Math 1030 Chapter 15

Reference.

Beezer, Ver 3.5 Section B (print version p233-238), Section D (print version p245-253)

Exercise.

- Exercises with solutions can be downloaded at http://linear.ups.edu/download/fcla-3.50-solution-manual.pdf (Replace $\mathbb{C}$ by $\mathbb{R}$)

  Section B p.88-92 C10, C11, C12, M20 Section D p.92-96 C21, C23, C30, C31, C35, C36, C37, M20, M21.

15.1 Basis

Definition 15.1. Let V be a vector space. Then a subset S of V is said to be a **basis** for V if

1. S is linearly independent.
2. $\langle S\rangle = V$, i.e. S spans V.

Remark. Most of the time V is a subspace of $\mathbb{R}^m$. Occasionally V is assumed to be a subspace of M_{mn} or P_n. It does not hurt to assume V is a subspace of $\mathbb{R}^m$.

Example 15.2. Let $V = \mathbb{R}^m$, then $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ is a basis for V. (recall all the entries of $\mathbf{e}_i$ is zero, except the i-th entry being 1).

It is called the **standard basis**: Obviously B is linearly independent. Also, for any $\mathbf{v} \in V$, $\mathbf{v} = [\mathbf{v}]_1\mathbf{e}_1 + \cdots + [\mathbf{v}]_m\mathbf{e}_m \in \langle B\rangle$. So $\langle B\rangle = V$.

Example 15.3. Math major only

Consider $V = M_{22}$. Let:

$$B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$B_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad B_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

Then $B = \{B_{11}, B_{12}, B_{21}, B_{22}\}$ is a basis for V.

Check: Obviously B is linearly independent (exercise). Also for any $A \in V$,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = aB_{11} + bB_{12} + cB_{21} + dB_{22}.$$

So $\langle B \rangle = M_{22}$.

Exercise 15.4. Math major only

Let $V = M_{mn}$.

For $1 \leq i \leq m$, $1 \leq j \leq n$, let B_{ij} be the $m \times n$ matrix with (i, j)-th entry equal to 1 and all other entries equal to 0.

Then $\{B_{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for V.

Example 15.5. Math major only

Let $V = P_n$. Then $1, x, x^2, \ldots, x^n$ is a basis. It is easy to show that $S = \{1, x, x^2, \ldots, x^n\}$ is linearly independent. Also any polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is a linear combinations of S.

Example 15.6. A vector space can have different bases.

Consider the vector space $V = \mathbb{R}^2$.

Then,

$$S = \{\mathbf{e}_1, \mathbf{e}_2\}$$

is a basis for V, and:

$$S' = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

is also a basis.

15.2 Bases for spans of column vectors

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of $\mathbb{R}^m$ Recall from lecture 14, that there are several methods to find a subset $T \subseteq \{S\}$.

Such that (i) T is linearly independent (ii) $\langle T \rangle = \langle S \rangle$. (In other words T is a basis for $\langle S \rangle$.)

Method 1 Let $A = [\mathbf{v}_1|\cdots|\mathbf{v}_n] \xrightarrow{\text{RREF}} B$.
Suppose $D = \{d_1, \ldots, d_r\}$ be the indexes of the pivot columns of B.
Let $T = \{\mathbf{v}_{d_1}, \ldots, \mathbf{v}_{d_r}\}$. Then T is a basis for $\langle S\rangle = \mathcal{C}(A)$
Method 2 Let $A = [\mathbf{v}_1|\cdots|\mathbf{v}_n]$. Suppose $A^t \xrightarrow{\text{RREF}} B$.
Let T be the nonzero columns of B^t. Then T is a basis for $\langle S\rangle = \mathcal{C}(A)$
This is an example from Lecture 14.

