MATH 1030 Chapter 9

The lecture is based on Beezer, A first course in Linear algebra. Ver 3.5 Downloadable at http://linear.ups.edu/download.html .

The print version can be downloaded at http://linear.ups.edu/download/fcla-3.50-print.pdf .

Reference.

Beezer, Ver 3.5 Section VO (print version p57 - p63)Subsection VS, EVS (print version p197-203) Strang, Section 2.1

Exercise.

Exercises with solutions can be downloaded at http://linear.ups.edu/download/fcla-3.50-solution-manual.pdfSection VO (p.28-31) All questions.C10-C15, T05-T07, T13, T17, T18, T30-T32 Section VS (p.75-77) Replace $\mathbb{C}$ (the set of complex numbers) by $\mathbb{R}$ (the set of real numbers) M11, M12, M13, M14, M15, M20.

9.1 Vectors

Notation : $\mathbb{R}$ is the set of real numbers. If X is a set, $x \in X$ means x is an element of the set X.

Definition 9.1 (Vector Space of Column Vectors)**.** The vector space $\mathbb{R}^m$ is the set of all column vectors of size m with entries from the set of real numbers, $\mathbb{R}$. $\mathbb{R}^m$ is also called the **Euclidean m-space**.

Definition 9.2 (Column Vector Equality)**.** Suppose that $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^m$. Then $\mathbf{u}$ and $\mathbf{v}$ are **equal**, written $\mathbf{u} = \mathbf{v}$ if

$$[\mathbf{u}]_i = [\mathbf{v}]_i \qquad\qquad 1 \le i \le m$$

That is,

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

if

$$u_i = v_i \qquad\qquad 1 \leq i \leq m.$$

Example 9.3. The system of linear equations:

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= -33 \\ 5x_1 + 5x_2 + 7x_3 &= 24 \\ x_1 + 4x_3 &= 5 \end{aligned}$$

can be rewritten as:

$$\begin{bmatrix} -7x_1 - 6x_2 - 12x_3 \\ 5x_1 + 5x_2 + 7x_3 \\ x_1 + 4x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}.$$

Definition 9.4 (Column Vector Addition)**.** Suppose that $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^m$. The **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v}$ defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i \qquad\qquad 1 \leq i \leq m.$$

That is

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_m + v_m \end{bmatrix}.$$

Example 9.5. Addition of two vectors in $\mathbb{R}^4$

If

$$\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 2 \end{bmatrix} \qquad\qquad \mathbf{v} = \begin{bmatrix} -1 \\ 5 \\ 2 \\ -7 \end{bmatrix}$$

then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \\ 2 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 + (-1) \\ -3 + 5 \\ 4 + 2 \\ 2 + (-7) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \\ -5 \end{bmatrix}$$

Definition 9.6 (Column Vector Scalar Multiplication)**.** Suppose $\mathbf{u} \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, then the **scalar multiple** of $\mathbf{u}$ by α is the vector $\alpha\mathbf{u}$ defined by

$$[\alpha\mathbf{u}]_i = \alpha\,[\mathbf{u}]_i \qquad\qquad 1 \leq i \leq m.$$

That is

$$\alpha \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \vdots \\ \alpha u_m \end{bmatrix}.$$

Example 9.7. If

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 4 \\ -1 \end{bmatrix}$$

and $\alpha = 6$, then

$$\alpha\mathbf{u} = 6 \begin{bmatrix} 3 \\ 1 \\ -2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 6(3) \\ 6(1) \\ 6(-2) \\ 6(4) \\ 6(-1) \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \\ -12 \\ 24 \\ -6 \end{bmatrix}.$$

Example 9.8. The system of linear equations

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= -33 \\ 5x_1 + 5x_2 + 7x_3 &= 24 \\ x_1 + 4x_3 &= 5 \end{aligned}$$

can be written as:

$$x_1 \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}.$$

9.2 Vector Space Properties

Warning : Read the statements of Theorem Theorem 9.9 (Vector Space Properties of Column Vectors) and skip the rest of this section **unless you are/going to be** a math major. The material skipped will not appear in the tests and the final exam. With definitions of vector addition and scalar multiplication we can state, and prove, several properties of each operation, and some properties that involve their interplay. We now collect ten of them here for later reference.

Theorem 9.9 (Vector Space Properties of Column Vectors)**.** *Suppose that $\mathbb{R}^m$ is the set of column vectors of size m with addition and scalar multiplication as defined in Definition Definition 9.4 (Column Vector Addition) and Definition Definition 9.6 (Column Vector Scalar Multiplication) . Then:*

1. **ACC** *Additive Closure, Column Vectors*

   If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$*, then* $\mathbf{u} + \mathbf{v} \in \mathbb{R}^m$.

2. **SCC** *Scalar Closure, Column Vectors*

   If $\alpha \in \mathbb{R}$ *and* $\mathbf{u} \in \mathbb{R}^m$*, then* $\alpha\mathbf{u} \in \mathbb{R}^m$.

