Math 1030 Chapter 18

Reference.

Beezer, Ver 3.5 Chapter D (print version p261-282)

Exercise.

Exercises with solutions can be downloaded at http://linear.ups.edu/download/fcla-3.50-solution-manual.pdf (Replace $\mathbb{C}$ by $\mathbb{R}$)Section DM p.98 - 101 all, Section PDM p.101-102 M30, T10, T15, T20

18.1 Definition of the determinant

The **determinant** is a function that take a square matrix as an input and produces a scalar as an output.

Suppose A is an $m \times n$ matrix. Then the **submatrix** $A(i|j)$ is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

Example 18.1. Suppose

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}.$$

Then

$$A(2|3) = \begin{bmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \end{bmatrix} \qquad A(3|1) = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 7 & 8 \end{bmatrix}$$

Example 18.2.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Then

$$A(3|2) = \begin{bmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{bmatrix} \qquad A(4|1) = \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Definition 18.3. Suppose A is a square matrix. Then its **determinant**, $\det(A)$ (or denoted by $|A|$), is an element of $\mathbb{R}$ defined recursively by:

1. If A is a 1×1 matrix, then $\det(A) = [A]_{11}$.
2. If A is a matrix of size n with $n \geq 2$, then

$$\begin{aligned}\det(A) = [A]_{11} \det(A(1|1)) - [A]_{12} \det(A(1|2)) + [A]_{13} \det(A(1|3)) - \\ [A]_{14} \det(A(1|4)) + \cdots + (-1)^{n+1} [A]_{1n} \det(A(1|n)) \\ = \sum_{k=1}^{n} (-1)^{k+1} A_{1k} \cdot \det(A(1|k))\end{aligned}$$

$$A = \left(\begin{array}{cccc|c|ccc} * & * & \cdots & * & A_{1k} & * & \cdots & * \\ \hline A_{21} & A_{22} & \cdots & A_{2(k-1)} & * & A_{2(k+1)} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & A_{3(k-1)} & * & A_{3(k+1)} & \vdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots & * & \vdots & \vdots & \vdots \\ A_{n1} & \cdots & \cdots & A_{n(k-1)} & * & A_{n(k+1)} & \cdots & A_{nn} \end{array}\right),$$

$$A(1|k) = \begin{pmatrix} A_{21} & A_{22} & \cdots & A_{2(k-1)} & A_{2(k+1)} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & A_{3(k-1)} & A_{3(k+1)} & \vdots & A_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n1} & \cdots & \cdots & A_{n(k-1)} & A_{n(k+1)} & \cdots & A_{nn} \end{pmatrix}$$

So to compute the determinant of a 5×5 matrix we must build 5 submatrices, each of size 4. To compute the determinants of each the 4×4 matrices we need to create 4 submatrices each, these now of size 3 and so on. To compute the determinant of a 10×10 matrix would require computing the determinant of $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 3,628,8001 \times 1$ matrices. Fortunately ,there are better ways.

Let us compute the determinant of a reasonably sized matrix by hand.

Theorem 18.4. *Suppose*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$\det(A) = ad - bc.$$

Proof. Theorem 18.4

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\det([d]) - b\det([c]) = ad - bc$$

□

Example 18.5. Suppose that we have the 3×3 matrix

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{bmatrix}$$

Then

$$\det(A) = |A| = \begin{vmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 6 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ -3 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 4 & 1 \\ -3 & -1 \end{vmatrix}$$

$$= 3\left(1\left|2\right| - 6\left|-1\right|\right) - 2\left(4\left|2\right| - 6\left|-3\right|\right) - \left(4\left|-1\right| - 1\left|-3\right|\right)$$

$$= 3\left(1(2) - 6(-1)\right) - 2\left(4(2) - 6(-3)\right) - \left(4(-1) - 1(-3)\right)$$

$$= 24 - 52 + 1$$

$$= -27$$

Theorem 18.6. *Suppose*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Then

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Proof. Theorem 18.6

$$\begin{aligned}
\det(A) &= a_{11}|A(1|1)| - a_{12}|A(1|2)| + a_{13}|A(1|3)| \\
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}
\end{aligned}$$

□

18.2 Computing Determinants

Theorem 18.7 (Determinant Expansion about Rows)**.** *Suppose that A is a square matrix of size n. Then for $1 \leq i \leq n$, we have:*

$$\begin{aligned}
\det(A) &= (-1)^{i+1}[A]_{i1}\det(A(i|1)) + (-1)^{i+2}[A]_{i2}\det(A(i|2)) \\
&\quad + (-1)^{i+3}[A]_{i3}\det(A(i|3)) + \cdots + (-1)^{i+n}[A]_{in}\det(A(i|n)) \\
&= \sum_{j=1}^{n}(-1)^{i}(-1)^{j}A_{ij}\det(A(i|j))
\end{aligned}$$

which is known as **expansion** *along the i-th row.*

The coeffient $(-1)^i(-1)^j$ means that the sign in front of each term of the expansion is equal to sign at the (i, j)-entry of the following matrix:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Proof. Determinant Expansion about Rows Skip the proof. If you are interested, see Beezer, p.266. □

Theorem 18.8 (Determinant of the Transpose)**.** *Suppose that A is a square matrix. Then* $\det(A^t) = \det(A)$.

