Math 1510 Chapter 1

1.1 Sets

A **set** is a collection of **elements** :

- Order does not matter:

$$\{1, 2, 3\} = \{3, 2, 1\}$$

- Representation does not matter:

$$\{x \ : \ x^2 = 1\} = \{-1, 1\} = \{x \mid x^2 = 1\} = \{-1, 1\}$$

  Here, ":" and "|" mean "such that".

Notation	Meaning
$x \in A$	x is an element of A
$x \notin A$	x is not an element of A
$A \subseteq B$	A is a **subset** of B , i.e., $x \in A \Rightarrow x \in B$
$\Rightarrow$	implies
$A \cap B$	$\{x \mid x \in A$ and $x \in B\}$ **intersection**
$A \cup B$	$\{x \mid x \in A$ or $x \in B\}$ **union**
$A \setminus B$	$\{x \in A \mid x \notin B\}$ **difference**

The followings are some symbols we will use to represent some of the standard sets:

$\varnothing = \{\}$	**empty set** (no element)
$\mathbb{N}$	the set of **natural numbers** , i.e.,
	$\{1, 2, 3, 4 \ldots\}$
$\mathbb{Z}$	the set of **integers** , i.e.,
	$\{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$
$\mathbb{Q}$	the set of **rational numbers** , i.e.,
	$\left\{\frac{a}{b} : a, b \in \mathbb{Z}\text{ and } b \neq 0\right\}$
$\mathbb{R}$	the set of **real numbers**

Clearly, we have:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

1.1.1 Intervals

(a, b)	$= \{x \in \mathbb{R} \mid a < x < b\}$ **open interval**
$(a, b]$	$= \{x \in \mathbb{R} \mid a < x \leq b\}$ **half-open interval**
$[a, b)$	$= \{x \in \mathbb{R} \mid a \leq x < b\}$ **half-open interval**
$[a, b]$	$= \{x \in \mathbb{R} \mid a \leq x \leq b\}$ **closed interval**
$(a, +\infty)$	$= \{x \in \mathbb{R} \mid x > a\}$ **open interval**
$(-\infty, a)$	$= \{x \in \mathbb{R} \mid x < a\}$ **open interval**
$(-\infty, +\infty)$	$= \mathbb{R}$ **open interval**

1.2 Functions

Definition 1.1. A function:

$$f : A \longrightarrow B$$

is a rule of correspondence from one set A (called the **domain**) to another set B (called the **codomain**).

Under this rule of correspondence, each element $x \in A$ corresponds to *exactly one* element $f(x) \in B$, called the **value** of f at x.

In the context of this course, the domain A is usually some subset (intervals, union of intervals) of $\mathbb{R}$, while the codomain B is often presumed to be $\mathbb{R}$.

Sometimes, the domain of a function is not explicitly given, and a function is simply defined by an expression in terms of an independent variable.

For example,

$$f(x) = \sqrt{\frac{x+1}{x-2}}$$

In this case, the domain of f is assumed to be the **implied domain** (or **natural domain**, **maximal domain**, **domain of definition**), namely the largest subset of $\mathbb{R}$ on which the expression defining f is well-defined.

Example 1.2. For the function:

$$f(x) = \sqrt{\frac{x+1}{x-2}},$$

the natural domain is:

$$\begin{aligned} \text{Domain}(f) &= \left\{ x \in \mathbb{R} \;\middle|\; \frac{x+1}{x-2} \geq 0 \right\} \\ &= (-\infty, -1] \cup (2, \infty). \end{aligned}$$

1.2.1 Algebraic Operations on Functions

Definition 1.3. Given two functions:

$$f, g : A \longrightarrow \mathbb{R},$$

- Their **sum/difference** is:

$$f \pm g : A \longrightarrow \mathbb{R},$$
$$(f+g)(a) := f(a) + g(a), \quad \text{for all } a \in A;$$
$$(f-g)(a) := f(a) - g(a), \quad \text{for all } a \in A;$$

- Their **product** is:

$$fg : A \longrightarrow \mathbb{R},$$
$$fg(a) := f(a)g(a), \quad \text{for all } a \in A;$$