3. **CC** *Commutativity, Column Vectors*

   If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$*, then* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

4. **AAC** *Additive Associativity, Column Vectors*

   If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^m$*, then* $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

5. **ZC** *Zero Vector, Column Vectors*

   There is a vector, $\mathbf{0}$*, called the* **zero vector***, such that* $\mathbf{u} + \mathbf{0} = \mathbf{u}$ *for all* $\mathbf{u} \in \mathbb{R}^m$.

6. **AIC** *Additive Inverses, Column Vectors*

   If $\mathbf{u} \in \mathbb{R}^m$*, then there exists a vector* $-\mathbf{u} \in \mathbb{R}^m$ *so that* $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

7. **SMAC** *Scalar Multiplication Associativity, Column Vectors*

   If $\alpha, \beta \in \mathbb{R}$ *and* $\mathbf{u} \in \mathbb{R}^m$*, then* $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.

8. **DVAC** *Distributivity across Vector Addition, Column Vectors*

   If $\alpha \in \mathbb{R}$ *and* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$*, then* $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

9. **DSAC** *Distributivity across Scalar Addition, Column Vectors*

   If $\alpha, \beta \in \mathbb{R}$ *and* $\mathbf{u} \in \mathbb{R}^m$*, then* $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.

10. **OC** *One, Column Vectors*

   If $\mathbf{u} \in \mathbb{R}^m$*, then* $1\mathbf{u} = \mathbf{u}$.

Proof of Vector Space Properties of Column Vectors. While some of these properties seem very obvious, they all require proof. However, the proofs are not very interesting, and border on tedious. We will prove one version of distributivity very carefully, and you can test your proof-building skills on some of the others. We need to establish an equality, so we will do so by beginning with one side of the equality, apply various definitions and theorems (listed to the right of each step) to massage the expression from the left into the expression on the right. Here we go with a proof of Definition 9.10Property DSAC in Theorem 9.9 (Vector Space Properties of Column Vectors). For $1 \leq i \leq m$,

$$\begin{aligned}
[(\alpha+\beta)\mathbf{u}]_i &= (\alpha+\beta)\,[\mathbf{u}]_i && \text{Column Vector Scalar Multiplication}\\
&= \alpha\,[\mathbf{u}]_i + \beta\,[\mathbf{u}]_i &&\\
&= [\alpha\mathbf{u}]_i + [\beta\mathbf{u}]_i && \text{Column Vector Scalar Multiplication}\\
&= [\alpha\mathbf{u} + \beta\mathbf{u}]_i && \text{Column Vector Addition}
\end{aligned}$$

Since the individual components of the vectors $(\alpha+\beta)\mathbf{u}$ and $\alpha\mathbf{u} + \beta\mathbf{u}$ are equal for all i, $1 \leq i \leq m$, Definition 9.2 (Column Vector Equality) tells us the vectors are equal. □

Many of the conclusions of our theorems can be characterized as **identities**, especially when we are establishing basic properties of operations such as those in this section. Most of the properties listed in Theorem Theorem 9.9 (Vector Space Properties of Column Vectors) are examples. So some advice about the style we use for proving identities is appropriate right now. Be careful with the notion of the vector $-\mathbf{u}$. This is a vector that we add to $\mathbf{u}$ so that the result is the particular vector $\mathbf{0}$. This is basically a property of vector addition. It happens that we can compute $-\mathbf{u}$ using the other operation, scalar multiplication. We can prove this directly by writing that

$$[-\mathbf{u}]_i = -\,[\mathbf{u}]_i = (-1)\,[\mathbf{u}]_i = [(-1)\mathbf{u}]_i$$

We will see later how to derive this property as a **consequence** of several of the ten properties listed in Theorem Theorem 9.9 (Vector Space Properties of Column Vectors). Similarly, we will often write something you would immediately recognize as **vector subtraction**. This could be placed on a firm theoretical foundation – as you can do yourself with exercise T30. A final note. Theorem 9.9 (Vector Space Properties of Column Vectors) Property **AAC** implies that we do not have to be careful about how we *parenthesize* the addition of vectors. In other words, there is nothing to be gained by writing $(\mathbf{u}+\mathbf{v}) + (\mathbf{w} + (\mathbf{x}+\mathbf{y}))$ rather than $\mathbf{u} + \mathbf{v} + \mathbf{w} + \mathbf{x} + \mathbf{y}$, since we get the same result no matter which order we choose to perform the four additions. So we will not be careful about using parentheses this way.

9.3 Vector Space

For math major only. Non-math major can skip the rest of this section. The material will not appear in the midterms or final In this section we will give an abstract definition of vector space.

Why do we need the abstract definitions? A lot of different algebraic objects (e.g. polynomials, matrices, sequences, functions) share similar properties with the set of column vectors. We can use the common properties to derive similar results. we therefore don't need to reproof and restate the results.