Proof. Determinant of the Transpose Skip the proof. If you are interested, see Beezer, p.267. □

Theorem 18.9 (Determinant Expansion about Columns)**.** *Suppose that A is a square matrix of size n. Then for $1 \le j \le n$, we have:*

$$\begin{aligned}\det(A) &= (-1)^{1+j}[A]_{1j}\det(A(1|j)) + (-1)^{2+j}[A]_{2j}\det(A(2|j)) \\ &\quad + (-1)^{3+j}[A]_{3j}\det(A(3|j)) + \cdots + (-1)^{n+j}[A]_{nj}\det(A(n|j)) \\ &= \sum_{i=1}^{n}(-1)^{j}(-1)^{i}A_{ij}\det(A(i|j))\end{aligned}$$

which is known as **expansion** *about column j.*

Proof. Determinant Expansion about Columns Skip the proof. If you are interested, see Beezer, p.268. □

Example 18.10. Let

$$A = \begin{bmatrix} -2 & 3 & 0 & 1 \\ 9 & -2 & 0 & 1 \\ 1 & 3 & -2 & -1 \\ 4 & 1 & 2 & 6 \end{bmatrix}$$

Then expanding about the fourth row yields,

$$\begin{aligned}|A| &= (4)(-1)^{4+1}\begin{vmatrix} 3 & 0 & 1 \\ -2 & 0 & 1 \\ 3 & -2 & -1 \end{vmatrix} + (1)(-1)^{4+2}\begin{vmatrix} -2 & 0 & 1 \\ 9 & 0 & 1 \\ 1 & -2 & -1 \end{vmatrix} \\ &\quad + (2)(-1)^{4+3}\begin{vmatrix} -2 & 3 & 1 \\ 9 & -2 & 1 \\ 1 & 3 & -1 \end{vmatrix} + (6)(-1)^{4+4}\begin{vmatrix} -2 & 3 & 0 \\ 9 & -2 & 0 \\ 1 & 3 & -2 \end{vmatrix} \\ &= (-4)(10) + (1)(-22) + (-2)(61) + 6(46) = 92\end{aligned}$$

Expanding about column 3 gives

$$\begin{aligned}|A| &= (0)(-1)^{1+3}\begin{vmatrix} 9 & -2 & 1 \\ 1 & 3 & -1 \\ 4 & 1 & 6 \end{vmatrix} + (0)(-1)^{2+3}\begin{vmatrix} -2 & 3 & 1 \\ 1 & 3 & -1 \\ 4 & 1 & 6 \end{vmatrix} + \\ &\quad (-2)(-1)^{3+3}\begin{vmatrix} -2 & 3 & 1 \\ 9 & -2 & 1 \\ 4 & 1 & 6 \end{vmatrix} + (2)(-1)^{4+3}\begin{vmatrix} -2 & 3 & 1 \\ 9 & -2 & 1 \\ 1 & 3 & -1 \end{vmatrix} \\ &= 0 + 0 + (-2)(-107) + (-2)(61) = 92\end{aligned}$$

Notice how much easier the second computation was. By choosing to expand about the third column, we have two entries that are zero, so two 3×3 determinants need not be computed at all!

When a matrix has all zeros above (or below) the diagonal, exploiting the zeros by expanding about the proper row or column makes computing a determinant insanely easy.

Theorem 18.11. *Suppose A is upper triangular matrix, i.e.*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Then

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

Proof. Theorem 18.11

$$\begin{aligned} \det(A) &= a_{11} \det\left(\begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}\right) && \text{expand along the first column.} \\ &= a_{11}a_{22} \det\left(\begin{bmatrix} a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix}\right) && \text{expand along the first column.} \\ &\cdots \\ &= a_{11}a_{22}\cdots a_{nn} \end{aligned}$$

$\square$

Theorem 18.12. *Suppose A is lower triangular matrix, i.e.*

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Then

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

Example 18.13. Suppose

$$T = \begin{bmatrix} 2 & 3 & -1 & 3 & 3 \\ 0 & -1 & 5 & 2 & -1 \\ 0 & 0 & 3 & 9 & 2 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Then, $\det(T) = 2(-1)(3)(-1)(5) = 30$.

When you consult other texts in your study of determinants, you may run into the terms **minor** and **cofactor**, especially in a discussion centered on expansion about rows and columns. We have chosen not to make these definitions formally since we have been able to get along without them. However, informally, a **minor** is a determinant of a submatrix, specifically $\det\left(A\left(i|j\right)\right)$ and is usually referenced as the minor of $[A]_{ij}$. A **cofactor** is a signed minor, specifically the cofactor of $[A]_{ij}$ is $(-1)^{i+j}\det\left(A\left(i|j\right)\right)$.

18.3 Properties of Determinants of Matrices

Theorem 18.14 (Determinant for Row or Column Multiples)**.** *Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a single row (say, row i) by the scalar α, or by multiplying a single column by the scalar α. Then* $\det\left(B\right) = \alpha\det\left(A\right)$.

Proof. Determinant for Row or Column Multiples Expand along row i, then

$$\begin{aligned}
\det\left(B\right) &= \sum_{k=1}^{n}(-1)^{i+1}[B]_{ik}\det\left(B(i|k)\right) \\
&= \sum_{k=1}^{n}(-1)^{i+1}\alpha[A]_{ik}\det\left(B(i|k)\right) \\
&= \alpha\sum_{k=1}^{n}(-1)^{i+1}[A]_{ik}\det\left(B(i|k)\right) \\
&= \alpha\det(A).
\end{aligned}$$

□

Example 18.15.

$$\det\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = 2\cdot\det\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = 2\cdot 1 = 2.$$

Example 18.16. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

with $\det(A) = 1$. Find

$$\begin{vmatrix} 2a_{11} & 3a_{12} & 4a_{13} \\ 2a_{21} & 3a_{22} & 4a_{23} \\ 2a_{31} & 3a_{32} & 4a_{33} \end{vmatrix}.$$

The above is

$$= 2 \begin{vmatrix} a_{11} & 3a_{12} & 4a_{13} \\ a_{21} & 3a_{22} & 4a_{23} \\ a_{31} & 3a_{32} & 4a_{33} \end{vmatrix} \qquad (\frac{1}{2}C_1)$$

$$= 2 \times 3 \begin{vmatrix} a_{11} & a_{12} & 4a_{13} \\ a_{21} & a_{22} & 4a_{23} \\ a_{31} & a_{32} & 4a_{33} \end{vmatrix} \qquad (\frac{1}{3}C_2)$$

$$= 2 \times 3 \times 4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (\frac{1}{4}C_3)$$

$$= 24.$$

Corollary 18.17 (Determinant with Zero Row or Column)**.** *Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then* $\det(A) = 0$.