- The **quotient function** $\dfrac{f}{g}$ is:

$$\frac{f}{g} : A' \longrightarrow \mathbb{R},$$
$$\frac{f}{g}(a) := \frac{f(a)}{g(a)}\ , \quad \text{for all } a \in A',$$

  where

$$A' = \{a \in A : g(a) \neq 0\}.$$

More generally, For:

$$f : A \longrightarrow \mathbb{R},$$
$$g : B \longrightarrow \mathbb{R},$$

we define $f \pm g$ and fg as follows:

$$f \pm g : A \cap B \longrightarrow \mathbb{R},$$
$$f \pm g(x) := f(x) \pm g(x), \quad x \in A \cap B.$$

$$fg : A \cap B \longrightarrow \mathbb{R},$$
$$fg(x) := f(x)g(x), \quad x \in A \cap B.$$

Similary, we define:

$$\frac{f}{g} : A \cap B' \longrightarrow \mathbb{R},$$
$$\frac{f}{g}(x) = \frac{f(x)}{g(x)}, \quad x \in A \cap B',$$

where $B' = \{b \in B : g(b) \neq 0\}$.

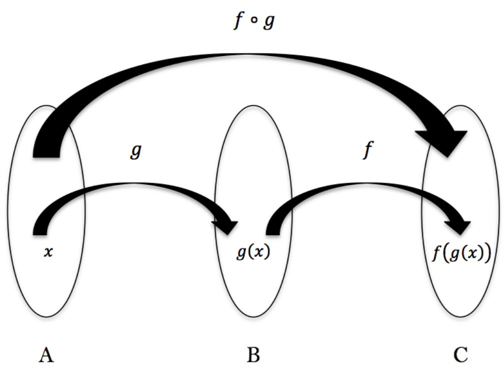
1.2.2 Composition of Functions

Given two functions:

$$g : A \longrightarrow B, \quad f : B \longrightarrow C,$$

the **composite function** $f \circ g$ is defined as follows:

$$f \circ g : A \longrightarrow C,$$
$$(f \circ g)(a) := f(g(a)), \quad \text{for all } a \in A.$$



When the codomain of g is not the same as the domain of f, the domain of $f \circ g$ is defined to be:

$$\text{Domain}(f \circ g) = \{a \in \text{Domain}(g) : g(a) \in \text{Domain}(f)\}.$$

Example 1.4. Find the implied domains of $f \circ g$ and $g \circ f$, where:

$$f(x) = x^2, \quad g(x) = \sqrt{x}.$$

1.2.3 Inverse of a Function

The **range** or **image** of a function $f : A \longrightarrow B$ is the set of all $b \in B$ such that $b = f(a)$ for some $a \in A$.

Notation.

$$\text{Image}(f) = \text{Range}(f) := \{b \in B \,:\, b = f(a) \text{ for some } a \in A\}.$$

Note that the range of f is not necessarily equal to the codomain B.

Definition 1.5. If $\text{Range}(f) = B$, we say that f is **surjective** or **onto** .

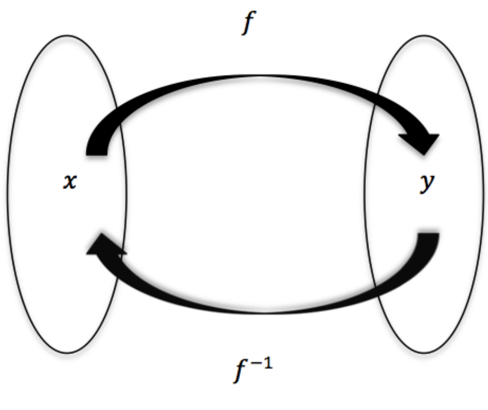
Definition 1.6. If $f(a) \neq f(a')$ for all $a, a' \in \text{Domain}(f)$ such that $a \neq a'$, we say that f is **injective** or **one-to-one** .

If $f : A \longrightarrow B$ is injective, then there exists an **inverse function**:

$$f^{-1} : \text{Range}(f) \longrightarrow A$$

such that $f^{-1} \circ f$ is the **identity function** on A, and $f \circ f^{-1}$ is the identity function on $\text{Range}(f)$, that is:

- $$f^{-1}(f(a)) = a, \qquad \text{for all } a \in A,$$
- $$f(f^{-1}(b)) = b, \qquad \text{for all } b \in \text{Range}(f).$$



