The **arclength** of $\vec{x}$ on $[a, b]$ is:

$$S = \int_a^b \|\vec{x}'(t)\| dt$$

Remark. In physics, if $\vec{x}(t)$ is the displacement of a moving particle at time t, then the arclength of $\vec{x}$ on $[a, b]$ is the **distance travelled** by the particle over the time period $[a, b]$.

If $\vec{x}(t) =$ displacement at time t.
Then, $\vec{x}'(t) =$ velocity
and $\|\vec{x}'(t)\| =$ speed.
$\int_a^b \|\vec{x}'(t)\| dt =$ distance travelled.
From a mathematical point of view, approximate a curve by line segments:
Take: $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. Then,

$$\begin{aligned} S &\approx \sum_{i=1}^{n} \|\vec{x}(t_i) - \vec{x}(t_{i-1})\| \qquad \left(\text{Recall } \vec{x}'(t) := \lim_{h \to 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h}\right) \\ &\approx \sum_{i=1}^{n} \|\vec{x}'(t_i)\| (t_i - t_{i-1}) \end{aligned}$$

Take Limit $\Rightarrow S = \int_a^b \|\vec{x}'(t)\| dt$

Example 2.21 (Helix)**.** $\vec{x}(t) = (\cos t, \sin t, t), t \in [0, 2\pi]$
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1. Find the tangent line of $\vec{x}$ at $t = \pi$
2. Find arclength of the helix.

Solution. 1.

$$\begin{aligned} \vec{x}(t) &= (\cos t, \sin t, t) \\ \vec{x}'(t) &= (-\sin t, \cos t, 1) \\ \vec{x}'(\pi) &= (0, -1, 1) \leftarrow \text{direction of tangent} \end{aligned}$$

Also, $\vec{x}(\pi) = (-1, 0, \pi) \leftarrow$ a point on tangent line

$\therefore$ Parametric form of tangent line

$$\vec{x} = (-1, 0, \pi) + t(0, -1, 1)$$

2.

$$\begin{aligned}
\|\vec{x}'(t)\| &= \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} \\
&= \sqrt{2} \\
\Rightarrow S &= \int_0^{2\pi} \|\vec{x}'(t)\|\, dt \\
&= \int_0^{2\pi} \sqrt{2}dt \\
&= [\sqrt{2}t]_0^{2\pi} \\
&= 2\sqrt{2}\pi
\end{aligned}$$

Theorem 2.22. *Arclength is independent of parametrization.*

Example 2.23.

$$\vec{x}(t) = (t, t) \ \ 0 \leqslant t \leqslant 4$$
$$\vec{y}(t) = (t^2, t^2) \ \ 0 \leqslant t \leqslant 2$$

$\vec{x}, \vec{y}$ are two parametrization of the same line segment:

$\vec{x}'(t) = (1, 1)$

arclength of $\vec{x}(t)$

$$\begin{aligned}
&= \int_0^4 \|\vec{x}'(t)\|dt \\
&= \int_0^4 \sqrt{2}dt \\
&= 4\sqrt{2}
\end{aligned}$$

$\vec{y}'(t) = (2t, 2t)$

arclength of $\vec{y}(t)$

$$\begin{aligned}
&= \int_0^2 \|\vec{y}'(t)\|dt \\
&= \int_0^2 \sqrt{(2t)^2 + (2t)^2}dt \\
&= \int_0^2 2\sqrt{2}dt \\
&= [\sqrt{2}t^2]_0^2 \\
&= 4\sqrt{2}
\end{aligned}$$