MATH 1510 Chapter 7

7.1 MVT for integrals

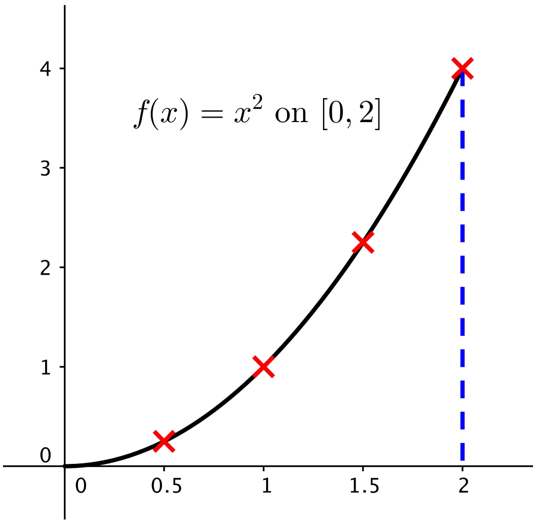
How should one define the "average value" of $f(x) = x^2$ over the interval $[0, 2]$? Let's start with approximating it by taking the function values at 4 points:

$$\text{Average} \approx \frac{1}{4}(f(0.5) + f(1) + f(1.5) + f(2)),$$

which can also be written as:

$$\frac{1}{2-0}(f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5)$$

Approximation of the (signed) area under the curve with 4 regular subintervals.



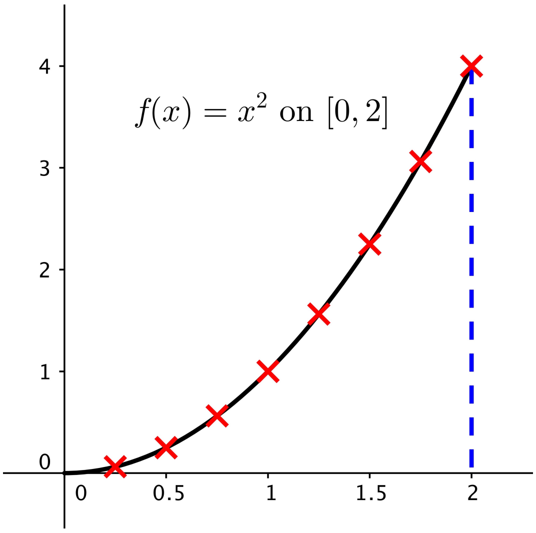
Naturally, we can get a better approximation by taking the function values at 8 points:

$$\begin{aligned}\text{Average} \approx\ & \frac{1}{8}(f(0.25) + f(0.5) + f(0.75) + f(1) \\ & + f(1.25) + f(1.5) + f(1.75) + f(2)),\end{aligned}$$

which can also be written as

$$\frac{1}{2-0}\left(f(0.25)0.25 + f(0.5)0.25 + f(0.75)0.25 + \cdots + f(1.75)0.25 + f(2)0.25\right)$$

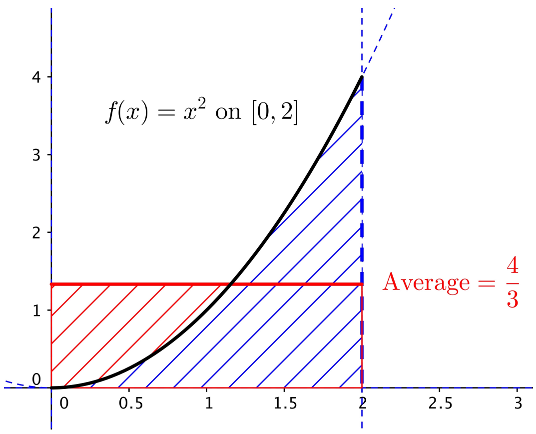
Approximation of the (signed) area under the curve with 8 regular subintervals.



Intuitively, yhe exact "average value" can then be found by dividing $[0, 2]$ into n regular subintervals and taking $n \to +\infty$.

Hence, by FTC, the "average value" of $f(x) = x^2$ over the interval $[0, 2]$ will then be:

$$\text{Average} = \frac{1}{2-0}\int_0^2 f(x)dx = \frac{4}{3}.$$



(One can immediately deduce that:

$$(2-0)\cdot \text{Average} = \int_0^2 f(x)\,dx.$$

That means the areas of the red box and the region shaded in blue are equal.)