One stone, kill many birds .

Definition 9.10. Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by $+$, and (2) **scalar multiplication**, which combines a real number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** over $\mathbb{R}$ if the following ten properties hold.

1. **AC** *Additive Closure*

   If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.

2. **SC** *Scalar Closure*

   If $\alpha \in \mathbb{R}$ and $\mathbf{u} \in V$, then $\alpha\mathbf{u} \in V$.

3. **C** *Commutativity*

   If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

4. **AA** *Additive Associativity*

   If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

5. **Z** *Zero Vector*

   There is a vector, $\mathbf{0}$, called the **zero vector**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.

6. **AI** *Additive Inverses*

   If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

7. **SMA** *Scalar Multiplication Associativity*

   If $\alpha,\ \beta \in \mathbb{R}$ and $\mathbf{u} \in V$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.

8. **DVA** *Distributivity across Vector Addition*

   If $\alpha \in \mathbb{R}$ and $\mathbf{u},\ \mathbf{v} \in V$, then $\alpha(\mathbf{u}+\mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

9. **DSA** *Distributivity across Scalar Addition*

   If $\alpha,\ \beta \in \mathbb{R}$ and $\mathbf{u} \in V$, then $(\alpha+\beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.

10. **O** *One*

    If $\mathbf{u} \in V$, then $1\mathbf{u} = \mathbf{u}$.

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.

Example 9.11. column vector space The set of column vectors $\mathbb{R}^n$ is a vector space.

Example 9.12. Row vector space

The set of row vector ($1 \times n$ matrices), is a vector space with the following operations:

- Vector addition: $[a_1\, a_2\, \ldots\, a_n] + [a_1\, b_2\, \ldots\, b_n] = [a_1+b_1\, a_2+b_2\, \ldots\, a_n+b_n]$
- Scalar multiplication $\alpha[a_1\, a_2\, \ldots, a_n] = [\alpha a_1\, \alpha a_2, \ldots, \alpha a_n]$

Example 9.13. Matrices

The set of $m \times n$ matrices, denoted by M_{mn}, is a vector space with the following operations:

- Vector addition:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

- Scalar multiplication

$$\alpha \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{bmatrix}$$

Property Z: The zero vector is

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Property AI:The inverse of

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is

$$\begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{m1} & -a_{m2} & \cdots & -a_{mn} \end{bmatrix}$$

You can try proving all other properties.

Example 9.14. The vector space of polynomials, P_n

The set of all polynomials of degree n or less in the variable x with coefficients from $\mathbb{R}$, denoted by P_n is a vector space.

- Vector Addition:

$$(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) + (b_0 + b_1x + b_2x^2 + \cdots + b_nx^n)$$

$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n$$

- Scalar Multiplication:

$$\alpha(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots + (\alpha a_n)x^n$$

This set, with these operations, will fulfill the ten properties, though we will not work all the details here. However, we will make a few comments and prove one of the properties. First, the zero vector (property Z) is what you might expect, and you can check that it has the required property.

$$\mathbf{0} = 0 + 0x + 0x^2 + \cdots + 0x^n$$

The additive inverse (Property AI) is also no surprise, though consider how we have chosen to write it.

$$-\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right) = (-a_0) + (-a_1)x + (-a_2)x^2 + \cdots + (-a_n)x^n$$

Now let us prove the associativity of vector addition (Property AA). This is a bit tedious, though necessary. Throughout, the plus sign (+) does triple-duty. You might ask yourself what each plus sign represents as you work through this proof.

$$\begin{aligned}
&\mathbf{u}+(\mathbf{v}+\mathbf{w})\\
&\quad= (a_0 + a_1x + \cdots + a_nx^n) + ((b_0 + b_1x + \cdots + b_nx^n) + (c_0 + c_1x + \cdots + c_nx^n))\\
&\quad= (a_0 + a_1x + \cdots + a_nx^n) + ((b_0 + c_0) + (b_1 + c_1)x + \cdots + (b_n + c_n)x^n)\\
&\quad= (a_0 + (b_0 + c_0)) + (a_1 + (b_1 + c_1))x + \cdots + (a_n + (b_n + c_n))x^n\\
&\quad= ((a_0 + b_0) + c_0) + ((a_1 + b_1) + c_1)x + \cdots + ((a_n + b_n) + c_n)x^n\\
&\quad= ((a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n) + (c_0 + c_1x + \cdots + c_nx^n)\\
&\quad= ((a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n)) + (c_0 + c_1x + \cdots + c_nx^n)\\
&\quad= (\mathbf{u}+\mathbf{v})+\mathbf{w}
\end{aligned}$$

You might try proving all the other properties.

Example 9.15. The vector space of functions

Let F be the set of functions for $\mathbb{R}$ to $\mathbb{R}$ Equality: $f = g$ if and only if $f(x) = g(x)$ for all $x \in \mathbb{R}$.

