A positive mass theorem
in 3-dimensional CR geometry

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Introduction

- Joint work with Chin-Yu Hsiao
- Solution of a tangential Kohn Laplacian $\Box_b$
- Difficulty: we will work on a non-compact CR manifold
  - e.g. Even on the Heisenberg group $\mathbb{H}^n$, $\Box_b$ may not have closed range, when it is extended as a closed linear operator
    
    $$\Box_b : L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)$$
  - Way out: use conformal equivalence, and extend $\Box_b$ instead as
    
    $$\Box_b : L^p \to L^q$$
    
    (Another possibility is to consider $\Box_b$ as an operator from a weighted $L^2$ space to itself, as in our earlier work; we will not pursue that today)
  - Application: a positive mass theorem in CR geometry, as was proposed by Cheng, Malchiodi and Yang.
Outline of the talk

- The (Riemannian) Yamabe problem
- The CR Yamabe problem
- A CR positive mass theorem in 3-dimensions
- Solution of $\Box_b$ on a certain class of non-compact 3-dimensional CR manifolds
The Yamabe problem

- $(M^n, g)$ compact Riemannian manifold of dimension $n \geq 2$
- A metric $\hat{g}$ is said to be conformally equivalent to $g$, if $\hat{g} = e^{2w}g$ for some smooth function $w$ on $M$.
- Question: Can one conformally change the metric $g$, such that the new metric $\hat{g}$ has constant scalar curvature?
- Answer: Yes.
  dimension $n = 2$: uniformization theorem
  dimension $n \geq 3$: contribution by Yamabe, Trudinger, Aubin, Scheon, Yau, ...
In dimension $n \geq 3$, write the conformal metric as

$$\hat{g} = u^{\frac{4}{n-2}} g$$

for some positive smooth function $u$ on $M$, and

$$L_g := c_n \Delta_g + R_g$$

for the conformal Laplacian. Then the Yamabe problem for $(M, g)$ reduces to the following PDE:

$$L_g u = R_{\hat{g}} u^{\frac{n+2}{n-2}}, \quad R_{\hat{g}} = \text{constant}.$$ 

This is a variational problem: it suffices to minimize the functional

$$E_g(u) := \frac{\int_M (|\nabla_g u|^2 + R_g u^2) d\text{vol}_g}{\left( \int_M |u|^{\frac{2n}{n-2}} d\text{vol}_g \right)^{\frac{n-2}{n}}}.$$
Define the Yamabe constant by

\[ Y(M, g) := \inf \{ E_g(u) : u \in C^\infty(M), u > 0 \} \].

It is known that for any compact Riemannian manifold \((M^n, g)\), we have

\[ Y(M, g) \leq Y(S^n, g_{std}) \]

and the inequality is strict unless \((M, [g]) \cong (S^n, [g_{std}])\).

If \(n = 3, 4, 5\) or \((M^n, g)\) is locally conformally flat, this last statement was established via a positive mass theorem.

This strictness of the inequality is important, because it is known that the Yamabe problem can be resolved in the affirmative when \(Y(M, g) < Y(S^n, g_{std})\).

We now discuss the analog of the Yamabe problem in 3-dimensional CR geometry.
The CR Yamabe problem: Set-up

- $M$: an orientable CR manifold of dimension 3, meaning that there exists a distinguished 1-dimensional subbundle $L$ of $\mathbb{C}TM$, with $L \cap \overline{L} = \{0\}$.
- Write $\xi = \text{Re} (L \oplus \overline{L})$.
- Assume that there exists a (real) contact form $\theta$ on $M$ (so $\theta \wedge d\theta \neq 0$ on $M$), such that

$$\text{kernel } \theta = \xi.$$ (In particular, this implies that $M$ is strongly pseudoconvex.)

- Replacing $\theta$ by $-\theta$ if necessary, one can define a Hermitian inner product on $L$, by

$$(Z, W)_\theta := 2i d\theta(Z \wedge \overline{W}), \quad Z, W \in \Gamma(L).$$

- We call such $(M, \theta)$ a pseudohermitian manifold, and think of $\theta \wedge d\theta$ as the natural volume form on $M$. 
\( (M, \theta) \): a pseudohermitian manifold

Then as was first shown by Tanaka and Webster, one can define an associated connection on \( TM \), that is compatible with the CR and pseudohermitian structures.