In general,

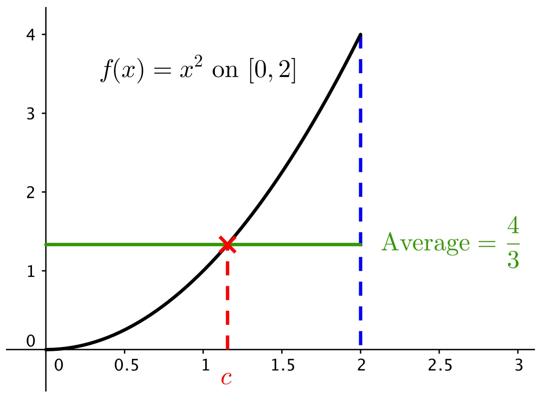
Definition 7.1 (Average Value of a Function)**.**

$$\text{Average value of } f(x) \text{ over } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Theorem 7.2 (Mean Value Theorem for Integrals)**.** *Suppose $f(x)$ is continuous on $[a,b]$. Then,*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \textit{for some } c \in (a,b).$$

(Basically, that means the average value will be achieved by some point in the interval.)



Proof of Mean Value Theorem for Integrals. Let:

$$F(x) = \int_a^x f(t)\,dt$$

By FTC, F is differentiable over $[a,b]$. By Lagrange's MVT, there exists $c \in (a,b)$ such that

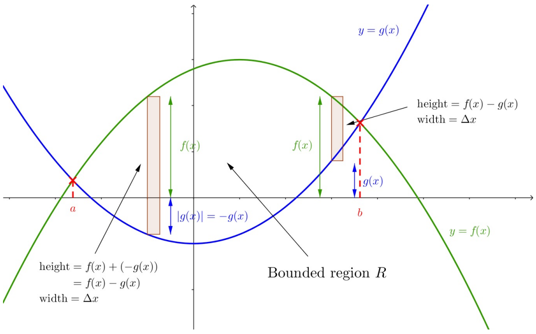
$$\frac{F(b)-F(a)}{b-a} = F'(c) \implies \frac{\int_a^b f(t)\,dt}{b-a} = f(c)$$

as desired. □

Example 7.3. Compute the average value of $f(x) = \sqrt{x}$ over $[1,4]$.

7.2 Area between Curves

Suppose $f(x), g(x)$ are two continuous functions and $f(x) \geq g(x)$ over $[a, b]$:



From the above graph, we can see that:

$$\text{Area of } R = \lim \sum (f(x) - g(x)) \, \Delta x.$$

Hence,

Proposition 7.4. *If $f(x), g(x)$ are continuous functions such that $f(x) \geq g(x)$ over $[a, b]$, then*

$$\textit{Area of the region bounded by } f(x), g(x) \textit{ over } [a, b] = \int_a^b (f(x) - g(x)) \, dx$$

Example 7.5. Consider the function $y = f(x) = x^3$ over the interval $[-1, 1]$.

Since $f(x) \geq 0$ when $x \in [0, 1]$ and $f(x) \leq 0$ when $x \in [-1, 0]$, to find the area of the region bounded by $y = f(x)$ and the x -axis, we need to split the interval $[-1, 1]$ into $[-1, 0]$ and $[0, 1]$:

$$\text{Area} = \int_{-1}^{0} \left(0 - x^3\right) dx + \int_0^1 \left(x^3 - 0\right) dx = \frac{1}{2}.$$

$$\left(\text{Note that: } \int_{-1}^{1} x^3 \, dx = 0.\right)$$

Example 7.6. Find the area of the region bounded by the curves:

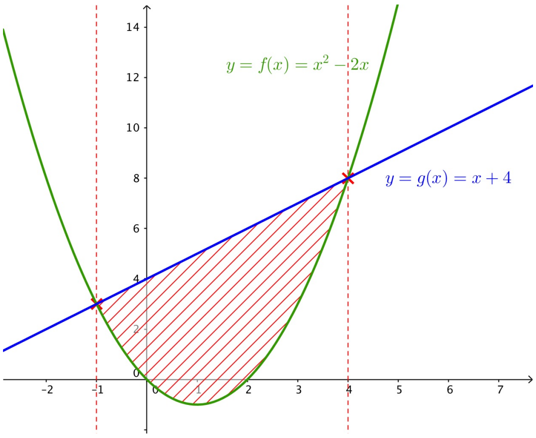
$$y = f(x) = x^2 - 2x$$
$$y = g(x) = x + 4$$