Proof. Determinant with Zero Row or Column This follows from Determinant for Row or Column Multiples with $\alpha = 0$. □

Theorem 18.18 (Determinant for Row or Column Swap)**.** *Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then* $\det(B) = -\det(A)$.

Proof. Determinant for Row or Column Swap Skip the proof. If you are interested, see Beezer p.273. □

Example 18.19. Suppose

$$A = \begin{bmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Find $\det(A)$.

$$\det(A) = -\begin{vmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{14} \end{vmatrix} \qquad (R_1 \leftrightarrow R_4)$$

$$= \begin{vmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{14} \end{vmatrix} \qquad (R_2 \leftrightarrow R_3)$$

$$= a_{41}a_{32}a_{23}a_{14}.$$

Corollary 18.20 (Determinant with Equal Rows or Columns)**.** *Suppose that A is a square matrix with two equal rows, or two equal columns. Then* $\det(A) = 0$*.*

Proof. Determinant with Equal Rows or Columns Switching the two equal rows (or columns) gives the same matrix A, so by the previous theorem we have:

$$\det(A) = -\det(A)$$

It follows that $\det(A) = 0$. □

Theorem 18.21 (Determinant for Row or Column Multiples and Addition)**.** *Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a row by the scalar α and then adding it to another row, or by multiplying a column by the scalar α and then adding it to another column. Then* $\det(B) = \det(A)$.

Proof. Determinant for Row or Column Multiples and Addition Suppose the row operation is $\alpha R_i + R_j$, expand along row j. For details, see Beezer p.275. □

Example 18.22. Suppose we want to compute the determinant of the 4×4 matrix

$$A = \begin{bmatrix} 2 & 0 & 2 & 3 \\ 1 & 3 & -1 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{bmatrix}$$

We will perform a sequence of row operations on this matrix, shooting for an upper triangular matrix, whose determinant will be simply the product of its diagonal entries. For each row operation, we will track the effect on the determinant via Theorem Determinant for Row or Column Swap Theorem Determinant for Row or Column Multiples and Theorem Determinant for Row or Column Multiples and Addition.

$$\begin{vmatrix} 2 & 0 & 2 & 3 \\ 1 & 3 & -1 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & -1 & 1 \\ 2 & 0 & 2 & 3 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{vmatrix} \qquad (R_1 \leftrightarrow R_2)$$

$$= -\begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & -6 & 4 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{vmatrix} \qquad (-2R_1 + R_2)$$

$$= -\begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & -6 & 4 & 1 \\ 0 & 4 & -2 & 3 \\ 3 & 5 & 4 & 0 \end{vmatrix} \qquad (1R_1 + R_3)$$

$$= -\begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & -6 & 4 & 1 \\ 0 & 4 & -2 & 3 \\ 0 & -4 & 7 & -3 \end{vmatrix} \qquad (-3R_1 + R_4)$$

$$= -\begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & -2 & 2 & 4 \\ 0 & 4 & -2 & 3 \\ 0 & -4 & 7 & -3 \end{vmatrix} \qquad (1R_3 + R_2)$$

$$= 2\begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 4 & -2 & 3 \\ 0 & -4 & 7 & -3 \end{vmatrix} \qquad (-\frac{1}{2}R_2)$$

$$= 2\begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 11 \\ 0 & -4 & 7 & -3 \end{vmatrix} \qquad (-4R_2 + R_3)$$

$$= 2\begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 11 \\ 0 & 0 & 3 & -11 \end{vmatrix} \qquad (-4R_2 + R_4)$$

$$= 2\begin{vmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 11 \\ 0 & 0 & 1 & -22 \end{vmatrix} \qquad (-1R_3 + R_4)$$

Example 18.23. Compute

$$\begin{vmatrix} 1 & a_1 & a_2 & a_3 \\ 1 & a_1+b_1 & a_2 & a_3 \\ 1 & a_1 & a_2+b_2 & a_3 \\ 1 & a_1 & a_2 & a_3+b_3. \end{vmatrix}$$

The above is

$$\begin{vmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & b_1 & 0 & 0 \\ 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & b_3 \end{vmatrix} \qquad (-1R_1+R_2, -1R_1+R_3, -1R_1+R_4)$$

$$= b_1 b_2 b_3 \qquad \text{(upper triangular matrix)}$$

Theorem 18.24. *Let* $\mathbf{A}_i$, $\mathbf{B}, \mathbf{C}$ *be row vectors with* n *components. Then:*

$$\det \begin{bmatrix} \mathbf{A}_1 \\ \hline \vdots \\ \hline \mathbf{A}_{i-1} \\ \hline \mathbf{B}+\mathbf{C} \\ \hline \mathbf{A}_{i-1} \\ \hline \vdots \\ \hline \mathbf{A}_n \end{bmatrix} = \det \begin{bmatrix} \mathbf{A}_1 \\ \hline \vdots \\ \hline \mathbf{A}_{i-1} \\ \hline \mathbf{B} \\ \hline \mathbf{A}_{i-1} \\ \hline \vdots \\ \hline \mathbf{A}_n \end{bmatrix} + \det \begin{bmatrix} \mathbf{A}_1 \\ \hline \vdots \\ \hline \mathbf{A}_{i-1} \\ \hline \mathbf{C} \\ \hline \mathbf{A}_{i-1} \\ \hline \vdots \\ \hline \mathbf{A}_n \end{bmatrix}$$