\( \rightarrow \) define the corresponding (scalar) curvature and torsion.

Write \( R_\theta \) for the scalar curvature associated to \( \theta \).

e.g. \((\mathbb{S}^3, \theta_{\text{std}})\): standard round sphere \( \{ |\zeta| = 1 \} \) in \( \mathbb{C}^2 \),

\[
L = \text{span} \left\{ \overline{\zeta}^2 \frac{\partial}{\partial \zeta^1} - \overline{\zeta}^1 \frac{\partial}{\partial \zeta^2} \right\}, \quad \theta_{\text{std}} := i(\overline{\partial} - \partial)|\zeta|^2.
\]

Then \( R_{\theta_{\text{std}}} \equiv 1 \).

e.g. \((\mathbb{H}^1, \theta_0)\): Heisenberg group \( \simeq \mathbb{C} \times \mathbb{R} \),

\[
L = \text{span} \left\{ \frac{\partial}{\partial z} + i\overline{z} \frac{\partial}{\partial t} \right\}, \quad \theta_0 := dt + i(zd\overline{z} - \overline{z}dz).
\]

Then \( R_{\theta_0} \equiv 0 \).
Various differential operators of interest on \((M, \theta)\)

- **The subgradient** \(\nabla_b\): 
  \[
  \nabla_b u = (Xu, Yu)
  \]
  where \(Z := \frac{1}{2}(X + iY)\) is a local section of \(L\) with \((Z, Z)_\theta = 1\).

- **The sublaplacian** \(\Delta_b\): 
  \[
  \Delta_b u = (X^* X + Y^* Y)u
  \]
  where \(X, Y\) are as above, and \(X^*, Y^*\) are their adjoint under \(L^2(\theta \wedge d\theta)\).

- **The Kohn Laplacian** \(\square_b\): 
  \[
  \square_b u = \overline{Z}^* \overline{Z} u
  \]
  where \(\overline{Z}\) is a local section of \(\overline{L}\) with \((\overline{Z}, \overline{Z})_\theta = 1\), and \(\overline{Z}^*\) is its adjoint under \(L^2(\theta \wedge d\theta)\).
The conformal sublaplacian $L_b$:

$$L_b f = (4\Delta_b + R_\theta)f.$$ 

It describes how the Tanaka-Webster scalar curvature changes under a conformal change of contact form: if $\hat{\theta} = u^2 \theta$, then

$$L_b u = R_{\hat{\theta}} u^3.$$ 

The CR Paneitz operator $P_b$:

$$P_b f = \frac{1}{4} \Box_b \Box_b f - iZ[\text{Tor}_\theta(T, \bar{Z})f]$$

where $\text{Tor}_\theta$ is the torsion of the Tanaka-Webster connection on $(M, \theta)$, $\bar{Z}$ is a local section of $\bar{L}$ with $(\bar{Z}, \bar{Z})_\theta = 1$, and $T$ is the Reeb vector field of the contact form $\theta$. It can be used to describe how a certain CR $Q$-curvature changes under a conformal change of the contact form.
The CR Yamabe problem

- $(M, \theta)$ 3-dimensional pseudohermitian.
- If $\hat{\theta} = u^2 \theta$ for some smooth function $u$ with $u > 0$, then
  \[(Z, W)_{\hat{\theta}} = u^2 (Z, W)_{\theta}, \quad Z, W \in \Gamma(L),\]
  and we say $\hat{\theta}$ is conformally equivalent to $\theta$.
- Question: If $(M, \theta)$ is compact, can we conformally change the contact form $\theta$, such that the new contact form $\hat{\theta}$ has Tanaka-Webster scalar curvature $R_{\hat{\theta}} = \text{constant}$?
- This is equivalent to solving the CR Yamabe equation on $M$:
  \[L_b u = R_{\hat{\theta}} u^3, \quad R_{\hat{\theta}} = \text{constant}.\]
The problem is again variational: it suffices to minimize the functional

\[ E_\theta(u) := \frac{\int_M (|\nabla_b u|^2 + R_\theta u^2) \theta \wedge d\theta}{(\int_M u^4 \theta \wedge d\theta)^{1/2}}. \]

Define the CR Yamabe constant by

\[ Y(M, \theta) := \inf\{ E_\theta(u) : u \in C^\infty(M), u > 0 \}. \]

It is an old result of Jerison and Lee, that for any compact 3-dimensional pseudohermitian manifolds \((M, \theta)\),

\[ Y(M, \theta) \leq Y(S^3, \theta_{\text{std}}). \]

Also, if strict inequality holds, then \(Y(M, \theta)\) is attained by a positive smooth function \(u\) on \(M\), and the CR Yamabe problem can be resolved in the affirmative.