First of all, we need to find the intersections:

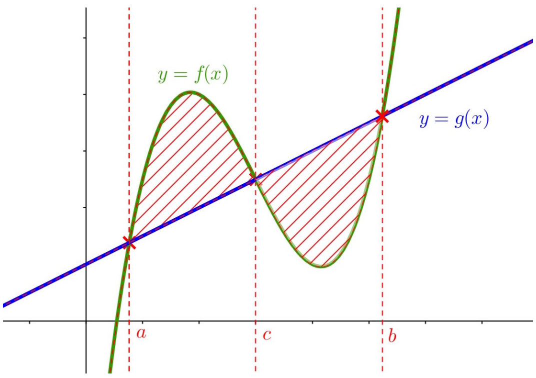
$$f(x) = g(x) \iff x = -1 \text{ or } 4.$$

From the sign chart for $g(x) - f(x)$, we know that $g(x) - f(x) \geq 0$ over the interval $[-1, 4]$. Therefore,

$$\text{Area} = \int_{-1}^{4} (g(x) - f(x))\, dx = \frac{125}{6}.$$



In general, the function $f(x)$ might not always be greater than $g(x)$:



In this case, $\int_a^b (f(x) - g(x))\, dx$ won't give us the desired result as there will be some cancellation of signed areas. Instead, we should split the interval $[a, b]$ into subintervals such that $f(x), g(x)$ won't change order within each subinterval:

$$\text{Area} = \underbrace{\int_a^c (f(x) - g(x))\, dx}_{f(x) \geq g(x) \text{ over } [a,c]} + \underbrace{\int_c^b (g(x) - f(x))\, dx}_{f(x) \leq g(x) \text{ over } [c,b]}$$

In fact, by taking absolute value inside, we will always be summing up the "positive areas of the rectangles". Hence,

Proposition 7.7. *If $f(x), g(x)$ are continuous functions over $[a, b]$, then:*

$$\textit{Area of the region bounded by } f(x), g(x) \textit{ over } [a,b] = \int_a^b |f(x) - g(x)|\, dx$$

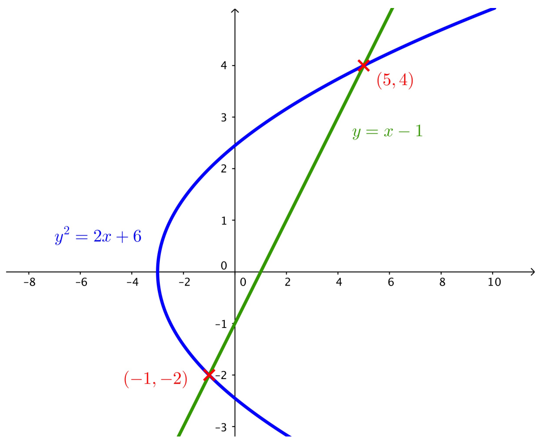
Example 7.8. Find the area of the region(s) bounded by the curves

$$y = f(x) = \sqrt{x}$$
$$y = g(x) = \frac{x}{2}$$

over the interval $[0, 5]$.

Example 7.9. Consider the curves

$$y = x - 1$$
$$y^2 = 2x + 6.$$



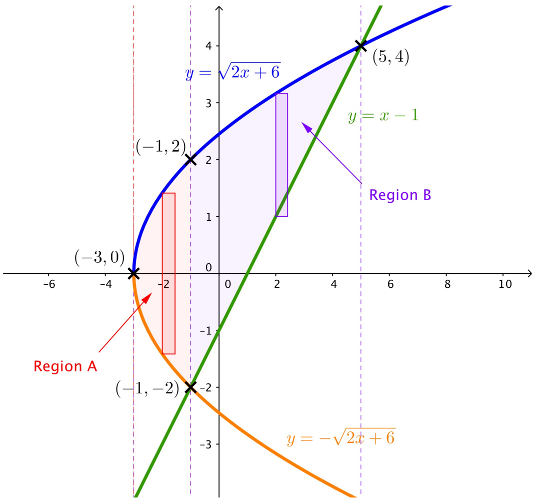
By some simple calculations, we know that they intersect at $(-1, -2)$ and $(5, 4)$. If we compute the area of the bounded region by summing up vertical rectangles like before, then