Similarly, for column vectors $\mathbf{A}_i$, $\mathbf{B}, \mathbf{C}$ *in* $\mathbb{R}^n$*, we have:*

$$\begin{aligned} \det[\mathbf{A}_1|\cdots|\mathbf{A}_{i-1}|\mathbf{B}+\mathbf{C}|\mathbf{A}_{i+1}|\cdots|\mathbf{A}_n] &= \det[\mathbf{A}_1|\cdots|\mathbf{A}_{i-1}|\mathbf{B}|\mathbf{A}_{i+1}|\cdots|\mathbf{A}_n] \\ &\quad + \det[\mathbf{A}_1|\cdots|\mathbf{A}_{i-1}|\mathbf{C}|\mathbf{A}_{i+1}|\cdots|\mathbf{A}_n] \end{aligned}$$

Proof. Theorem 18.24 Expand along row i (or column i). □

18.4 Examples

Example 18.25. Compute

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}.$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 0 & b(b-a) & c(c-a) \end{vmatrix} \qquad (-aR_2 + R_3)$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b(b-a) & c(c-a) \end{vmatrix} \qquad (-aR_1 + R_2)$$

$$= \begin{vmatrix} b-a & c-a \\ b(b-a) & c(c-a) \end{vmatrix} \qquad \text{(expand along the first column)}$$

$$= (b-a) \begin{vmatrix} 1 & c-a \\ b & c(c-a) \end{vmatrix} \qquad \text{(pull out } b-a \text{ from column 1)}$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix} \qquad \text{(pull out } c-a \text{ from column 2)}$$

$$= (b-a)(c-a)(c-b).$$

More generally:

Example 18.26 (Vandermonde Determinant)**.** Let

$$V_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{n-2} & a_2^{n-2} & a_3^{n-2} & \cdots & a_n^{n-2} \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{bmatrix}$$

1. $\det(V_n) = \det(V_{n-1}) \prod_{i=1}^{n-1}(a_n - a_i)$.
2. $\det(V_n) = \prod_{1 \leq i < j \leq n}(a_j - a_i)$ for $n \geq 2$.

Proof. Vandermonde Determinant

1. Performing the row operations:

$$-a_n R_{n-1} + R_n, -a_n R_{n-2} + R_{n-1}, \ldots, -a_n R_1 + R_2,$$

we have: $\det V_n =$

$$= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 - a_n & a_2 - a_n & a_3 - a_n & \cdots & 0 \\ a_1^2 - a_1 a_n & a_2^2 - a_2 a_n & a_3^2 - a_3 a_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{n-2} - a_1^{n-3} a_n & a_2^{n-2} - a_2^{n-3} a_n & a_3^{n-2} - a_3^{n-3} a_n & \cdots & 0 \\ a_1^{n-1} - a_1^{n-2} a_n & a_2^{n-1} - a_2^{n-2} a_n & a_3^{n-1} - a_3^{n-2} a_n & \cdots & 0 \end{vmatrix}$$

(expanding along the last column)

$$= (-1)^{1+n} \begin{vmatrix} a_1 - a_n & a_2 - a_n & a_3 - a_n & \cdots & a_{n-1} - a_n \\ a_1(a_1 - a_n) & a_2(a_2 - a_n) & a_3(a_3 - a_n) & \cdots & a_{n-1}(a_{n-1} - a_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{n-3}(a_1 - a_n) & a_2^{n-3}(a_2 - a_n) & a_3^{n-3}(a_3 - a_n) & \cdots & a_{n-1}^{n-3}(a_{n-1} - a_n) \\ a_1^{n-2}(a_1 - a_n) & a_2^{n-2}(a_2 - a_n) & a_3^{n-2}(a_3 - a_n) & \cdots & a_{n-1}^{n-2}(a_{n-1} - a_n) \end{vmatrix}$$

(pull out factor $a_1 - a_n$ from column 1, $a_2 - a_n$ from column 2, …, $a_{n-1} - a_n$ from column $n-1$)

$$= (-1)^{n-1}(a_1 - a_n)(a_2 - a_n)\cdots(a_{n-1} - a_n) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_{n-1}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^{n-2} & a_2^{n-2} & a_3^{n-2} & \cdots & a_{n-1}^{n-2} \end{vmatrix}$$

$$= (a_n - a_1)\cdots(a_n - a_{n-1}) \det(V_{n-1}) = \det(V_{n-1}) \prod_{i=1}^{n-1} (a_n - a_i).$$

2. Again by mathematical induction:

   Step 1 When $n = 2$,

   $$\begin{vmatrix} 1 & 1 \\ a_1 & a_2 \end{vmatrix} = a_2 - a_1.$$

   So the formula is valid for $n = 2$.

   Step 2 Suppose the statement is true for $n = k$, i.e.

   $$\det(V_k) = \prod_{1 \le i < j \le k} (a_j - a_i).$$

Then for $n = k+1$

$$\det(V_{k+1}) = \det(V_k) \prod_{i=1}^{k} (a_{k+1} - a_i)$$

$$= \prod_{1 \le i < j \le k} (a_j - a_i) \prod_{i=1}^{k} (a_{k+1} - a_i)$$

$$= \prod_{1 \le i < j \le k+1} (a_j - a_i)$$

$$= \prod_{1 \le 1 < j \le n} (a_j - a_i).$$

The formula is valid for $n = k+1$.

Step 3 By mathematical induction, the formula is valid for all $n \geq 2$. Or without mathematical induction, you can simple repeat the steps again and again until $n = 2$.