Example 15.7. Column space from row operations
Let

$$S = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 8 \\ 0 \\ -4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 9 \\ -4 \\ 8 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 7 \\ -13 \\ 12 \\ -31 \end{bmatrix}, \mathbf{v}_7 = \begin{bmatrix} -9 \\ 7 \\ -8 \\ 37 \end{bmatrix} \right.$$

Find a basis for $\langle S\rangle$.

$$A = [\mathbf{v}_1|\cdots|\mathbf{v}_7] = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}.$$

Method 1

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & \boxed{1} & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & \boxed{1} & 2 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let

$$T = \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix} \right\}.$$

Then T is a basis for $\langle S\rangle = \mathcal{C}(A)$.
Method 2 The transpose of A is

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 4 & 8 & 0 & -4 \\ 0 & -1 & 2 & 2 \\ -1 & 3 & -3 & 4 \\ 0 & 9 & -4 & 8 \\ 7 & -13 & 12 & -31 \\ -9 & 7 & -8 & 37 \end{bmatrix}.$$

Row-reduced this becomes,

$$D = \begin{bmatrix} \boxed{1} & 0 & 0 & -\frac{31}{7} \\ 0 & \boxed{1} & 0 & \frac{12}{7} \\ 0 & 0 & \boxed{1} & \frac{13}{7} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we can take

$$T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{31}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{12}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{13}{7} \end{bmatrix} \right\}.$$

T is a basis for $\mathcal{C}(A) = \langle S \rangle$.

Theorem 15.8. *Let S be a finite subset of $\mathbb{R}^m$. Then, a basis for $\langle S \rangle$ exists.*

In fact, there exists a subset T of S such that T is a basis for $\langle S \rangle$ (see Basis of the Column Space).

15.3 Bases and nonsingular matrices

Theorem 15.9. *Suppose that A is a square matrix of size m.*

Then, the columns of A is a basis for $\mathbb{R}^m$ if and only if A is nonsingular.

Proof. This is a direct consequence of the theorem Nonsingular Matrix Equivalences, Round 2:

If columns of A form a basis, then in particular they are linearly independent. So, item 5 of the theorem holds.

It now follows from the theorem that item 1, namely that A is nonsingular, also holds.

Conversely, suppose A is nonsingular.

Then, by Item 5 of Nonsingular Matrix Equivalences, Round 2 the columns of A are linearly independent, and by Item 4 they span $\mathbb{R}^m$. Hence, the columns of A is a basis for $\mathbb{R}^m$. □

In fact, we may further extend Nonsingular Matrix Equivalences, Round 2 as follows:

Theorem 15.10 (Nonsingular Matrix Equivalences)**.** *Suppose that A is an $m \times m$ square matrix. The following are equivalent:*

1. *A is nonsingular.*
2. *A row-reduces to the identity matrix.*
3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
4. *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*
5. *The columns of A form a linearly independent set.*
6. *The columns of A form a basis for $\mathbb{R}^m$.*

Example 15.11. Consider $S' = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Let

$$A = [\mathbf{v}_1 | \mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Exercise. The matrix A is nonsingular.

Hence, S' is a basis for $\mathbb{R}^2$.

Example 15.12.

$$A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}.$$

It may be shown that A is row equivalent to the 3×3 identity matrix.

Hence A is nonsingular, so the columns of A form a basis for $\mathbb{R}^3$.

15.4 Dimension

Definition 15.13 (Dimension)**.** Let V be a vector space.

Suppose a finite set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_t\}$ is a basis for V.

Then, we say that V is a **finite dimensional vector space**.

The number t (namely the number of vectors in the basis) is called the **dimension** of V.

The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be 0.

Remark. It is a non-trivial fact that the dimension is well-defined, i.e., If both $\{\mathbf{v}_1, \ldots, \mathbf{v}_t\}$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_s\}$ are bases for V, then $s = t$.

Theorem 15.14. *Suppose that $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \ldots,\ \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V. Then any set of $t + 1$ or more vectors from V is linearly dependent.*

Proof. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be m vectors in V, where $m \geq t + 1$. Let $A = [\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_t]$. Since S spans V, for every $\mathbf{u}_i$ ($1 \leq i \leq m$) there exists $\mathbf{w}_i \in \mathbb{R}^t$ such that:

$$A\mathbf{w}_i = \mathbf{u}_i.$$

Now, consider the matrix:

$$B = [\mathbf{w}_1|\mathbf{w}_2|\cdots|\mathbf{w}_m].$$

This is a $t \times m$ matrix. In particular, it has more columns than rows, due to the assumption that $m > t$.

Hence, the homogeneous linear system $\mathcal{LS}(B, \mathbf{0})$ has a non-trivial solution $\mathbf{x} \in \mathbb{R}^m$. That is:

$$B\mathbf{x} = \mathbf{0}.$$

The above equation implies that:

$$A\,(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}.$$

By the associativity of matrix multiplication, we have:

$$A\,(B\mathbf{x}) = (AB)\,\mathbf{x}.$$

On the other hand:

$$\begin{aligned} AB &= A[\mathbf{w}_1|\mathbf{w}_2|\cdots|\mathbf{w}_m] \\ &= [A\mathbf{w}_1|A\mathbf{w}_2|\cdots|A\mathbf{w}_m] \\ &= [\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_m] \end{aligned}$$

Hence,

$$(AB)\,\mathbf{x} = \mathbf{0}$$

is equivalent to:

$$[\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_m]\mathbf{x} = \mathbf{0}$$

which is in turn equivalent to:

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \cdots x_m\mathbf{u}_m = \mathbf{0}.$$

Since, $\mathbf{x}$ is not the zero vector, not all the x_i's are equal to zero. We conclude that the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are linearly dependent. □

Theorem 15.15. *Suppose that V is a vector space with a finite basis B and a second basis C.*

Then B and C have the same size.

Proof. Denote the size of B by t. If C has $\geq t+1$ vectors, then by the previous theorem, C is linearly dependent, in contradiction to the condition that C is a basis.

By the same reasoning, the linearly independent set B must also not have more vectors than C.

So, B and C have the same number of vectors. □

Remark. The above theorem shows that the dimension is well-defined. No matter which basis we choose, the size is always the same.

Example 15.16. It follows from Example 15.2 that:

$$\dim \mathbb{R}^m = m.$$

Example 15.17. Math major only

$\dim M_{mn} = mn$. See example 3.

Example 15.18. Math major only

$\dim P_n = n + 1$. See example 4.

Example 15.19. Math major only

Let S_2 be the set of 2×2 symmetric matrices. For $A \in S_2$,

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We can show that:

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for S_2. Hence $\dim S_2 = 3$.

Example 15.20. Math major only

Let P be the set of all real polynomials. As $\{1, x, x^2, x^3, \ldots\}$ is linearly independent, so $\dim P$ does not exists (or we can write $\dim P = \infty$).

Lemma 15.21. *Let V be a vector space and $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u} \in V$.*

Suppose $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent and $\mathbf{u} \notin \langle S \rangle$. Then $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u}\}$ is linearly independent.

Proof. Let the relation of linear dependence of S' be

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k + \alpha\mathbf{u} = \mathbf{0}.$$

Suppose $\alpha \neq 0$, then

$$\mathbf{u} = -\frac{\alpha_1}{\alpha}\mathbf{v}_1 - \cdots - \frac{\alpha_k}{\alpha}\mathbf{v}_k \in \langle S \rangle .$$

Contradiction.

So $\alpha = 0$, then

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k = \mathbf{0}.$$

By the linear independence of S, $\alpha_i = 0$ for all i. Hence the above relation of dependence of S' is trivial. □

Theorem 15.22. *Let V be a nonzero subspace of $\mathbb{R}^m$. (That is, $V \neq \{\mathbf{0}\}$.)*

Then, there exists a basis for V.

Proof. Let V be a nonzero vector space. Let $\mathbf{v}_1$ be a nonzero vector in V. If $V = \langle\{\mathbf{v}_1\}\rangle$, we can take $S = \{\mathbf{v}_1\}$. Then obviously $\{\mathbf{v}_1\}$ is linearly independent and hence S is a basis for V.

Otherwise, let $\mathbf{v}_2 \in V$ but not in $\langle\{\mathbf{v}_1\}\rangle$.

By the previous lemma, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. If $V = \langle\{\mathbf{v}_1, \mathbf{v}_2\}\rangle$, we can take $S = \{\mathbf{v}_1, \mathbf{v}_2\}$.

So S is a basis for V.

Otherwise, let $\mathbf{v}_3 \in V$ but not in $\langle\{\mathbf{v}_1, \mathbf{v}_2\}\rangle$.

By the previous lemma, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Repeat the above process, inductive we can define $\mathbf{v}_{k+1}$ as following: If $V = \langle\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}\rangle$, we can take $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

Because $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent, S is a basis for V.

Otherwise defined $\mathbf{v}_{k+1} \notin \langle\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}\rangle$.

By the previous lemma, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k+1}\}$ is linearly independent.

If the process stops, say at step k, i.e., $V = \langle\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}\rangle$.

Then we can take $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

Because $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent, it is a basis for V.

This completes the proof.

