MATH 1030 Chapter 12

The lecture is based on Beezer, A first course in Linear algebra. Ver 3.5 Downloadable at http://linear.ups.edu/download.html .

The print version can be downloaded at http://linear.ups.edu/download/fcla-3.50-print.pdf .

Reference.

- Beezer, Ver 3.5 Section LDS (print version p105 - p113)
- Strang, Sect 2.3

Exercise

- Exercises with solutions can be downloaded at http://linear.ups.edu/download/fcla-3.50-solution-manual.pdf Section LI (p.48-51) (Replace $\mathbb{C}$ by $\mathbb{R}$ in the following questions) C20, C40, C50, C51, C52, C55, C70, M10, T40.
- Strang, Sect 2.3

12.1 Linearly Independent Sets of Vectors

Definition 12.1 (Relation of Linear Dependence)**.** Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$, an equality of the form

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S. If this equality is formed in a trivial fashion, i.e., $\alpha_i = 0$, $1 \leq i \leq n$, then we say that it is the **trivial relation of linear dependence** on S.

Definition 12.2 (Linear Independence)**.** The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the only relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Remark. In short, a set of vectors $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_n\}$ is linearly independent if and only if the only solution to:

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots + x_n\mathbf{u}_n = \mathbf{0}$$

is:

$$x_1 = x_2 = \cdots = x_n = 0.$$

Theorem 12.3 (Linearly Independent Vectors and Homogeneous Systems)**.** *Suppose that $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \ldots,\ \mathbf{v}_n\} \subseteq \mathbb{R}^m$ is a set of vectors and that A is the $m\times n$ matrix whose columns are the vectors in S. Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A,\mathbf{0})$ has a unique solution.*

Proof of Linearly Independent Vectors and Homogeneous Systems. ($\Leftarrow$) Suppose that $\mathcal{LS}(A,\mathbf{0})$ has a unique solution. Since it is a homogeneous system, this solution must be the trivial solution, $\mathbf{x} = \mathbf{0}$. This means that the only relation of linear dependence on S is the trivial one. So S is linearly independent.

($\Rightarrow$) We will prove the contrapositive. Suppose that $\mathcal{LS}(A,\mathbf{0})$ does not have a unique solution. Since it is a homogeneous system, it is consistent. And so must have infinitely many solutions. One of these infinitely many solutions must be nontrivial (in fact, almost all of them are); choose one. This nontrivial solution will give a nontrivial relation of linear dependence on S. We therefore conclude that S is a linearly dependent set. $\square$

Since the above theorem is an "if-and-only-if" statement, we can use it to determine the linear independence or dependence of any set of column vectors, just by creating a matrix and analyzing its row-reduced echelon form. Let us illustrate this with two more examples.

Example 12.4. Linearly dependent set in $\mathbb{R}^5$

Consider the following set of $n = 4$ vectors in $\mathbb{R}^5$:

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix},\ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix},\ \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix},\ \begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

To determine linear independence, we first form an arbitrary relation of linear dependence,

$$\alpha_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

We know that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ is a solution to this equation, but that is of no interest whatsoever. That is always the case, no matter what four vectors we might have chosen. We are curious to know if there are other, nontrivial, solutions.

In other words, are there nontrivial solutions to the homogeneous linear system $\mathcal{LS}(A, \mathbf{0})$, where the columns of A consist of the vectors in S.

Row-reducing the matrix A gives:

$$A = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & -2 \\ 0 & \boxed{1} & 0 & 4 \\ 0 & 0 & \boxed{1} & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We could solve the corresponding homogeneous system completely, but for this example all we need is one nontrivial solution. Setting the lone free variable to any nonzero value, such as $x_4 = 1$, yields the nontrivial solution:

$$\mathbf{x} = \begin{bmatrix} 2 \\ -4 \\ 3 \\ 1 \end{bmatrix}.$$

Hence,

$$2 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} + (-4) \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}.$$

This is a relation of linear dependence on S that is not trivial, so we conclude that S is **linearly dependent** .

Example 12.5. Linearly independent set in $\mathbb{R}^5$

Consider the following set of $n = 4$ vectors in $\mathbb{R}^5$:

$$T = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

To determine linear independence we first form an arbitrary relation of linear dependence,

$$\alpha_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} -6 \\ 7 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}.$$

We want to know if there are solutions to the equation above besides the trivial one: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.

Row-reducing the associated matrix gives:

$$B = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the form of this matrix, we see that there are no free variables. Hence the associated homogeneous linear system has only the trivial solution. So we now know that there is but one way to combine the four vectors of T into a relation of linear dependence, and that this one way is the easy and obvious way. Hence, the set T is **linearly independent** .

12.1.1 More Examples

Example 12.6. Linearly independent

Is the set of vectors:

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -4 \\ 5 \\ 1 \end{bmatrix} \right\}$$

linearly independent or linearly dependent?

Solution. The above theorem suggests that we study the matrix A whose columns are the vectors in S. Specifically, we are interested in the size of the solution set of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. Row-reducing A gives:

$$A = \begin{bmatrix} 2 & 6 & 4 \\ -1 & 2 & 3 \\ 3 & -1 & -4 \\ 4 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have $r = 3$, so there are $n - r = 3 - 3 = 0$ free variables. Hence $\mathcal{LS}(A, \mathbf{0})$ has a unique solution. By the above theorem, the set S is linearly independent.

Example 12.7. Linearly dependent

Is the set of vectors:

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -4 \\ -1 \\ 2 \end{bmatrix} \right\}$$

linearly independent or linearly dependent?

Solution. Theorem Theorem 12.3 (Linearly Independent Vectors and Homogeneous Systems) suggests that we study the matrix A whose columns are the vectors in S. Specifically, we are interested in the size of the solution set of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. Row-reducing A gives

$$A = \begin{bmatrix} 2 & 6 & 4 \\ -1 & 2 & 3 \\ 3 & -1 & -4 \\ 4 & 3 & -1 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have $r = 2$, so there are $n - r = 3 - 2 = 1$ free variables. Hence $\mathcal{LS}(A, \mathbf{0})$ has infinitely many solutions. By Theorem Theorem 12.3 (Linearly Independent Vectors and Homogeneous Systems), the set S is linearly dependent.

Theorem Theorem 12.3 (Linearly Independent Vectors and Homogeneous Systems) gives us a straightforward way to determine if a set of vectors is linearly independent or dependent.

Review the previous two examples. They are very similar, differing only in the last two slots of the third vector. This resulted in slightly different matrices when row-reduced, and different values of r, the number of nonzero rows. Notice, too, that we are less interested in the actual solution set, and more interested in its form or size. These observations allow us to make a slight improvement on Theorem Theorem 12.3 (Linearly Independent Vectors and Homogeneous Systems).

12.2 Linearly Dependent Sets and Spans

If we use a linearly dependent set to construct a span, then we can always create the same infinite set by starting with a set that is one vector smaller in size. We will illustrate this behaviour in Example 12.9. However, this will not be possible if we build a span from a linearly independent set. So, in a certain sense, using a linearly independent set to formulate a span is the best possible way – there are no any extra vectors being used to build up all the necessary linear combinations. OK, here is the theorem, and then the example.

Theorem 12.8 (Dependency in Linearly Dependent Sets)**.** *Suppose that* $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ *is a set of vectors. Then* S *is a linearly dependent set if and only if there is an index* t*,* $1 \leq t \leq n$*, such that* $\mathbf{u_t}$ *is a linear combination of the vectors* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \ldots, \mathbf{u}_n$.

Proof of Dependency in Linearly Dependent Sets. ($\Rightarrow$) Suppose that S is linearly dependent. Then there exists a nontrivial relation of linear dependence (Definition 12.1 (Relation of Linear Dependence)). That is, there are scalars, α_i, $1 \leq i \leq n$, not all of which are zero, such that

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}.$$

Suppose that α_t is nonzero. Then,

$$\begin{aligned}
\mathbf{u}_t &= \frac{-1}{\alpha_t}(-\alpha_t\mathbf{u}_t) \\
&= \frac{-1}{\alpha_t}(\alpha_1\mathbf{u}_1 + \cdots + \alpha_{t-1}\mathbf{u}_{t-1} + \alpha_{t+1}\mathbf{u}_{t+1} + \cdots + \alpha_n\mathbf{u}_n) \\
&= \frac{-\alpha_1}{\alpha_t}\mathbf{u}_1 + \cdots + \frac{-\alpha_{t-1}}{\alpha_t}\mathbf{u}_{t-1} + \frac{-\alpha_{t+1}}{\alpha_t}\mathbf{u}_{t+1} + \cdots + \frac{-\alpha_n}{\alpha_t}\mathbf{u}_n.
\end{aligned}$$

Since $\frac{\alpha_i}{\alpha_t}$ is again a scalar, we have expressed $\mathbf{u}_t$ as a linear combination of the other elements of S.

($\Leftarrow$) Assume that the vector $\mathbf{u}_t$ is a linear combination of the other vectors in S. Write such a linear combination as

$$\mathbf{u_t} = \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_{t-1}\mathbf{u}_{t-1} + \beta_{t+1}\mathbf{u}_{t+1} + \cdots + \beta_n\mathbf{u}_n.$$

Then we have

$$\begin{aligned}
&\beta_1\mathbf{u}_1 + \cdots + \beta_{t-1}\mathbf{u}_{t-1} + (-1)\mathbf{u}_t + \beta_{t+1}\mathbf{u}_{t+1} + \cdots + \beta_n\mathbf{u}_n \\
&\qquad = \mathbf{u}_t + (-1)\mathbf{u}_t \\
&\qquad = (1 + (-1))\,\mathbf{u}_t \\
&\qquad = 0\mathbf{u}_t \\
&\qquad = \mathbf{0}.
\end{aligned}$$

So the scalars $\beta_1,\ \beta_2,\ \beta_3,\ \ldots,\ \beta_{t-1},\ \beta_t = -1, \beta_{t+1},\ \ldots,\ \beta_n$ provide a nontrivial relation of linear dependence of the vectors in S, thus establishing that S is a linearly dependent set. $\square$

This theorem can be used, sometimes repeatedly, to whittle down the size of a set of vectors used in a span construction. In the next example we will examine some of the subtleties.

Example 12.9. Reducing the generating set of a span in $\mathbb{R}^5$

Consider the following set of $n = 4$ vectors in $\mathbb{R}^5$,

$$R = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -7 \\ 6 \\ -11 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \\ 6 \end{bmatrix} \right\}.$$

Define $V = \operatorname{Span} R$.

We form a 5×4 matrix, D, and row-reduce it to understand the solutions to the homogeneous system $\mathcal{LS}(D, \mathbf{0})$:

$$D = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 1 & -7 & 1 \\ -1 & 3 & 6 & 2 \\ 3 & 1 & -11 & 1 \\ 2 & 2 & -2 & 6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 4 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can find infinitely many solutions to the system $\mathcal{LS}(D, \mathbf{0})$, most of which are nontrivial. Choose any nontrivial solution to build a nontrivial relation of linear dependence on R. Let us begin with $x_4 = 1$, to find the solution

$$\begin{bmatrix} -4 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

The corresponding relation of linear dependence is

$$(-4)\mathbf{v}_1 + 0\mathbf{v}_2 + (-1)\mathbf{v}_3 + 1\mathbf{v}_4 = \mathbf{0}.$$

The theorem above guarantees that we can solve this relation of linear dependence for some vector in R, but the choice of which one is up to us. Notice however that

$\mathbf{v}_2$ has a zero coefficient. In this case, we cannot choose to solve for $\mathbf{v}_2$. Maybe some other relation of linear dependence would produce a nonzero coefficient for $\mathbf{v}_2$ if we just had to solve for this vector. Unfortunately, this example has been engineered to always produce a zero coefficient here, as you can see from solving the homogeneous system. Every solution has $x_2 = 0$!

OK, if we are convinced that we cannot solve for $\mathbf{v}_2$, let us instead solve for $\mathbf{v}_3$:

$$\mathbf{v}_3 = (-4)\mathbf{v}_1 + 0\mathbf{v}_2 + 1\mathbf{v}_4 = (-4)\mathbf{v}_1 + 1\mathbf{v}_4$$

We claim that this particular equation will allow us to write

$$V = \operatorname{Span} R = \operatorname{Span}\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}_4\} = \operatorname{Span}\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_4\},$$

in essence declaring $\mathbf{v}_3$ as surplus for the task of building V as a span of R. This claim is an equality of two sets. Let $R' = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_4\}$ and $V' = \operatorname{Span} R'$. We want to show that $V = V'$.

First show that $V' \subseteq V$. Since every vector of R' is in R, any vector we can construct in V' as a linear combination of vectors from R' can also be constructed as a vector in V by the same linear combination of the same vectors in R. That was easy, now turn it around.

Next show that $V \subseteq V'$. Choose any $\mathbf{v}$ from V. So there are scalars $\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4$ such that

$$\begin{aligned}
\mathbf{v} &= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 \\
&= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\left((-4)\mathbf{v}_1 + 1\mathbf{v}_4\right) + \alpha_4\mathbf{v}_4 \\
&= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \left((-4\alpha_3)\mathbf{v}_1 + \alpha_3\mathbf{v}_4\right) + \alpha_4\mathbf{v}_4 \\
&= \left(\alpha_1 - 4\alpha_3\right)\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \left(\alpha_3 + \alpha_4\right)\mathbf{v}_4.
\end{aligned}$$

This equation says that $\mathbf{v}$ can be written as a linear combination of the vectors in R' and hence qualifies for membership in V'. So $V \subseteq V'$ and we have established that $V = V'$.

If R' was also linearly dependent (in fact, it is not), we could reduce the set R' even further. Notice that we could have chosen to eliminate any one of $\mathbf{v}_1$, $\mathbf{v}_3$ or $\mathbf{v}_4$, but somehow $\mathbf{v}_2$ is essential to the creation of V since it cannot be replaced by any linear combination of $\mathbf{v}_1$, $\mathbf{v}_3$ or $\mathbf{v}_4$.

12.3 Relation between Linear Independence and the Number of Pivot Columns

Theorem 12.10 (Linearly Independent Vectors r and n)**.** *Suppose that*

$$S = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_n\} \subseteq \mathbb{R}^m$$

is a set of vectors and that A is the $m \times n$ matrix whose columns are the vectors in S. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of pivot columns in B. Then S is linearly independent if and only if $n = r$.

Proof of Linearly Independent Vectors, r and n. Theorem Theorem 12.3 (Linearly Independent Vectors and Homogeneous Systems) says the linear independence of S is equivalent to the homogeneous linear system $\mathcal{LS}(A, \mathbf{0})$ having a unique solution. Since the zero vector is a solution of $\mathcal{LS}(A, \mathbf{0})$, $\mathcal{LS}(A, \mathbf{0})$ is consistent. We can therefore can apply Theorem 5.21 (Consistent Systems, r and n) to see that the solution is unique exactly when $n = r$. ☐

Here is an example of the most straightforward way to determine if a set of column vectors is linearly independent or linearly dependent. While this method can be quick and easy, do not forget the logical progression from the definition of linear independence through homogeneous system of equations which makes it possible.

Example 12.11. Linear dependence, r and n

Is the set of vectors:

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ -6 \\ -2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 4 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 1 \\ 4 \\ 3 \\ 2 \end{bmatrix} \right\}$$

linearly independent or linearly dependent?

Solution. Theorem Theorem 12.10 (Linearly Independent Vectors, r and n) suggests that we take the vectors of S as the columns of a matrix and then analyze its reduced row-echelon form:

$$\begin{bmatrix} 2 & 9 & 1 & -3 & 6 \\ -1 & -6 & 1 & 1 & -2 \\ 3 & -2 & 1 & 4 & 1 \\ 1 & 3 & 0 & 2 & 4 \\ 0 & 2 & 0 & 1 & 3 \\ 3 & 1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we need only compute that $r = 4 < 5 = n$ to recognize, via Theorem Theorem 12.10 (Linearly Independent Vectors, r and n), that S is a linearly dependent set. Boom!

Example 12.12. Large linearly dependent set in $\mathbb{R}^4$

Consider the set of $n = 9$ vectors from $\mathbb{R}^4$,

$$R = \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \\ 9 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

To employ Theorem Theorem 12.3 (Linearly Independent Vectors and Homogeneous Systems), we form a 4×9 matrix C whose columns are the vectors in R:

$$C = \begin{bmatrix} -1 & 7 & 1 & 0 & 5 & 2 & 3 & 1 & -6 \\ 3 & 1 & 2 & 4 & -2 & 1 & 0 & 1 & -1 \\ 1 & -3 & -1 & 2 & 4 & -6 & -3 & 5 & 1 \\ 2 & 6 & -2 & 9 & 3 & 4 & 1 & 3 & 1 \end{bmatrix}.$$

To determine if the homogeneous system $\mathcal{LS}(C, \mathbf{0})$ has a unique solution or not, we would normally row-reduce this matrix. But in this particular example, we can do better:

Since the system is homogeneous with $n = 9$ variables in $m = 4$ equations, and $n > m$, there are infinitely many solutions. Since there is not a unique solution, Theorem Theorem 12.3 (Linearly Independent Vectors and Homogeneous Systems) says the set R is linearly dependent.

The following theorem generalizes the previous example.

Theorem 12.13 (More Vectors than Size implies Linear Dependence)**.** *Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\} \subseteq \mathbb{R}^m$ and $n > m$. Then S is a linearly dependent set.*

Proof of More Vectors than Size implies Linear Dependence. Form the $m \times n$ matrix A whose columns are $\mathbf{u}_i$, $1 \le i \le n$. Consider the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. By Theorem 7.6 this system has infinitely many solutions. Since the system does not have a unique solution, Theorem Theorem 12.3 (Linearly Independent Vectors and Homogeneous Systems) says the columns of A form a linearly dependent set, as desired. $\square$

12.4 Linear Independence and Nonsingular Matrices

We will now specialize to sets of n vectors in $\mathbb{R}^n$.

Example 12.14. Linearly dependent columns
Do the columns of the matrix

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

form a linearly independent or dependent set?

Solution. We can show that A is singular. According to the definition of nonsingular matrices, the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has infinitely many solutions. So, by Theorem Theorem 12.3 (Linearly Independent Vectors and Homogeneous Systems), the columns of A form a linearly dependent set.

Example 12.15. Linearly independent columns
Do the columns of this matrix

$$B = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$$

form a linearly independent or dependent set?

Solution. We can show that B is nonsingular. According to the definition of nonsingular matrices, the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution. So, by Theorem Theorem 12.3 (Linearly Independent Vectors and Homogeneous Systems), the columns of B form a linearly independent set.

That the previous two examples have opposite properties for the columns of their coefficient matrices is no accident. Here is the theorem, and then we will update our equivalences for nonsingular matrices.

Theorem 12.16 (Nonsingular Matrices have Linearly Independent Columns)**.** *Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.*

Proof of Nonsingular Matrices have Linearly Independent Columns. This is a proof where we can chain together equivalences, rather than proving the two halves separately.

$$\begin{aligned} A \text{ nonsingular} &\iff \mathcal{LS}(A, \mathbf{0}) \text{ has a unique solution} \\ &\iff A\vec{x} = \vec{0} \text{ has a unique solution } \vec{x} \\ &\iff \text{columns of } A \text{ are linearly independent} \end{aligned}$$

□

Here is the update to Theorem 7.25 (Nonsingular Matrix Equivalences)

Theorem 12.17 (Nonsingular Matrix Equivalences Round 2)**.** *Suppose that A is a square matrix. The following are equivalent.*

1. *A is nonsingular.*
2. *A row-reduces to the identity matrix.*
3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
4. *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*
5. *A is invertible.*
6. *The columns of A form a linearly independent set.*

Proof of Nonsingular Matrix Equivalences, Round 2. This follows directly from Theorem 12.16 (Nonsingular Matrices have Linearly Independent Columns) and Theorem 7.25 (Nonsingular Matrix Equivalences). □

12.5 Uniqueness of RREF

Math Major only. You can skip this section. Similar concept appears in the classworks.

Example 12.18. Entries of RREF B gives relationship of columns of A
Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 8 & 1 & 17 \\ 1 & 2 & 2 & 13 & 3 & 37 \\ 1 & 2 & 0 & 3 & -2 & -10 \end{bmatrix}.$$

Then A can be row reduced to

$$B = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

Let $\mathbf{A}_i$ (resp. $\mathbf{B}_i$) be the i-th column of A (resp. B) for $i = 1, \ldots, 6$. By the equivalence of system of linear equation $\mathcal{LS}(A, \mathbf{0})$ and $\mathcal{LS}(B, \mathbf{0})$, we have

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_6\mathbf{A}_6 = \mathbf{0} \qquad (12.1)$$

if and only if

$$x_1\mathbf{B}_1 + x_2\mathbf{B}_2 + \cdots + x_6\mathbf{B}_6 = \mathbf{0}. \tag{12.2}$$

Step 1 First of all, if $(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, 0, 0, 0, 0, 0)$ is a solution of (12.2), then

$$x_1\mathbf{B}_1 = \mathbf{0}.$$

So x_1 is zero. This is equivalent to

$$x_1\mathbf{A}_1 = \mathbf{0}.$$

It has only the trivial solution, i.e. $\{\mathbf{A}_1\}$ is linearly independent. Hence $d_1 = 1$ is a pivot column.

Step 2 Let's move to x_2. Suppose that $(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, x_2, 0, 0, 0, 0)$. Then

$$x_1\mathbf{B}_1 + x_2\mathbf{B}_1 = \mathbf{0}$$

has nontrivial solution. Say $(x_1, x_2) = (-2, 1)$.

These can also be seen as

$$-2\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{0}$$

or equivalently

$$\mathbf{A}_2 = 2\mathbf{A}_1.$$

Step 3 Consider x_3. Let $(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, 0, x_3, 0, 0, 0)$. Then

$$x_1\mathbf{B}_1 + x_3\mathbf{B}_3 = \mathbf{0}$$

has only trivial solution. Equivalently $\{\mathbf{A}_1, \mathbf{A}_3\}$ is linearly independent.Column 3 of B is a pivot column.

Step 4 Consder

$$\mathbf{B}_4 = 3\mathbf{B}_1 + 5\mathbf{B}_3,$$

or equivalently

$$\mathbf{A}_4 = 3\mathbf{A}_1 + 5\mathbf{A}_3.$$

The relation of columns of A gives the entries of the column 4 of B.

Step 5 $\mathbf{B}_5$ is not in span of $\mathbf{B}_1$ and $\mathbf{B}_3$. Equivalently $\mathbf{A}_5$ is not in span of $\mathbf{A}_1$ and $\mathbf{A}_3$. Column 5 of B is a pivot column.

Step 6 Consider

$$\mathbf{B}_6 = 4\mathbf{B}_1 + 6\mathbf{B}_3 + 7\mathbf{B}_5.$$

Equivalently

$$\mathbf{A}_6 = 4\mathbf{A}_1 + 6\mathbf{A}_3 + 7\mathbf{A}_5.$$

The relation of columns of A gives the entries of the column 6 of B.

Example 12.19. Relationship of columns of A determine entries of B

Row reduce

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 & 4 \\ 2 & 1 & 5 & 1 & 1 & 2 & 7 \\ 1 & -1 & 1 & 2 & 1 & -3 & 10 \\ 1 & 3 & 5 & 1 & -1 & 1 & 1 \end{bmatrix}$$

to a RREF B by the above technique. Let $\mathbf{A}_i$ (resp. $\mathbf{B}_i$) be the i-th column of A (resp. B) for $i = 1, \ldots, 7$.

Step 1 $\mathbf{A}_1$ is nonzero column. So the index $d_1 = 1$ corresponds to a pivot column. We have

$$\mathbf{B}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Step 2 $\mathbf{A}_2$ is not in $\mathrm{Span}\,\{\mathbf{A}_{d_1}\}$. So the index $d_2 = 2$ corresponds to a pivot column. We have

$$\mathbf{B}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Step 3 Consider

$$\mathbf{A}_3 = 2\mathbf{A}_{d_1} + \mathbf{A}_{d_2}.$$

So we have

$$\mathbf{B}_3 = 2\mathbf{B}_{d_1} + \mathbf{B}_{d_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Step 4 $\mathbf{A}_4$ is not in $\mathrm{Span}\,\{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}\}$.
So the index $d_3 = 4$ corresponds to a pivot column. We have

$$\mathbf{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Step 5 $\mathbf{A}_5$ is not in $\mathrm{Span}\,\{\mathbf{A}_{d_1}, \mathbf{A}_{d_3}, \mathbf{A}_{d_3}\}$.
So the index $d_4 = 5$ corresponds to a pivot column. We have

$$\mathbf{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Step 6 Consider

$$\mathbf{A}_6 = \mathbf{A}_{d_1} + \mathbf{A}_{d_2} - 2\mathbf{A}_{d_3} + \mathbf{A}_{d_4}.$$

So, we have

$$\mathbf{B}_6 = \mathbf{B}_{d_1} + \mathbf{B}_{d_2} - 2\mathbf{B}_{d_3} + \mathbf{B}_{d_4} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

Step 7 Consider

$$\mathbf{A}_7 = 2\mathbf{A}_{d_1} - \mathbf{A}_{d_2} + 3\mathbf{A}_{d_3} + \mathbf{A}_{d_4}.$$

So, we have

$$\mathbf{B}_7 = \mathbf{B}_{d_1} + \mathbf{B}_{d_2} - 2\mathbf{B}_{d_3} + \mathbf{B}_{d_4} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}$$

Hence the RREF of A is

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Important remark: from the above computation, the entries of B are uniquely determined by A.

So the RREF B is unique.