MATH 1030 Chapter 1

Reference:

1. Gilbert Strang, *Linear Algebra and Its Applications* . Section 1.3.
2. Robert A. Beezer, *A First Course in Linear Algebra* . Section Solving Systems of Linear Equations .

1.1 Introduction

In this section we give examples of systems of linear equations and solve one example.

Example 1.1. Solve the system of equations:

$$\begin{aligned} 2x_1 + 3x_2 &= 3 && (1.1) \\ x_1 - x_2 &= 4 && (1.2) \\ & && (1.3) \end{aligned}$$

Solution. Adding Equation (1.1) to $-2 \times$ Equation (1.2) gives:

$$5x_2 = -5.$$

So $x_2 = -1$. Substituting $x_2 = -1$ into Equation (1.1) gives

$$2x_1 + 3(-1) = 3$$

Hence, $x_1 = 3$ and $x_2 = -1$ is a solution.

Indeed:

$$2(3) + 3(-1) = 3$$

$$(3) - (-1) = 4.$$

In fact, it is the unique solution.

Main goal: One of the main goals of this course is to solve systems of linear equations with more variables and more equations.

Example 1.2.

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 4 \\ x_1 + 3x_2 + 3x_3 &= 5 \\ 2x_1 + 6x_2 + 5x_3 &= 6 \end{aligned}$$

Example 1.3.

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 7 \\ x_1 + x_2 + x_3 - x_4 &= 3 \\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 \end{aligned}$$

1.2 System of linear equations of two unknowns

You should all be very familiar with the procedure of solving a system equations of two unknowns.

1.2.1 Substitution

Example 1.4.

$$3x + 4y = 2 \tag{1.4}$$

$$4x + 5y = 3 \tag{1.5}$$

Use (1.5), we can solve for y in terms of x:

$$y = \frac{3}{5} - \frac{4}{5}x. \tag{1.6}$$

Substitution this into (1.4):

$$\begin{aligned} 3x + 4(\frac{3}{5} - \frac{4}{5}x) &= 2 \\ \frac{12}{5} - \frac{x}{5} &= 2 \\ x &= 2. \end{aligned}$$

Substituting $x = 2$ into (1.6), we can solve for y:

$$y = \frac{3}{5} - \frac{4}{5} \times 2 = -1.$$

So the solution is $x = 2$, $y = -1$.

Remark. There are other ways to use substitution, for example

1. Solve x by (1.5) in terms of y and substitute it into (1.4)
2. Solve y by (1.4) in terms of x and substitute it into (1.5)
3. Solve x by (1.4) in terms of y and substitute it into (1.5)

But you **cannot** get the solution by

1. Solve y by (1.5) in terms of x and substitute it into (1.5) (No substitution back to the original equation)
2. Solve y by (1.4) in terms of x and substitute it into (1.4)
3. Solve x by (1.5) in terms of y and substitute it into (1.5)
4. Solve x by (1.4) in terms of y and substitute it into (1.4)

1.2.2 Elimination

Example 1.5. Again, let's solve the system of linear equations from the previous example.

$$3x + 4y = 2 \tag{1.7}$$

$$4x + 5y = 3 \tag{1.8}$$

Consider (1.7) $- \frac{3}{4}$(1.8).

$$\begin{array}{rlll} & 3x & +4y & =2 \\ -) & 3x & +\frac{15}{4}y & =\frac{9}{4} \\ \hline & & \frac{1}{4}y & =-\frac{1}{4} \end{array}$$

Thus $y = -1$. Substituting it into (1.7):

$$3x + 4(-1) = 2$$

$$x = 2.$$

So we obtain the solution $x = 2$, $y = -1$.

Remark. 1. The number $\frac{3}{4}$ is so chosen such that the coefficient of x is eliminated.