- Vector Addition: $f + g$ is the function with outputs defined by $(f + g)(x) = f(x) + g(x)$.
- Scalar Multiplication: αf is the function with outputs defined by $(\alpha f)(x) = \alpha f(x)$.

The zero vector is a function z whose definition is $z(x) = 0$ for every input $x \in \mathbb{R}$. Try proving all the other properties.

Example 9.16. The crazy vector space

Let $C = \{\, (x_1,\, x_2) \mid x_1,\, x_2 \in \mathbb{R}\}$.

1. Vector Addition: $(x_1,\, x_2) + (y_1,\, y_2) = (x_1 + y_1 + 1,\, x_2 + y_2 + 1)$.
2. Scalar Multiplication: $\alpha(x_1,\, x_2) = (\alpha x_1 + \alpha - 1,\, \alpha x_2 + \alpha - 1)$.

I am free to define my set and my operations any way I please. They may not look natural, or even useful, but we will now verify that they provide us with another example of a vector space. We will check all it satisfies all the definition of vector spaces.

- **Property AC, SC**

  The result of each operation is a pair of complex numbers, so these two closure properties are fulfilled.

- **Property C**

$$
\begin{aligned}
\mathbf{u} + \mathbf{v} &= (x_1,\, x_2) + (y_1,\, y_2) = (x_1 + y_1 + 1,\, x_2 + y_2 + 1) \\
&= (y_1 + x_1 + 1,\, y_2 + x_2 + 1) = (y_1,\, y_2) + (x_1,\, x_2) \\
&= \mathbf{v} + \mathbf{u}
\end{aligned}
$$

- **Property AA**

$$
\begin{aligned}
\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (x_1,\, x_2) + ((y_1,\, y_2) + (z_1,\, z_2)) \\
&= (x_1,\, x_2) + (y_1 + z_1 + 1,\, y_2 + z_2 + 1) \\
&= (x_1 + (y_1 + z_1 + 1) + 1,\, x_2 + (y_2 + z_2 + 1) + 1) \\
&= (x_1 + y_1 + z_1 + 2,\, x_2 + y_2 + z_2 + 2) \\
&= ((x_1 + y_1 + 1) + z_1 + 1,\, (x_2 + y_2 + 1) + z_2 + 1) \\
&= (x_1 + y_1 + 1,\, x_2 + y_2 + 1) + (z_1,\, z_2) \\
&= ((x_1,\, x_2) + (y_1,\, y_2)) + (z_1,\, z_2) \\
&= (\mathbf{u} + \mathbf{v}) + \mathbf{w}
\end{aligned}
$$

- **Property Z**

  The zero vector is $\mathbf{0} = (-1,\, -1)$ (**not** $(0, 0)$)

  $\mathbf{u} + \mathbf{0} = (x_1,\, x_2) + (-1,\, -1) = (x_1 + (-1) + 1,\, x_2 + (-1) + 1) = (x_1,\, x_2) = \mathbf{u}$

- **Property AI**

  For each vector, $\mathbf{u}$, we must locate an additive inverse, $-\mathbf{u}$. Here it is, $-(x_1,\, x_2) = (-x_1 - 2,\, -x_2 - 2)$. As odd as it may look, I hope you are withholding judgment. Check:

$$
\begin{aligned}
\mathbf{u} + (-\mathbf{u}) &= (x_1,\, x_2) + (-x_1 - 2,\, -x_2 - 2) \\
&= (x_1 + (-x_1 - 2) + 1,\, -x_2 + (x_2 - 2) + 1) = (-1,\, -1) = \mathbf{0}
\end{aligned}
$$

- **Property SMA**

$$\begin{aligned}\alpha(\beta\mathbf{u}) &= \alpha(\beta(x_1,\, x_2)) \\ &= \alpha(\beta x_1 + \beta - 1,\, \beta x_2 + \beta - 1) \\ &= (\alpha(\beta x_1 + \beta - 1) + \alpha - 1,\, \alpha(\beta x_2 + \beta - 1) + \alpha - 1) \\ &= ((\alpha\beta x_1 + \alpha\beta - \alpha) + \alpha - 1,\, (\alpha\beta x_2 + \alpha\beta - \alpha) + \alpha - 1) \\ &= (\alpha\beta x_1 + \alpha\beta - 1,\, \alpha\beta x_2 + \alpha\beta - 1) \\ &= (\alpha\beta)(x_1,\, x_2) \\ &= (\alpha\beta)\mathbf{u}\end{aligned}$$

- **Property DVA**

  If you have hung on so far, here is where it gets even wilder. In the next two properties we mix and mash the two operations.

