SOME SPECIAL ISOMORPHISMS OF LIE ALGEBRAS
IN LOW DIMENSIONS

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In this note, we present a more geometric construction of some special isomorphisms between Lie algebras in low dimensions. For simplicity our Lie algebras will be defined over \( \mathbb{C} \); the statements and the proofs will all go through if \( \mathbb{C} \) is replaced by an algebraically closed field \( k \) with \( \text{char} \, k \neq 2 \).

First we recall the definitions of some standard matrix Lie algebras:

\[
\text{sl}_n = \{ x \in \text{gl}_n : \text{tr} \, x = 0 \}
\]

\[
\text{so}_n = \{ x \in \text{gl}_n : x + x^t = 0 \}
\]

\[
\text{sp}_{2n} = \{ x \in \text{gl}_{2n} : xJ_{2n} + J_{2n}x^t = 0 \}
\]

where

\[
J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}
\]

and \( I_n \) is the \( n \times n \) identity matrix. It follows that

\[
\text{sp}_{2n} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \text{gl}_n, a = -d^t, b = b^t, c = c^t \right\}.
\]

Next, let \( V \) be a finite dimensional vector space over \( \mathbb{C} \). A symmetric bilinear form \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \) is said to be non-degenerate, if for every non-zero \( v \in V \), there exists some \( w \in V \) such that \( \langle v, w \rangle \neq 0 \). It is known that all non-degenerate symmetric bilinear forms on \( V \) are equivalent: if \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \) are two non-degenerate symmetric bilinear forms on \( V \), then there exists a linear isomorphism \( T : V \to V \) such that \( \langle v, w \rangle_1 = \langle Tv, Tw \rangle_2 \) for all \( v, w \in V \). In particular, if \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \) is a non-degenerate symmetric bilinear form on \( V \), then there exists a basis \( \{ e_1, \ldots, e_n \} \) of \( V \) such that \( \langle \cdot, \cdot \rangle \) becomes diagonal in this basis, i.e.

\[
\langle e_j, e_k \rangle = \delta_{jk}.
\]

Now suppose \( g \) is a complex Lie algebra, and \( V \) is a complex vector space with a non-degenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \). Suppose we also have a representation \( \rho : g \to \text{gl}(V) \) of \( g \) preserving \( \langle \cdot, \cdot \rangle \), i.e.

\[
\langle \rho(x)v, w \rangle + \langle v, \rho(x)w \rangle = 0 \quad \text{for all} \ x \in g \ \text{and all} \ v, w \in V.
\]

Then picking a special basis \( \{ e_1, \ldots, e_n \} \) as above, so that \( \langle \cdot, \cdot \rangle \) becomes diagonal in this basis, one can identify \( V \) with \( \mathbb{C}^n \), and identify \( \rho \) as a representation \( \rho : g \to \text{so}_n \). We will make repeated use of this fact below.

**Theorem 1.** \( \text{sl}_2 = \text{sp}_2 \simeq \text{so}_3 \).

**Proof.** From definition of \( \text{sl}_2 \) and \( \text{sp}_2 \), it is clear that the two are identical.

Now let \( V = \text{sl}_2 \). On \( V \) there is a non-degenerate symmetric bilinear form

\[
\langle y, z \rangle = \text{tr} \, (yz).
\]
(This is a multiple of the Killing form of $\mathfrak{sl}_2$.) The adjoint action $\text{ad}: \mathfrak{sl}_2 \to \mathfrak{gl}(\mathfrak{sl}_2)$ preserves this non-degenerate symmetric bilinear form:
\[
\langle \text{ad}(x)y, z \rangle + \langle y, \text{ad}(x)z \rangle = 0 \quad \text{for all } x, y, z \in \mathfrak{sl}_2.
\]
In fact,
\[
\langle \text{ad}(x)y, z \rangle + \langle y, \text{ad}(x)z \rangle = \text{tr} \left( \left( xy - yx \right) z + y(xz - zx) \right) = \text{tr} \left( x(yz) - (yz)x \right) = 0
\]
for all $x, y, z \in \mathfrak{sl}_2$. It follows that the adjoint action $\text{ad}$ induces a Lie homomorphism of $\mathfrak{sl}_2$ into $\mathfrak{so}_3$. This is an injective homomorphism, since its kernel is a proper ideal of $\mathfrak{sl}_2$, and $\mathfrak{sl}_2$ is simple; since both $\mathfrak{sl}_2$ and $\mathfrak{so}_3$ are 3-dimensional, it follows that this is an isomorphism of Lie algebras. \hfill \Box

**Theorem 2.** $\mathfrak{sl}_4 \simeq \mathfrak{so}_6$.

**Proof.** Let $V = \Lambda^2 \mathbb{C}^4$ be the vector space of skew-symmetric 2-tensors on $\mathbb{C}^4$. In other words, $V$ is the span of $z \wedge w$ over all $z, w \in \mathbb{C}^4$, where $z \wedge w := z \otimes w - w \otimes z$. Then $V$ is 6-dimensional. Furthermore, there is a natural non-degenerate symmetric bilinear form on $V$: if $\iota: \Lambda^4 \mathbb{C}^4 \to \mathbb{C}$ is an isomorphism of the vector space of alternating 4-tensors on $\mathbb{C}^4$ with $\mathbb{C}$, then one can define a non-degenerate symmetric bilinear form on $V$ by
\[
\langle u, v \rangle = \iota(u \wedge v) \quad \text{for } u, v \in V.
\]
(Both symmetry and non-degeneracy of the bilinear form can be checked by hand easily.) Now the vector representation of $\mathfrak{sl}_4$ on $\mathbb{C}^4$ naturally induces a representation $\rho: \mathfrak{sl}_4 \to \mathfrak{gl}(V)$. Moreover, this representation preserves the non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$:
\[
\langle \rho(x)u, v \rangle + \langle u, \rho(x)v \rangle = 0 \quad \text{for all } x \in \mathfrak{sl}_4 \text{ and all } u, v \in V.
\]
In fact, by linearity, it suffices to check this when $u = z_1 \wedge z_2$ and $v = z_3 \wedge z_4$, where each $z_i$ is one of the standard basis vectors $e_1, e_2, e_3, e_4$ of $\mathbb{C}^4$. Then
\[
\langle \rho(x)u, v \rangle + \langle u, \rho(x)v \rangle = \iota \left[ \left( xz_1 \right) \wedge z_2 \wedge z_3 \wedge z_4 + z_1 \wedge \left( xz_2 \right) \wedge z_3 \wedge z_4 + z_1 \wedge z_2 \wedge \left( xz_3 \right) \wedge z_4 + z_1 \wedge z_2 \wedge z_3 \wedge \left( xz_4 \right) \right];
\]
Here $xz_i$ is the natural action of $x \in \mathfrak{sl}_4$ on $z_i \in \mathbb{C}^4$. Hence by skew-symmetry, this is zero unless $\{z_1, z_2, z_3, z_4\}$ is a re-ordering of $\{e_1, e_2, e_3, e_4\}$. By relabelling the basis $\{e_1, e_2, e_3, e_4\}$, we may assume that $z_i = e_i$ for $i = 1, \ldots, 4$. In that case,
\[
\langle \rho(x)u, v \rangle + \langle u, \rho(x)v \rangle = \iota \left[ \left( \text{tr} x \right) e_1 \wedge e_2 \wedge e_3 \wedge e_4 \right] = 0
\]
as desired as well. Hence $\rho$ induces a representation of $\mathfrak{sl}_4$ into $\mathfrak{so}_6$. By simplicity of $\mathfrak{sl}_4$, the latter is an injective Lie homomorphism; since both $\mathfrak{sl}_4$ and $\mathfrak{so}_6$ are 15 dimensional, it follows that they are isomorphic. \hfill \Box