$$\text{Total area} = \text{Area of A} + \text{Area of B}$$

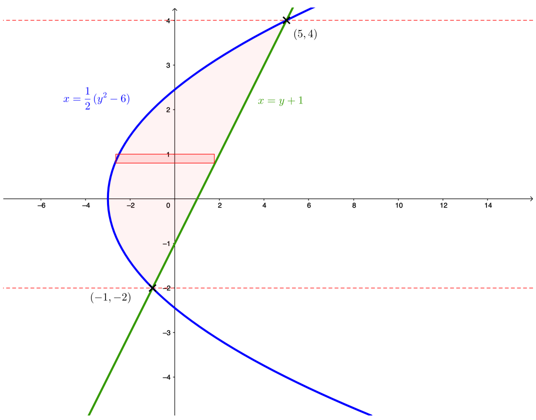
where

$$\text{Area of A} = \int_{-3}^{-1} (\sqrt{2x+6} - (-\sqrt{2x+6}))\, dx,$$

$$\text{Area of B} = \int_{-1}^{5} (\sqrt{2x+6} - (x-1))\, dx.$$



Or, we could sum up horizontal rectangles instead:



$$\begin{aligned}\text{Total area} &= \int_{-2}^{4} \left((y+1) - \frac{1}{2}(y^2 - 6) \right)\, dy \\ &= 18.\end{aligned}$$

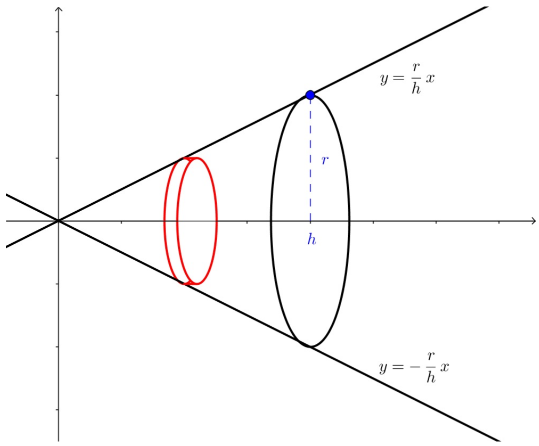
7.3 Volume

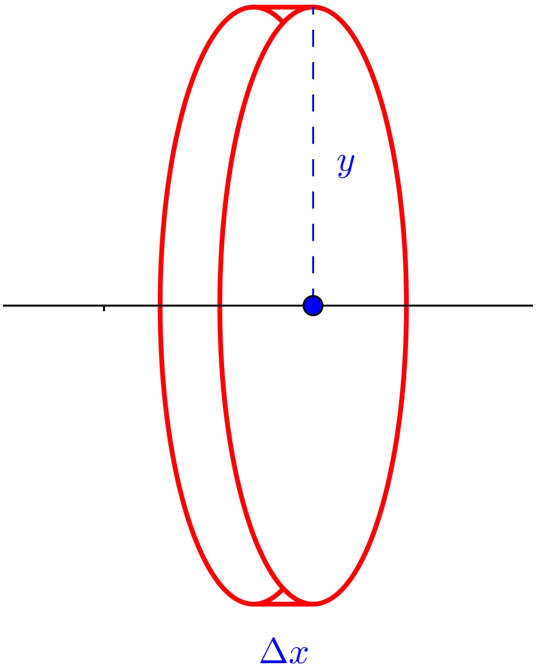
The volume of a right circular cone is:

$$V = \frac{1}{3}\pi r^2 h.$$

But why?