→ Focus only on the case \(Y(M, \theta) > 0\).
The Green’s function of the conformal sublaplacian

- \((M, \theta)\) 3-dimensional compact pseudohermitian, \(Y(M, \theta) > 0\).
- Fix a point \(p \in M\).
- We study the Green’s function \(G_p\) of the conformal sublaplacian of \((M, \theta)\) with pole \(p\): in other words, \(G_p\) is singular at \(p\), with

\[L_b G_p = 16\delta_p.\]

- Write \(\rho(q)\) for a suitable non-isotropic distance from \(q\) to \(p\).
- Also, let \(\mathcal{O}^j\) be the set of all smooth functions \(f\) on \(M \setminus \{p\}\), with

\[|f(q)| \lesssim \rho(q)^j,
\]

and \(|\nabla_b^k f(q)| \lesssim \rho(q)^{j-k}\) for \(k = 1, 2, \ldots\).
By first conformally changing the contact form on $M$ if necessary, for $q \in M$ near $p$, the Green’s function admits an expansion

$$G_p(q) = \frac{1}{2\pi} \rho(q)^{-2} + A + \text{error}, \quad \text{error} \in \mathcal{O}^1.$$ 

where $A$ is a constant.

This is the analog of the conformal normal coordinates in CR geometry.

We will assume our contact form $\theta$ has been chosen already, so that the above expansion of $G_p$ is valid near $p$.

The constant $A$ will be a positive multiple of the mass of a certain blow-up of $(M, \theta)$. Its sign will be important in the CR Yamabe problem in 3 dimensions.
A CR positive mass theorem

Theorem (Cheng-Malchiodi-Yang)

Suppose \((M, \theta)\) is a 3-dimensional compact pseudohermitian CR manifold. Suppose in addition

(i) \(Y(M, \theta) > 0\), and
(ii) the Paneitz operator \(P_b\) is non-negative, in the sense that
\[
\int_M v \cdot P_b v \theta \wedge d\theta \geq 0 \text{ for all } v \in C^\infty(M).
\]

For any \(p \in M\), let \(G_p\) be the Green’s function of the conformal sublaplacian \(L_b\) at \(p\), and \(A\) be the constant term in the expansion of \(G_p\) in CR conformal normal coordinates. Then

(a) \(A \geq 0\);
(b) If \(A = 0\) at some point \(p \in M\), then \(M\) is CR equivalent to \(S^3\), and \([\theta] = [\theta_{\text{std}}]\).
It follows that under the same assumptions, unless \((M, [\theta]) \simeq (\mathbb{S}^3, [\theta_{\text{std}}])\), we have \(A > 0\) in the expansion of \(G_p\).

But when \(A > 0\), one can construct a suitable test function \(u\) on \(M\), to show that

\[ E_\theta(u) < Y(\mathbb{S}^3, \theta_{\text{std}}). \]

\((u\) is obtained by gluing \(G_p\) to a standard bubble on \((\mathbb{H}^1, \theta_0)\).\)

Hence under the assumptions of the above theorem, we have

\[ Y(M, \theta) < Y(\mathbb{S}^3, \theta_{\text{std}}) \]

unless \((M, [\theta]) \simeq (\mathbb{S}^3, [\theta_{\text{std}}])\), and the CR Yamabe quotient \(Y(M, \theta)\) is achieved by some positive smooth minimizer.

See also Gamara and Gamara-Jacoub, where they solved the CR Yamabe problem by seeking critical points of the functional \(E_\theta\) that are not necessarily minimizers.
Theorem (Cheng-Malchiodi-Yang)

Suppose $(M, \theta)$ is a 3-dimensional compact pseudohermitian CR manifold. Suppose in addition

(i) $Y(M, \theta) > 0$, and

(ii) the Paneitz operator $P_b$ is non-negative, in the sense that
\[ \int_M v \cdot \overline{P_b v} \theta \wedge d\theta \geq 0 \]
for all $v \in C^\infty(M)$.