2. At some point, we still need to use substitution to get the solution.

1.2.3 Substitution

Example 1.6. Solve the following system of linear equations

$$x + 2y + 2z = 4 \tag{1.9}$$

$$x + 3y + 3z = 5 \tag{1.10}$$

$$2x + 6y + 5z = 6 \tag{1.11}$$

Find x in terms of y, z by (1.9) :

$$x = 4 - 2y - 2z. \tag{1.12}$$

Substituting (1.12) into (1.10), we obtain

$$(4 - 2y - 2z) + 3y + 3z = 5$$

i.e.,

$$y + z = 1. \tag{1.13}$$

Substituting (1.12) into (1.11), we obtain

$$2(4 - 2y - 2z) + 6y + 5z = 6,$$

i.e.,

$$2y + z = -2 \tag{1.14}$$

The equations are reduced to solving a linear system of equations with two unknowns:

$$\begin{cases} y + z = 1 \\ 2y + z = -2 \end{cases}$$

Solve y in terms of z by (1.13):

$$y = 1 - z \tag{1.15}$$

Then substitute $y = 1 - z$ into (1.14):

$$2(1 - z) + z = -2$$

i.e.

$$z = 4.$$

By (1.15)

$$y = 1 - z = -3.$$

Substitute $y = -3$, $z = 4$ into (1.12),

$$x = 4 - 2y - 2z = 4 - 2 \times (-3) - 2 \times 4 = 2.$$

Hence $x = 2, y = -3, z = 4$ is a solution.

1.2.4 Elimination

Using substitution all the way to solve linear equations is not the best way. Instead, we can use elimination to simplify the system of linear equations first.

Example 1.7. We solve the following system by a sequence of equation operations.

$$x + 2y + 2z = 4 \tag{1}$$

$$x + 3y + 3z = 5 \tag{2}$$

$$2x + 6y + 5z = 6 \tag{3}$$

-1 times equation 1, add to equation 2:

$$x + 2y + 2z = 4$$
$$0x + 1y + 1z = 1$$
$$2x + 6y + 5z = 6$$

-2 times equation 1, add to equation 3:

$$x + 2y + 2z = 4$$
$$0x + 1y + 1z = 1$$
$$0x + 2y + 1z = -2$$

-2 times equation 2, add to equation 3:

$$x + 2y + 2z = 4$$
$$0x + 1y + 1z = 1$$
$$0x + 0y - 1z = -4$$

-1 times equation 3:

$$x + 2y + 2z = 4$$
$$0x + 1y + 1z = 1$$
$$0x + 0y + 1z = 4$$

which can be written more clearly as:

$$x + 2y + 2z = 4$$
$$y + z = 1$$
$$z = 4$$

The third equation requires that $z = 4$ to be true. Making this substitution into equation 2 we arrive at $y = -3$, and finally, substituting these values of y and z into the first equation, we find that $x = 2$.

Remark. We can add several more eliminations to solve x, y, z without substitution:

$$x + 2y + 2z = 4$$
$$0x + 1y + 1z = 1$$
$$0x + 0y + 1z = 4$$

-1 times equation 3, add to equation 2 and -2 times equation 3, add to equation 1

$$x + 2y + 0z = -4$$
$$0x + 1y + 0z = -3$$
$$0x + 0y + 1z = 4$$

-2 times equation 2, add to equation 1

$$x + 0y + 0z = 2$$
$$0x + 1y + 0z = -3$$
$$0x + 0y + 1z = 4$$

So $x = 2, y = -3, z = 4$ is a solution.

1.3 More Examples

Example 1.8. Solve:

$$x_1 - 5x_2 + 3x_3 = 1 \tag{1}$$
$$2x_1 - 4x_2 + x_3 = 0 \tag{2}$$
$$x_1 + x_2 - 2x_3 = -1 \tag{3}$$

-2 times equation 1, add to equation 2:

$$x_1 - 5x_2 + 3x_3 = 1$$
$$0x_1 + 6x_2 - 5x_3 = -2$$
$$x_1 + x_2 - 2x_3 = -1$$

-1 times equation 1, add to equation 3:

$$x_1 - 5x_2 + 3x_3 = 1$$
$$0x_1 + 6x_2 - 5x_3 = -2$$
$$0x_1 + 6x_2 - 5x_3 = -2$$

-1 times equation 2, add to equation 3:

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= 1 \\ 0x_1 + 6x_2 - 5x_3 &= -2 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{aligned}$$

$\frac{5}{6}$ times equation 2, add to equation 1:

$$\begin{aligned} x_1 + 0x_2 - \frac{7}{6}x_3 &= -\frac{2}{3} \\ 0x_1 + 6x_2 - 5x_3 &= -2 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{aligned}$$

We can express x_1, x_2 in terms of x_3:

$$x_1 = -\frac{2}{3} + \frac{7}{6}x_3$$

$$x_2 = -\frac{1}{3} + \frac{5}{6}x_3$$

The solution set is:

$$\left\{ (-\frac{2}{3} + \frac{7}{6}a, -\frac{1}{3} + \frac{5}{6}a, a) \mid a \text{ real numbers.} \right\}$$

Example 1.9. Solve:

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= 1 \\ 2x_1 - 4x_2 + x_3 &= 0 \\ x_1 + x_2 - 2x_3 &= -2 \end{aligned}$$

-2 times equation 1, add to equation 2:

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= 1 \\ 0x_1 + 6x_2 - 5x_3 &= -2 \\ x_1 + x_2 - 2x_3 &= -2 \end{aligned}$$

-1 times equation 1, add to equation 3:

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= 1 \\ 0x_1 + 6x_2 - 5x_3 &= -2 \\ 0x_1 + 6x_2 - 5x_3 &= -3 \end{aligned}$$

-1 times equation 2, add to equation 3:

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= 1 \\ 0x_1 + 6x_2 - 5x_3 &= -2 \\ 0x_1 + 0x_2 + 0x_3 &= -1 \end{aligned}$$

The last equation, $0 = -1$ has no solution. So the system of linear equations has no solution.

Example 1.10. Solve:

$$\begin{aligned} x_1 + 2x_2 + 0x_3 + x_4 &= 7 \\ x_1 + x_2 + x_3 - x_4 &= 3 \\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 \end{aligned}$$

-1 times equation 1, add to equation 2:

$$\begin{aligned} x_1 + 2x_2 + 0x_3 + x_4 &= 7 \\ 0x_1 - x_2 + x_3 - 2x_4 &= -4 \\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 \end{aligned}$$

-3 times equation 1, add to equation 3:

$$\begin{aligned} x_1 + 2x_2 + 0x_3 + x_4 &= 7 \\ 0x_1 - x_2 + x_3 - 2x_4 &= -4 \\ 0x_1 - 5x_2 + 5x_3 - 10x_4 &= -20 \end{aligned}$$

-5 times equation 2, add to equation 3:

$$\begin{aligned} x_1 + 2x_2 + 0x_3 + x_4 &= 7 \\ 0x_1 - x_2 + x_3 - 2x_4 &= -4 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \end{aligned}$$

-1 times equation 2:

$$\begin{aligned} x_1 + 2x_2 + 0x_3 + x_4 &= 7 \\ 0x_1 + x_2 - x_3 + 2x_4 &= 4 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \end{aligned}$$

-2 times equation 2, add to equation 1:

$$\begin{aligned} x_1 + 0x_2 + 2x_3 - 3x_4 &= -1 \\ 0x_1 + x_2 - x_3 + 2x_4 &= 4 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \end{aligned}$$

which can be written more clearly as:

$$\begin{aligned} x_1 + 2x_3 - 3x_4 &= -1 \\ x_2 - x_3 + 2x_4 &= 4 \\ 0 &= 0 \end{aligned}$$

The last equation $0 = 0$ is always true, so we can ignore it and only consider the first two equations. We can analyze the second equation without consideration of the variable x_1. It would appear that there is considerable latitude in how we can choose x_2, x_3, x_4 and make this equation true. Let us choose x_3 and x_4 to be **anything** we please, say $x_3 = a$ and $x_4 = b$.

Now we can take these arbitrary values for x_3 and x_4, substitute them in equation 1, to obtain

$$x_1 + 2a - 3b = -1$$

$$x_1 = -1 - 2a + 3b$$

Similarly, equation 2 becomes

$$x_2 - a + 2b = 4$$

$$x_2 = 4 + a - 2b$$

So our arbitrary choices of values for x_3 and x_4 (a and b) translate into specific values of x_1 and x_2.

