MATH 1030 Chapter 3

The lecture is based on Beezer, A first course in Linear algebra. Ver 3.5 Downloadable at http://linear.ups.edu/download.html .

The print version can be downloaded at http://linear.ups.edu/download/fcla-3.50-print.pdf .

Reference.

Beezer, Ver 3.5 Sect SSLE (print version p7 - p14)

3.1 Introduction

Definition 3.1 (System of Linear Equations)**.** A **system of linear equations** is a collection of m equations in the variables $x_1,\ x_2,\ x_3, \ldots, x_n$ of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij}, b_i and x_j, $1 \leq i \leq m$, $1 \leq j \leq n$, are real numbers.

Definition 3.2. $(s_1, s_2, \ldots, s_n)$ is a **solution** of a system of linear equations in n variables if every equation in the system is valid for $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$. The **solution set** of a linear system of equations is the set consisting of all solutions to the system, and nothing more.

Example 3.3. The following system of linear equations

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 7 \\ x_1 + x_2 + x_3 - x_4 &= 3 \\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 \end{aligned}$$

can be rewritten as:

$$\begin{aligned}1x_1 + 2x_2 + 0x_3 + 1x_4 &= 7\\ 1x_1 + 1x_2 + 1x_3 - 1x_4 &= 3\\ 3x_1 + 1x_2 + 5x_3 - 7x_4 &= 1\end{aligned}$$

So it is a system of linear equations, with $n = 4$ variables and $m = 3$ equations. Also,

$$\begin{array}{lllll} a_{11} = 1 & a_{12} = 2 & a_{13} = 0 & a_{14} = 1 & b_1 = 7\\ a_{21} = 1 & a_{22} = 1 & a_{23} = 1 & a_{24} = -1 & b_2 = 3\\ a_{31} = 3 & a_{32} = 1 & a_{33} = 5 & a_{34} = -7 & b_3 = 1\end{array}$$

One solution is given by $x_1 = -2$, $x_2 = 4$, $x_3 = 2$, $x_4 = 1$. In fact, from the previous lecture this system of equations has infinitely many solutions.

The solution set may be described as follows:

$$\{(-1 - 2a + 3b,\ 4 + a - 2b,\ a,\ b) \mid a, b \in \mathbb{R}\}.$$

3.2 Possibilities for Solution Sets

Definition 3.4. A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

Example 3.5.

1. The following system of linear equations has **only one solution**.

$$\begin{aligned}2x_1 + 3x_2 &= 3\\ x_1 - x_2 &= 4\end{aligned}$$

The solution set is $(x_1, x_2) = (3, -1)$.

Since the solution set is nonempty, the system is **consistent**.

2. The following system of linear equations has **infinite many solutions**.

$$\begin{aligned}2x_1 + 3x_2 &= 3\\ 4x_1 + 6x_2 &= 6\end{aligned}$$

The solution set is $\{(x_1, x_2) = (t, \frac{3-2t}{3})\}$, where t is any real number.

Since the solution set is nonempty, the system is **consistent**.

3. The following system of linear equations has **no solutions**.

$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ 4x_1 + 6x_2 &= 10 \end{aligned}$$

   The solution set is empty.

   So, the system is **inconsistent**.

Theorem 3.6. *A system of linear equations can have (1) a unique solution or (2) infinitely many solutions or (3) no solutions.*

Remark. For example it is impossible for a system of linear equation to have exactly 2 solutions.

3.3 Equivalent Systems and Equation Operations

Definition 3.7 (Equivalent Systems)**.** Two systems of linear equations are **equivalent** if their solution sets are equal.

Definition 3.8 (Equation Operations)**.** Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an **equation operation**:

1. Swap the locations of two equations in the list of equations.
2. Multiply each term of an equation by a nonzero number.
3. Multiply each term of one equation by some number, and add these terms to a second equation, on both sides of the equality.

Theorem 3.9 (Equation Operations Preserve Solution Sets)**.** *If we apply one of the three equation operations of Definition Definition 3.8 (Equation Operations) to a system of linear equations, then the original system and the transformed system are equivalent.*

Proof of Equation Operations Preserve Solution Sets. This will be clear after we have established Theorem Theorem 4.17 (Row-Equivalent Matrices represent Equivalent Systems) in a later chapter. □

Example 3.10. The following two systems equations are equivalent:

$$\begin{cases} 2x_1 + 3x_2 = 3 \\ x_1 - x_2 = 4 \end{cases}$$

and

$$\begin{cases} 5x_2 = -5 \\ x_1 - x_2 = 4 \end{cases}$$

In fact, the second system of linear equations is obtained by applying operation 3 on equation 1 (namley, replace equation 1 with [equation 1] - 2 $\times$ [equation 2]).