**Theorem 3.** $\mathfrak{sp}_4 \simeq \mathfrak{so}_5$.

**Proof.** The isomorphism between $\mathfrak{sp}_4$ and $\mathfrak{so}_5$ is obtained by restricting the isomorphism in the previous theorem. In fact, $\mathfrak{sp}_4 \subset \mathfrak{sl}_4$, so if $V = \Lambda^2 \mathbb{C}^4$, $\langle \cdot, \cdot \rangle$ and $\rho: \mathfrak{sl}_4 \to \mathfrak{gl}(V)$ is as in the previous theorem, then it induces a representation $\rho_0: \mathfrak{sp}_4 \to \mathfrak{gl}(V)$ preserving $\langle \cdot, \cdot \rangle$. Now let
\[
v_0 = e_1 \wedge e_3 + e_2 \wedge e_4.
\]
If \( x \in \mathfrak{sp}_4 \), then \( \rho_0(x) v_0 \) is a multiple of \( v_0 \). Hence if \( W \) is the orthogonal complement of \( v_0 \) in \( V \), i.e.
\[
W = \{ w \in V : \langle w, v_0 \rangle = 0 \},
\]
then \( \rho_0(x) \) restricts to a map from \( W \) into \( W \) for all \( x \in \mathfrak{sp}_4 \). It follows that \( \rho_0 \) induces a representation \( \rho_1 : \mathfrak{sp}_4 \to \mathfrak{gl}(W) \). Furthermore, one can restrict \( \langle \cdot, \cdot \rangle \) to \( W \), and the restriction gives a non-degenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle_1 \) on the 5-dimensional vector space \( W \). Since \( \rho_1 \) preserves \( \langle \cdot, \cdot \rangle_1 \), it induces a Lie homomorphism of \( \mathfrak{sp}_4 \) into \( \mathfrak{so}_5 \). Since \( \mathfrak{sp}_4 \) is simple, the kernel of this map is trivial; since \( \mathfrak{sp}_4 \) and \( \mathfrak{so}_5 \) are both 10-dimensional, it follows that they are isomorphic. \[ \square \]