□

Reference: https://en.wikipedia.org/wiki/Vandermonde_matrix

Example 18.27. Compute

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 & 4 \end{vmatrix}$$

By $-1R_1 + R_2$, $-1R_1 + R_3$, $-1R_1 + R_4$, $-1R_1 + R_5$, the above is

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix}.$$

Expand along column 3, the above is

$$(-1)^{1+3} \times 1 \times \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 1 \times 1 \times 2 \times 3 = 6.$$

Example 18.28. Compute

$$\det(A) = \begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a. \end{vmatrix}$$

By $-1R_1 + R_2$, $-1R_1 + R_3$, $-1R_1 + R_4$, the above is

$$\begin{vmatrix} a & 1 & 1 & 1 \\ 1-a & a-1 & 0 & 0 \\ 1-a & 0 & a-1 & 0 \\ 1-a & 0 & 0 & a-1 \end{vmatrix}$$

Take out the common factor $a-1$ of row 2, row 3 and row 4, the above is

$$(a-1)^3 \begin{vmatrix} a & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix}$$

$$= (a-1)^3 \begin{vmatrix} a+3 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} \qquad (-1R_4 + R_1, -1R_3 + R_1, -1R_2 + R_1)$$

$$= (a+3)(a-1)^3.$$

Example 18.29. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Let

$$C = \begin{bmatrix} A & \mathcal{O}_{32} \\ \mathcal{O}_{23} & B \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ 0 & 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & 0 & b_{21} & b_{22} \end{bmatrix}$$

Show that

$$\det(C) = \det(A)\det(B).$$

Expand C along the last row, $\det(C) =$

$$(-1)^{5+4} b_{21} \begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & b_{12} \end{vmatrix} + (-1)^{5+5} b_{22} \begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & b_{11} \end{vmatrix}.$$

For each 4×4 submatrix, expand along the last row, the above is

$$(-1)^{5+4} b_{21} (-1)^{4+4} b_{12} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + (-1)^{5+5} b_{22} (-1)^{4+4} b_{11} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (b_{11}b_{22} - b_{21}b_{12})\det(A) = \det(A)\det(B).$$

Remark: The result is also valid when A is a square matrix of size n and B is a square matrix of size m.

18.5 More properties of determinants

Corollary 18.30. *1. Let* $I_n \xrightarrow{R_i \leftrightarrow R_j} J$*, then* $\det(J) = -1$.

2. Let $I_n \xrightarrow{\alpha R_i} J$*, then* $\det(J) = \alpha$.

3. Let $I_n \xrightarrow{\alpha R_i + R_j} J$*, then* $\det(J) = 1$.

Corollary 18.31. *Let A be a square matrix, apply row operation on A and obtain a new matrix B. Let J be obtained by applying the same row operation on I_n. By lecture 13, $B = JA$. Then* $\det(B) = \det(JA) = \det(J)\det(A)$.

Theorem 18.32. *A is nonsingular if and only if* $\det(A) \neq 0$.

Proof. Theorem 18.32 Let B be the RREF of an $n \times n$ square matrix A. Then A is nonsinuglar if and only if $B = I_n$.

Moreover, if A is singular, then B must contain a zero row, which implies that $\det(B) = 0$.

By Theorem 5.37, there is a sequence of elementary matrices J_i, corresponding to row operations, such that:

$$J_k \cdots J_2 J_1 A = B.$$

Applying the previous corollary repeatedly, we have:

$$\det(J_k) \cdots \det(J_2) \det(J_1) \det(A) = \det(B).$$

Since, each $\det(J_i) \neq 0$, we have:

$$\det(A) \neq 0 \Leftrightarrow \det(B) \neq 0.$$

On the other hand, B is an $n \times n$ RREF matrix, so $\det(B) \neq 0$ if and only if $B = I_n$. We conclude that A is nonsingular if and only if $\det(A) \neq 0$. $\square$

Corollary 18.33. *The columns of an $n \times n$ matrix A are linearly independent if and only if* $\det(A) \neq 0$.

Example 18.34. Find x such that

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & x \\ 0 & 2 & 3 & 1 \end{bmatrix}$$

is singular.

Expand along the first column

$$\det(A) = 2 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & x \\ 2 & 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix}.$$

Expand along the second row

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & x \\ 2 & 3 & 1 \end{vmatrix} = -x \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -x.$$

Finally

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} = -1.$$

Hence

$$\det(A) = -2x - 1.$$

It is singular if and only if $\det(A) = 0$ if and only if $x = -\frac{1}{2}$.

Example 18.35. Let

$$A = \begin{bmatrix} a & b & c & d \\ e & 0 & 0 & 0 \\ f & 0 & 0 & 0 \\ g & 0 & 0 & 0 \end{bmatrix}$$

find $\det(A)$.

Method 1

$$a \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} - b \begin{vmatrix} e & 0 & 0 \\ f & 0 & 0 \\ g & 0 & 0 \end{vmatrix} + c \begin{vmatrix} e & 0 & 0 \\ f & 0 & 0 \\ g & 0 & 0 \end{vmatrix} - d \begin{vmatrix} e & 0 & 0 \\ f & 0 & 0 \\ g & 0 & 0 \end{vmatrix}$$

In each of the above matrices, there is one zero columns, so all the determinants of the 3×3 submatrices must be zero. Therefore the above is

$$a0 - b0 + c0 - d0 = 0.$$

Method 2 If $c = 0$, then column 3 is the zero column, so $\det(A) = 0$. Otherwise

$$A \begin{bmatrix} 0 \\ 1 \\ -b/c \\ 0 \end{bmatrix} = \mathbf{0}.$$

So A is singular and hence $\det(A) = 0$.