Now we can easily and quickly find many more (infinitely more). Suppose we choose $a = 5$ and $b = -2$, then we compute

$$x_1 = -1 - 2(5) + 3(-2) = -17$$

$$x_2 = 4 + 5 - 2(-2) = 13$$

and you can verify that $(x_1,\ x_2,\ x_3,\ x_4) = (-17,\ 13,\ 5,\ -2)$ makes all three equations true. The entire solution set is written as

$$\{(-1 - 2a + 3b,\ 4 + a - 2b,\ a,\ b) \mid a, b \text{ real numbers}\}.$$

Example 1.11. Solve the following system of linear equations:

$$\begin{aligned} x_2 + \ \ x_3 + 2x_4 + 2x_5 &= 2 \\ x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 &= 4 \\ -2x_1 - \ \ x_2 - 3x_3 + 3x_4 + \ \ x_5 &= 3 \end{aligned}$$

Swap equation 1 and equation 2:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 &= 4 \\ x_2 + \ \ x_3 + 2x_4 + 2x_5 &= 2 \\ -2x_1 - \ \ x_2 - 3x_3 + 3x_4 + \ \ x_5 &= 3 \end{aligned}$$

2 times equation 1, and add it to equation 3:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 &= \ \ 4 \\ x_2 + \ \ x_3 + 2x_4 + 2x_5 &= \ \ 2 \\ 3x_2 + 3x_3 + 7x_4 + 7x_5 &= 11 \end{aligned}$$

-3 times equation 2 and add it to equation 3:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 &= 4 \\ x_2 + \ \ x_3 + 2x_4 + 2x_5 &= 2 \\ x_4 + \ \ x_5 &= 5 \end{aligned}$$

Now the system of linear equations looks like an "inverted stair". We can then solve the system of linear equations by substitution. (A better method well be given later.)

By the last equation:

$$x_4 = 5 - x_5.$$

Solve x_2 in terms of other variables by equation 2:

$$x_2 = 2 - x_3 - 2x_4 - 2x_5$$

$$= 2 - x_3 - 2(5 - x_5) - 2x_5$$

$$= -8 - x_3$$

Solve x_1 in terms of other variables by equation 1:

$$x_1 = 4 - 2x_2 - 3x_3 - 2x_4 - 3x_5$$

$$= 4 - 2(-8 - x_3) - 3x_3 - 2(5 - x_5) - 3x_5$$

$$= 10 - x_3 - x_5.$$

x_3, x_5 can be taken as any values.

Set $x_3 = a$, $x_5 = b$, the solution set can be given by

$$\{(10 - a - b, -8 - a, a, 5 - b, b) \mid a, b \text{ real numbers}\}.$$

A better method: Instead of subsitution, we could use more elimination:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 &= 4 \\ x_2 + \ \ x_3 + 2x_4 + 2x_5 &= 2 \\ x_4 + \ \ x_5 &= 5 \end{aligned}$$

-2 times equation 3 and add it to equation 2:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 &= \ \ 4 \\ x_2 + \ \ x_3 \qquad\qquad &= -8 \\ x_4 + \ \ x_5 &= \ \ 5 \end{aligned}$$

-2 times equation 3 and add it to equation 1:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + \quad + x_5 &= -6 \\ x_2 + \ \ x_3 \qquad\qquad &= -8 \\ x_4 + x_5 &= \ \ 5 \end{aligned}$$

-2 times equation 2 and add it to equation 1:

$$\begin{aligned} x_1 + \quad + x_3 \quad + x_5 &= \ 10 \\ x_2 + x_3 \qquad &= -8 \\ x_4 + x_5 &= \ \ 5 \end{aligned}$$

Notice the following:

1. The system of equations looks like an ”inverted” stairs.
2. The leftmost variables in the equations are x_1, x_2 and x_4.
3. Only the first equation has variable x_1.
4. Only the second equation has variable x_2.
5. Only the third equation has variable x_4.

Move x_3, x_5 to another side.

$$\begin{aligned} x_1 &= 10 - x_3 - x_5 \\ x_2 &= -8 - x_3 \\ x_4 &= 5 - x_5 \end{aligned}$$

The right hand sides have x_3, x_5 as variables only and x_3, x_5 can be taken as any values. Set $x_3 = a$, $x_5 = b$, the solution set can be given by

$$\{(10 - a - b, -8 - a, a, 5 - b, b) \mid a, b \text{ real numbers}\}.$$