**Theorem 4.** \( \mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \).

Proof. Let \( V = \mathfrak{gl}_2 \) be our 4-dimensional vector space. First, \( \mathfrak{sl}_2 \) acts on \( V \) on the left. In other words, there is a representation \( \rho_1 : \mathfrak{sl}_2 \to \mathfrak{gl}(V) \), given by
\[
\rho_1(x)v = xv \quad \text{for all } x \in \mathfrak{sl}_2 \text{ and all } v \in V.
\]
Similarly, \( \mathfrak{sl}_2 \) acts on \( V \) on the right. In other words, there is a representation \( \rho_2 : \mathfrak{sl}_2 \to \mathfrak{gl}(V) \), defined by
\[
\rho_2(y)v = -vy \quad \text{for all } y \in \mathfrak{sl}_2 \text{ and all } v \in V.
\]
Note \( [\rho_1(x), \rho_2(y)] = 0 \) for all \( x, y \in \mathfrak{sl}_2 \). Thus one can define a representation \( \rho : \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathfrak{gl}(V) \), namely
\[
\rho(x, y) = \rho_1(x) + \rho_2(y) \quad \text{for all } (x, y) \in \mathfrak{sl}_2 \oplus \mathfrak{sl}_2.
\]
More explicitly,
\[
\rho(x, y)v = xv - vy \quad \text{for all } (x, y) \in \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \text{ and all } v \in V.
\]
Now let
\[
J = J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]
and define a bilinear form on \( V \) by\footnote{More explicitly, if
\[
v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad w = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]
then this bilinear form is given by
\[
\langle v, w \rangle = -aD + bC + cB - dA.
\]}
\[
(\langle v, w \rangle) = \text{tr} (vJw^t).
\]
This bilinear form is symmetric, since
\[
\langle v, w \rangle = \text{tr} (vJw^t) = \text{tr} (vJw^t) = \text{tr} (wJv^t) = \text{tr} (wJv^t) = \langle w, v \rangle.
\]
Furthermore, this bilinear form is non-degenerate on \( V \), because the bilinear form
\[
(v, w) := \text{tr} (vw) \text{ is non-degenerate, and the map } w \mapsto Jw^t J \text{ is a linear isomorphism of } V \text{ onto itself. We claim that } \rho \text{ preserves this non-degenerate symmetric bilinear form } \langle \cdot, \cdot \rangle \text{. In fact, by definition of } \rho , \text{ it suffices to show that both } \rho_1 \text{ and } \rho_2 \text{ preserves } \langle \cdot, \cdot \rangle \text{. To see the latter, note that for any } x \in \mathfrak{sl}_2 \text{ and any } v, w \in V , \text{ we have}
\]
\[
\langle \rho_1(x)v, w \rangle = \text{tr} (xvJw^t) = \text{tr} (vJw^t Jx),
\]
and
\[
\langle v, \rho_1(x)w \rangle = \text{tr} (vJ(xw)^t ) = \text{tr} (vJw^t x^t J).
\]
But from $x \in \mathfrak{sl}_2 = \mathfrak{sp}_2$, we have $xJ + Jx^t = 0$. Hence
\[
\langle \rho_1(x)v, w \rangle + \langle v, \rho_1(x)w \rangle = \text{tr} \left( vJw^t(x^tJ + Jx) \right) = 0
\]
as desired. Similarly, for any $y \in \mathfrak{sl}_2$ and any $v, w \in V$, we have
\[
\langle \rho_2(y)v, w \rangle + \langle v, \rho_2(y)w \rangle = -\text{tr} \left( vyJw^tJ \right) - \text{tr} \left( vJ(uy)^tJ \right)
= -\text{tr} \left( v(yJ + Jy^t)w^tJ \right) = 0.
\]
Thus $\rho$ preserves $\langle \cdot, \cdot \rangle$, and induces a map $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathfrak{so}_4$. The kernel of this map
is an ideal of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, and by simplicity of $\mathfrak{sl}_2$ can only be $\{0\}, \mathfrak{sl}_2 \oplus \{0\}, \{0\} \oplus \mathfrak{sl}_2$, or $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. It is then clear that the kernel of this map is trivial, and since both
$\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathfrak{so}_4$ are 6-dimensional, it follows that they are isomorphic.

We remark that one could rephrase the above proof by identifying $V = \mathfrak{gl}_2$ naturally as $\mathbb{C}^2 \otimes (\mathbb{C}^2)^*$. In fact, from the vector representation of $\mathfrak{sl}_2$ on $\mathbb{C}^2$, one can induce naturally an action of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ on $\mathbb{C}^2 \otimes (\mathbb{C}^2)^*$, and that induced representation agrees with the representation $\rho$ we defined in (1). Furthermore, the bilinear form on $V$ defined by (2) is just the one defined by
\[
\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = -\omega(v_1, v_2)\omega(w_1, w_2)
\]
for all $v_1, v_2 \in \mathbb{C}^2$ and all $w_1, w_2 \in (\mathbb{C}^2)^*$, where $\omega$ is the symplectic form on $\mathbb{C}^2$
(and on $(\mathbb{C}^2)^*$ by abuse of notation). Now $\langle \cdot, \cdot \rangle$ is symmetric on $V$ since $\omega$ is anti-
symmetric on $\mathbb{C}^2$, and $\langle \cdot, \cdot \rangle$ is non-degenerate on $V$ since $\omega$ is non-degenerate on $\mathbb{C}^2$.
Furthermore, $\langle \cdot, \cdot \rangle$ is preserved by the action of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, since $\omega$ is preserved by $\mathfrak{sl}_2 = \mathfrak{sp}_2$. This gives us a more conceptual way of presenting the above argument. \qed