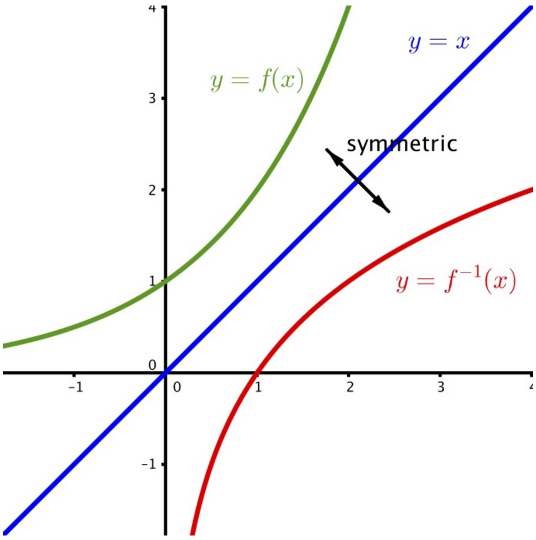
It may be shown that:

Proposition 1.7. *If f has an inverse f^{-1}, then:*

$$\textit{Domain}(f^{-1}) = \textit{Range}(f)$$

$$\textit{Range}(f^{-1}) = \textit{Domain}(f)$$

Geometrically, the graph of f^{-1} is the reflection of the graph of f over the diagonal line $y = x$:



Example 1.8. Find the inverse of:

- $$f(x) = \frac{2x - 1}{1 - x}$$

- $$f(x) = x^2 + x \text{ with domain } D = [0, +\infty)$$

1.3 Piecewise Defined Functions

Example 1.9.

- $$f(x) = \begin{cases} -x + 1 & \text{if } -2 \leq x < 0 \\ 3x & \text{if } 0 \leq x \leq 5 \end{cases}$$

- The **absolute value function**

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Example 1.10. Consider,

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ |x-2| - 1 & \text{if } x \geq 1 \end{cases}$$

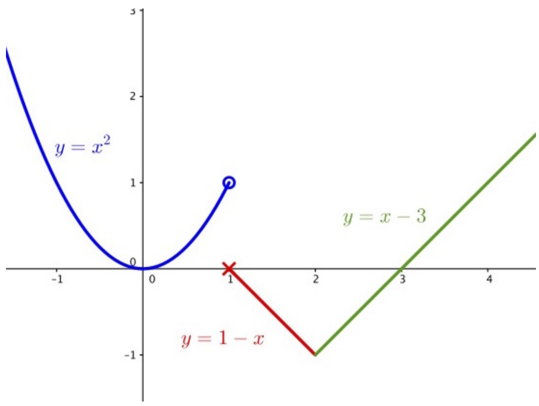
Then, for example,

$$\begin{aligned} f(-1) &= & (-1)^2 = 1 \\ f(0) &= & 0^2 = 0 \\ f(1) &= & |1-2| - 1 = 0 \\ f(2) &= & |2-2| - 1 = -1 \end{aligned}$$

We can rewrite f as:

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 1 - x & \text{if } 1 \leq x < 2 \\ x - 3 & \text{if } x \geq 2 \end{cases}$$

The graph $y = f(x)$ of f is as follows:



Exercise 1.11. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by:

$$f(x) = -3x + 4 - |x+1| - |x-1|$$

for any $x \in \mathbb{R}$.

1. Express the 'explicit formula' of the function f as that of a piecewise defined function, with one 'piece' for each of $(-\infty, -1)$, $[-1, 1)$, $[1, +\infty)$.
2. Sketch the graph of the function f.
3. Is f an injective function on $\mathbb{R}$? Justify your answer.
4. What is the image of $\mathbb{R}$ under the function f?

Solution.

1.
$$f(x) = \begin{cases} -x+4 & \text{if} \quad x < -1 \\ -3x+2 & \text{if} \quad -1 \leq x < 1 \\ -5x+4 & \text{if} \quad x \geq 1 \end{cases}$$

2. **Open in browser**

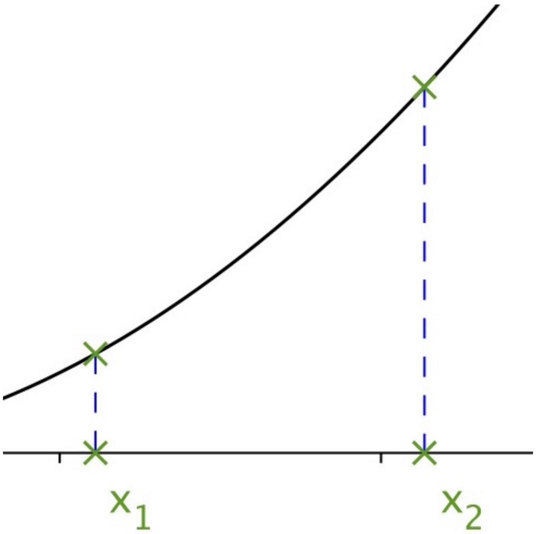
3. f is strictly decreasing on $\mathbb{R}$. Hence, f is injective on $\mathbb{R}$.

