Sobolev inequalities for $(0, q)$ forms on CR manifolds of finite type

Po-Lam Yung

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Goal: to study Sobolev inequalities for differential forms

3 parts of the talk:
1. Known result: the exterior derivative $d$ in $\mathbb{R}^N$ (elliptic complex)
2. Corresponding result for $\overline{\partial}_b$ complex (subelliptic)
3. A key element in the proof: a decomposition lemma

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The elliptic complex

- Work of Bourgain-Brezis, Lanzani-Stein and van Schaftingen
  - Set-up: Introduce componentwise $L^p$ norm on the space of $q$ forms on $\mathbb{R}^N$
  - $d$: Hodge de-Rham exterior derivative
    $d : q \text{ forms } \rightarrow (q + 1) \text{ forms}$
  - $d^*$: adjoint of $d$ under the Euclidean inner product
    $d^* : q \text{ forms } \rightarrow (q - 1) \text{ forms}$
  - Question: Suppose $u$ is a $q$ form on $\mathbb{R}^N$ and $du, d^*u \in L^1$. What can we say about $u$?
  - If $q = 0$, $du$ is just the gradient of $u$, so
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More generally

**Theorem (Sobolev inequality for Hodge $d$)**

If $u$ is a compactly supported smooth $q$ form on $\mathbb{R}^N$, and if $q \neq 1$ nor $N - 1$, then

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\|u\|_{L^{\frac{N}{N-1}}} \leq C (\|du\|_{L^1} + \|d^*u\|_{L^1}).
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- Result not true if $q = 1$ or $N - 1$ (‘the forbidden degrees’, dual to each other)
- Essence of the theorem is contained in the following $L^1$-duality inequality:
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Theorem \((L^1\text{-duality inequality})\)

If \(f = (f_1, \ldots, f_N)\) is a divergence free vector field on \(\mathbb{R}^N\), i.e. if

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\sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j} = 0
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with \(f_j \in C_c^\infty\), then for any \(\Phi \in C_c^\infty\),

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\left| \int_{\mathbb{R}^N} f_1 \Phi \right| \leq C \| f \|_{L^1} \| \nabla \Phi \|_{L^N}.
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- Remedy of failure of embedding of \(W^{1,N}\) into \(L^\infty\) on \(\mathbb{R}^N\).
- Relevant to previous Sobolev inequality for \(q\) forms because every component of \(du\) and \(d^*u\) is a component of a divergence free vector field, to which we can apply this duality inequality.
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Example: $q = 0$, $u$ is a function, $du = \sum \frac{\partial u}{\partial x_j} dx_j$.

Each component of $du$ is a component of a divergence free vector field: e.g. $\frac{\partial u}{\partial x_2}$ satisfies

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\frac{\partial}{\partial x_1} \left( \frac{\partial u}{\partial x_2} \right) + \frac{\partial}{\partial x_2} \left( - \frac{\partial u}{\partial x_1} \right) = 0.
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This is because $d \circ d = 0$.

Similar phenomenon for $d^* u$, since $d^* \circ d^* = 0$.

Works as long as $du$ is not top form and $d^* u$ is not a function, which is why we needed $q \neq 1$ nor $N - 1$. 
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- **$M$:** boundary of a bounded smooth pseudoconvex domain in $\mathbb{C}^{n+1}$, $n \geq 2$
- Question: Suppose $u$ is $(0, q)$ form on $M$, and $\partial_b u, \partial^*_b u \in L^1$. What can you say about $u$?
- Problem is subelliptic in nature: $\partial_b u, \partial^*_b u \in L^p$, $1 < p < \infty$ does NOT imply $u \in W^{1,p}$
- Will associate to $M$ a non-isotropic dimension $Q > \dim_\mathbb{R}(M)$ and obtain a corresponding Sobolev inequality
- Recall that in Sobolev inequalities, the bigger the dimension, the less one gains in exponent
- But this is in the nature of subelliptic analysis, and we cannot hope to gain as much as in the elliptic setting
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We have the following Sobolev inequality for $\overline{\partial}_b$ on $M$:

