Math 1030 Chapter 14

Reference.

- Beezer, Ver 3.5 Section CRS (print version p167-178)

Exercise.
Exercises with solutions can be downloaded at http://linear.ups.edu/download/fcla-3.50-solution-manual.pdf (Replace $\mathbb{C}$ by $\mathbb{R}$) Section CRS p.66-71 C20, C30-C35, M10, M20, M21, T40, T41, T45.

14.1 Column Spaces and Systems of Equations

Definition 14.1 (Column Space of a Matrix)**.** Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$. Then the **column space** of A, written $\mathcal{C}(A)$, is the subset of $\mathbb{R}^m$ containing all linear combinations of the columns of A,

$$\mathcal{C}(A) = \langle\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}\rangle$$

Theorem 14.2 (Column Spaces and Consistent Systems)**.** *Suppose A is an $m \times n$ matrix and $\mathbf{b}$ is a vector of size m. Then $\mathbf{b} \in \mathcal{C}(A)$ if and only if $\mathcal{LS}(A, \mathbf{b})$ is consistent.*

Proof. Column Spaces and Consistent Systems ($\Rightarrow$) Suppose $\mathbf{b} \in \mathcal{C}(A)$. Then we can write $\mathbf{b}$ as some linear combination of the columns of A. Then by Recognizing Consistency of a Linear System we can use the scalars from this linear combination to form a solution to $\mathcal{LS}(A, \mathbf{b})$, so this system is consistent.

($\Leftarrow$) If $\mathcal{LS}(A, \mathbf{b})$ is consistent, there is a solution that may be used with Recognizing Consistency of a Linear System to write $\mathbf{b}$ as a linear combination of the columns of A. This qualifies $\mathbf{b}$ for membership in $\mathcal{C}(A)$. $\square$

This theorem tells us that asking if the system $\mathcal{LS}(A, \mathbf{b})$ is consistent is exactly the same question as asking if $\mathbf{b}$ is in the column space of A. Or equivalently, it tells us that the column space of the matrix A is precisely those vectors of constants, $\mathbf{b}$, that can be paired with A to create a system of linear equations $\mathcal{LS}(A, \mathbf{b})$ that is consistent.

We can form the chain of equivalences

$$\mathbf{b} \in \mathcal{C}(A) \iff \mathcal{LS}(A, \mathbf{b}) \text{ is consistent} \iff A\mathbf{x} = \mathbf{b} \text{ for some } \mathbf{x}$$

Thus, an alternative (and popular) definition of the column space of an $m \times n$ matrix A is

$$\mathcal{C}(A) = \{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\} = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

Example 14.3. Consider the column space of the 3×4 matrix A,

$$A = \begin{bmatrix} 3 & 2 & 1 & -4 \\ -1 & 1 & -2 & 3 \\ 2 & -4 & 6 & -8 \end{bmatrix}$$

Show that $\mathbf{v} = \begin{bmatrix} 18 \\ -6 \\ 12 \end{bmatrix}$ is in the column space of A, $\mathbf{v} \in \mathcal{C}(A)$. The above theorem says that we need to check the consistency of $\mathcal{LS}(A, v)$. From the augmented matrix and row-reduce,

$$\begin{bmatrix} 3 & 2 & 1 & -4 & 18 \\ -1 & 1 & -2 & 3 & -6 \\ 2 & -4 & 6 & -8 & 12 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & -2 & 6 \\ 0 & \boxed{1} & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the last column is not a pivot column, so the system is consistent and hence $v \in \mathcal{C}(A)$. In fact, we have

$$\mathbf{v} = 6\mathbf{A}_1.$$

Next we show that $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ is not in the column space of A, $\mathbf{w} \notin \mathcal{C}(A)$. The above theorem says that we need to check the consistency of $\mathcal{LS}(A, v)$. From the augmented matrix and row-reduce,

$$\begin{bmatrix} 3 & 2 & 1 & -4 & 2 \\ -1 & 1 & -2 & 3 & 1 \\ 2 & -4 & 6 & -8 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & -2 & 0 \\ 0 & \boxed{1} & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Since the final column is a pivot column, the system is inconsistent and therefore $\mathbf{w} \notin \mathcal{C}(A)$.

The next two examples illustrate the main idea of describing $\mathcal{C}(A)$.