4. The image of $\mathbb{R}$ under f is $\mathbb{R}$.

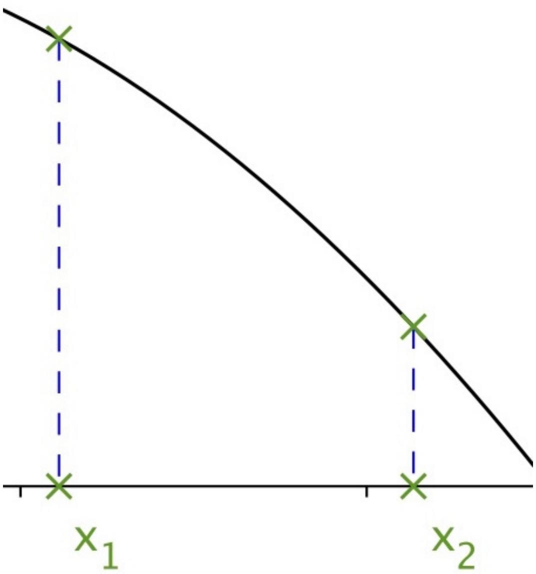
1.4 Properties of Functions

For a function f, we say that: f is **increasing** ($\nearrow$) if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$

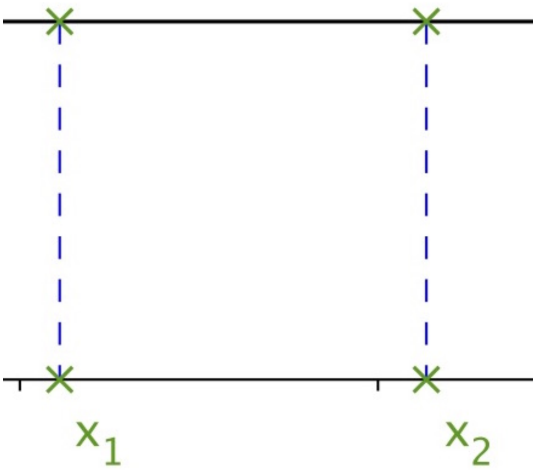
f is **strictly increasing** ($\nearrow$) if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$



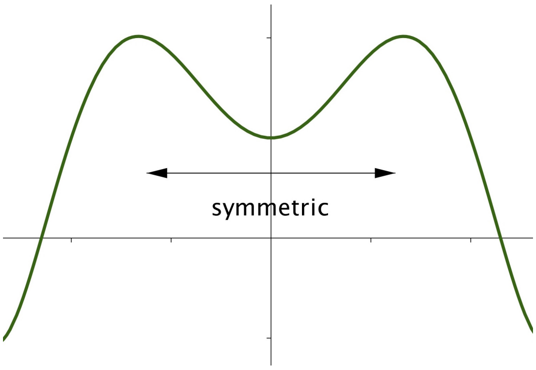
f is **decreasing** ($\searrow$) if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$
f is **strictly decreasing** ($\searrow$) if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$



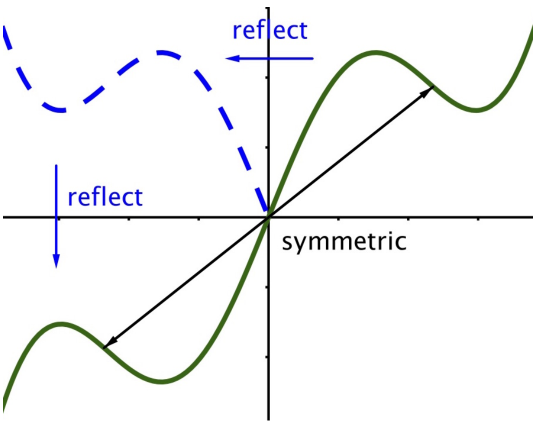
f is **constant** if $f(x_1) = f(x_2)$ for all x_1, x_2



We say that f is an **even** function if $f(-x) = f(x)$ for all $x \in \text{Domain}(f)$



symmetric about the y -axis
We say that f is an **odd** function if $f(-x) = -f(x)$ for all $x \in \text{Domain}(f)$



symmetric about the origin. (It is possible for a function to be neither even nor odd.)