Otherwise, the process continues infinitely, in particular, we can take $k = m + 1$ and $V \neq \langle\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m+1}\}\rangle$ and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m+1}\}$ is linearly independent.

Since $\langle\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}\rangle = \mathbb{R}^m$, by Theorem 15.14 the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{m+1}\}$ are linearly dependent. Contradiction. □

Proposition 15.23. *Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq \mathbb{R}^m$. Then*

$$\dim \langle S \rangle \leq n.$$

Proof. By Theorem Theorem 15.8, there exists a subset T of S such that T is a basis for $\langle S \rangle$.

$$\dim \langle S \rangle = \text{number of vectors in } T \leq \text{number of vectors in } S = n.$$

□

Remark. Both Theorem 15.8 and Proposition 15.23 is valid if we replace $\mathbb{R}^m$ by P_n, M_{mn} or any finite dimensional vector space.

Theorem 15.24. *Suppose a vector space V has dimension n. Then, any linearly independent set with n vectors in V is a basis for V.*

Theorem 15.25. *Suppose a vector space V has dimension n. Suppose S is a set of n vectors in V which spans V (That is, $\langle S \rangle = V$).*

Then, S is a basis for V.

15.5 Rank and nullity of a matrix

Definition 15.26 (Nullity of a matrix)**.** Suppose that $A \in M_{mn}$. Then the **nullity** of A is the dimension of the null space of A, $n(A) = \dim(\mathcal{N}(A))$.

Definition 15.27 (Rank of a matrix)**.** Suppose that $A \in M_{mn}$. Then the **rank** of A is the dimension of the column space of A, $r(A) = \dim(\mathcal{C}(A))$.

Example 15.28. Rank and nullity of a matrix

Let us compute the rank and nullity of

$$A = \begin{bmatrix} 2 & -4 & -1 & 3 & 2 & 1 & -4 \\ 1 & -2 & 0 & 0 & 4 & 0 & 1 \\ -2 & 4 & 1 & 0 & -5 & -4 & -8 \\ 1 & -2 & 1 & 1 & 6 & 1 & -3 \\ 2 & -4 & -1 & 1 & 4 & -2 & -1 \\ -1 & 2 & 3 & -1 & 6 & 3 & -1 \end{bmatrix}$$

To do this, we will first row-reduce the matrix since that will help us determine bases for the null space and column space.

$$\begin{bmatrix} \boxed{1} & -2 & 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & 3 & 0 & -2 \\ 0 & 0 & 0 & \boxed{1} & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this row-equivalent matrix in reduced row-echelon form we record $D = \{1,\, 3,\, 4,\, 6\}$ and $F = \{2,\, 5,\, 7\}$.

By Basis of the Column Space, for each index in D, we can create a single basis vector. In fact $T = \{\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_6\}$ is a basis for $\mathcal{C}(A)$. In total the basis will have 4 vectors, so the column space of A will have dimension 4 and we write $r\,(A) = 4$.

By Theorem 11.10, for each index in F, we can create a single basis vector. In total the basis will have 3 vectors, so the null space of A will have dimension 3 and we write $n\,(A) = 3$. In fact:

$$R = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\mathcal{N}(A)$.

Theorem 15.29 (Computing rank and nullity)**.** *Suppose $A \in M_{mn}$ and $A \xrightarrow{RREF} B$. Let r denote the number of pivot columns (= number of nonzero rows). Then $r\,(A) = r$ and $n\,(A) = n - r$.*

Proof. Let $D = \{d_1, \ldots, d_r\}$ be the indexes of the pivot columns of B. By Basis of the Column Space, $\{\mathbf{A}_{d_1}, \ldots, \mathbf{A}_{d_r}\}$ is a basis for $\mathcal{C}(A)$. So $r\,(A) = r$.

By Theorem 11.10, each free variable corresponding to a single basis vector for the null space. So $n\,(A)$ is the number of free variables $= n - r$. $\square$

Corollary 15.30 (Dimension formula)**.** *Suppose $A \in M_{mn}$, then*

$$r\,(A) + n\,(A) = n.$$

Theorem 15.31. *Let A be a $m \times n$ matrix. Then*

$$r\,(A) = r\left(A^t\right).$$

Equivalently

$$\dim \mathcal{C}\,(A) = \dim \mathcal{R}(A)\,.$$

Proof. Let $A \xrightarrow{\text{RREF}} B$.