Consider the line segment defined by the equation $y = \frac{r}{h}x$ over the interval $[0, h]$. If we rotate it about the x -axis, we obtain the same right circular cone. To find its volume, we "scan" in the x -direction, cut the cone into infinitely many slices and approximate each slice by a cylinder:





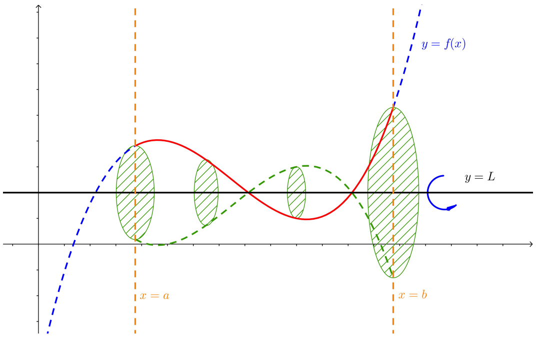
$$\Delta V = \pi y^2 \Delta x.$$

Hence,

$$\begin{aligned}
\text{Volume} = V &= \lim \sum \Delta V \\
&= \lim \sum \pi y^2 \, \Delta x \\
&= \lim \sum \pi \left(\frac{r}{h}x\right)^2 \, \Delta x \\
&= \int_0^h \pi \left(\frac{r}{h}x\right)^2 \, dx \\
&= \frac{1}{3}\pi r^2 h
\end{aligned}$$

as desired.

If the segment of a curve $y = f(x)$ over the interval $[a, b]$ is rotated about a line $y = L$:

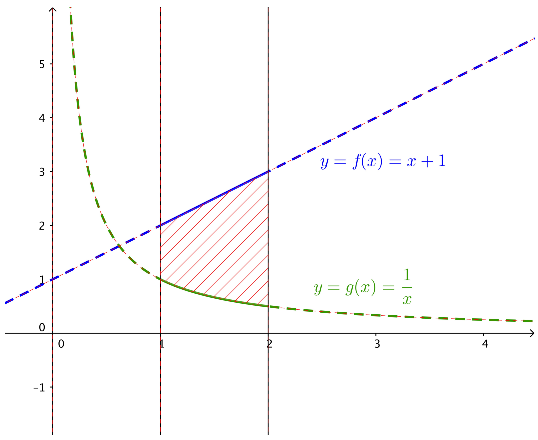


then we obtain a **solid of revolution**.
As before, we can deduce that

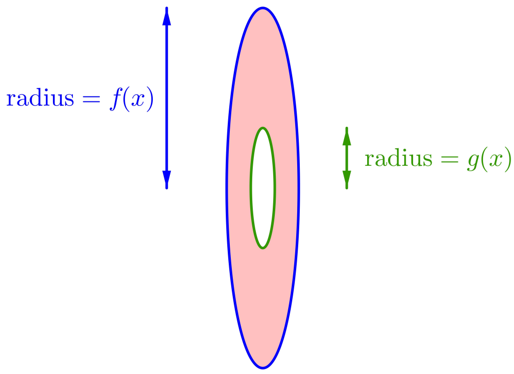
$$\begin{aligned} \text{Volume} &= \lim \sum \Delta V \\ &= \lim \sum \pi (f(x) - L)^2 \, \Delta x \\ &= \int_a^b \pi (f(x) - L)^2 \, dx \end{aligned}$$

Example 7.10. Find the volume of the solid obtained by revolving the curve $y = f(x) = x^2$ over $[0, 2]$ about the line $y = 1$. Express it as the integral of a function (You do not need to evaluate the integrals).

If a region is rotated about a line to form a solid of revolution, it's possible to have hole(s). Consider the region bounded by the curves $f(x) = x + 1, g(x) = \dfrac{1}{x}$ over the interval $[1, 2]$:



If it's rotated about the x -axis to form a solid, its cross section will look like:



and its volume will then be:

$$V = \int_1^2 (\pi f(x)^2 - \pi g(x)^2)\, dx = \int_1^2 \left(\pi(x+1)^2 - \pi \left(\frac{1}{x}\right)^2 \right) dx = \frac{35}{6}\pi.$$

Example 7.11. Consider the region bounded by the curve $y = x^3$ and the line $y = 1$ over the interval $[0, 1]$. Find the volume of the solid defined by rotating it about:

- the line $y = 1$;
- x -axis;
- y -axis.

Express it as the integral of a function (You do not need to evaluate the integrals).