**Theorem (Y. 2009)**

- Assume $M$ is of finite commutator type $m$ at every point i.e. Commutators of $Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n$ of length $\leq m$ span the tangent space to $M$, where $Z_1, \ldots, Z_n$ is a basis of holomorphic vector fields tangent to $M$
  
  e.g. strongly pseudoconvex $\Rightarrow$ commutator type 2

- Also assume $M$ satisfy condition $D(q_0)$ for some $1 \leq q_0 \leq n/2$ i.e. there is a constant $C > 0$ such that for any point $x \in M$, the sum of any $q_0$ eigenvalues of the Levi form at $x$ is bounded by $C$ times any other such sum.
  
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  i.e. there is a constant $C > 0$ such that for any point $x \in M$, the sum of any $q_0$ eigenvalues of the Levi form at $x$ is bounded by $C$ times any other such sum.
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Po-Lam Yung  
Sobolev inequalities for $(0, q)$ forms
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**Theorem (Y. 2009)**

- Assume $M$ is of finite commutator type $m$ at every point i.e. Commutators of $Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n$ of length $\leq m$ span the tangent space to $M$, where $Z_1, \ldots, Z_n$ is a basis of holomorphic vector fields tangent to $M$
  
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Po-Lam Yung

**Sobolev inequalities for $(0, q)$ forms**
Let $Q = 2n + m$.

(a) Let $u = \text{smooth } (0, q)$ form on $M$ orthogonal to $\text{Kernel}(\square_b)$, where $q_0 \leq q \leq n - q_0$ and $q \neq 1$ nor $n - 1$. Then

$$\|u\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\partial_b u\|_{L^1(M)} + \|\partial_b^* u\|_{L^1(M)}.$$ 

(b) Let $v = \text{smooth } (0, q_0 - 1)$ form orthogonal to $\text{Kernel}(\partial_b)$. Then

$$\|v\|_{L^{\frac{Q}{Q-1}}(M)} \lesssim \|\partial_b v\|_{L^1(M)}.$$ 

(c) A similar inequality for $(0, n - q_0 + 1)$ forms orthogonal to $\text{Kernel}(\partial_b^*)$ by duality.
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Corollary

- \( M \): boundary of a bounded smooth strongly pseudoconvex domain in \( \mathbb{C}^{n+1} \), \( n \geq 2 \)
- \( q \neq 1 \) nor \( n - 1 \)
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\[
\| u \|_{L^{Q^{-1}} (M)} \lesssim \| \bar{\partial} b u \|_{L^1 (M)} + \| \bar{\partial}^* b u \|_{L^1 (M)}
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for all smooth functions \( u \) orthogonal to Kernel\((\bar{\partial} b)\)
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Remarks

- There is also a version of these Sobolev inequalities for abstract CR manifolds.
- The proof of the Sobolev inequality for $\overline{\partial}_b$ relies on a subelliptic version of $L^1$-duality inequality (to be stated on the next page), and the fact that $\overline{\partial}_b \circ \overline{\partial}_b = 0$.
- We assumed $n \geq 2$ because our method does not allow $q = 1$ or $n - 1$.
- The conditions of finite commutator type and $D(q_0)$ were made to ensure maximal subellipticity of the solution operator to $\Box_b$ in the $L^p$ sense.
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Theorem (Y. 2009)

- \( X_1, \ldots, X_n \) smooth real vector fields near 0 on \( \mathbb{R}^N \)
- Assume they are linearly independent at 0, and their commutators of length \( \leq r \) span at 0.
- Let \( V_j(0) \) be the span of the restrictions of the commutators of \( X_1, \ldots, X_n \) of length \( \leq j \) to 0.
- Let \( Q = \sum_{j=1}^{r} j \cdot (\text{dim} V_j(0) - \text{dim} V_{j-1}(0)) \)
- Then there is a neighborhood \( U \) of 0 and \( C > 0 \) such that if
  \[
  X_1 f_1 + \cdots + X_n f_n = 0
  \]
on \( U \) with \( f_1, \ldots, f_n \in C^\infty(U) \) and \( \Phi \in C^\infty_c(U) \), then
  \[
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Remarks