Let r denote the number of pivot columns (= number of nonzero rows).

Then by the above discussion $r = r\,(A)$. By Basis for the Row Space, the first r columns of B^t form a basis for $\mathcal{R}(A) = \mathcal{C}(A^t)$. Hence $r = r\,(A^t)$. This completes the proof. $\square$

Let us take a look at the rank and nullity of a square matrix.

Example 15.32. The matrix

$$E = \begin{bmatrix} 0 & 4 & -1 & 2 & 2 & 3 & 1 \\ 2 & -2 & 1 & -1 & 0 & -4 & -3 \\ -2 & -3 & 9 & -3 & 9 & -1 & 9 \\ -3 & -4 & 9 & 4 & -1 & 6 & -2 \\ -3 & -4 & 6 & -2 & 5 & 9 & -4 \\ 9 & -3 & 8 & -2 & -4 & 2 & 4 \\ 8 & 2 & 2 & 9 & 3 & 0 & 9 \end{bmatrix}$$

is row-equivalent to the matrix in reduced row-echelon form,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

With $n = 7$ columns and $r = 7$ nonzero rows tells us the rank is $r(E) = 7$ and the nullity is $n(E) = 7 - 7 = 0$.

The value of either the nullity or the rank are enough to characterize a nonsingular matrix.

Theorem 15.33 (Rank and Nullity of a Nonsingular Matrix)**.** *Suppose that A is a square matrix of size n. The following are equivalent.*

1. *A is nonsingular.*
2. *The rank of A is n, $r(A) = n$.*
3. *The nullity of A is zero, $n(A) = 0$.*

Proof. ($1 \Rightarrow 2$) If A is nonsingular then $\mathcal{C}(A) = \mathbb{R}^n$.

If $\mathcal{C}(A) = \mathbb{R}^n$, then the column space has dimension n, so the rank of A is n.

($2 \Rightarrow 3$) Suppose $r(A) = n$. Then the dimension formula gives

$$\begin{aligned} n(A) &= n - r(A) \\ &= n - n \\ &= 0 \end{aligned}$$

($3 \Rightarrow 1$) Suppose $n(A) = 0$, so a basis for the null space of A is the empty set. This implies that $\mathcal{N}(A) = \{\mathbf{0}\}$ and hence A is nonsingular. □

With a new equivalence for a nonsingular matrix, we can update our list of equivalences which now becomes a list requiring double digits to number.

Theorem 15.34. *Suppose that A is a square matrix of size n. The following are equivalent.*

1. *A is nonsingular.*
2. *A row-reduces to the identity matrix.*
3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
4. *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*
5. *The columns of A are a linearly independent set.*
6. *A is invertible.*
7. *The column space of A is $\mathbb{R}^n$, $\mathcal{C}(A) = \mathbb{R}^n$.*
8. *The columns of A are a basis for $\mathbb{R}^n$.*
9. *The rank of A is n, $r(A) = n$.*
10. *The nullity of A is zero, $n(A) = 0$.*

15.6 Linear relation of P_n and M_{mn}

You can skip this section. It is for math major only

In this section, we discuss the linear relation of P_n or M_{mn} by using the techniques used for the vector space $\mathbb{R}^k$.

Let $V = P_n$ and $f_1, \ldots, f_m, g \in P_n$.

Write

$$f_i(x) = a_{i0} + a_{i1}x + \cdots + a_{in}x^n,$$

$$g(x) = b_0 + b_1 x + \cdots + b_n x^n.$$

By comparing coefficients,

$$g(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_m f_m(x)$$

if and only if

$$\alpha_1 a_{10} + \alpha_2 a_{20} + \cdots + \alpha_m a_{m0} = b_0,$$

$$\alpha_1 a_{11} + \alpha_2 a_{21} + \cdots + \alpha_m a_{m1} = b_1,$$

$$\vdots$$

$$\alpha_1 a_{1n} + \alpha_2 a_{2n} + \cdots + \alpha_m a_{mn} = b_n$$

if and only if

$$\alpha_1 \begin{bmatrix} a_{10} \\ a_{11} \\ \vdots \\ a_{1n} \end{bmatrix} + \alpha_2 \begin{bmatrix} a_{20} \\ a_{21} \\ \vdots \\ a_{2n} \end{bmatrix} + \cdots + \alpha_m \begin{bmatrix} a_{m0} \\ a_{m1} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

The above motivates us to define

$$\mathbf{v}_1 = \begin{bmatrix} a_{10} \\ a_{11} \\ \vdots \\ a_{1n} \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} a_{20} \\ a_{21} \\ \vdots \\ a_{2n} \end{bmatrix}, \cdots, \mathbf{v}_m = \begin{bmatrix} a_{m0} \\ a_{m1} \\ \vdots \\ a_{mn} \end{bmatrix}, \mathbf{u} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

The entries of $\mathbf{v}_i$ are the coefficients of f_i.