$$\begin{aligned}&\alpha(\mathbf{u} + \mathbf{v}) \\ &\quad = \alpha\left((x_1,\, x_2) + (y_1,\, y_2)\right) \\ &\quad = \alpha(x_1 + y_1 + 1,\, x_2 + y_2 + 1) \\ &\quad = (\alpha(x_1 + y_1 + 1) + \alpha - 1,\, \alpha(x_2 + y_2 + 1) + \alpha - 1) \\ &\quad = (\alpha x_1 + \alpha y_1 + \alpha + \alpha - 1,\, \alpha x_2 + \alpha y_2 + \alpha + \alpha - 1) \\ &\quad = (\alpha x_1 + \alpha - 1 + \alpha y_1 + \alpha - 1 + 1,\, \alpha x_2 + \alpha - 1 + \alpha y_2 + \alpha - 1 + 1) \\ &\quad = ((\alpha x_1 + \alpha - 1) + (\alpha y_1 + \alpha - 1) + 1,\, (\alpha x_2 + \alpha - 1) + (\alpha y_2 + \alpha - 1) + 1) \\ &\quad = (\alpha x_1 + \alpha - 1,\, \alpha x_2 + \alpha - 1) + (\alpha y_1 + \alpha - 1,\, \alpha y_2 + \alpha - 1) \\ &\quad = \alpha(x_1,\, x_2) + \alpha(y_1,\, y_2) \\ &\quad = \alpha\mathbf{u} + \alpha\mathbf{v}\end{aligned}$$

- **Property DSA**

$$\begin{aligned}&(\alpha + \beta)\mathbf{u} \\ &\quad = (\alpha + \beta)(x_1,\, x_2) \\ &\quad = ((\alpha + \beta)x_1 + (\alpha + \beta) - 1,\, (\alpha + \beta)x_2 + (\alpha + \beta) - 1) \\ &\quad = (\alpha x_1 + \beta x_1 + \alpha + \beta - 1,\, \alpha x_2 + \beta x_2 + \alpha + \beta - 1) \\ &\quad = (\alpha x_1 + \alpha - 1 + \beta x_1 + \beta - 1 + 1,\, \alpha x_2 + \alpha - 1 + \beta x_2 + \beta - 1 + 1) \\ &\quad = ((\alpha x_1 + \alpha - 1) + (\beta x_1 + \beta - 1) + 1,\, (\alpha x_2 + \alpha - 1) + (\beta x_2 + \beta - 1) + 1) \\ &\quad = (\alpha x_1 + \alpha - 1,\, \alpha x_2 + \alpha - 1) + (\beta x_1 + \beta - 1,\, \beta x_2 + \beta - 1) \\ &\quad = \alpha(x_1,\, x_2) + \beta(x_1,\, x_2) \\ &\quad = \alpha\mathbf{u} + \beta\mathbf{u}\end{aligned}$$

- **Property O**

  After all that, this one is easy, but no less pleasing.

  $$1\mathbf{u} = 1(x_1,\, x_2) = (x_1 + 1 - 1,\, x_2 + 1 - 1) = (x_1,\, x_2) = \mathbf{u}$$

That is it, C is a vector space, as crazy as that may seem. Notice that in the case of the zero vector and additive inverses, we only had to propose possibilities and then verify that they were the correct choices. You might try to discover how you would arrive at these choices, though you should understand why the process of discovering them is not a necessary component of the proof itself.

9.4 Basic Properties of Vector Spaces

For math major only. Non-math major can skip the rest of this section. The material will not appear in the midterms or final

Theorem 9.17 (Cancellation Law for Vector Addition)**.** *if* $\mathbf{v}$*,* $\mathbf{u}$ *and* $\mathbf{w}$ *are vectors in a vector space* V *such that*

$$\mathbf{v} + \mathbf{w} = \mathbf{u} + \mathbf{w},$$

then $\mathbf{v} = \mathbf{u}$

Proof of Cancellation Law for Vector Addition. By Property AI, there exists a vector $-\mathbf{w}$ such that $\mathbf{w} + (-\mathbf{w}) = \mathbf{0}$. Thus,

$$\begin{aligned} (\mathbf{v} + \mathbf{w}) + (-\mathbf{w}) &= (\mathbf{u} + \mathbf{w}) + (-\mathbf{w}) && \\ \mathbf{v} + (\mathbf{w} + (-\mathbf{w})) &= \mathbf{u} + (\mathbf{w} + (-\mathbf{w})) && \text{Property AA} \\ \mathbf{v} + \mathbf{0} &= \mathbf{u} + \mathbf{0} && \text{Property AI} \\ \mathbf{v} &= \mathbf{u} && \text{Property Z} \end{aligned}$$

□

Theorem 9.18 (Uniqueness of the zero vector)**.** *Let* V *be a vector space. The vector* $\mathbf{0}$ *described in Property Z is unique.*

Proof of Uniqueness of the zero vector. Suppose both $\mathbf{0}_1$ and $\mathbf{0}_2$ satisfy the property described in Property Z. Let $\mathbf{w}$ be an element in V.