Theorem 18.36. *If A and B are square matrices. Then*

$$\det(AB) = \det(A)\det(B).$$

Proof. Theorem 18.36 Suppose A or B is singular. Then, accordingly $\det(A)$ or $\det(B)$ is equal to zero. By Nonsingular Product has Nonsingular Terms the matrix AB is also singular, hence:

$$\det(AB) = 0 = \det(A)\det(B).$$

If both A and B are nonsingular, then there are elementary matrices, J_i and K_i, coresponding to row operations, such that:

$$A = J_1 J_2 \cdots J_k I_n,$$

$$B = K_1 K_2 \cdots K_l I_n.$$

This implies that:

$$AB = (J_1 J_2 \cdots J_k)(K_1 K_2 \cdots K_l) I_n$$

By Corollary 18.31, we have:

$$\begin{aligned}
\det(A) &= \det(J_1)\det(J_2)\cdots\det(J_k),\\
\det(B) &= \det(K_1)\det(K_2)\cdots\det(K_l),\\
\det(AB) &= \det(J_1)\det(J_2)\cdots\det(J_k)\\
&\quad\cdot\det(K_1)\det(K_2)\cdots\det(K_l).
\end{aligned}$$

Hence, $\det(AB) = \det(A)\det(B)$. □

Theorem 18.37. *If* $\det(A) \neq 0$, *then* A *is invertible and*

$$\det\left(A^{-1}\right) = \frac{1}{\det(A)}. \tag{18.1}$$

Proof. Theorem 18.37 It follows from Theorem 18.32 that A^{-1} exists. The identity (18.1) then follows from:

$$AA^{-1} = I_n$$

and Theorem 18.36. □

Theorem 18.38 (Cramers rule)**.** *Let* A *be a invertible square matrix of size* n. *Let* $\mathbf{b} \in \mathbb{R}^n$. *Let* M_k *be the square matrix by replacing the* k*-th column of* A *by* $\mathbf{b}$. *If*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is a solution of $A\mathbf{x} = \mathbf{b}$, *then*

$$x_k = \frac{\det(M_k)}{\det(A)}$$

where $k = 1, \ldots, n$.

Proof. Cramer's rule Because A is invertible, $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$. Let X_k be matrix obtained from the identity matrix I_n by replacing column k with $\mathbf{x}$. Then

$$A\mathbf{e}_i = \mathbf{A}_i \text{ if } i \neq k \qquad A\mathbf{x} = \mathbf{b} \text{ if } i = k.$$

Hence

$$AX_k = M_k.$$

Expanding X_k along the row k, we have

$$\det(X_k) = x_k \det(I_{n-1}) = x_k.$$

So

$$\det(M_k) = \det(AX_k) = \det(A)\det(X_k) = \det(A)\,x_k.$$

Therefore

$$x_k = \frac{\det(M_k)}{\det(A)}.$$

$\square$

Example 18.39. Using Cramer's rule to solve the following system of linear equation.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2 \\ x_1 \qquad\;\; + x_3 &= 3 \\ x_1 + x_2 - x_3 &= 1 \end{aligned}$$

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

$$\det(A) = 6.$$

$$M_1 = \begin{bmatrix} 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \det(M_1) = 15.$$

$$x_1 = \frac{\det(M_1)}{\det(A)} = \frac{15}{6} = \frac{5}{2}.$$

$$M_2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \det(M_2) = -6.$$

$$x_2 = \frac{\det(M_2)}{\det(A)} = \frac{-6}{6} = -1.$$

$$M_3 = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \det(M_3) = 3.$$

$$x_3 = \frac{\det(M_3)}{\det(A)} = \frac{3}{6} = \frac{1}{2}.$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}$$

is a solution.

Theorem 18.40 (Formula for inverse)**.** *Suppose A is an invertible matrix. Then*

$$[A^{-1}]_{ji} = \frac{(-1)^{i+j}\det(A(i|j))}{\det(A)}.$$

Pay attention to the order of the indexes i and j.

Proof. Formula for inverse Let $B = A^{-1}$. Let $\mathbf{B}_i$ be the i-th column of B. Then

$$A\mathbf{B}_i = \mathbf{e}_i.$$

The vector $\mathbf{B}_i$ is a solution of $A\mathbf{x} = \mathbf{e}_i$. We can use the previous theorem to find $\mathbf{B}_i$. Let M_j be the square matrix by replacing the j-th column of A by $\mathbf{e}_i$. Expand along the j-th column of M_j, we have

$$\det(M_j) = (-1)^{i+j}\det(M_j(i|j)) = (-1)^{i+j}\det(A(i|j)).$$

Then the j-th coordinate of $\mathbf{B}_i$ is given by

$$B_{ji} = [\mathbf{B}_i]_j = \frac{\det(M_j)}{\det(A)} = \frac{(-1)^{i+j}\det(A(i|j))}{\det(A)}.$$

□

Example 18.41. By the above formula, find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix},$$

$$\det(A) = 6.$$

$$A(1|1) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \det(A(1|1)) = -1,$$

$$A(1|2) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \det(A(1|2)) = -2,$$

$$A(1|3) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \det(A(1|3)) = 1,$$

$$A(2|1) = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}, \qquad \det(A(2|1)) = -5,$$

$$A(2|2) = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}, \qquad \det(A(2|2)) = -4,$$

$$A(2|3) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad \det(A(2|3)) = -1,$$

$$A(3|1) = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, \qquad \det(A(3|1)) = 2,$$

$$A(3|2) = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}, \qquad \det(A(3|2)) = -2,$$

$$A(3|3) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \qquad \det(A(3|3)) = -2.$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} \det(A(1|1)) & -\det(A(2|1)) & \det(A(3|1)) \\ -\det((A(1|2)) & \det(A(2|2)) & -\det(A(3|2)) \\ \det(A(1|3)) & -\det(A(2|3)) & \det(A(3|3)) \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \end{bmatrix}$$

18.6 More examples

Example 18.42. Let A_n be a $n \times n$ matrix

$$\underbrace{\begin{bmatrix} x & 1 & 1 & \cdots & 1 \\ 1 & x & 1 & \cdots & 1 \\ 1 & 1 & x & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & 1 \\ 1 & 1 & 1 & \cdots & x \end{bmatrix}}_{n} \left.\vphantom{\begin{bmatrix} x \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}\right\} n$$

Find $\det(A_n)$.