Example 1.12. Determine if the following function is even, odd or neither:

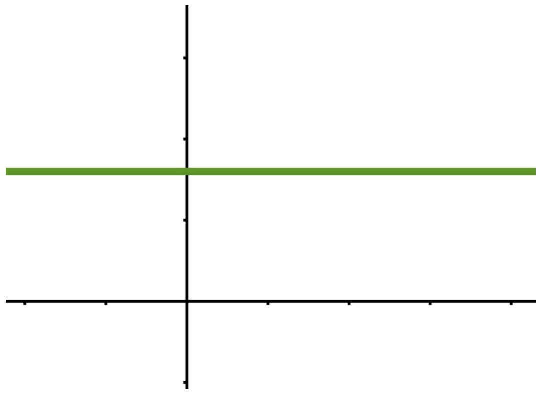
- $$f(x) = x^2 - x^{2/3}$$

- $$g(x) = \sin x - \tan x$$

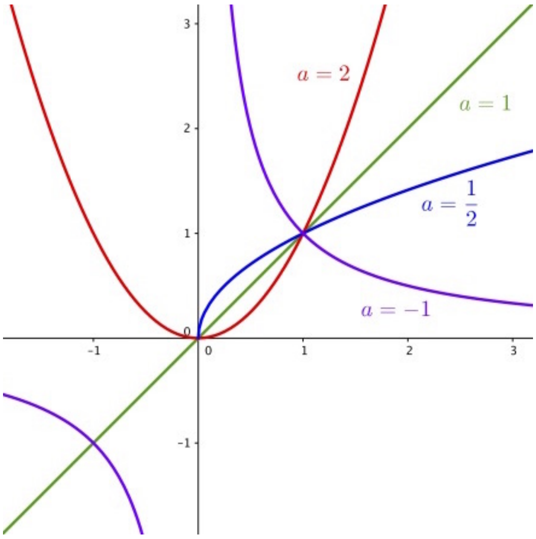
- $$h(x) = x - 1$$

1.5 Elementary functions

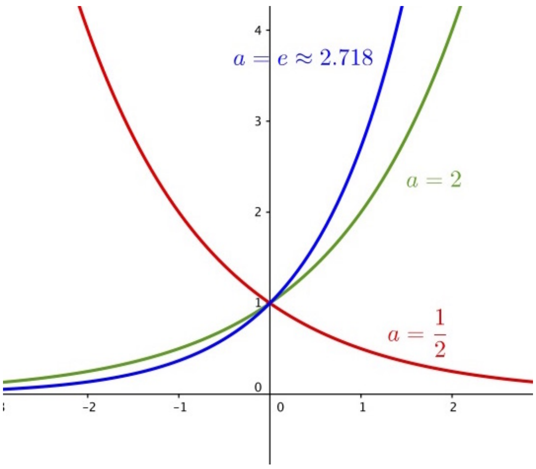
- Constant: $f(x) = c$



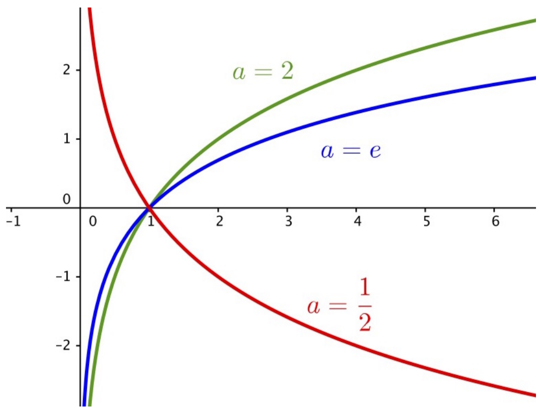
- Power: $f(x) = x^a$



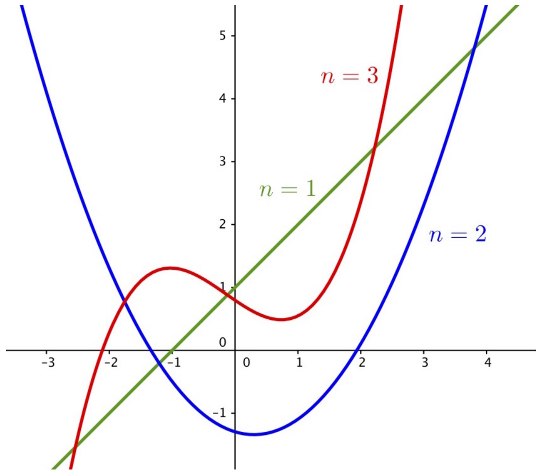
- Exponential: $f(x) = a^x$ where $a > 0$ increasing if $a > 1$ decreasing if $0 < a < 1$



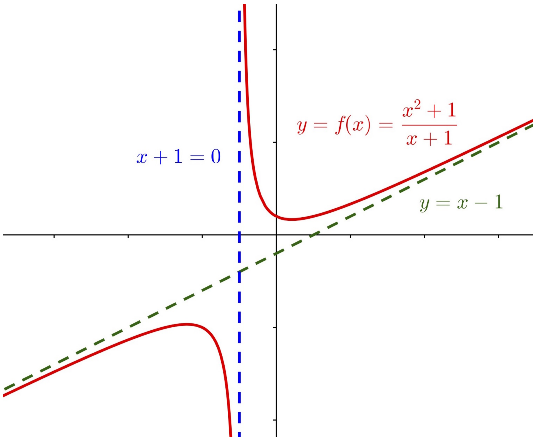
- Logarithmic: $f(x) = \log_a x$ where $a > 0$ "log " : $a = 10$ "ln " : $a = e \approx 2.718...$



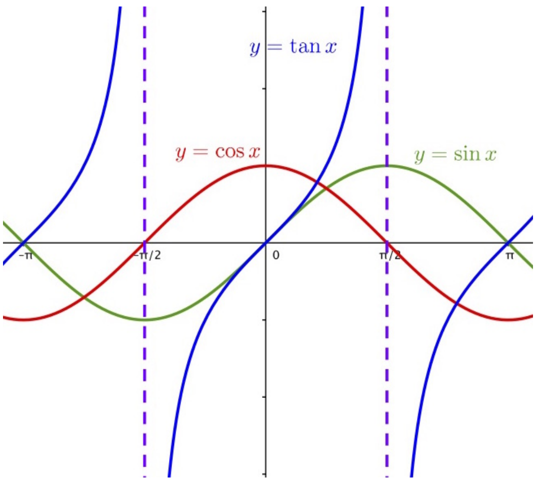
- Polynomial: $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ where $a_i \in \mathbb{R}$ are the coefficients and $n \geq 0$ (integer) is the degree (provided that $a_n \neq 0$)



- Rational: $f(x) = \dfrac{P(x)}{Q(x)}$ where P, Q are polynomials and $Q \neq 0$



- Trigonometric: $f(x) = \sin x, \cos x, \tan x, \sec x, \csc x$ or $\cot x$



1.6 Parametric Equations