$$\mathbf{0}_1 + \mathbf{w} = \mathbf{w} = \mathbf{0}_2 + \mathbf{w} \qquad \text{Property Z}$$

$$\mathbf{0}_1 = \mathbf{0}_2 \qquad \text{by the previous theorem}$$

□

Theorem 9.19 (Uniqueness of the additive inverse)**.** *Let V be a vector space and $\mathbf{v}, \mathbf{u}, \mathbf{w}$ are vectors of V. If both $\mathbf{v}$ and $\mathbf{u}$ satisfies*

$$\mathbf{v} + \mathbf{w} = \mathbf{0},$$

$$\mathbf{u} + \mathbf{w} = \mathbf{0},$$

i.e., both $\mathbf{u}$ and $\mathbf{v}$ are additive inverse of $\mathbf{w}$ in Property AI, then

$$\mathbf{v} = \mathbf{u}.$$

This shows that the additive inverse is unique.

Proof of Uniqueness of the additive inverse.

$$\mathbf{v} + \mathbf{w} = \mathbf{0} = \mathbf{u} + \mathbf{w}.$$

By Theorem 9.17 (Cancellation Law for Vector Addition),

$$\mathbf{v} = \mathbf{u}.$$

□

Theorem 9.20. *Let V be a vector space, α a real number, $\mathbf{v}$ a vector in V. We have the following statement.*

1. $0\mathbf{v} = \mathbf{0}$.
2. $a\mathbf{0} = \mathbf{0}$.
3. $(-\alpha)\mathbf{v} = -(\alpha\mathbf{v}) = \alpha(-\mathbf{v})$.

Proof of Theorem 9.20. **For math major only**

1.

$$\begin{aligned} 0\mathbf{v} + 0\mathbf{v} &= (0+0)\mathbf{v} && \text{Property DSA} \\ 0\mathbf{v} + 0\mathbf{v} &= 0\mathbf{v} = \mathbf{0} + 0\mathbf{v} && \text{Property Z} \end{aligned}$$

Hence,

$$0\mathbf{v} = \mathbf{0},$$

by Theorem 9.17 (Cancellation Law for Vector Addition).

2.

$$\begin{aligned} \alpha\mathbf{0} + \alpha\mathbf{0} &= \alpha(\mathbf{0} + \mathbf{0}) && \text{Property DVA} \\ &= \alpha\mathbf{0} && \text{Property Z} \\ &= \mathbf{0} + \alpha\mathbf{0} && \text{Property Z} \end{aligned}$$

By Theorem 9.17 (Cancellation Law for Vector Addition),

$$\alpha\mathbf{0} = \mathbf{0}$$

3.

$$\begin{aligned} \alpha\mathbf{v} + (-\alpha)\mathbf{v} &= (\alpha + (-\alpha))\mathbf{v} && \text{Property DSA} \\ &= 0\mathbf{v} \\ &= \mathbf{0} && \text{item 1} \end{aligned}$$

By Definition 9.10 and the uniqueness of the additive inverse (Theorem 9.19 (Uniqueness of the additive inverse)),

$$(-\alpha)\mathbf{v} = -\alpha\mathbf{v}.$$

Next

$$\begin{aligned} \alpha\mathbf{v} + \alpha(-\mathbf{v}) &= \alpha(\mathbf{v} + (-\mathbf{v})) && \text{Property DVA} \\ &= \alpha\mathbf{0} && \text{Property AI} \\ &= \mathbf{0} && \text{item 2} \end{aligned}$$

By Property AI and the uniqueness of the additive inverse (Theorem 9.19 (Uniqueness of the additive inverse)),

$$\alpha(-\mathbf{v}) = -\alpha\mathbf{v}.$$

□

9.5 Subspaces

Definition 9.21. Let V be vector space. A subset W of V is said to be a **subspace** of V if:

1. W is nonempty.
2. For $\mathbf{v}, \mathbf{w} \in W$, then $\mathbf{v} + \mathbf{w} \in W$.

3. For $\alpha \in \mathbf{R}$, $\mathbf{v} \in W$, then $\alpha\mathbf{v} \in W$.

We will prove several theorem first before we give examples.

Proposition 9.22. *Let V be a vector space and W a subspace of V. Then $\mathbf{0}$ is in W.*

Proof of Proposition 9.22. By Definition 9.21, Condition 1, W is nonempty. Let $\mathbf{w} \in W$. By Definition 9.21, Condition 3, with $\alpha = 0$, $0\mathbf{w} \in W$. On the other hand, by Theorem 9.20, $0\mathbf{w} = \mathbf{0}$. Hence, the zero vector $\mathbf{0}$ lies in W. □

Theorem 9.23. *Let V be a vector space and W a subset of V, then W is a subspace if and only if*

1. *W is nonempty.*

2. *For any $\alpha \in \mathbf{R}$, $\mathbf{v}, \mathbf{w} \in W$, $\alpha\mathbf{v} + \mathbf{w} \in W$.*

Proof of Theorem 9.23. ($\Rightarrow$) Suppose W is a subspace.

By Definition 9.21, Condition 1, W is nonempty.