Add columns $C_2, C_3, \ldots, C_n$ to C_1:

$$\det(A_n) = \underbrace{\begin{vmatrix} x+(n-1) & 1 & 1 & \cdots & 1 \\ x+(n-1) & x & 1 & \cdots & 1 \\ x+(n-1) & 1 & x & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ x+(n-1) & 1 & 1 & \cdots & x \end{vmatrix}}_{n}$$

$$= (x+(n-1)) \underbrace{\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & x & 1 & \cdots & 1 \\ 1 & 1 & x & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 1 & 1 & 1 & \cdots & x \end{vmatrix}}_{n}$$

Performing the following sequence of column operations:

$$-C_1 + C_2, -C_1 + C_3, \ldots, -C_1 + C_n,$$

we conclude that the determinant is equal to:

$$(x+n-1) \underbrace{\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x-1 & 0 & \cdots & 0 \\ 1 & 0 & x-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & 0 & 0 & \cdots & x-1 \end{vmatrix}}_{n} \left.\vphantom{\begin{vmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{vmatrix}}\right\} n = (x+n-1)(x-1)^{n-1}.$$

The last step follows by the fact that the matrix on the left hand side is the lower triangular matrix.

Example 18.43. Let B_n be a $n \times n$ matrix in the form

$$\begin{bmatrix} 1-a_1 & a_2 & 0 & \cdots & 0 & 0 \\ -1 & 1-a_2 & a_3 & \cdots & 0 & 0 \\ 0 & -1 & 1-a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-a_{n-1} & a_n \\ 0 & 0 & 0 & \cdots & -1 & 1-a_n \end{bmatrix}$$

1. Show that $\det(B_n) = \det(B_{n-1}) + (-1)^n(a_1a_2\cdots a_n)$.

2. Hence show $\det(B_n) = 1 + \sum_{i=1}^{n}(-1)^i(a_1a_2\cdots a_i)$.

Solution. 1. Adding rows $R_1, \ldots, R_{n-1}$ to R_n, we have: $\det(B_n) =$

$$\begin{vmatrix} 1-a_1 & a_2 & 0 & \cdots & 0 & 0 \\ -1 & 1-a_2 & a_3 & \cdots & 0 & 0 \\ 0 & -1 & 1-a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-a_{n-1} & a_n \\ -a_1 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}$$

Expand along the last row, the determinant above is equal to:

$$(-1)^{n+1}(-a_1)\begin{vmatrix} a_2 & 0 & \cdots & 0 & 0 \\ 1-a_2 & a_3 & \cdots & 0 & 0 \\ -1 & 1-a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1-a_{n-1} & a_n \end{vmatrix} +$$

$$+(-1)^{n+n}\begin{vmatrix} 1-a_1 & a_2 & 0 & \cdots & 0 \\ -1 & 1-a_2 & a_3 & \cdots & 0 \\ 0 & -1 & 1-a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1-a_{n-1} \end{vmatrix}$$

The first matrix is an lower triangular matrix, so the determinant is the product of the diagonal entries, the second matrix is B_{n-1}.

$$= (-1)^n(a_1\cdots a_n) + \det(B_{n-1}).$$

2. We prove the result by **mathematical induction** :

   Step 1 : The formula is valid for $n = 1$: $\det(B_1) = 1 - a_1$.

   Step 2 : Suppose the formula is true for $n = k$, we want to show that the formula is true for $n = k+1$:

$$B_{k+1} = (-1)^{k+1}(a_1\cdots a_{k+1}) + \det(B_k)$$

$$= 1 + \sum_{i=1}^{k}(-1)^i(a_1a_2\cdots a_i) + (-1)^{k+1}(a_1\cdots a_{k+1})$$

$$= 1 + \sum_{i=1}^{k+1} (-1)^i (a_1 \cdots a_i)$$

$$= 1 + \sum_{i=1}^{n} (-1)^i (a_1 \cdots a_i).$$

The formula is true for $n = k + 1$.

Step 3 : By mathematical induction, the formula is valid for all positive integer. **Explanation**: the formula is true for $k = 1$, then it is true for $k + 1 = 2$, so true for $k + 1 = 3$, etc. Hence true for all integers.

Example 18.44. Let C_n be a $n \times n$ matrix given by

$$C_n = \left.\underbrace{\begin{bmatrix} x & a & a & \cdots & a & a \\ -a & x & a & \cdots & a & a \\ -a & -a & x & \cdots & a & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a & -a & -a & \cdots & -a & x \end{bmatrix}}_{n}\right\} n$$

1. Show that $\det(C_n) = a(x + a)^{n-1} + (x - a)\det(C_{n-1})$.
2. Show that $\det(C_n) = \frac{1}{2}((x + a)^n + (x - a)^n)$.