- This generalizes the $L^1$-duality inequality we stated at the beginning.
- Chanillo-van Schaftingen has proved the theorem above when the underlying space is a homogeneous group and $X_1, \ldots, X_n$ is a basis of vector fields of degree 1 on that group.
- Difficulty in the current theorem: getting the best (i.e. smallest) possible value of $Q$. The one we had given is the best possible. Thus $Q$ should be thought of as the non-isotropic dimension of 0 in such a situation.
- In fact we have the following subelliptic Sobolev inequality with the best possible exponent:
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- Then there exists a neighborhood $U$ of 0 and $C > 0$ such that if $u$ is a smooth function on $U$ and $1 \leq p < Q$, then

$$\|u\|_{L^{p^*}(U)} \leq C \left( \sum_{j=1}^{n} \|X_j u\|_{L^p(U)} + \|u\|_{L^p(U)} \right)$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q}$.

Moreover the inequality cannot hold for any bigger value of $p^*$.

This generalizes a result of Caponga, Danielli and Garofalo.
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- On $\mathbb{R}^2$, use coordinates $(x, t)$, and let $X = \frac{\partial}{\partial x}$, $Y = x \frac{\partial}{\partial t}$.
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Po-Lam Yung | Sobolev inequalities for $(0, q)$ forms
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Po-Lam Yung

Sobolev inequalities for $(0, q)$ forms
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If $Xf_1 + Yf_2 = 0$ on $\mathbb{R}^2$, with $f_1, f_2 \in C^\infty_c$, then for all $\Phi \in C^\infty_c$,

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Decomposition Lemma

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Subelliptic \( L^1\text{-duality inequality} \Rightarrow \text{Sobolev inequality for } \overline{\partial}_b \)

because \( d \circ d = 0 \) and \( \overline{\partial}_b \circ \overline{\partial}_b = 0 \).

- We have also seen the subelliptic \( L^1\text{-duality inequality} \) in a model example (\( X = \frac{\partial}{\partial x}, \ Y = x\frac{\partial}{\partial t} \) on \( \mathbb{R}^2 \)).

- We now turn to the proof of the inequality in this model case.

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The key is a decomposition lemma:
Lemma (Euclidean Decomposition Lemma)

Given any function $\Phi \in \mathcal{C}_c^\infty(\mathbb{R}^{N-1})$ and any $\lambda > 0$, there exists a decomposition $\Phi = \Phi_1 + \Phi_2$ such that

$$\|\Phi_1\|_{L^\infty} \leq C\lambda^{\frac{1}{N}}\|\nabla \Phi\|_{L^N}$$

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- The original $L^1$-duality inequality then follows by ‘freezing variables’.
Recall that the $L^1$-duality inequality says that if $f_j \in C^\infty_c$ on $\mathbb{R}^N$ and $\sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j} = 0$ then for any $\Phi \in C^\infty_c$,

$$\left| \int_{\mathbb{R}^N} f_1 \Phi \, dx \right| \leq C \| f \|_{L^1} \| \nabla \Phi \|_{L^N}.$$ 

Now

$$\int_{\mathbb{R}^N} f_1 \Phi \, dx = \int_{-\infty}^{\infty} \int_{\mathbb{R}^{N-1}} f_1 \Phi \, dx' \, dx_1.$$

Freeze $x_1 = a$, restrict $\Phi$ to the hyperplane $\{x_1 = a\}$ and for any $\lambda > 0$ decompose $\Phi|_{\{x_1 = a\}} = \Phi_1^a + \Phi_2^a$.

$$\left| \int_{\{x_1 = a\}} f_1 \Phi_1^a \right| \leq \| f_1 \|_{L^1(\{x_1 = a\})} \| \Phi_1^a \|_{L^\infty(\{x_1 = a\})}$$

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\[
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\]

\[
= \int_{-\infty}^{a} \int_{\mathbb{R}^{N-1}} \frac{\partial f_1}{\partial x_1}(x_1, x') \Phi_2^a(a, x') \, dx' \, dx_1
\]

\[
= \int_{-\infty}^{a} \int_{\mathbb{R}^{N-1}} - \sum_{j=2}^{N} \frac{\partial f_j}{\partial x_j}(x_1, x') \Phi_2^a(a, x') \, dx' \, dx_1
\]

\[
= \sum_{j=2}^{N} \int_{-\infty}^{a} \int_{\mathbb{R}^{N-1}} f_j(x_1, x') \frac{\partial \Phi_2^a}{\partial x_j}(a, x') \, dx' \, dx_1
\]

\[
\leq \|f\|_{L^1(\mathbb{R}^N)} \|\nabla \Phi_2^a\|_{L^\infty(\{x_1=a\})}.
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and \(\|\nabla \Phi_2^a\|_{L^\infty(\{x_1=a\})}\) can be estimated by the lemma.

Optimize \(\lambda\), integrate in \(a\) and get the desired estimate.
\[
\int_{\mathbb{R}^{N-1}} f_1(a, x') \Phi_2^a(a, x') \, dx' = \int_{-\infty}^{a} \int_{\mathbb{R}^{N-1}} \frac{\partial f_1}{\partial x_1} (x_1, x') \Phi_2^a(a, x') \, dx' \, dx_1
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\[\Rightarrow\] Optimize \(\lambda\), integrate in \(a\) and get the desired estimate.
\[
\begin{align*}
\int_{\mathbb{R}^{N-1}} & f_1(a, x') \Phi_2^a(a, x') dx' \\
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= & \int_{-\infty}^{a} \int_{\mathbb{R}^{N-1}} - \sum_{j=2}^{N} \frac{\partial f_j}{\partial x_j}(x_1, x') \Phi_2^a(a, x') dx' dx_1 \\
= & \sum_{j=2}^{N} \int_{-\infty}^{a} \int_{\mathbb{R}^{N-1}} f_j(x_1, x') \frac{\partial \Phi_2^a}{\partial x_j}(a, x') dx' dx_1 \\
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The elliptic complex

The subelliptic complex

Decomposition Lemma

Euclidean case

Subelliptic case via model example

Next

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\int_{\mathbb{R}^{N-1}} f_1(a, x') \Phi_2^a(a, x') dx' \\
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To prove Euclidean Decomposition Lemma, it suffices to observe that

- the decomposition is dilation invariant
  → reduces to the case $\lambda = 1$
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To prove the subelliptic $L^1$-duality inequality in the model case, we need instead

**Lemma (Subelliptic Decomposition Lemma in model example)**

Given $\Phi \in C^\infty_c(\mathbb{R}^2)$, for each $a \in \mathbb{R}$ and $\lambda > 0$, there is a decomposition $\Phi = \Phi_1^a + \Phi_2^a$ on the hyperplane $\{x = a\}$ and an extension of $\Phi_2^a$ into the whole $\mathbb{R}^2$ such that

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\| \Phi_1^a \|_{L^\infty(\{x=a\})} \leq C \lambda^{\frac{1}{3}} M I(a)
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Po-Lam Yung

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and $M$ is the standard Hardy-Littlewood maximal function on $\mathbb{R}$.
Key idea in its proof: *lifting*
(also important for the general case)

On $\mathbb{R}^3$ use coordinates $(x, y, t)$. Consider the map

$$\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad (x, y, t) \mapsto (x, t)$$

The vector fields $X = \frac{\partial}{\partial x}$, $Y = x \frac{\partial}{\partial t}$ on $\mathbb{R}^2$ can be lifted to vector fields

$$\tilde{X} := \frac{\partial}{\partial x}, \quad \tilde{Y} := \frac{\partial}{\partial y} + x \frac{\partial}{\partial t}$$
on $\mathbb{R}^3$

such that $d\pi(\tilde{X}) = X$, $d\pi(\tilde{Y}) = Y$.

Any function $\Phi$ on $\mathbb{R}^2$ can be pulled back to another function $\tilde{\Phi}$ on $\mathbb{R}^3$ by letting

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Clearly $\tilde{X}\tilde{\Phi} = X\tilde{\Phi}$ and $\tilde{Y}\tilde{\Phi} = Y\tilde{\Phi}$

Why is this good? Because $\mathbb{R}^3$ can be endowed with the structure of a *Lie group* such that $\tilde{X}, \tilde{Y}$ are left-invariant vector fields: in fact we can define

$$(x, y, t) \cdot (u, v, w) := (x + u, y + v, t + w + xv)$$

(Heisenberg group)

One advantage of having a group structure is that we can then define convolutions:

$$(F \ast G)(x, y, t) := \int_{\mathbb{R}^3} F((x, y, t) \cdot (u, v, w)) G(u, v, w) du dv dw$$

Since $\tilde{X}, \tilde{Y}$ are left-invariant, they are very compatible with convolutions: e.g.

$$(\tilde{X}F) \ast G = -F \ast (\tilde{X}G), \quad (\tilde{Y}F) \ast G = -F \ast (\tilde{Y}G)$$

(Cannot do these on the underlying $\mathbb{R}^2$!)

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(Cannot do these on the underlying \( \mathbb{R}^2 \)!)}
Another observation is that we actually obtained a homogeneous group, i.e. a group that carries an automorphic dilation

\[ \lambda \cdot (x, y, t) := (\lambda x, \lambda y, \lambda^2 t) \]

Define a dilation \( I_\lambda \) on functions that preserves \( L^1 \) norm:

\[(I_\lambda \eta)(x, y, t) := \lambda^{-4} \eta(\lambda^{-1} x, \lambda^{-1} y, \lambda^{-2} t)\]

Recall now the decomposition lemma:

Given \( \Phi \in C^\infty_c(\mathbb{R}^2) \), for each \( a \in \mathbb{R} \) and \( \lambda > 0 \), there is a decomposition \( \Phi \mid_{\{x=a\}} = \Phi_1^a + \Phi_2^a \) on the hyperplane \( \{x = a\} \) and an extension of \( \Phi_2^a \) into the whole \( \mathbb{R}^2 \) such that

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$$\|\nabla_b \Phi_2^a\|_{L^\infty(\mathbb{R}^2)} \leq C \lambda^{-\frac{2}{3}} MI(a)$$
Another observation is that we actually obtained a \textit{homogeneous group}, i.e. a group that carries an automorphic dilation

$$\lambda \cdot (x, y, t) := (\lambda x, \lambda y, \lambda^2 t)$$

Define a dilation $I_\lambda$ on functions that preserves $L^1$ norm:

$$(I_\lambda \eta)(x, y, t) := \lambda^{-4} \eta(\lambda^{-1} x, \lambda^{-1} y, \lambda^{-2} t)$$

Recall now the decomposition lemma:
Given $\Phi \in C^\infty_c(\mathbb{R}^2)$, for each $a \in \mathbb{R}$ and $\lambda > 0$, there is a decomposition $\Phi|_{\{x=a\}} = \Phi^a_1 + \Phi^a_2$ on the hyperplane $\{x = a\}$ and an extension of $\Phi^a_2$ into the whole $\mathbb{R}^2$ such that

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\[ \| \Phi_1^a \|_{L^\infty(\{x=a\})} \leq C \lambda^{\frac{1}{3}} M_\mathcal{I}(a) \]

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To prove lemma, fix $\lambda > 0$, $a \in \mathbb{R}$.

Let $\eta \in C_c^\infty$ be a bump function on the group $\mathbb{R}^3$, $\int \eta = 1$. The desired decomposition of $\Phi|_{\{x=a\}}$ is given by

$$\Phi^a_2(a, t) := \tilde{\Phi} * I_\lambda \eta(a, y, t) \quad \text{for all } t$$

(the right hand side actually does not depend on $y$) and

$$\Phi^a_1(a, t) := \Phi(a, t) - \Phi^a_2(a, t)$$

The desired extension of $\Phi^a_2$ is given by

$$\Phi^a_2(a + s, t) := \tilde{\Phi} * I_{\sqrt{\lambda^2 + s^2}} \eta(a, y, t) \quad \text{for all } s, t$$

Difficulty: Need to integrate away the variable we added during the lifting process.
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To illustrate the proof of the desired estimates, consider $X\Phi^a_2$.

Recall $\Phi^a_2(a + s, t) := \tilde{\Phi} * l_{\sqrt{\lambda^2 + s^2}}^\eta(a, y, t)$

$$(X\Phi^a_2)(a + s, t) = \frac{d}{ds}\Phi^a_2(a + s, t) = \tilde{\Phi} * \frac{d}{ds}l_{\sqrt{\lambda^2 + s^2}}^\eta(a, y, t)$$

\[
\frac{d}{ds}l_{\sqrt{\lambda^2 + s^2}}^\eta = \frac{d}{d\tau}l^\eta_{\tau} \bigg|_{\tau=\sqrt{\lambda^2 + s^2}} \cdot \frac{s}{\sqrt{\lambda^2 + s^2}}
\]

\[
\frac{d}{d\tau}l^\eta_{\tau} = \tilde{X}(l^\eta_{\tau_1}) + \tilde{Y}(l^\eta_{\tau_2}) \quad \text{for some } \eta_1, \eta_2 \in C_c^\infty
\]

\[
|(X\Phi^a_2)(a + s, t)| \leq |\tilde{\Phi} * (\tilde{X}l^\eta_{\tau_1} + \tilde{Y}l^\eta_{\tau_2})|(a, y, t)
\]

\[
\leq |\tilde{X}\tilde{\Phi} * l^\eta_{\tau_1}| + |\tilde{Y}\tilde{\Phi} * l^\eta_{\tau_2}|(a, y, t), \quad \tau = \sqrt{\lambda^2 + s^2}.
\]
To illustrate the proof of the desired estimates, consider $X \Phi_2^a$.

Recall $\Phi_2^a(a + s, t) := \tilde{\Phi} \ast I \sqrt{\lambda^2 + s^2} \eta(a, y, t)$

\[
(X \Phi_2^a)(a + s, t) = \frac{d}{ds} \Phi_2^a(a + s, t) = \tilde{\Phi} \ast \frac{d}{ds} I \sqrt{\lambda^2 + s^2} \eta(a, y, t) = \tilde{\Phi} \ast I \sqrt{\lambda^2 + s^2} \eta(a, y, t)
\]

\[
\frac{d}{ds} I \sqrt{\lambda^2 + s^2} \eta = \left. \frac{d}{d\tau} I_\tau \eta \right|_{\tau = \sqrt{\lambda^2 + s^2}} \cdot \frac{s}{\sqrt{\lambda^2 + s^2}}
\]

\[
\frac{d}{d\tau} I_\tau \eta = \tilde{X}(I_\tau \eta_1) + \tilde{Y}(I_\tau \eta_2) \quad \text{for some } \eta_1, \eta_2 \in C^\infty_c
\]

\[
| (X \Phi_2^a)(a + s, t) | \\
\leq | \tilde{\Phi} \ast (\tilde{X} I_\tau \eta_1 + \tilde{Y} I_\tau \eta_2) | (a, y, t) \\
\leq | \tilde{X} \tilde{\Phi} \ast I_\tau \eta_1 | + | \tilde{Y} \tilde{\Phi} \ast I_\tau \eta_2 | (a, y, t), \quad \tau = \sqrt{\lambda^2 + s^2}.
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$$\frac{d}{ds} l_{\sqrt{\lambda^2 + s^2}} = \frac{d}{d\tau} I_{\tau} \eta \bigg|_{\tau = \sqrt{\lambda^2 + s^2}} \cdot \frac{s}{\sqrt{\lambda^2 + s^2}}$$

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\[ |\tilde{X}\tilde{\Phi} \ast l_{\tau} \eta_1| (a, y, t) \]
\[ |\hat{X}\Phi \ast I_T \eta_1|(a, y, t) \]
\[ |\tilde{X}\Phi \ast l_\tau \eta_1|(a, y, t) \]

\[ = \int_{\mathbb{R}^3} |X\Phi|(a + u, t + w + av) \left| \eta_1\left(\frac{u}{\tau}, \frac{v}{\tau}, \frac{w}{\tau^2}\right) \right| \frac{1}{\tau^4} dudvdw \]
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Holder in \( w \):

\[ \leq \int_{\mathbb{R}^{2}} \left\| X\Phi(a + u, w) \right\|_{L^{3}(dw)} \left\| \eta_{1}(\frac{u}{T}, \frac{v}{T}, w) \right\|_{L^{3/2}(dw)} T^{-4 + \frac{4}{3}} dudv \]
\[ |\tilde{X}\Phi * l_T \eta_1|(a, y, t) = \int_{\mathbb{R}^3} |X\Phi|(a + u, t + w + av) \left| \eta_1\left(\frac{u}{t}, \frac{v}{t}, \frac{w}{t^2}\right) \right| \frac{1}{t^4} dudvdw \]

Holder in \( w \):

\[ \leq \int_{\mathbb{R}^2} I(a + u) \|\eta_1\left(\frac{u}{t}, \frac{v}{t}, w\right)\|_{L^{3/2}(dw)^{T^{-4+\frac{4}{3}}} dudv} \]

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Sobolev inequalities for \((0, q)\) forms
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\[ \leq \int_{\mathbb{R}^2} I(a + u) \left\| \eta_1\left(\frac{u}{T}, \frac{v}{T}, \frac{w}{T^2}\right) \right\|_{L^{3/2}(dw)T^{-4+\frac{4}{3}}} \, dudv \]

Integrate in \( v \): (Important!)
\[ \leq \int_{\mathbb{R}} I(a + u) \left\| \eta_1\left(\frac{u}{T}, v, \frac{w}{T^2}\right) \right\|_{L^{3/2}(dw)L^{1}(dv)T^{-4+\frac{4}{3}+1}} \, du \]
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Hölder in \( w \):

\[
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\]

Integrate in \( v \): (Important!)

\[
\leq \int_{\mathbb{R}} \mathcal{I}(a + u) \|\eta_1(\frac{u}{T}, v, w)\|_{L^{3/2}(dw)L^1(dv)} \tau^{-4 + \frac{4}{3} + 1} du
\]

Estimate by maximal function:

\[
\leq C \frac{1}{\tau} \int_{-C\tau}^{C\tau} \mathcal{I}(a + u) du \cdot \tau^{-4 + \frac{4}{3} + 1 + 1}
\]
\[
|\tilde{X}\Phi \ast l_\tau \eta_1|(a, y, t)
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Holder in \(w\):

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\leq \int_{\mathbb{R}^2} \mathcal{I}(a + u) \left\| \eta_1\left(\frac{u}{T}, \frac{v}{T}, w\right) \right\|_{L^{3/2}(dw)} T^{-4 + \frac{4}{3}} \, dudv
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Integrate in \(v\): (Important!)

\[
\leq \int_{\mathbb{R}} \mathcal{I}(a + u) \left\| \eta_1\left(\frac{u}{T}, v, w\right) \right\|_{L^{3/2}(dw)L^1(dv)} T^{-4 + \frac{4}{3} + 1} \, du
\]

Estimate by maximal function:

\[
\leq C \frac{1}{\tau} \int_{-C\tau}^{C\tau} \mathcal{I}(a + u) du \cdot T^{-4 + \frac{4}{3} + 1 + 1} \leq CMI(a)\lambda^{-\frac{2}{3}} \quad \text{because } \lambda \leq \tau.
\]
This basically completes the proof of the model case.

Some difficulties in the general case are:

- In general one cannot expect the lifted vector fields be left-invariant under a group law; rather one can only approximate the lifted vector fields by left-invariant vector fields of a homogeneous group. Need to take care of error terms that arise.

- In general it is not possible to put a coordinate system on $\mathbb{R}^N$ so that $X_2, \ldots, X_n$ are all tangent to level sets of $x_1$. When $X_1, \ldots, X_n$ are linearly independent, a perturbation argument would work, but it is not clear whether the condition of linear independence is necessary.
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Further directions of exploration:

- Sobolev inequality for $d$ on bounded smooth domains with boundaries
- Sobolev inequality for $\bar{\partial}$ on bounded pseudoconvex domains of finite type
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Thank you!