Next, for $\alpha \in \mathbb{R}$, $\mathbf{v}, \mathbf{w} \in W$. By Definition 9.21, Condition 3, $\alpha\mathbf{v} \in W$. By Definition 9.21, Condition 2, $\alpha\mathbf{v} + \mathbf{w} \in W$.

($\Leftarrow$) Suppose W satisfies Conditions 1 and 2 of the theorem.

By Condition 1 of the theorem, Definition 9.21 Condition 1 is true.

Now we want to check Definition 9.21 Condition 2.

Suppose $\mathbf{v}, \mathbf{w} \in W$. In Condition 2 of the theorem, let $\alpha = 1$, then $\mathbf{v} + \mathbf{w} = \alpha\mathbf{v} + \mathbf{w} \in W$.

Next we want to check Definition 9.21 Condition 3.

First, since W is nonempty, there exists some element $\mathbf{x} \in W$. Let $\mathbf{v} = \mathbf{w} = \mathbf{x}$ and $\alpha = -1$ in Condition 2 of the theorem. Then, $\mathbf{0} = (-1)\mathbf{w} + \mathbf{w} \in W$.

Now, for any $\mathbf{v} \in W$, $\alpha \in \mathbb{R}$, let $\mathbf{w} = \mathbf{0} \in W$. Then, $\alpha\mathbf{v} = \alpha\mathbf{v} + \mathbf{w} \in W$, by Condition 2 of the theorem. □

9.5.1 Examples

Example 9.24. Let $V = \mathbb{R}^3$. Let:

$$W = \left\{ \begin{bmatrix} w_1 \\ 0 \\ w_3 \end{bmatrix} \;\middle|\; w_1, w_3 \in \mathbb{R} \right\}.$$

We now show that W is a subspace of V:

1. Clearly, $\begin{bmatrix}0\\0\\0\end{bmatrix}$ lies in W, hence W is nonempty.

2. Given any $\vec{v}$ and $\vec{w}$ in W, by definition of W we have:

$$\vec{v} = \begin{bmatrix}v_1\\0\\v_3\end{bmatrix}, \quad \vec{w} = \begin{bmatrix}w_1\\0\\w_3\end{bmatrix}, \quad v_1, v_2, w_1, w_3 \in \mathbb{R}.$$

   Hence,

$$\vec{v}+\vec{w} = \begin{bmatrix}v_1+w_1\\0+0\\v_3+w_3\end{bmatrix} = \begin{bmatrix}v_1+w_1\\0\\v_3+w_3\end{bmatrix} \in W.$$

3. Given any $\alpha \in \mathbb{R}$ and $\vec{w}$ in W, by definition of W we have:

$$\vec{w} = \begin{bmatrix}w_1\\0\\w_3\end{bmatrix}, \quad w_1, w_3 \in \mathbb{R}.$$

   Hence,

$$\alpha\vec{w} = \begin{bmatrix}\alpha w_1\\\alpha\cdot 0\\\alpha w_3\end{bmatrix} = \begin{bmatrix}\alpha w_1\\0\\\alpha w_3\end{bmatrix} \in W.$$

Example 9.25. $V = \mathbb{R}^3$.

$$W = \left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x+2y+3z=0 \right\}$$

Theorem 9.26. *Let $A \in M_{mn}$, then $W = \mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.*

Proof of Theorem 9.26. Because $\mathbf{0} \in \mathcal{N}(A)$, so W is nonempty.
For $\alpha \in \mathbb{R}$, $\mathbf{v}, \mathbf{w} \in W$. Then

$$A\mathbf{v} = \mathbf{0}, \qquad A\mathbf{w} = \mathbf{0}.$$

Then

$$A(\alpha\mathbf{v}+\mathbf{w}) = \alpha A\mathbf{v} + A\mathbf{w} = \alpha\mathbf{0}+\mathbf{0} = \mathbf{0}.$$

So

$$\alpha\mathbf{v}+\mathbf{w} \in \mathcal{N}(A).$$

Thus by Theorem 9.23, $W = \mathcal{N}(A)$ is a subspace. $\square$

Example 9.27. Skip for now, until you learn the definition of column space. Let $A \in M_{mn}$, then $\mathcal{C}(A)$ is a subspace of $\mathbb{R}^m$.

