THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2023-2024 Term 1
Suggested Solutions of WeBWork Coursework 9

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math1010@math.cuhk.edu.hk

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1. (1 point) Evaluate the indefinite integral.

$$\int \sin^3(3x)\cos^{10}(3x)dx$$

________________________ $+C$

Solution. Recall that through the Pythagorean Identity $\sin^2(x) = 1-\cos^2(x)$. Then we know that $\sin^3(3x) = \sin(3x)\big(\sin^2(3x)\big) = \sin(3x)\big(1-\cos^2(3x)\big)$. Substituting this into the integral we see

$$\int \sin^3(3x)\cos^{10}(3x)dx = \int \sin(3x)\big(1-\cos^2(3x)\big)\cos^{10}(3x)dx.$$

Distributing just the cosines, this becomes

$$\int \sin(3x)\big(\cos^{10}(3x)-\cos^{12}(3x)\big)dx.$$

Now use the substitution $t=\cos(3x) \Rightarrow dt = d\cos(3x) = -3\sin(3x)dx$, i.e., $\sin(3x)dx = -\frac{1}{3}dt$, then the integral becomes

$$\int \big(\cos^{10}(3x)-\cos^{12}(3x)\big)\sin(3x)dx = \int (t^{10}-t^{12})(-\frac{1}{3})dt = \frac{1}{3}(\frac{t^{13}}{13}-\frac{t^{11}}{11})+C.$$

Reordering and back-substituting with $t=\cos(3x)$, we get

$$\int \sin^3(3x)\cos^{10}(3x)dx = \frac{1}{3}(\frac{\cos^{13}(3x)}{13}-\frac{\cos^{11}(3x)}{11})+C.$$

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2. (1 point)
Evaluate the integral

$$\int \sin^4(x)\,dx.$$

Note: Use an upper-case ”C” for the constant of integration.

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Solution. This integral is mostly about clever rewriting of your functions. As a rule of thumb, if the power is even, we use the double angle formula. The double angle formula says

$$\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta)).$$

If we split up our integral like this

$$\int \sin^2(x)\cdot\sin^2(x)dx$$

We can use the double angle formula twice:

$$\int \frac{1}{2}(1-\cos(2x))\cdot\frac{1}{2}(1-\cos(2x))dx$$

Both parts are the same, so we can just put it as a square:

$$\int \left(\frac{1}{2}(1-\cos(2x))\right)^2 dx$$

Expanding, we get:

$$\int \frac{1}{4}\left(1-2\cos(2x)+\cos^2(2x)\right)dx$$

We can then use the other double angle formula

$$\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$$

to rewrite the last term as follows:

$$\begin{aligned}
&\frac{1}{4}\int 1-2\cos(2x)+\frac{1}{2}(1+\cos(4x))dx\\
=&\frac{1}{4}\left(\int 1dx-\int 2\cos(2x)dx+\frac{1}{2}\int 1+\cos(4x)dx\right)\\
=&\frac{1}{4}\left(x-\int 2\cos(2x)dx+\frac{1}{2}\left(x+\int \cos(4x)dx\right)\right)
\end{aligned}$$

We will call the left integral in the parenthesis Integral 1, and the right on Integral 2.

Integral 1: $\int 2\cos(2x)dx$

Looking at the integral, we have the derivative of the inside, 2 outside of the function, and this should immediately ring a bell that you should use u-substitution. If we let $u = 2x$, the derivative becomes 2, so we divide through by 2 to integrate with respect to u

$$\int \frac{\cancel{2}\cos(u)}{\cancel{2}}du = \int \cos(u)du,$$

Integral 2: $\int \cos(4x)dx$

$$\int \cos(u)du = \sin(u) = \sin(2x).$$

It's not as obvious here, but we can also use u-substitution here. We can let $u = 4x$, and the derivative will be 4

$$\frac{1}{4}\int \cos(u)dx = \frac{1}{4}\sin(u) = \frac{1}{4}\sin(4x).$$

Completing the original integral Now that we know Integral 1 and Integral 2, we can plug them back into our original expression to get the final answer

$$\begin{aligned}
&\frac{1}{4}\left(x-\sin(2x)+\frac{1}{2}\left(x+\frac{1}{4}\sin(4x)\right)\right)+C\\
=&\frac{1}{4}\left(x-\sin(2x)+\frac{1}{2}x+\frac{1}{8}\sin(4x)\right)+C\\
=&\frac{1}{4}x-\frac{1}{4}\sin(2x)+\frac{1}{8}x+\frac{1}{32}\sin(4x)+C\\
=&\frac{3}{8}x-\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C.
\end{aligned}$$

3. (1 point)

Suppose that $f(1)=-9$, $f(4)=1$, $f'(1)=8$, $f'(4)=7$, and f'' is continuous. Find the value of $\int_1^4 xf''(x)\,dx$.

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Solution. Integrate by parts using the formula $\int udv = uv - \int vdu$, where $u = x$ and $dv = f''(x)$. Then we have

$$\begin{aligned}
&\int_1^4 xf''(x)dx \\
=&xf'(x)|_1^4 - \int_1^4 f'(x)dx \\
=&xf'(x)|_1^4 - f(x)|_1^4 \\
=&\big(4\cdot f'(4) - 1\cdot f'(1)\big) - \big(f(4) - f(1)\big) \\
=&(4\cdot 7 - 1\cdot 8) - (1-(-9)) \\
=&28 - 8 - 10 \\
=&10
\end{aligned}$$

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4. (1 point)

A rumor is spread in a school. For $0 < a < 1$ and $b > 0$, the time t at which a fraction p of the school population has heard the rumor is given by

$$t(p) = \int_a^p \frac{b}{x(1-x)}\,dx.$$

(a) Evaluate the integral to find an explicit formula for $t(p)$. Write your answer so it has only one ln term.

$\int_a^p \frac{b}{x(1-x)}\,dx =$ ____________________

(b) At time $t = 0$, three percent of the school population ($p = 0.03$) has heard the rumor. What is a?

$a =$ _____

(c) At time $t = 1$, fifty-six percent of the school population ($p = 0.56$) has heard the rumor. What is b?

$b =$ ____________

(d) At what time has ninety-one percent of the population ($p = 0.91$) heard the rumor?

$t =$ ____________

Solution:

(a) We integrate to find

$$\int \frac{b}{x(1-x)}\,dx = b\int\left(\frac{1}{x}+\frac{1}{1-x}\right)dx = b(\ln|x| - \ln|1-x|) + C = b\ln|\frac{x}{1-x}| + C,$$

so

$$t(p) = \int_a^p \frac{b}{x(1-x)}\,dx = b\ln\left(\frac{p}{1-p}\right) - b\ln\left(\frac{a}{1-a}\right) = b\ln\left(\frac{p(1-a)}{a(1-p)}\right).$$

(b) We know that $t(0.03) = 0$, so

$$0 = b\ln\left(\frac{0.03(1-a)}{0.97a}\right).$$

But $b > 0$ and $\ln x = 0$ means $x = 1$, so

$$\frac{0.03(1-a)}{0.97a} = 1, \qquad \text{or} \qquad 0.03(1-a) = 0.97a.$$

Solving, $a = 0.03$.

(c) We know that $t(0.56) = 1$ so

$$1 = b\ln\left(\frac{0.56\cdot 0.97}{0.44\cdot 0.03}\right) = b\ln\frac{0.5432}{0.0132},$$

so $b = \frac{1}{\ln \frac{0.5432}{0.0132}}$.

(d) We have

$$t(0.91) = \int_{0.03}^{0.91} \frac{b}{x(1-x)}\, dx = \frac{1}{\ln \frac{0.5432}{0.0132}} \ln\left(\frac{0.91(1-0.03)}{0.03(1-0.91)}\right) \approx 1.5575.$$

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5. (1 point) Given

$$f(x) = \int_0^x \frac{t^2-16}{1+\cos^2(t)} dt$$

At what value of x does the local max of $f(x)$ occur?

$x =$ __________

Solution:

First, note that we don't need to do any computation to compute the first derivative, which we will use to check for local maxima and minima. By applying the Fundamental Theorem of Calculus, we see that:

$$f'(x) = \frac{x^2-16}{1+\cos^2(x)}$$

Now, we can use this derivative to find the critical points of the function. We set this to zero and solve for x to get:

$$\frac{x^2-16}{1+\cos^2(x)} = 0$$

$$x^2 - 16 = 0$$

$$(x+4)(x-4) = 0$$

$$x = 4 \text{ or } x = -4$$

Checking on either side of these two points shows that -4 is the local maximum for which we are looking.