We then have the following theorem:

Theorem 15.35. *1. $\{f_1, \ldots, f_m\}$ is linearly independent if and only if $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly independent.*

2. g is a linearly combination of $f_1, \ldots, f_m$ if and only if $\mathbf{u}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_m$

Problems regarding polynomials can therefore be transformed to problems regarding column vectors.

Similarly given $m \times n$ matrices $A_1, \ldots, A_k, B$. Let

$$\mathbf{v}_1 = \begin{bmatrix} [A_1]_1 \\ [A_1]_2 \\ \vdots \\ [A_1]_n \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} [A_2]_1 \\ [A_2]_2 \\ \vdots \\ [A_2]_n \end{bmatrix}, \cdots, \mathbf{u} = \begin{bmatrix} [B]_1 \\ [B]_2 \\ \vdots \\ [B]_n \end{bmatrix}.$$

We have the following:

Theorem 15.36. *1. $\{A_1, \dots, A_k\}$ is linearly independent if and only if $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent.*

2. B is a linearly combination of $A_1, \dots, A_k$ if and only if $\mathbf{u}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$

Again, problems regarding polynomials can be transformed to problems regarding column vectors.

Example 15.37. 1. Determine if

$$A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & -1 \\ 5 & 6 \end{bmatrix}, A_3 = \begin{bmatrix} -2 & 0 \\ -3 & -4 \end{bmatrix}$$

is linearly independent or not.

2. Express

$$B = \begin{bmatrix} -3 & 0 \\ 4 & 1 \end{bmatrix}$$

as a linear combination of A_1, A_2, A_3.

Solution. 1. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 5 \\ -1 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ -3 \\ 0 \\ -4 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \end{bmatrix}.$$

$$[\mathbf{v}_1 | \mathbf{v}_2 | \mathbf{v}_3] = \begin{bmatrix} 1 & 1 & -2 \\ 3 & 5 & -3 \\ 2 & -1 & 0 \\ 4 & 6 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Obviously the columns of the RREF is linearly independent, hence $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Therefore $\{A_1, A_2, A_3\}$ is linearly independent.

2. Next

$$[\mathbf{v}_1 | \mathbf{v}_2 | \mathbf{v}_3 | \mathbf{b}] = \begin{bmatrix} 1 & 1 & -2 & -3 \\ 3 & 5 & -3 & 4 \\ 2 & -1 & 0 & 0 \\ 4 & 6 & -5 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\mathbf{u} = \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$. Hence $B = A_1 + 2A_2 + 3A_3$.

Example 15.38. Let $f_1(x) = 1 + x + x^3$, $f_2(x) = 2 + x + x^2$, $f_3(x) = 4 + 3x + x^2 + 2x^3$, $f_4(x) = 2x^2 + x^3$, $f_5(x) = 3 + 2x + 3x^2 + 2x^3$.

Find a basis for $\langle\{f_1, f_2, f_3, f_4, f_5\}\rangle$.

Solution. Let

$$\mathbf{v}_1 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}2\\1\\1\\0\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}4\\3\\1\\2\end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix}0\\0\\2\\1\end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix}3\\2\\3\\2\end{bmatrix}.$$

Then

$$A = [\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3|\mathbf{v}_4|\mathbf{v}_5] = \begin{bmatrix}1 & 2 & 4 & 0 & 3\\1 & 1 & 3 & 0 & 2\\0 & 1 & 1 & 2 & 3\\1 & 0 & 2 & 1 & 2\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1 & 0 & 2 & 0 & 1\\0 & 1 & 1 & 0 & 1\\0 & 0 & 0 & 1 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for $\langle\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}\rangle$.

So $\{f_1, f_2, f_4\}$ is a basis for $\langle\{f_1, f_2, f_3, f_4, f_5\}\rangle$.