MATH 1030 Chapter 2

The lecture is based on Beezer, A first course in Linear algebra. Ver 3.5 Downloadable at http://linear.ups.edu/download.html .

The print version can be downloaded at http://linear.ups.edu/download/fcla-3.50-print.pdf .

2.1 Intersections between Lines and Planes

2.1.1 Two Dimensions

Example 2.1. Solve the equations:

$$\begin{cases} 2x - 3y = -5, \\ x + y = 5. \end{cases}$$

There are two variables and we can draw these equations in a (2-dimensional) plane. Each (linear) equation represented by a straight line. For the line that corresponds to $2x - 3y = -5$, it passes through the point $(0, 5/3)$ (take $x = 0$ and one gets $y = 5/3$) and another the point $(-5/2, 0)$. Hence it is determined. Similarly, the second straight line is the one passes through the points $(0, 5)$ and (5.0).

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Intersection between lines The solution of the two equations corresponds to the unique intersection $(2, 3)$ between the two straight lines.

Example 2.2. The following system of equations has no solution.

$$\begin{cases} 2x - 3y = -5, \\ 2x - 3y = 0. \end{cases}$$

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Parallel lines This is because the corresponding lines are parallel to each other and there is no intersections.

Example 2.3. How about the system of equations:

$$\begin{cases} 2x - 3y = -5, \\ 2x - 3y = -5. \end{cases}$$

Here, there are infinite solutions: For any real number t,

$$(x, y) = (t, \frac{2t+5}{3})$$

is a solution. The two lines coincide and their intersection is just the whole line.

2.1.2 Three Dimensions

Example 2.4. Solve the equations:

$$\begin{cases} 2x + 3y + 3z = 3, \\ 2x - y + 0z = 0. \end{cases}$$

Intersection of planes

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There are three variables and we work in 3-dimensional space. Each (linear) equation corresponds to a plane. For instance, the ones above correspond to the blue and red planes in the figure above.

Their intersection is the thick straight line, where the points on it can be expressed as

$$\left\{ \left(\frac{3-3t}{8}, \frac{3-3t}{4}, t \right) \mid t \in \mathbb{R} \right\}.$$

Example 2.5. The following system of equations has no solutions, since the corresponding planes are parallel.

$$\begin{cases} 2x - y + 0z = 2, \\ 2x - y + 0z = 0. \end{cases}$$

Parallel Planes

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Example 2.6 (Different Cases)**.** Suppose that there are three (linear) equations with three variables that corresponds to three distinct planes. If they do not share a unique intersection, then either they share no intersections or their intersection is a line. More precisely, there are five cases:

1. All three planes are parallel to each other.

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2. Only two planes are parallel to each other.

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3. The intersection of each pair of planes is a line and three such lines are parallel to each other.

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4. Their intersection is a line.

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5. Their intersection is a point, e.g. the xy-plane, yz-plane and zx-plane intersect at $(0, 0, 0)$.

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Remark. Higher Dimensions How does this row picture extend into n dimensions? The n equations will contain n unknowns. The first equation still determines a plane. It is no longer a two dimensional plane in 3-space; somehow it has dimension $n - 1$. It must be flat and extremely thin within n-dimensional space, although it would look solid to us.

If time is the fourth dimension, then the plane t = 0 cuts through four-dimensional space and produces the three-dimensional universe we live in (or rather, the universe as it was at $t = 0$). Another plane is $z = 0$, which is also three-dimensional; it is the ordinary $x - y$ plane taken over all time. Those three-dimensional planes will intersect! They share the ordinary $x - y$ plane at $t = 0$. We are down to two dimensions, and the next plane leaves a line. Finally a fourth plane leaves a single point. It is the intersection point of 4 planes in 4 dimensions, and it solves the 4 underlying equations.

I will be in trouble if that example from relativity goes any further. The point is that linear algebra can operate with any number of equations. The first equation produces an $(n - 1)$-dimensional plane in n dimensions, The second plane intersects it (we hope) in a smaller set of dimension $n - 2$. Assuming all goes well, every new plane (every new equation) reduces the dimension by one. At the end, when all n planes are accounted for, the intersection has dimension zero. It is a point, it lies on all the planes, and its coordinates satisfy all n equations. It is the solution!