For any $p \in M$, let $G_p$ be the Green’s function of the conformal sublaplacian $L_b$ at $p$, and $A$ be the constant term in the expansion of $G_p$ in CR conformal normal coordinates. Then

(a) $A \geq 0$;

(b) If $A = 0$ at some point $p \in M$, then $M$ is CR equivalent to $S^3$, and $[\theta] = [\theta_{std}]$. 
The theorem is about understanding the Green’s function $G_p$.

To do so, one first construct the blow-up $(M^\# , \theta^\#)$ of $(M, \theta)$, where

$$M^\# := M \setminus \{p\}, \quad \theta^\# := G_p^2 \theta.$$

Then $(M^\#, \theta^\#)$ becomes a non-compact pseudohermitian manifold with infinite volume.

Under a further change of coordinates, if $U$ is a sufficiently small neighborhood of $p$ in $M$, then one can identify

$$U \setminus \{p\} \subset M^\# \leftrightarrow \text{a neighborhood of infinity on } \mathbb{H}^1.$$

Since $\mathbb{H}^1$ is flat, this allows one to identify $M^\#$ as an asymptotically flat pseudohermitian manifold.
Example:

\[ M = S^3 \subset \mathbb{C}^2, \quad \theta = \theta_{\text{std}} = i(\bar{\partial} - \partial)|\zeta|^2, \quad p = (0, -1) \]

The Green's function of conformal sublaplacian on \( M \) with pole \( p \) is then \( G_p = |h| \), where

\[ h(\zeta_1, \zeta_2) = \frac{1}{1 + \zeta_2}. \]

Then \( (M^\#, \theta^\#) := (M \setminus \{p\}, G_p^2 \theta) \) is isometric to the Heisenberg group \( (\mathbb{H}^1, \theta_0) \), where \( \theta_0 = dt + izd\bar{z} - \bar{z}dz \); in fact the 'stereographic projection' map

\[ \zeta \in S^3 \setminus \{p\} \mapsto (z, t) \in \mathbb{H}^1 \]

\[ z = \frac{\zeta_1}{1 + \zeta_2}, \quad t = -\text{Re} \frac{1 - \zeta_2}{1 + \zeta_2} \]

is an isometry between \( (M^\#, \theta^\#) \) and \( (\mathbb{H}^1, \theta_0) \).
Back to our general setting, where \((M^\#, \theta^\#)\) is asymptotically flat; in particular, there exists a compact subset \(K\) of \(M^\#\), where we identify \(M^\# \setminus K\) with a neighborhood of infinity on \(\mathbb{H}^1\).

It turns out one can define the mass of such \((M^\#, \theta^\#)\), by means of an integral of certain geometric quantities on a ‘sphere at infinity’ on \(\mathbb{H}^1\).

**Proposition (Cheng-Malchiodi-Yang)**

*Suppose \((M^\#, \theta^\#)\) arises from the blow-up of a compact 3-dimensional pseudohermitian manifold \((M, \theta)\) as described above at some point \(p \in M\). Then its mass satisfies*

\[
m(M^\#, \theta^\#) = 48\pi^2 A,
\]

*where \(A\) is the constant in the expansion of the Green’s function \(G_p\) of \(L_b\) on \((M, \theta)\) at \(p\), in CR conformal normal coordinates.*
Proposition (continued)

Furthermore, there exists some function \( w \in \mathcal{O}^{-1} \) on \( M^\# \), with \( \Box_b^\# w \in \mathcal{O}^4 \), such that the mass of \((M^\#, \theta^\#)\) satisfies

\[
m(M^\#, \theta^\#) = -\frac{3}{2} \int_{M^\#} |\Box_b^\# w|^2 \theta^\# \wedge d\theta^\# + 3 \int_{M^\#} |\nabla^\# \overline{Z}^\# \nabla^\# w|^2 \theta^\# \wedge d\theta^# + \frac{3}{4} \int_{M^\#} w \cdot \overline{P_b^\#} w \theta^\# \wedge d\theta^#.
\]