Sometimes, it's preferable to express the coordinates of points (x, y) in 2D (or (x, y, z) in 3D) in terms of an independent variable t. That is,

$$(x, y) = (f(t), g(t))$$

where $f(t), g(t)$ are both functions of t . The equation displayed above in fact consists of two equations:

$$x = f(t)$$
$$y = g(t)$$

They are called **parametric equations**, and t is called a **parameter**.

Example 1.13. Suppose the coordinates of an object at time t is given by:

$$\begin{cases} x &= f(t) = \cos(36° t) \\ y &= g(t) = \sin(36° t) \end{cases}$$

Then its coordinates at different times t are:

t	0	1	2	2.5	5	10
(x, y)	$(1, 0)$	$(\cos 36°, \sin 36°)$	$(\cos 72°, \sin 72°)$	$(0, 1)$	$(-1, 0)$	$(1, 0)$

To represent this object geometrically, it's often useful to consider an equation in x, y which is satisfied by all points (x, y) which satisfy $x = f(t), y = g(t)$ for some t. (The set of all such points is called the **locus** of the equation).

In this example, we have:

$$x^2 + y^2 = \cos^2(36° t) + \sin^2(36° t)$$
$$x^2 + y^2 = 1,$$

which is a circle. Then, by finding out the coordinates of the object at a few different times, we can draw some arrows to indicate the movement of the object along its locus: