MATH 1030 Chapter 10

The lecture is based on Beezer, A first course in Linear algebra. Ver 3.5 Downloadable at http://linear.ups.edu/download.html .

The print version can be downloaded at http://linear.ups.edu/download/fcla-3.50-print.pdf .

Reference.

Beezer, Ver 3.5 Section LC (print version p65 - p81)Strang: Section 2.3

Exercise.

Exercises with solutions can be downloaded at http://linear.ups.edu/download/fcla-3.50-solution-manual.pdf

Section LC (p.32-33) C40, C41, M10, M11

10.1 Linear Combinations

Definition 10.1 (Linear Combination of Column Vectors)**.** Given n vectors $\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3,\ \ldots,\ \mathbf{u}_n$ in $\mathbb{R}^m$ and n scalars $\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots,\ \alpha_n$, their **linear combination** is the vector:

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n$$

in $\mathbb{R}^m$.

Example 10.2. Two linear combinations in $\mathbb{R}^6$

Suppose that:

$$\alpha_1 = 1 \quad \alpha_2 = -4 \quad \alpha_3 = 2 \quad \alpha_4 = -1$$

and

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 1 \\ 2 \\ 9 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 6 \\ 3 \\ 0 \\ -2 \\ 1 \\ 4 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} -5 \\ 2 \\ 1 \\ 1 \\ -3 \\ 0 \end{bmatrix} \qquad \mathbf{u}_4 = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 7 \\ 1 \\ 3 \end{bmatrix}.$$

The resulting linear combination is:

$$\alpha_1\mathbf{u_1} + \alpha_2\mathbf{u_2} + \alpha_3\mathbf{u_3} + \alpha_4\mathbf{u_4}$$
$$= (1)\begin{bmatrix}2\\4\\-3\\1\\2\\9\end{bmatrix} + (-4)\begin{bmatrix}6\\3\\0\\-2\\1\\4\end{bmatrix} + (2)\begin{bmatrix}-5\\2\\1\\1\\-3\\0\end{bmatrix} + (-1)\begin{bmatrix}3\\2\\-5\\7\\1\\3\end{bmatrix}$$
$$= \begin{bmatrix}2\\4\\-3\\1\\2\\9\end{bmatrix} + \begin{bmatrix}-24\\-12\\0\\8\\-4\\-16\end{bmatrix} + \begin{bmatrix}-10\\4\\2\\2\\-6\\0\end{bmatrix} + \begin{bmatrix}-3\\-2\\5\\-7\\-1\\-3\end{bmatrix} = \begin{bmatrix}-35\\-6\\4\\4\\-9\\-10\end{bmatrix}$$

A different linear combination, but with the same set of vectors, can be formed with different scalars. Taking

$$\beta_1 = 3 \quad \beta_2 = 0 \quad \beta_3 = 5 \quad \beta_4 = -1$$

we can form the linear combination:

$$\beta_1\mathbf{u_1} + \beta_2\mathbf{u_2} + \beta_3\mathbf{u_3} + \beta_4\mathbf{u_4}$$
$$= (3)\begin{bmatrix}2\\4\\-3\\1\\2\\9\end{bmatrix} + (0)\begin{bmatrix}6\\3\\0\\-2\\1\\4\end{bmatrix} + (5)\begin{bmatrix}-5\\2\\1\\1\\-3\\0\end{bmatrix} + (-1)\begin{bmatrix}3\\2\\-5\\7\\1\\3\end{bmatrix}$$
$$= \begin{bmatrix}6\\12\\-9\\3\\6\\27\end{bmatrix} + \begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}-25\\10\\5\\5\\-15\\0\end{bmatrix} + \begin{bmatrix}-3\\-2\\5\\-7\\-1\\-3\end{bmatrix} = \begin{bmatrix}-22\\20\\1\\1\\-10\\24\end{bmatrix}.$$

Notice how we could keep our set of vectors fixed but use a different set of scalars to construct different vectors. Can you create the following vector $\mathbf{w}$ with a suitable choice of four scalars?

$$\mathbf{w} = \begin{bmatrix} 13 \\ 15 \\ 5 \\ -17 \\ 2 \\ 25 \end{bmatrix}$$

Going further, can you create any possible vector from $\mathbb{R}^6$ by choosing the proper scalars?

Example 10.3. The system of linear equation:

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= -33 \\ 5x_1 + 5x_2 + 7x_3 &= 24 \\ x_1 + 4x_3 &= 5 \end{aligned}$$

or equivalently

$$\begin{bmatrix} -7x_1 - 6x_2 - 12x_3 \\ 5x_1 + 5x_2 + 7x_3 \\ x_1 + 4x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix},$$

can be rewritten as:

$$\begin{bmatrix} -7x_1 \\ 5x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -6x_2 \\ 5x_2 \\ 0x_2 \end{bmatrix} + \begin{bmatrix} -12x_3 \\ 7x_3 \\ 4x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}$$

or:

$$x_1 \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}.$$

The solution is:

$$x_1 = -3 \quad x_2 = 5 \quad x_3 = 2.$$

So, in the context of this example, we can express the fact that these values of the variables are a solution by writing a linear combination:

$$(-3) \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix} + (5) \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}$$

Furthermore, these are the only three scalars that will accomplish this equality, since they come from a unique solution.

Notice how the three vectors in this example are the columns of the coefficient matrix of the system of equations. This is our first hint of the important interplay between the vectors that form the columns of a matrix, and the matrix itself.

Example 10.4. The system of linear equations

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 8 \\ x_1 + x_2 &= 5 \end{aligned}$$

can be written as:

$$\begin{bmatrix} x_1 - x_2 + 2x_3 \\ 2x_1 + x_2 + x_3 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}.$$

This vector equation is equivalent to:

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}.$$

Row-reducing the augmented matrix for the system leads to the conclusion that the system is consistent and has free variables, hence infinitely many solutions. So for example, the two solutions:

$$\begin{aligned} x_1 &= 2 & x_2 &= 3 & x_3 &= 1 \\ x_1 &= 3 & x_2 &= 2 & x_3 &= 0 \end{aligned}$$

can be used together to say that:

$$(2) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (3) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix} = (3) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (2) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + (0) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Ignore the middle of this equation, and move all the terms to the left-hand side:

$$(2) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (3) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + (-0) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Regrouping gives:

$$(-1)\begin{bmatrix}1\\2\\1\end{bmatrix}+(1)\begin{bmatrix}-1\\1\\1\end{bmatrix}+(1)\begin{bmatrix}2\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Notice that the three vectors on the left hand side are the columns of the coefficient matrix for the system of equations. This equality says that there is a linear combination of those columns that equals the vector of all zeros. Give it some thought, but this says that:

$$x_1 = -1 \qquad x_2 = 1 \qquad x_3 = 1$$

is a nontrivial solution to the homogeneous system of equations with the coefficient matrix of the original system. In particular, this demonstrates that this coefficient matrix is singular.

Theorem 10.5 (Solutions to Linear Systems are Linear Combinations)**.** *Denote the columns of the $m \times n$ matrix A as vectors $\mathbf{A}_1,\ \mathbf{A}_2,\ \mathbf{A}_3,\ \ldots,\ \mathbf{A}_n$. Then $\mathbf{x} \in \mathbb{R}^n$ is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if $\mathbf{b}$ is equal to the linear combination of the columns of A formed with the entries of $\mathbf{x}$,*

$$[\mathbf{x}]_1\,\mathbf{A}_1 + [\mathbf{x}]_2\,\mathbf{A}_2 + [\mathbf{x}]_3\,\mathbf{A}_3 + \cdots + [\mathbf{x}]_n\,\mathbf{A}_n = \mathbf{b}$$

Equivalently, $\mathbf{x} \in \mathbb{R}^n$ satisfies $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{b}$ is a linear combination of $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$.

Proof of Solutions to Linear Systems are Linear Combinations. If $\mathbf{x} \in \mathbb{R}^n$ is a solution of $\mathcal{LS}(A, \mathbf{b})$, then

$$\mathbf{b} = A\mathbf{x} = [\mathbf{x}]_1\,\mathbf{A}_1 + [\mathbf{x}]_2\,\mathbf{A}_2 + [\mathbf{x}]_3\,\mathbf{A}_3 + \cdots + [\mathbf{x}]_n\,\mathbf{A}_n.$$

Hence $\mathbf{b}$ is a linear combination of the columns of A.

Conversely, if $\mathbf{b}$ is a linear combinations of the columns of A, say:

$$\mathbf{b} = [\mathbf{x}]_1\,\mathbf{A}_1 + [\mathbf{x}]_2\,\mathbf{A}_2 + [\mathbf{x}]_3\,\mathbf{A}_3 + \cdots + [\mathbf{x}]_n\,\mathbf{A}_n,$$

then

$$\mathbf{b} = A\mathbf{x}.$$

So $\mathbf{x}$ is a solution of $\mathcal{LS}(A, \mathbf{b})$. □

Computational question: Determine if $\mathbf{u}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$. That is, determine whether or not the system of linear equations:

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{u}$$

has a solution. The augmented matrix is:

$$[\mathbf{v}_1|\cdots|\mathbf{v}_n|\mathbf{u}].$$

We can now solve the corresponding system of linear questions using row operations.

Example 10.6. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

1. Determine if $\mathbf{u}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. If yes, find the linear combination.

2. Determine if $\mathbf{u}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. If yes, find the linear combination.

Solution. 1. To determine if $\mathbf{u}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, we need to solve

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{u},$$

i.e.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + x + 2 + x_3 &= 1 \\ 3x_1 + 3x_3 &= 3. \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 0 & 3 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Because the last column of the RREF is a pivot column, the system of linear equations is not solvable. Hence $\mathbf{u}$ is not a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

2. To determine if $\mathbf{u}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, we need to solve

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{u}.$$

The augmented matrix is:

$$[\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3|\mathbf{v}_4|\mathbf{u}] = \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 3 & 0 & 3 & 1 & 3 \end{array}\right].$$

Using the standard method, we find one solution (there are infinitely many):

$$x_1 = \frac{5}{6} \qquad x_2 = -\frac{2}{3} \qquad x_3 = 0 \qquad x_4 = \frac{1}{2}$$

i.e.

$$\mathbf{u} = \frac{5}{6}\mathbf{v}_1 - \frac{2}{3}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_4.$$

10.2 Vector Form of Solution Sets

Example 10.7. Consider the linear system

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 8 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= -12 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 4. \end{aligned}$$

Row-reducing the augmented matrix yields

$$\begin{bmatrix} \boxed{1} & 0 & 3 & -2 & 4 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

from which we see that there are $r = 2$ pivot columns. Also, $D = \{1, 2\}$, so that the dependent variables are x_1 and x_2, and $F = \{3, 4, 5\}$, so that the free variables are x_3 and x_4. We will express a generic solution for the system by two slightly different methods, though both arrive at the same conclusion.

Rearranging each equation represented in the row-reduced form of the augmented matrix by solving for the dependent variable in each row yields the vector equality,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 - 3x_3 + 2x_4 \\ -x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_4 \\ 3x_4 \\ 0 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

We will develop the same linear combination a bit quicker, using three steps. While the method above is instructive, the method below will be our preferred approach.

Step 1. Write the vector of variables as a fixed vector plus a linear combination of $n - r$ vectors, using the free variables as the scalars:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} + x_3 \begin{bmatrix} \\ \\ \\ \end{bmatrix} + x_4 \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Step 2. Use 0's and 1's to ensure equality for the entries of the vectors with indices in F (corresponding to the free variables):

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \\ \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \\ \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \\ \\ 0 \\ 1 \end{bmatrix}.$$

Step 3. For each dependent variable, use the augmented matrix to formulate an equation expressing the dependent variable as a constant plus a linear combination of the free variables. Convert this equation into entries of the vectors that ensure equality for each dependent variable, one at a time.

$$x_1 = 4 - 3x_3 + 2x_4 \Rightarrow \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ \\ 0 \\ 1 \end{bmatrix}$$

$$x_2 = 0 - 1x_3 + 3x_4 \Rightarrow \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

While this form is useful for quickly creating solutions, it is even better because it tells us exactly what every solution looks like. We know the solution set is infinite, which is pretty big, but now we can say that a solution is some multiple of $\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ plus a multiple of $\begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ plus the fixed vector $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Period. So it only takes us three vectors to describe the entire infinite solution set, provided we also agree on how to combine the three vectors into a linear combination.

Example 10.8. Consider a linear system of $m = 5$ equations in $n = 7$ variables, having the augmented matrix

$$A = \begin{bmatrix} 2 & 1 & -1 & -2 & 2 & 1 & 5 & 21 \\ 1 & 1 & -3 & 1 & 1 & 1 & 2 & -5 \\ 1 & 2 & -8 & 5 & 1 & 1 & -6 & -15 \\ 3 & 3 & -9 & 3 & 6 & 5 & 2 & -24 \\ -2 & -1 & 1 & 2 & 1 & 1 & -9 & -30 \end{bmatrix}.$$

Row-reducing we obtain the matrix

$$B = \begin{bmatrix} \boxed{1} & 0 & 2 & -3 & 0 & 0 & 9 & 15 \\ 0 & \boxed{1} & -5 & 4 & 0 & 0 & -8 & -10 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & -6 & 11 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 7 & -21 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and we see that there are $r = 4$ pivot columns. Also, $D = \{1, 2, 5, 6\}$ so the dependent variables are x_1, x_2, x_5, and x_6. Similarly, $F = \{3, 4, 7, 8\}$ and the $n - r = 3$ free variables are x_3, x_4 and x_7. We will express a generic solution for the system by two different methods: both a decomposition and a construction.

Rearranging each equation represented in the row-reduced echelon form of the augmented matrix by solving for the dependent variable in each row yields the vector equality

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 - 2x_3 + 3x_4 - 9x_7 \\ -10 + 5x_3 - 4x_4 + 8x_7 \\ x_3 \\ x_4 \\ 11 + 6x_7 \\ -21 - 7x_7 \\ x_7 \end{bmatrix}.$$

$$= \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_3 \\ 5x_3 \\ x_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_4 \\ -4x_4 \\ 0 \\ x_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -9x_7 \\ 8x_7 \\ 0 \\ 0 \\ 6x_7 \\ -7x_7 \\ x_7 \end{bmatrix}$$

$$= \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -9 \\ 8 \\ 0 \\ 0 \\ 6 \\ -7 \\ 1 \end{bmatrix}.$$

We will now develop the same linear combination a bit quicker, using three steps. While the method above is instructive, the method below will be our preferred approach.

Step 1. Write the vector of variables as a fixed vector, plus a linear combination of $n - r$ vectors, using the free variables as the scalars:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} + x_3 \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} + x_4 \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} + x_7 \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Step 2. Use 0's and 1's to ensure equality for the entries of the vectors with indices in F (corresponding to the free variables):

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \\ \\ 0 \\ 0 \\ \\ \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \\ \\ 1 \\ 0 \\ \\ \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \\ \\ 0 \\ 1 \\ \\ \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} \\ \\ 0 \\ 0 \\ \\ \\ 1 \end{bmatrix}$$

Step 3. For each dependent variable, use the augmented matrix to formulate an equation expressing the dependent variable as a constant plus multiples of the free variables. Convert this equation into entries of the vectors that ensure equality for each dependent variable, one at a time.

$$x_1 = 15 - 2x_3 + 3x_4 - 9x_7 \ \Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ \\ 0 \\ 0 \\ \\ \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ \\ 1 \\ 0 \\ \\ \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ \\ 0 \\ 1 \\ \\ \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -9 \\ \\ 0 \\ 0 \\ \\ \\ 1 \end{bmatrix}$$

$$x_2 = -10 + 5x_3 - 4x_4 + 8x_7 \ \Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ \\ \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ \\ \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ \\ \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -9 \\ 8 \\ 0 \\ 0 \\ \\ \\ 1 \end{bmatrix}$$

$$x_5 = 11 + 6x_7 \ \Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \\ \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 0 \\ \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -9 \\ 8 \\ 0 \\ 0 \\ 6 \\ \\ 1 \end{bmatrix}$$

$$x_6 = -21 - 7x_7 \ \Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -9 \\ 8 \\ 0 \\ 0 \\ 6 \\ -7 \\ 1 \end{bmatrix}$$

This final form of a typical solution is especially pleasing and useful. For example, we can build solutions quickly by choosing values for our free variables, and then compute a linear combination. For example

$$x_3 = 2,\ x_4 = -4,\ x_7 = 3 \qquad \Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-4) \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (3) \begin{bmatrix} -9 \\ 8 \\ 0 \\ 0 \\ 6 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} -28 \\ 40 \\ 2 \\ -4 \\ 29 \\ -42 \\ 3 \end{bmatrix}$$

or perhaps,

$$x_3 = 5,\ x_4 = 2,\ x_7 = 1 \qquad \Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} + (5) \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} -9 \\ 8 \\ 0 \\ 0 \\ 6 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ 5 \\ 2 \\ 17 \\ -28 \\ 1 \end{bmatrix}$$

or even,

$$x_3 = 0,\ x_4 = 0,\ x_7 = 0 \qquad \Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} + (0) \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (0) \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (0) \begin{bmatrix} -9 \\ 8 \\ 0 \\ 0 \\ 6 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix}.$$

So we can compactly express all of the solutions to this linear system with just 4 fixed vectors, provided we agree how to combine them in a linear combinations to create solution vectors.

Suppose you were told that the vector $\mathbf{w}$ below was a solution to this system of equations. Could you turn the problem around and write $\mathbf{w}$ as a linear combination of the four vectors $\mathbf{c}$, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$?

$$\mathbf{w} = \begin{bmatrix} 100 \\ -75 \\ 7 \\ 9 \\ -37 \\ 35 \\ -8 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} \qquad \mathbf{u}_1 = \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} -9 \\ 8 \\ 0 \\ 0 \\ 6 \\ -7 \\ 1 \end{bmatrix}$$

Theorem 10.9 (Vector Form of Solutions to Linear Systems)**.** *Suppose that* $[A|\mathbf{b}]$ *is the augmented matrix for a consistent linear system* $\mathcal{LS}(A, \mathbf{b})$ *of* m *equations in* n *variables. Let* B *be a row-equivalent* $m \times (n+1)$ *matrix in reduced row-echelon form. Suppose that* B *has* r *pivot columns, with indices* $D = \{d_1, d_2, d_3, \ldots, d_r\}$*, while the* $n - r$ *non-pivot columns have indices in* $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$*. Define vectors* $\mathbf{c}$*,* $\mathbf{u}_j$*,* $1 \leq j \leq n - r$ *of size* n *by*

$$[\mathbf{c}]_i = \begin{cases} 0 & \text{if } i \in F \\ [B]_{k,n+1} & \text{if } i \in D,\ i = d_k \end{cases}$$

$$[\mathbf{u}_j]_i = \begin{cases} 1 & \text{if } i \in F,\ i = f_j \\ 0 & \text{if } i \in F,\ i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D,\ i = d_k \end{cases}.$$

Then the set of solutions to the system of equations $\mathcal{LS}(A, \mathbf{b})$ *is*

$$S = \{\mathbf{c} + \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_{n-r}\mathbf{u}_{n-r} \mid \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-r} \in \mathbb{R}.\}$$

Proof of Vector Form of Solutions to Linear Systems. **You can skip this proof for now, as long as you understand the examples** First, the equation $\mathcal{LS}(A, \mathbf{b})$ is equivalent to the linear system of equations that has the matrix B as its augmented matrix. So we need only show that S is the solution set for the system with B as its augmented matrix. The conclusion of this theorem is that the solution set is equal to the set S.

We begin by showing that every element of S is indeed a solution to the system. Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-r}$ be one choice of the scalars used to describe elements of S. So an arbitrary element of S, which we will consider as a proposed solution, is

$$\mathbf{x} = \mathbf{c} + \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_{n-r}\mathbf{u}_{n-r}.$$

When $r + 1 \leq \ell \leq m$, row ℓ of the matrix B is a zero row, so the equation represented by that row is always true, no matter which solution vector we propose.

So concentrate on rows representing equations $1 \leq \ell \leq r$. We evaluate equation ℓ of the system represented by B with the proposed solution vector $\mathbf{x}$ and refer to the value of the left-hand side of the equation as β_ℓ:

$$\beta_\ell = [B]_{\ell 1} [\mathbf{x}]_1 + [B]_{\ell 2} [\mathbf{x}]_2 + [B]_{\ell 3} [\mathbf{x}]_3 + \cdots + [B]_{\ell n} [\mathbf{x}]_n$$

Since $[B]_{\ell d_i} = 0$ for all $1 \leq i \leq r$, except that $[B]_{\ell d_\ell} = 1$, we see that β_ℓ simplifies to

$$\beta_\ell = [\mathbf{x}]_{d_\ell} + [B]_{\ell f_1} [\mathbf{x}]_{f_1} + [B]_{\ell f_2} [\mathbf{x}]_{f_2} + [B]_{\ell f_3} [\mathbf{x}]_{f_3} + \cdots + [B]_{\ell f_{n-r}} [\mathbf{x}]_{f_{n-r}}.$$

Notice that for $1 \leq i \leq n - r$

$$\begin{aligned}
[\mathbf{x}]_{f_i} &= [\mathbf{c}]_{f_i} + \alpha_1 [\mathbf{u}_1]_{f_i} + \alpha_2 [\mathbf{u}_2]_{f_i} + \cdots + \alpha_i [\mathbf{u}_i]_{f_i} + \cdots + \alpha_{n-r} [\mathbf{u}_{\mathbf{n-r}}]_{f_i} \\
&= 0 + \alpha_1(0) + \alpha_2(0) + \cdots + \alpha_i(1) + \cdots + \alpha_{n-r}(0) \\
&= \alpha_i.
\end{aligned}$$

So β_ℓ simplifies further, and we expand the first term

$$\begin{aligned}
\beta_\ell &= [\mathbf{x}]_{d_\ell} + [B]_{\ell f_1} \alpha_1 + [B]_{\ell f_2} \alpha_2 + [B]_{\ell f_3} \alpha_3 + \cdots + [B]_{\ell f_{n-r}} \alpha_{n-r} \\
&= [\mathbf{c} + \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_{n-r} \mathbf{u}_{n-r}]_{d_\ell} + \\
&\qquad [B]_{\ell f_1} \alpha_1 + [B]_{\ell f_2} \alpha_2 + [B]_{\ell f_3} \alpha_3 + \cdots + [B]_{\ell f_{n-r}} \alpha_{n-r} \\
&= [\mathbf{c}]_{d_\ell} + \alpha_1 [\mathbf{u}_1]_{d_\ell} + \alpha_2 [\mathbf{u}_2]_{d_\ell} + \alpha_3 [\mathbf{u}_3]_{d_\ell} + \cdots + \alpha_{n-r} [\mathbf{u}_{\mathbf{n-r}}]_{d_\ell} + \\
&\qquad [B]_{\ell f_1} \alpha_1 + [B]_{\ell f_2} \alpha_2 + [B]_{\ell f_3} \alpha_3 + \cdots + [B]_{\ell f_{n-r}} \alpha_{n-r} \\
&= [B]_{\ell, n+1} + \\
&\qquad \alpha_1(-[B]_{\ell f_1}) + \alpha_2(-[B]_{\ell f_2}) + \alpha_3(-[B]_{\ell f_3}) + \cdots + \alpha_{n-r}(-[B]_{\ell f_{n-r}}) + \\
&\qquad [B]_{\ell f_1} \alpha_1 + [B]_{\ell f_2} \alpha_2 + [B]_{\ell f_3} \alpha_3 + \cdots + [B]_{\ell f_{n-r}} \alpha_{n-r} \\
&= [B]_{\ell, n+1}.
\end{aligned}$$

So β_ℓ began as the left-hand side of equation ℓ of the system represented by B and we now know it equals $[B]_{\ell, n+1}$, the constant term for equation ℓ of this system. So the arbitrarily chosen vector from S makes every equation of the system true, and therefore is a solution to the system. So all the elements of S are solutions to the system.

For the second half of the proof, assume that $\mathbf{x}$ is a solution vector for the system having B as its augmented matrix. For convenience and clarity, denote the entries of $\mathbf{x}$ by x_i. In other words, $x_i = [\mathbf{x}]_i$. We desire to show that this solution vector is also an element of the set S. Begin with the observation that the entries of a solution vector make equation ℓ of the system true for all $1 \leq \ell \leq m$:

$$[B]_{\ell,1}\, x_1 + [B]_{\ell,2}\, x_2 + [B]_{\ell,3}\, x_3 + \cdots + [B]_{\ell,n}\, x_n = [B]_{\ell,n+1}$$

When $\ell \leq r$, the pivot columns of B have zero entries in row ℓ with the exception of column d_ℓ, which will contain a 1. So for $1 \leq \ell \leq r$, equation ℓ simplifies to

$$1x_{d_\ell} + [B]_{\ell,f_1}\, x_{f_1} + [B]_{\ell,f_2}\, x_{f_2} + [B]_{\ell,f_3}\, x_{f_3} + \cdots + [B]_{\ell,f_{n-r}}\, x_{f_{n-r}} = [B]_{\ell,n+1}\,.$$

This allows us to write,

$$\begin{aligned}
[\mathbf{x}]_{d_\ell} &= x_{d_\ell} \\
&= [B]_{\ell,n+1} - [B]_{\ell,f_1}\, x_{f_1} - [B]_{\ell,f_2}\, x_{f_2} - [B]_{\ell,f_3}\, x_{f_3} - \cdots - [B]_{\ell,f_{n-r}}\, x_{f_{n-r}} \\
&= [\mathbf{c}]_{d_\ell} + x_{f_1}\, [\mathbf{u}_1]_{d_\ell} + x_{f_2}\, [\mathbf{u}_2]_{d_\ell} + x_{f_3}\, [\mathbf{u}_3]_{d_\ell} + \cdots + x_{f_{n-r}}\, [\mathbf{u}_{n-r}]_{d_\ell} \\
&= \left[\mathbf{c} + x_{f_1}\mathbf{u}_1 + x_{f_2}\mathbf{u}_2 + x_{f_3}\mathbf{u}_3 + \cdots + x_{f_{n-r}}\mathbf{u}_{n-r}\right]_{d_\ell}.
\end{aligned}$$

This tells us that the entries of the solution vector $\mathbf{x}$ corresponding to dependent variables (indices in D) are equal to those of a vector in the set S. We still need to check the other entries of the solution vector $\mathbf{x}$ corresponding to the free variables (indices in F) to see if they are equal to the entries of the same vector in the set S. To this end, suppose $i \in F$ and $i = f_j$. Then

$$\begin{aligned}
[\mathbf{x}]_i &= x_i = x_{f_j} \\
&= 0 + 0x_{f_1} + 0x_{f_2} + 0x_{f_3} + \cdots + 0x_{f_{j-1}} + 1x_{f_j} + 0x_{f_{j+1}} + \cdots + 0x_{f_{n-r}} \\
&= [\mathbf{c}]_i + x_{f_1}\, [\mathbf{u}_1]_i + x_{f_2}\, [\mathbf{u}_2]_i + x_{f_3}\, [\mathbf{u}_3]_i + \cdots + x_{f_j}\, [\mathbf{u}_j]_i + \cdots + x_{f_{n-r}}\, [\mathbf{u}_{n-r}]_i \\
&= \left[\mathbf{c} + x_{f_1}\mathbf{u}_1 + x_{f_2}\mathbf{u}_2 + \cdots + x_{f_{n-r}}\mathbf{u}_{n-r}\right]_i.
\end{aligned}$$

So entries of $\mathbf{x}$ and $\mathbf{c} + x_{f_1}\mathbf{u}_1 + x_{f_2}\mathbf{u}_2 + \cdots + x_{f_{n-r}}\mathbf{u}_{n-r}$ are equal and therefore they are equal vectors. Since $x_{f_1}, x_{f_2}, x_{f_3}, \ldots, x_{f_{n-r}}$ are scalars, this shows us that $\mathbf{x}$ qualifies for membership in S. So the set S contains all of the solutions to the system. $\square$