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MATH1510 Calculus for Engineers
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Appendix 5: Integration

Definition 1

Given a bounded function f over $[a, b]$. For a partition

$$P : a = x_0 < x_1 < \cdots < x_n = b$$

we define

$$\operatorname{norm} P = \max\{|x_k - x_{k+1}| \mid 0 \le k \le n-1\}$$

$$S(P, f) = \sum_{k=1}^{n} \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}(x_k - x_{k-1})$$

$$s(P, f) = \sum_{k=1}^{n} \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}(x_k - x_{k-1})$$

$$\text{Upper Riemann sum} = \overline{\int_a^b} f(x)\,dx = \inf_P S(P, f)$$

$$\text{Lower Riemann sum} = \underline{\int_a^b} f(x)\,dx = \sup_P s(P, f)$$

We say that f is **Riemann integrable** if both upper and lower Riemann sum exist and are equal. In that case, we denote

$$\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

Proposition 1

For any bounded function f over $[a, b]$ and partitions P, Q, we have

$$s(P, f) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le S(Q, f)$$

Proof

Suppose Q is a partition of $[c, d] \subseteq [a, b]$:

$$Q : c = y_0 < y_1 < \cdots < y_n = d$$

Then we have

$$s([c, d], f) \le s(Q, f) \le S(Q, f) \le S([c, d], f)$$

Therefore, for any refinement P' of P,

$$s(P,f) \leq s(P',f) \leq S(P',f) \leq S(P,f)$$

So, if P, Q are two partitions of $[a,b]$, there exists a refinement R of them and

$$s(P,f) \leq s(R,f) \leq S(R,f) \leq S(Q,f)$$

Hence, both $\underline{\int_a^b} f(x)\,dx, \overline{\int_a^b} f(x)\,dx$ exist and

$$s(P,f) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq S(Q,f)$$

□

Theorem 1
If f is a piecewise continuous function over $[a,b]$, then f is Riemann integrable.

Proof

Given sufficiently large $n \in \mathbb{Z}^+$, we will construct a partition P_n.
Suppose f is continuous over $(c,d) \subseteq [a,b]$. By continuity,

$$\forall x \in (c,d), \exists \delta_x > 0, \forall y \in (c,d) \text{ such that } |x-y| < \delta_x, \quad |f(x)-f(y)| < \frac{1}{n}$$

WLOG, we may assume $(x-\delta_x, x+\delta_x) \subseteq (c,d)$. So, as long as $\frac{1}{n} < d-c$

$$[c, c+\frac{1}{n}) \cup (d-\frac{1}{n}, d] \cup \bigcup_{x\in(c,d)} (x-\delta_x, x+\delta_x)$$

defines an open cover of $[c,d]$. Since $[c,d]$ is a compact set, we have a finite subcover:

$$[c, c+\frac{1}{n}) \cup (x_1-\delta_{x_1}, x_1+\delta_{x_1}) \cup \cdots \cup (x_m-\delta_{x_m}, x_m+\delta_{x_m}) \cup (d-\frac{1}{n}, d]$$

where $c < x_1 < x_2 < \cdots < x_m < d$. Take

$$z_0 = c, z_1 \in (x_1-\delta_{x_1}, c+\frac{1}{n}), z_2 \in (x_2-\delta_{x_2}, x_1+\delta_{x_1}), \ldots,$$

$$z_m \in (x_m-\delta_{x_m}, x_{m-1}+\delta_{x_{m-1}}), z_{m+1} \in (d-\frac{1}{n}, x_m+\delta_{x_m}), z_{m+2} = d$$

Then,

$$Q_n : c = z_0 < z_1 < \cdots < z_{m+2} = d$$

defines a partition of $[c,d]$ such that

$$\forall 1 \leq k \leq m, \quad z_k, z_{k+1} \in (x_k-\delta_{x_k}, x_k+\delta_{x_k}) \implies [z_k, z_{k+1}] \subseteq (x_k-\delta_{x_k}, x_k+\delta_{x_k})$$

$$\implies \forall s,t \in [z_k, z_{k+1}], \quad |f(s)-f(t)| \leq |f(s)-f(x_k)| + |f(x_k)-f(t)| < \frac{2}{n}$$

Therefore, if A, B are upper, lower bounds of f over $[a,b]$ respectively, then

$$\begin{aligned}
& S(Q_n, f) - s(Q_n, f) \\
= & \left(\sum_{k=0}^{m+1} \sup\{f(x) \mid x \in [z_k, z_{k+1}]\}(z_{k+1} - z_k)\right) - \left(\sum_{k=0}^{m+1} \inf\{f(x) \mid x \in [z_k, z_{k+1}]\}(z_{k+1} - z_k)\right) \\
= & (\sup\{f(x) \mid x \in [c, z_1]\} - \inf\{f(x) \mid x \in [c, z_1]\})(z_1 - c) \\
& + (\sup\{f(x) \mid x \in [z_{m+1}, d]\} - \inf\{f(x) \mid x \in [z_{m+1}, d]\})(d - z_{m+1}) \\
& + \sum_{k=1}^{m} (\sup\{f(x) \mid x \in [z_k, z_{k+1}]\} - \inf\{f(x) \mid x \in [z_k, z_{k+1}]\})(z_{k+1} - z_k) \\
\leq & (A - B)\frac{2}{n} + \sum_{k=1}^{m} \frac{2}{n}(z_{k+1} - z_k) \leq \frac{2(A-B)}{n} + \frac{2(d-c)}{n}
\end{aligned}$$