Here \( \Box_b^\# \), \( \nabla^\# \) and \( P_b^\# \) are the Kohn Laplacian, the Tanaka-Webster connection, and CR Paneitz operator with respect to \((M^\#, \theta^\#)\), and \( \overline{Z}^\# \) is a section of \( \overline{L} \) on \( M^\# \) with \((\overline{Z}^\#, \overline{Z}^\#)_{\theta^\#} = 1\).

\[\blacktriangleright\] This is a version of Bochner's formula; one gets this by integrating by parts twice in the term involving \( P_b^\# \).
Proposition (continued)

\[ m(M^\#, \theta^\#) = -\frac{3}{2} \int_{M^\#} |\Box^\#_b w|^2 \theta^\# \wedge d\theta^\# + 3 \int_{M^\#} |\nabla^\#_Z \nabla^\#_Z w|^2 \theta^\# \wedge d\theta^\# \\
+ \frac{3}{4} \int_{M^\#} w \cdot P^\#_b w \theta^\# \wedge d\theta^\#. \]

In addition, the same continues to hold, when \( w \) is replaced by any \( v \) on \( M^\# \), with \( v - w \in \mathcal{O}^{1+\delta} \) and \( \Box^\#_b v \in \mathcal{O}^{3+\delta} \) for some \( \delta > 0 \).

Theorem (Hsiao-Y.)

Under the assumptions of the 3-dim CR positive mass theorem, namely that \( Y(M, \theta) > 0 \) and \( P_b \geq 0 \) on \((M, \theta)\), there exists a smooth function \( v \) on \( M^\# \), such that

\[ v - w \in \mathcal{O}^{1+\delta} \text{ for all } \delta \in (0, 1), \text{ and } \Box^\#_b v = 0. \]
As a result, the formula for mass simplifies:

\[
m(M^\#, \theta^\#) = 3 \int_{M^\#} |\nabla^\#_Z \nabla^\#_Z v|^2 \theta^\# \wedge d\theta^\# + \frac{3}{4} \int_{M^\#} v \cdot \overline{P^\#_b v} \theta^\# \wedge d\theta^\#.
\]

With a little more work to bring the integral involving \(P^\#_b\) under control, we can show that \(m(M^\#, \theta^\#) \geq 0\).

(In fact the integral involving \(P^\#_b\) can be written as the sum of a non-negative term with \(-\frac{4}{3} m(M^\#, \theta^\#)\), the latter of which can be reabsorbed into the left hand side.)

Recalling the relation between \(m(M^\#, \theta^\#)\) and the constant term \(A\) in the expansion of the Green’s function \(G_p\) at \(p\), one sees that

\[
A = \frac{1}{48\pi^2} m(M^\#, \theta^\#) \geq 0.
\]

Further work then allows one to characterize when \(A\) is zero at some point \(p\).
Solving $\square_b^\#$

- Recall the statement of our theorem: $w \in \mathcal{O}^{-1}$ is a given function on $M$, with $\square_b^\# w \in \mathcal{O}^4$.

Theorem (Hsiao-Y.)

If $Y(M, \theta) > 0$ and $P_b \geq 0$ on $(M, \theta)$, then there exists a smooth function $v$ on $M^\#$, such that

$$v - w \in \mathcal{O}^{1+\delta} \quad \text{for all} \ \delta \in (0, 1), \ \text{and} \quad \square_b^\# v = 0.$$

- To prove this, let $f = \square_b^\# w \in \mathcal{O}^{3+\delta}$ for all $\delta \in (0, 1)$.
- We solve $\square_b^\# u = f$ for $u \in \mathcal{O}^{1+\delta}$ with estimates.
- Hence taking $v = w - u$, we have all conclusions of our theorem, namely $v - w \in \mathcal{O}^{1+\delta}$, and $\square_b^\# v = 0$.
- Thus the key is to solve the Kohn Laplacian on $(M^\#, \theta^\#)$. This is done via the conformal equivalence between $\theta^\#$ with $\theta$. 
A toy problem

- We saw how \((\mathbb{H}^1, \theta_0)\) arises as the blow-up of \((\mathbb{S}^3, \theta_{\text{std}})\).
- We know very well how one could solve the Kohn Laplacian \(\Box_b\) on \((\mathbb{S}^3, \theta_{\text{std}})\).
- Question: Can we use this knowledge to solve \(\Box^#_b u = f\) on \((\mathbb{H}^1, \theta_0)\)?
- The key here turns out to be that not only \(\theta_0 = G_p^2 \theta_{\text{std}}\), but also there exists a CR function \(h\) on \(\mathbb{S}^3 \setminus \{p\}\), i.e. one with \(\overline{\partial} h = 0\), such that \(G_p = |h|\).