Proof of Example 9.27. $\mathcal{C}(A) = \langle\{\mathbf{A}_1, \ldots, \mathbf{A}_n\}\rangle$. So by the previous theorem, $\mathcal{C}(A)$ is a subspace of $\mathbb{R}^m$. **Alternate proof**: Suppose For $\alpha \in \mathbb{R}$, $\mathbf{v}, \mathbf{w} \in W = \mathcal{C}(A)$. Recall

$$\mathcal{C}(A) = \{A\mathbf{x} \,|\, x \in \mathbb{R}^m\}.$$

Then there exist $\mathbf{x}, \mathbf{y}$ such that $A\mathbf{x} = \mathbf{v}$, $A\mathbf{y} = \mathbf{w}$.

$$\alpha\mathbf{v} + \mathbf{w} = \alpha A\mathbf{x} + A\mathbf{y} = A(\alpha\mathbf{x} + \mathbf{y}) \in \mathcal{C}(A)\,.$$

Thus by Theorem 9.23, W is a subspace. □

Example 9.28. For math majors only

Let S_n be the set of symmetric matrices of M_{nn}. Then S_n is a subspace of M_{nn}. Check that $W = S_n$ is a subspace: Because $\mathcal{O} \in W$, so W is nonempty. Suppose $\alpha \in \mathbb{R}$, $A, B \in W$. Then $A^t = A$, $B^t = B$.

$$(\alpha A + B)^t = \alpha A^t + B^t = \alpha A + B.$$

Thus $\alpha A + B \in S_n$. Hence S_n is a subspace by Theorem 9.23.

Example 9.29. For math majors only

Let

$$F = \{f(x) \in P_n \,|\, f(1) = 0\}.$$

Then F is a subspace of P_n Because $0 \in E$, so E is nonempty. Suppose $\alpha \in \mathbb{R}$, $f(x), g(x) \in E$. Then $f(1) = g(1) = 0$. Let $h = \alpha f + g$. Then

$$h(1) = \alpha f(1) + g(1) = \alpha 0 + 0 = 0.$$

So $h \in F$. Hence F is a subspace by Theorem 9.23.

Example 9.30. For math majors only

Let

$$E = \{f(x) \in P_n \,|\, f(x) = f(-x)\}.$$

Then E is a subspace of P_n: Because $0 \in E$, so E is nonempty. Suppose $\alpha \in \mathbb{R}$, $f(x), g(x) \in E$. Then $f(x) = f(-x)$, $g(x) = g(-x)$. Let $h = \alpha f + g$. Then

$$h(-x) = \alpha f(-x) + g(-x) = \alpha f(x) + g(x) = h(x).$$

So $h \in E$. Hence E is a subspace.

9.5.2 Non-Examples

To show that W is **not** a subspace of V, it suffices to show that it violates Definition 9.21 conditions 1, 2 or 3. This can be done by finding counter examples to either condition. Usually before checking those conditions, we quickly check if $\mathbf{0}_V \in W$ (see Proposition 9.22).

Example 9.31. $V = \mathbb{R}^m$, $W = \{\mathbf{v} \in V \,|\, [\mathbf{v}]_1 = 1\}$.

Method 1 Obviously $\mathbf{0}$ is not in W. So by Proposition 9.22, W is not a subspace.

Method 2 For Suppose $\mathbf{v}, \mathbf{w} \in W$. Then $[\mathbf{v} + \mathbf{w}]_1 = [\mathbf{v}]_1 + [\mathbf{w}]_1 = 2$. So $\mathbf{v}+\mathbf{w} \notin W$. So W violates Definition 9.21 condition 1 and hence not a subspace.

Example 9.32. $V = \mathbb{R}^m$, $W = \{\mathbf{v} \in V \,|\, \sum_{i=1}^{m} [\mathbf{v}]_i = 1\}$.

Method 1 (the easiest method) Obviously $\mathbf{0}$ is not in W. So by Proposition 9.22, W is not a subspace.

Method 2 We will find an explicit counter example, let

$$\mathbf{v} = \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then both $\mathbf{v}$ and $\mathbf{w}$ are in W.

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} 2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Obvious $\mathbf{v}+\mathbf{w} \notin W$. Therefore W violates Definition 9.21 condition 1 and hence not a subspace.

Example 9.33. $V = \mathbf{R}^2$, $W = \{\mathbf{v} \in V \,|\, [\mathbf{v}]_1 \geq 0\}$. Let $\alpha = -1$ and

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then $[\alpha\mathbf{v}]_1 = \alpha[\mathbf{v}]_1 = \alpha = -1 < 0$. So $\alpha\mathbf{v} \notin W$. Thus W violates Definition 9.21 condition 3 and hence not a subspace.

Example 9.34. $V = \mathbb{R}^2$, $W = \{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in V \,|\, v_1 v_2 \geq 0\}$. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \in W$. $\mathbf{v}+\mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Because $1 \times (-1) = -1 < 0$. So $\mathbf{v}+\mathbf{w} \notin W$. Thus W violates Definition 9.21 condition 2 and hence not a subspace. In fact, we can show that W satisfies Definition 9.21 condition 3 but fails condition 2.

Example 9.35. For math majors only

Let $V = P_n$. Let G be the set of polynomial with degree exactly equal to n. Let $f(x) = x^n$, $g(x) = -x^n + 1$. Both f and g have degree exactly equal to n. But

$$f(x) + g(x) = 1$$

is a polynomial with degree 0. So $f + g$ is not in G. Thus W violates Definition 9.21 condition 2 and hence not a subspace.