Since f is piecewise continuous, we have

$$a = c_1 < d_1 = c_2 < d_2 \cdots c_{p-1} < d_{p-1} = c_p < d_p = b$$

such that f is continuous over (c_k, d_k) for all $1 \leq k \leq p$. As long as $\frac{1}{n} < d_k - c_k$, we can construct a partition Q_n of $[c_k, d_k]$ as above. Thus, for sufficiently large n, we can construct a partition P_n of $[a,b]$ by assembling all the partitions Q_n of $[c_k, d_k]$. So,

$$0 \leq S(P_n, f) - s(P_n, f) \leq \frac{2(A-B)p}{n} + \sum_{k=1}^{p} \frac{2(d_k - c_k)}{n} = \frac{2(A-B)p}{n} + \frac{2(b-a)}{n}$$

$$\implies \lim_{n\to\infty} (S(P_n, f) - s(P_n, f)) = 0$$

By squeeze theorem,

$$S(P_n, f) - s(P_n, f) \geq S(P_n, f) - \overline{\int_a^b} f(x)\,dx, \underline{\int_a^b} f(x)\,dx - s(P_n, f) \geq 0$$

$$\implies \lim_{n\to\infty} S(P_n, f) = \overline{\int_a^b} f(x)\,dx \quad \text{and} \quad \lim_{n\to\infty} s(P_n, f) = \underline{\int_a^b} f(x)\,dx$$

$$\implies \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} (S(P_n, f) - s(P_n, f)) = 0$$

□

Theorem 2

If a bounded function f over $[a,b]$ is Riemann integrable, then for any sequence of partitions

$$P_n : a = x_{n,0} < x_{n,1} < \cdots < x_{n,m_n} = b$$

such that $\lim_{n\to\infty} \operatorname{norm} P_n = 0$ and any choices $t_{n,k} \in [x_{n,k-1}, x_{n,k}]$

$$\lim_{n\to\infty} \sum_{k=1}^{m_n} f(t_{n,k})(x_{n,k} - x_{n,k-1}) = \lim_{n\to\infty} S(P_n, f) = \lim_{n\to\infty} s(P_n, f) = \int_a^b f(x)\,dx$$

Proof

Let $I = \int_a^b f(x)\,dx$. Suppose $A > f(x) > B$ for all $x \in [a,b]$. Given $\epsilon > 0$, there exist partitions Q_1, Q_2 of $[a,b]$ such that

$$S(Q_1, f) - I, I - s(Q_2, f) < \frac{\epsilon}{4}$$

Let

$$Q : a = x_0 < x_1 < \cdots < x_{m+1} = b$$

be a refinement of Q_1, Q_2 (WLOG, $m \geq 1$). Then,

$$S(Q, f) - s(Q, f) \leq S(Q_1, f) - s(Q_2, f) < \frac{\epsilon}{2}$$

Notice that

$$\lim_{n\to\infty} \operatorname{norm} P_n = 0 \implies \exists N \in \mathbb{Z}^+, \forall n > N, \quad \operatorname{norm} P_n < \frac{\epsilon}{4m(A-B)} = \epsilon_2$$

Therefore, for all $n > N$, let Q_n be the partition defined by combining P_n, Q. Since we are, at worst, inserting $x_1, \ldots, x_m$ into m different intervals of P_n with width $< \epsilon_2$,

$$S(P_n, f) - S(Q_n, f) < m\epsilon_2(A-B) \quad \text{and} \quad s(Q_n, f) - s(P_n, f) < m\epsilon_2(A-B)$$

$$\implies S(P_n, f) - s(P_n, f) < S(Q_n, f) - s(Q_n, f) + 2m\epsilon_2(A-B) \leq S(Q, f) - s(Q, f) + \frac{\epsilon}{2} < \epsilon$$

Thus,

$$\lim_{n\to\infty} (S(P_n, f) - s(P_n, f)) = 0$$

By squeeze theorem,

$$S(P_n, f) - s(P_n, f) \geq S(P_n, f) - I, I - s(P_n, f) \geq 0$$

$$\implies \lim_{n\to\infty} S(P_n, f) = \lim_{n\to\infty} s(P_n, f) = I$$

Finally, the result follows by applying squeeze theorem on

$$S(P_n, f) \geq \sum_{k=1}^{m_n} f(t_{n,k})(x_{n,k} - x_{n,k-1}) \geq s(P_n, f)$$

□

Proposition 2

For any Riemann integrable functions f, g (over the corresponding interval), $\forall a, b, c, \alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}
\int_a^b f(x)\,dx &= -\int_b^a f(x)\,dx; \\
\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx; \\
f(x) \le g(x) \text{ on } [a,b] &\implies \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.
\end{aligned}$$

$(\alpha f(x) + \beta g(x))$ is Riemann integrable over $[a,b]$ and

$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

Proof

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Directly from definition.

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Trivial if $a = b, a = c$ or $b = c$. If $a < c < b$, then define the partitions

$$\begin{aligned}
P_n &: a < a + d_1 < a + 2d_1 < \cdots < a + nd_1 = c \quad \text{where } d_1 = \frac{c-a}{n} \\
Q_n &: c < c + d_2 < c + 2d_2 < \cdots < c + nd_2 = b \quad \text{where } d_2 = \frac{b-c}{n} \\
R_n &: a < a + d_1 < a + 2d_1 < \cdots < a + nd_1 = c < c + d_2 < c + 2d_2 < \cdots < c + nd_2 = b
\end{aligned}$$

Then,

$$\lim_{n\to\infty} \operatorname{norm} P_n = \lim_{n\to\infty} \operatorname{norm} Q_n = \lim_{n\to\infty} \operatorname{norm} R_n = 0$$

Hence,

$$S(R_n, f) = S(P_n, f) + S(Q_n, f)$$

$$\implies \int_a^b f(x)\,dx = \lim_{n\to\infty} S(R_n, f) = \lim_{n\to\infty} S(P_n, f) + \lim_{n\to\infty} S(Q_n, f) = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

If $a < b < c$, then

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx - \int_b^c f(x)\,dx = \int_a^b f(x)\,dx$$

The arguments regarding the other orders of a, b, c are similar.

$f(x) \le g(x)$ on $[a,b] \implies \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

Let $(P_n)_{n\in\mathbb{Z}^+}$ be a sequence of partitions of $[a,b]$ such that $\lim_{n\to\infty} \operatorname{norm} P_n = 0$. Then,

$$S(P_n, f) \le S(P_n, g) \implies \int_a^b f(x)\,dx = \lim_{n\to\infty} S(P_n, f) \le \lim_{n\to\infty} S(P_n, g) = \int_a^b g(x)\,dx$$

$\int_a^b (\alpha f(x) + \beta g(x))\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx$

Let $I = \displaystyle\int_a^b f(x)\, dx$ and $J = \displaystyle\int_a^b g(x)\, dx$.

Let $(P_n)_{n\in\mathbb{Z}^+}, (P'_n)_{n\in\mathbb{Z}^+}, (Q_n)_{n\in\mathbb{Z}^+}$ and $(Q'_n)_{n\in\mathbb{Z}^+}$ be four sequences of partitions such that $S(P_n, f), S(Q_n, g)$ are decreasing sequences, $s(P'_n, f), s(Q'_n, g)$ are increasing sequences and

$$\lim_{n\to\infty} S(P_n, f) = \lim_{n\to\infty} s(P'_n, f) = I \quad \text{and} \quad \lim_{n\to\infty} S(Q_n, g) = \lim_{n\to\infty} s(Q'_n, g) = J$$

Let R_n be a refinement of P_n, P'_n, Q_n, Q'_n. Then, $S(R_n, f), S(R_n, g)$ are decreasing sequences, $s(R_n, f), s(R_n, g)$ are increasing sequences and

$$\lim_{n\to\infty} S(R_n, f) = \lim_{n\to\infty} s(R_n, f) = I \quad \text{and} \quad \lim_{n\to\infty} S(R_n, g) = \lim_{n\to\infty} s(R_n, g) = J$$

Suppose $\alpha, \beta \geq 0$.

$$\alpha s(R_n, f) + \beta s(R_n, g) \leq s(R_n, \alpha f + \beta g) \leq S(R_n, \alpha f + \beta g) \leq \alpha S(R_n, f) + \beta S(R_n, g)$$

By taking $n \to \infty$, we know that $\alpha f + \beta g$ is Riemann integrable and

$$\int_a^b (\alpha f(x) + \beta g(x))\, dx = \lim_{n\to\infty} s(R_n, \alpha f + \beta g) = \alpha I + \beta J$$

For the special case $\alpha = -1, \beta = 0$,

$$-S(R_n, f) \leq s(R_n, -f) \leq S(R_n, -f) \leq -s(R_n, f)$$

Then, again, $-f$ is Riemann integrable and

$$\int_a^b (-f(x))\, dx = -\int_a^b f(x)\, dx$$

If $\alpha \geq 0, \beta < 0$, then

$$-g \text{ Riemann integrable} \implies \alpha f + \beta g = \alpha f + |\beta|(-g) \text{ Riemann integrable}$$

$$\begin{aligned}
\int_a^b (\alpha f(x) + \beta g(x))\, dx &= \int_a^b (\alpha f(x) + |\beta|(-g(x)))\, dx \\
&= \alpha \int_a^b f(x)\, dx + |\beta| \int_a^b (-g(x))\, dx \\
&= \alpha \int_a^b f(x)\, dx - |\beta| \int_a^b g(x)\, dx = \alpha I + \beta J
\end{aligned}$$

The case when $\alpha < 0, \beta \geq 0$ is handled by interchanging f, g. The case when $\alpha, \beta < 0$ can be handled similarly. □