In fact, as we saw before, in this case one can take \(h\) to be

\[ h(\zeta_1, \zeta_2) = \frac{1}{1 + \zeta_2}. \]
Let $\mathcal{Z}$ be a section of $\mathcal{L}$ on $S^3$ with $(\mathcal{Z}, \mathcal{Z})_{\theta_{\text{std}}} = 1$.

Write $\mathcal{Z}^*$ for its formal adjoint under $L^2(S^3, \theta_{\text{std}} \wedge d\theta_{\text{std}})$.

Then $\mathcal{Z}^\# := h^{-1} \mathcal{Z}$ is a section of $\mathcal{L}$ on $H^1$, with $(\mathcal{Z}^\#, \mathcal{Z}^\#)_{\theta_0} = 1$.

Also, the formal adjoint of $\mathcal{Z}^\#$ under $L^2(H^1, \theta_0 \wedge d\theta_0)$ is given by

$$(\mathcal{Z}^\#)^* v = |h|^{-4} \mathcal{Z}^*(h|h|^2 v);$$

this follows since $\theta_0 \wedge d\theta_0 = |h|^4 \theta_{\text{std}} \wedge d\theta_{\text{std}}$. In fact,

$$\int \mathcal{Z}^\# u \cdot \bar{v} \theta_0 \wedge d\theta_0 = \int h^{-1} \mathcal{Z} u \cdot \bar{v} |h|^4 \theta_{\text{std}} \wedge d\theta_{\text{std}}$$

$$= \int u \cdot \mathcal{Z}^*(h|h|^2 v) \theta_{\text{std}} \wedge d\theta_{\text{std}}$$

$$= \int u \cdot |h|^{-4} \mathcal{Z}^*(h|h|^2 v) \theta_0 \wedge d\theta_0.$$
\[ \bar{Z}^\# u = h^{-1}Z u, \quad (\bar{Z}^\#)^* v = |h|^{-4} \bar{Z}^* (h|h|^2 v), \quad \Box_b^\# = (\bar{Z}^\#)^* \bar{Z}^\#. \]

Hence

\[ \Box_b^\# u = |h|^{-4} \bar{Z}^* (h|h|^2 \cdot h^{-1}Z u) = |h|^{-4} \bar{h} \bar{z}^* \bar{z}(hu), \]

the last equality following from the commutativity about \( \bar{Z} \) and \( h \). In other words,

\[ \Box_b^\# u = \bar{h}^{-1} h^{-2} \Box_b (hu). \]

Thus to solve \( \Box_b^\# u = f \) on \( \mathbb{H}^1 \), one could solve instead

\[ \Box_b(hu) = \bar{h} h^2 f \quad \text{on} \quad S^3; \]

one can do this using standard theory about solutions of \( \Box_b \).
The general case

- Back to the general case, where $M^\# = M \setminus \{p\}$, and $\theta^\# = G_p^2 \theta$. Then it is not necessarily true that

$$G_p = |h|$$

for some CR function $h$.

- Good news: one can still construct a CR function $h$, so that

$$|h|^2 G_p^{-2} = 1 + a,$$

for some error $a \in \mathcal{O}^2$.

- Bad news: The error $a$ may not be smooth across $p$. 
A tale of 3 different $\Box_b$’s

- Goal: to solve $\Box_b^\#$ on $M^\#$
- Step 1: Introduce $\tilde{\Box}_b$ on $M$, such that $\Box_b^\#$ is conjugate to $\tilde{\Box}_b$.
- Problem: $\tilde{\Box}_b$ will in general have non-smooth coefficients
- Way out: Construct $\hat{\Box}_b$, with smooth coefficients, that approximates $\tilde{\Box}_b$
Let $\overline{Z}$ be a local section of $\overline{L}$ on $M$, with $(\overline{Z}, \overline{Z})_{\theta} = 1$.

Let $\overline{Z}^\# := G_p^{-1} \overline{Z}$, and define its Hilbert space closure

$$\overline{Z}^\# : L^2(\theta^\# \wedge d\theta^\#) \to L^2(\theta^\# \wedge d\theta^\#).$$

Let $(\overline{Z}^\#)^*$ be its adjoint. Then

$$\Box^\#_b = (\overline{Z}^\#)^* \overline{Z}^\#.$$
Define two (possibly non-smooth) measures

\[ \tilde{m}_0 = (1 + \chi a)^{-1} \theta \wedge d\theta, \quad \tilde{m}_1 = G_p^2 |h|^{-2} \theta \wedge d\theta. \]

Here \( \chi \) is a smooth function, which is identically 1 near \( p \), and vanishes outside a small neighborhood of \( p \).

\( \tilde{m}_0 \) and \( \tilde{m}_1 \) are finite measures on \( M \), which we think of as perturbations of \( \theta \wedge d\theta \). In fact

\[ \tilde{m}_0 = \tilde{m}_1 = \theta \wedge d\theta \quad \text{when} \quad a = 0. \]

Let \( \tilde{Z} := G_p \bar{Z}^\# \), and define its Hilbert space closure

\[ \tilde{Z} : L^2(\tilde{m}_0) \to L^2(\tilde{m}_1). \]

Let \( \tilde{Z}^* \) be its adjoint. Define

\[ \Box_b := \tilde{Z}^* \tilde{Z}. \]
One can check that for any function $u$,

$$\square^\#_b u = (1 + \chi a)^{-1} G_p^{-4} \bar{h} \tilde{\square}_b (h^{-1} u).$$

Hence solving $\square^\#_b u = f$ is the same as solving

$$\tilde{\square}_b (h^{-1} u) = (1 + \chi a) G_p^4 \bar{h}^{-1} f.$$

Problem: $\tilde{\square}_b$ is defined using two possibly non-smooth measures $\tilde{m}_0$ and $\tilde{m}_1$. The standard theory of Kohn Laplacians do not cover this!

The way out: construct a smooth Kohn Laplacian $\hat{\square}_b$, which approximates $\tilde{\square}_b$. 

Define two new measures
\[ \hat{m}_0 = \theta \land d\theta, \quad \hat{m}_1 = (1 + \chi a) G_p^2 |h|^{-2} \theta \land d\theta. \]
so that near \( p \),
\[ \hat{m}_1 = (1 + a) G_p^2 |h|^{-2} \theta \land d\theta = \theta \land d\theta. \]

In particular, \( \hat{m}_0 \) and \( \hat{m}_1 \) are both smooth across \( p \).

Let \( \overline{Z} \) be as before with \( (\overline{Z}, \overline{Z})_\theta = 1 \), and \( \hat{\overline{Z}} := \overline{Z} \).

We extend \( \hat{\overline{Z}} \) to its Hilbert space closure
\[ \hat{\overline{Z}} : L^2(m_0) \rightarrow L^2(m_1). \]

Let \( \hat{\overline{Z}}^* \) be its adjoint. Define
\[ \hat{\Box}_b := \hat{\overline{Z}}^* \hat{\overline{Z}}. \]
\(\hat{\square}_b\) is not quite the standard Kohn Laplacian \(\square_b\) on \(M\), since the adjoint \(\hat{Z}^*\) is taken with respect to two different measures; but the standard theory of Kohn Laplacians carry over easily.

By a result of Chanillo-Chiu-Yang, the conditions \(Y(M, \theta) > 0\) and \(P_b \geq 0\) implies that

\[
\hat{\square}_b : L^2(\theta \wedge d\theta) \rightarrow L^2(\theta \wedge d\theta) \text{ has closed range.}
\]

So we know in principle how to solve \(\hat{\square}_b\).

But one can check that there exists a function \(g \in O^1\), with a sufficiently small support near \(p\), such that

\[
\tilde{\square}_b = \hat{\square}_b + g\overline{Z}.
\]

One can then solve \(\tilde{\square}_b\) using the solution operator for \(\hat{\square}_b\), by adding up a suitable Neumann series. The key is the estimates of various solution operators in \(L^p(\theta \wedge d\theta)\) and \(O^\alpha\).