Example 14.4. Describe $\mathcal{C}(A)$ as a null space

Let

$$A = \begin{bmatrix} 1 & 2 & 7 & 1 & -1 \\ 1 & 1 & 3 & 1 & 0 \\ 3 & 2 & 5 & -1 & 9 \\ 1 & -1 & -5 & 2 & 0 \end{bmatrix}.$$

Find $\mathcal{C}(A)$. Let's determine if $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_4 \end{bmatrix} \in \langle S \rangle$.

Applying Gauss-Jordan elimination to the augmented matrix

$$\begin{bmatrix} 1 & 2 & 7 & 1 & -1 & v_1 \\ 1 & 1 & 3 & 1 & 0 & v_2 \\ 3 & 2 & 5 & -1 & 9 & v_3 \\ 1 & -1 & -5 & 2 & 0 & v_4 \end{bmatrix},$$

we obtain

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 3 & -3v_1 + 5v_2 - v_4 \\ 0 & 1 & 4 & 0 & -1 & v_1 - v_2 \\ 0 & 0 & 0 & 1 & -2 & 2v_1 - 3v_2 + v_4 \\ 0 & 0 & 0 & 0 & 0 & 9v_1 - 16v_2 + v_3 + 4v_4 \end{bmatrix}$$

If $9v_1 - 16v_2 + v_3 + 4v_4 = 0$, the above is a RREF. The last column is not a pivot columns. So $\mathbf{v} \in \langle S \rangle$. If $9v_1 - 16v_2 + v_3 + 4v_4 \neq 0$, the equation corresponding to the last row is

$$9v_1 - 16v_2 + v_3 + 4v_4 = 0.$$

So the corresponding system of linear equations is inconsistent. So $\mathbf{v} \in \langle S \rangle$. Hence $\mathbf{v} \in \langle S \rangle$ if and only if $9v_1 - 16v_2 + v_3 + 4v_4 = 0$. Therefore

$$\mathcal{C}(A) = \mathcal{N}([9 \ -16 \ 1 \ 4]).$$

Example 14.5. Describe $\mathcal{C}(A)$ by basis

Let

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix},$$

find $\mathcal{C}(A)$.

$$A \xrightarrow{\text{RREF}} B = \begin{bmatrix} \boxed{1} & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & \boxed{1} & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & \boxed{1} & 2 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The indexes of the pivot columns are $D = \{1, 3, 4\}$. Hence $\mathcal{C}(A) = \langle A \rangle = \langle \{\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4\} \rangle$.

14.2 Column Space Spanned by Original Columns

So we have a foolproof, automated procedure for determining membership in $\mathcal{C}(A)$. While this works just fine a vector at a time, we would like to have a more useful description of the set $\mathcal{C}(A)$ as a whole. The next example will preview the first of two fundamental results about the column space of a matrix.

Example 14.6. Consider the 5×7 matrix A,

$$\begin{bmatrix} 2 & 4 & 1 & -1 & 1 & 4 & 4 \\ 1 & 2 & 1 & 0 & 2 & 4 & 7 \\ 0 & 0 & 1 & 4 & 1 & 8 & 7 \\ 1 & 2 & -1 & 2 & 1 & 9 & 6 \\ -2 & -4 & 1 & 3 & -1 & -2 & -2 \end{bmatrix}$$

The column space of A is

$$\mathcal{C}(A) = \left\langle \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 8 \\ 9 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 7 \\ 6 \\ -2 \end{bmatrix} \right\} \right\rangle$$

While this is a concise description of an infinite set, we might be able to describe the span with fewer than seven vectors. Now we row-reduce,

$$\begin{bmatrix} 2 & 4 & 1 & -1 & 1 & 4 & 4 \\ 1 & 2 & 1 & 0 & 2 & 4 & 7 \\ 0 & 0 & 1 & 4 & 1 & 8 & 7 \\ 1 & 2 & -1 & 2 & 1 & 9 & 6 \\ -2 & -4 & 1 & 3 & -1 & -2 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & \boxed{1} & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are $D = \{1, 3, 4, 5\}$, so we can create the set

$$T = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

and know that $\mathcal{C}(A) = \langle T \rangle$ and T is a linearly independent set of columns from the set of columns of A.

The following theorem is a direct consequence of Basis of a Span:

Theorem 14.7 (Basis of the Column Space)**.** *Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r pivot columns. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the set of indices for the pivot columns of B. Let $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$. Then*

1. *T is a linearly independent set.*
2. *$\mathcal{C}(A) = \langle T \rangle$.*

14.3 Column Space of a Nonsingular Matrix

Theorem 14.8 (Column Space of a Nonsingular Matrix)**.** *Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{C}(A) = \mathbb{R}^n$.*

Proof. Column Space of a Nonsingular Matrix See Theorem 11.9. □

Example 14.9. Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 4 \end{bmatrix}.$$

We can show that A is nonsingular as $A \xrightarrow{\text{RREF}} I_4$. So $\mathcal{C}(A) = \mathbb{R}^4$.

14.4 Row Space of a Matrix

Definition 14.10 (Row Space of a Matrix)**.** Suppose A is an $m \times n$ matrix. The **row space** of A, $\mathcal{R}(A)$ is column space $\mathcal{C}(A^t)$ of A^t.

Informally, the row space is the set of all linear combinations of the rows of A. However, we write the rows as column vectors, thus the necessity of using the transpose to make the rows into columns. Additionally, with the row space defined in terms of the column space, all of the previous results of this section can be applied to row spaces.

Notice that if A is a rectangular $m \times n$ matrix, then $\mathcal{C}(A) \subseteq \mathbb{R}^m$, while $\mathcal{R}(A) \subseteq \mathbb{R}^n$ and the two sets are not comparable since they do not even hold objects of the same type. However, when A is square of size n, both $\mathcal{C}(A)$ and $\mathcal{R}(A)$ are subsets of $\mathbb{R}^n$, though usually the sets will not be equal.

Example 14.11. Find $\mathcal{R}(A)$ for

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}.$$

To build the row space, we transpose the matrix,

$$A^t = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 4 & 8 & 0 & -4 \\ 0 & -1 & 2 & 2 \\ -1 & 3 & -3 & 4 \\ 0 & 9 & -4 & 8 \\ 7 & -13 & 12 & -31 \\ -9 & 7 & -8 & 37 \end{bmatrix}$$

Then the columns of this matrix are used in a span to build the row space,

$$\mathcal{R}(A) = \mathcal{C}\left(A^t\right) = \left\langle \left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \\ -1 \\ 0 \\ 7 \\ -9 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ -1 \\ 3 \\ 9 \\ -13 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \\ -4 \\ 12 \\ -8 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 2 \\ 4 \\ 8 \\ -31 \\ 37 \end{bmatrix} \right\} \right\rangle.$$

First, row-reduce A^t,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & -\frac{31}{7} \\ 0 & \boxed{1} & 0 & \frac{12}{7} \\ 0 & 0 & \boxed{1} & \frac{13}{7} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the pivot columns have indices $D = \{1,\, 2,\, 3\}$, the column space of A^t can be spanned by just the first three columns of A^t,

$$\mathcal{R}(A) = \mathcal{C}\left(A^t\right) = \left\langle \left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \\ -1 \\ 0 \\ 7 \\ -9 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ -1 \\ 3 \\ 9 \\ -13 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \\ -4 \\ 12 \\ -8 \end{bmatrix} \right\} \right\rangle.$$

Theorem 14.12 (Row-Equivalent Matrices have Equal Row Spaces)**.** *Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.*

Proof. Row-Equivalent Matrices have Equal Row Spaces Observe that if B is obtained from A via a row operation of the type $R_i \leftrightarrow R_j$, then the rows of B are the same as the rows of A, and hence the columns of B^t are still the same as the columns of A^t, only with the order changed. Hence,

$$\mathcal{R}(B) = \mathcal{C}\left(B^t\right) = \mathcal{C}\left(A^t\right) = \mathcal{R}(A).$$

If B is obtained from A via a row operation of the type αR_i ($\alpha \neq 0$), then the i-th column of B^t is equal to α times the i-th column of A^t, and the other columns remain the same as those of A^t with the corresponding indices.

In paricular, the i-th column of B^t is a linear combination of the columns of A^t.

Hence, the columns of B^t all lie in $\mathcal{C}(A^t)$, which in turn implies that:

$$\mathcal{R}(B) = \mathcal{C}\left(B^t\right) \subseteq \mathcal{C}\left(A^t\right) = \mathcal{R}(A).$$

On the other hand, if B is obtained from A via αR_i, then A is obtained from B via $\left(\frac{1}{\alpha}\right) R_i$. So, by the same argument as before we have:

$$\mathcal{R}(A) = \mathcal{C}\left(A^t\right) \subseteq \mathcal{C}\left(B^t\right) = \mathcal{R}(B).$$

Hence, $\mathcal{R}(B) = \mathcal{R}(A)$.

If B is obtained from A via a row operation of the type $\alpha R_i + R_j$, then:

$$\left[B^t\right]_j = \alpha \left[A^t\right]_i + \left[A^t\right]_j,$$

and the other columns of B^t remain the same as those of A^t with the corresponding indices.

In paricular, the i-th column of B^t is a linear combination of the columns of A^t.

Hence, the columns of B^t all lie in $\mathcal{C}(A^t)$, which in turn implies that:

$$\mathcal{R}(B) = \mathcal{C}\left(B^t\right) \subseteq \mathcal{C}\left(A^t\right) = \mathcal{R}(A).$$

On the other hand, if B is obtained from A via $\alpha R_i + R_j$, then A is obtained from B via $(-\alpha)R_i + R_j$. So, by the same argument as before we have:

$$\mathcal{R}(A) = \mathcal{C}\left(A^t\right) \subseteq \mathcal{C}(B^t) = \mathcal{R}(B).$$

Hence, $\mathcal{R}(B) = \mathcal{R}(A)$.

We now see that the row space of a matrix remains unchanged after any application of a row operation.

Hence, $\mathcal{R}(B) = \mathcal{R}(A)$ if B is row-equivalent to A, since by the definition of row-equivalence (Row-Equivalent Matrices) B is obtained by A via a series of row operations. □

Example 14.13. Row spaces of two row-equivalent matrices

The matrices

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 5 & 2 & -2 & 3 \\ 1 & 1 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 & 6 \\ 3 & 0 & -2 & -9 \\ 2 & -1 & 3 & 4 \end{bmatrix}$$

are row-equivalent via a sequence of two row operations.

Hence by the above theorem

$$\mathcal{R}(A) = \left\langle \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 6 \end{bmatrix} \right\} \right\rangle = \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ -9 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix} \right\} \right\rangle = \mathcal{R}(B)$$

Theorem 14.14 (Basis for the Row Space)**.** *Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t. Then*

1. $\mathcal{R}(A) = \langle S \rangle$.
2. *S is a linearly independent set.*

Proof. Basis for the Row Space From Theorem Row-Equivalent Matrices have Equal Row Spaces. we know that $\mathcal{R}(A) = \mathcal{R}(B)$. If B has any zero rows, these are columns of B^t that are the zero vector. We can safely toss out the zero vector in the span construction, since it can be recreated from the nonzero vectors by a linear combination where all the scalars are zero. So $\mathcal{R}(A) = \langle S \rangle$.

Suppose B has r nonzero rows and let $D = \{d_1,\, d_2,\, d_3,\, \ldots,\, d_r\}$ denote the indices of the pivot columns of B. Denote the r column vectors of B^t, the vectors in S, as $\mathbf{B}_1,\, \mathbf{B}_2,\, \mathbf{B}_3,\, \ldots,\, \mathbf{B}_r$. To show that S is linearly independent, start with a relation of linear dependence

$$\alpha_1\mathbf{B}_1 + \alpha_2\mathbf{B}_2 + \alpha_3\mathbf{B}_3 + \cdots + \alpha_r\mathbf{B}_r = \mathbf{0}$$

Now consider this vector equality in location d_i. Since B is in reduced row-echelon form, the entries of column d_i of B are all zero, except for a leading 1 in row i. Thus, in B^t, row d_i is all zeros, excepting a 1 in column i. So, for $1 \leq i \leq r$,

$$\begin{aligned}
0 &= [\mathbf{0}]_{d_i} \\
&= [\alpha_1\mathbf{B}_1 + \alpha_2\mathbf{B}_2 + \alpha_3\mathbf{B}_3 + \cdots + \alpha_r\mathbf{B}_r]_{d_i} \\
&= [\alpha_1\mathbf{B}_1]_{d_i} + [\alpha_2\mathbf{B}_2]_{d_i} + [\alpha_3\mathbf{B}_3]_{d_i} + \cdots + [\alpha_r\mathbf{B}_r]_{d_i} \\
&= \alpha_1 [\mathbf{B}_1]_{d_i} + \alpha_2 [\mathbf{B}_2]_{d_i} + \alpha_3 [\mathbf{B}_3]_{d_i} + \cdots + \alpha_r [\mathbf{B}_r]_{d_i} \\
&= \alpha_1(0) + \alpha_2(0) + \alpha_3(0) + \cdots + \alpha_i(1) + \cdots + \alpha_r(0) \\
&= \alpha_i
\end{aligned}$$

So we conclude that $\alpha_i = 0$ for all $1 \leq i \leq r$, establishing the linear independence of S. $\square$

Example 14.15. Improving a span

Suppose in the course of analyzing a matrix (its column space, its null space, its ...) we encounter the following set of vectors, described by a span

$$X = \left\langle \left\{ \begin{bmatrix} 1\\2\\1\\6\\6 \end{bmatrix},\, \begin{bmatrix} 3\\-1\\2\\-1\\6 \end{bmatrix},\, \begin{bmatrix} 1\\-1\\0\\-1\\-2 \end{bmatrix},\, \begin{bmatrix} -3\\2\\-3\\6\\-10 \end{bmatrix} \right\} \right\rangle$$

Let A be the matrix whose rows are the vectors in X, so by design $X = \mathcal{R}(A)$,

$$A = \begin{bmatrix} 1 & 2 & 1 & 6 & 6 \\ 3 & -1 & 2 & -1 & 6 \\ 1 & -1 & 0 & -1 & -2 \\ -3 & 2 & -3 & 6 & -10 \end{bmatrix}$$

Row-reduce A to form a row-equivalent matrix in reduced row-echelon form,

$$B = \begin{bmatrix} \boxed{1} & 0 & 0 & 2 & -1 \\ 0 & \boxed{1} & 0 & 3 & 1 \\ 0 & 0 & \boxed{1} & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the above theorem says we can grab the nonzero columns of B^t and write

$$X = \mathcal{R}(A) = \mathcal{R}(B) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 5 \end{bmatrix} \right\} \right\rangle$$

These three vectors provide a much-improved description of X. There are fewer vectors, and the pattern of zeros and ones in the first three entries makes it easier to determine membership in X.

Theorem 14.16 (Column Space Row Space Transpose)**.** *Suppose A is a matrix. Then $\mathcal{C}(A) = \mathcal{R}(A^t)$.*

Proof. Column Space, Row Space, Transpose

$$\mathcal{C}(A) = \mathcal{C}\left(\left(A^t\right)^t\right) = \mathcal{R}\left(A^t\right)$$

□

Example 14.17. Column space from row operations

Find the column space of A in Example 14.11.

Method 1

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & \boxed{1} & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & \boxed{1} & 2 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let

$$T = \{\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix} \right\}.$$

Then T is linear independent and $\mathcal{C}(A) = \langle T \rangle$.

Method 2 The transpose of A is

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 4 & 8 & 0 & -4 \\ 0 & -1 & 2 & 2 \\ -1 & 3 & -3 & 4 \\ 0 & 9 & -4 & 8 \\ 7 & -13 & 12 & -31 \\ -9 & 7 & -8 & 37 \end{bmatrix}.$$

Row-reduced this becomes,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & -\frac{31}{7} \\ 0 & \boxed{1} & 0 & \frac{12}{7} \\ 0 & 0 & \boxed{1} & \frac{13}{7} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, using Theorem Column Space, Row Space, Transpose and Theorem Basis for the Row Space,

$$\mathcal{C}(A) = \mathcal{R}(A^t) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{31}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{12}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{13}{7} \end{bmatrix} \right\} \right\rangle.$$

This is a very nice description of the column space. Fewer vectors than the 7 involved in the definition, and the pattern of the zeros and ones in the first 3 slots can be used to advantage. For example, let's check if

$$\mathbf{b} = \begin{bmatrix} 3 \\ 9 \\ 1 \\ 4 \end{bmatrix}$$

is in $\mathcal{C}(A)$ or not.

If it is, then

$$\mathbf{b} = \begin{bmatrix} 3 \\ 9 \\ 1 \\ 4 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{31}{7} \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{12}{7} \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{13}{7} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ -\frac{31}{7}x + \frac{12}{7}y + \frac{13}{7}z \end{bmatrix}.$$

From the first three coordinate $x = 3, y = 9, z = 1$. Let's check the last coordinate:

$$-\frac{31}{7} \times 3 + \frac{12}{7} \times 9 + \frac{13}{7} \times 1 = 4.$$

So

$$\mathbf{b} = \begin{bmatrix} 3 \\ 9 \\ 1 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{31}{7} \end{bmatrix} + 9 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{12}{7} \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{13}{7} \end{bmatrix}$$

and hence $\mathbf{b} \in \mathcal{C}(A)$.

Remark. Both methods describe algorithms to find bases (i.e., linear independent set the generate the column space) for the column space. Here are the differences.

1. In method 1, we find a subset of columns that forms a basis. However in method 2, the basis is not a subset of columns.

2. Given a vector $\mathbf{b} \in \mathcal{C}(A)$, it is easier to express it as a linear combination of the basis given by method 2.