Solution. 1. The last column can be written as

$$\begin{bmatrix} a \\ a \\ a \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \\ \vdots \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ x - a \end{bmatrix}.$$

Then $\det(C_n) =$

$$\underbrace{\begin{vmatrix} x & a & a & \cdots & a & a \\ -a & x & a & \cdots & a & a \\ -a & -a & x & \cdots & a & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a & -a & -a & \cdots & -a & a \end{vmatrix}}_{n} + \underbrace{\begin{vmatrix} x & a & a & \cdots & a & 0 \\ -a & x & a & \cdots & a & 0 \\ -a & -a & x & \cdots & a & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a & -a & -a & \cdots & -a & x - a \end{vmatrix}}_{n}$$

For the first determinant, pulling out a from the last column, it is equal to:

$$a\underbrace{\begin{vmatrix} x & a & a & \cdots & a & 1 \\ -a & x & a & \cdots & a & 1 \\ -a & -a & x & \cdots & a & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a & -a & -a & \cdots & -a & 1 \end{vmatrix}}_{n}$$

Then, performing the row operations:

$$-1R_n + R_1, \ldots, -1R_{n-1} + R_{n-1},$$

the determinant above is equal to:

$$a\underbrace{\begin{vmatrix} x+a & 2a & 2a & \cdots & 2a & 0 \\ 0 & x+a & 2a & \cdots & 2a & 0 \\ 0 & 0 & x+a & \cdots & 2a & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a & -a & -a & \cdots & -a & 1 \end{vmatrix}}_{n}$$

$$= (-1)^{n+n}a\left.\underbrace{\begin{vmatrix} x+a & 2a & 2a & \cdots & 2a \\ 0 & x+a & 2a & \cdots & 2a \\ 0 & 0 & x+a & \cdots & 2a \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x+a \end{vmatrix}}_{n-1}\right\} n-1$$

(Expand along the last column.)

$$= a(x+a)^{n-1}$$

(Determinant of upper triangular matrix.)

For the second determinant,

$$\left.\underbrace{\begin{vmatrix} x & a & a & \cdots & a & 0 \\ -a & x & a & \cdots & a & 0 \\ -a & -a & x & \cdots & a & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a & -a & -a & \cdots & -a & x-a \end{vmatrix}}_{n}\right\} n$$

$$= (x-a)\underbrace{\left.\begin{vmatrix} x & a & a & \cdots & a \\ -a & x & a & \cdots & a \\ -a & -a & x & \cdots & a \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a & -a & -a & \cdots & x-a \end{vmatrix}\right\}}_{n-1} n-1 \quad \text{(expand along the last column)}$$

(Expand along the last column.)

$$= (x-a)\det(C_{n-1}).$$

Adding the results

$$\det(C_n) = a(x+a)^{n-1} + (x-a)\det(C_{n-1}).$$

2. We will prove the formula by induction.

   Step 1 : When $n = 1$, $C_1 = [x]$, $\det(C_1) = x = \frac{1}{2}((x+a)+(x-a))$. So the formula is valid for $n = 1$.

   Step 2 : Suppose the formula is valid for $n = k$, i.e.

   $$\det(C_k) = \frac{1}{2}((x+a)^k + (x-a)^k).$$

   Then for $n = k+1$,

   $$\begin{aligned} \det(C_{k+1}) &= a(x+a)^k + (x-a)\det(C_k) \\ &= a(x+a)^k + (x-a)\frac{1}{2}((x+a)^k + (x-a)^k) \\ &= \frac{1}{2}(x+a)^k(2a+x-a) + \frac{1}{2}(x-a)^{k+1} \\ &= \frac{1}{2}((x+a)^{k+1} + (x-a)^{k+1}) \\ &= \frac{1}{2}((x+a)^n + (x-a)^n). \end{aligned}$$

   So the formula is valid for $n = k+1$.

   Step 3 : By mathematical induction, the formula is valid for all integers $n \geq 1$.

18.7 Properties of Determinant (summary)

Let A be a square matrix with size n.

1.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

2.

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

3. Expand along row i

$$\begin{aligned} \det(A) &= (-1)^{i+1} [A]_{i1} \det(A(i|1)) + (-1)^{i+2} [A]_{i2} \det(A(i|2)) \\ &\quad + (-1)^{i+3} [A]_{i3} \det(A(i|3)) + \cdots + (-1)^{i+n} [A]_{in} \det(A(i|n)) \end{aligned}$$

4. Expand along column j

$$\begin{aligned} \det(A) &= (-1)^{1+j} [A]_{1j} \det(A(1|j)) + (-1)^{2+j} [A]_{2j} \det(A(2|j)) \\ &\quad + (-1)^{3+j} [A]_{3j} \det(A(3|j)) + \cdots + (-1)^{n+j} [A]_{nj} \det(A(n|j)) \end{aligned}$$

5. $\det(A^t) = \det(A)$

6. Determinant of upper/lower triangular matrix.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}.$$

$$\begin{vmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}.$$

7. Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then $\det(A) = 0$.

8. Suppose that A is a square matrix with two equal rows, or two equal columns, i.e., $R_i = R_j$ or $C_i = C_j$ for $i \neq j$. Then $\det(A) = 0$.

9. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns, i.e., $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$, $i \neq j$. Then $\det(B) = -\det(A)$.

10. Let B be the square matrix obtained from A by multiplying a single row (say, row i) by the scalar α, or by multiplying a single column by the scalar α, i.e., αR_i or αC_i. Then $\det(B) = \alpha \det(A)$.

11. Let B be the square matrix obtained from A by multiplying a row by the scalar α and then adding it to another row, or by multiplying a column by the scalar α and then adding it to another column, i.e., $\alpha R_i + R_j$ or $\alpha C_i + C_j$ for $i \neq j$. Then $\det(B) = \det(A)$.

12.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ b_1 & b_2 & \cdots & b_n \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ c_1 & c_2 & \cdots & c_n \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Similarly

$$\begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & b_1 + c_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,i-1} & b_2 + c_2 & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & b_n + c_n & a_{n,i+1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & c_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,i-1} & c_2 & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & c_n & a_{n,i+1} & \cdots & a_{nn} \end{vmatrix}$$

13. If A and B are square matrices, then

$$\det(AB) = \det(A)\det(B).$$