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6. (1 point) Let $F(x) = \displaystyle\int_0^x \frac{1-t}{t^2+15}\, dt$ for $-\infty < x < +\infty$

(a) Find the value of x where F obtains its maximum value.

$x =$ _____

(b) Find the intervals over which F is only increasing or decreasing. Use interval notation using U for union and enter "none" if no interval.

Intervals where F is increasing: __________

Intervals where F is decreasing: __________

(c) Find open intervals over which F is only concave up or concave down. Use interval notation using U for union and enter "none" if no interval.

Intervals where F is concave up: __________

Intervals where F is concave down: __________

Solution:

(a) $F'(x) = \frac{d}{dx}\int_0^x \frac{1-t}{t^2+15}\, dt = \frac{1-x}{x^2+15} = 0$ when $x = 1$ which is the only critical point. From sign analysis of F' we see this is a maximum.

(b) F is increasing on $(-\infty, 1]$ and decreasing on $[1, +\infty)$.

(c) $F''(x) = \frac{d}{dx}\left[\frac{1-x}{x^2+15}\right] = \frac{x^2-2x-15}{\left(x^2+15\right)^2} = \frac{(x-5)(x+3)}{\left(x^2+15\right)^2} = 0$, when $x = -3, 5$. Sign analysis of F'' shows that F is concave up on $(-\infty, -3)$ and $(5, +\infty)$ and concave down on $(-3, 5)$.

7. (1 point)
Evaluate the integral

$$\int_{-1}^{2} (5x - 3|x|)\, dx$$

Integral = ____________

Solution. We split the interval $[-1,2]$ to $[-1,0]$ and $[0,2]$. Then we get

$$\begin{aligned}
\int_{-1}^{2}(5x-3|x|)dx =& \int_{-1}^{0}(5x-3|x|)dx + \int_{0}^{2}(5x-3|x|)dx \\
=& \int_{-1}^{0}(5x+3x)dx + \int_{0}^{2}(5x-3x)dx \\
=& \int_{-1}^{0} 8xdx + \int_{0}^{2} 2xdx \\
=& 4x^2|_{-1}^{0} + x^2|_0^2 \\
=& 0 - 4\cdot(-1)^2 + 2^2 - 0 \\
=& 0
\end{aligned}$$

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8. (1 point)
Evaluate the definite integral (if it exists)

$$\int_0^{\pi} -5\sec^2(t/4)\, dt$$

If the integral does not exist, type "DNE".

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Solution.

$$\begin{aligned}
\int_0^{\pi} -5\sec^2(t/4)\, dt =& (-5)\int_0^{\pi} \sec^2(t/4)\, dt \\
=& (-5)(4\tan(\frac{t}{4}))|_0^{\pi} \\
=& (-20)(\tan(\frac{t}{4}))|_0^{\pi} \\
=& (-20)(\tan(\frac{\pi}{4}) - \tan(0)) \\
=& (-20)\cdot(1-0) \\
=& -20
\end{aligned}$$

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9. (1 point) Let

$$I = \int_0^{\pi/4} \tan^{10}(x)\sec(x)\; dx.$$

Express the value of

$$\int_0^{\pi/4} \tan^{12}(x)\sec(x)\; dx$$

in terms of I.

$\int_0^{\pi/4} \tan^{12}(x)\sec(x)\; dx =$ __________.

Solution: Let

$$J = \int_0^{\pi/4} \tan^{12}(x)\sec(x)\; dx.$$

Factorizing the integrand as displayed below, and then integrating by parts, gives

$$\begin{aligned} J &= \int_0^{\pi/4} \tan^{11}(1x)\sec(1x)\tan(1x)\,dx \\ &= \tfrac{1}{1}\tan^{11}(1x)\sec(1x)\Big|_0^{\pi/4} - 11\int_0^{\pi/4}\tan^{10}(1x)\sec^3(1x)\,dx. \end{aligned}$$

Now

$$\begin{aligned} \tfrac{1}{1}\tan^{11}(1x)\sec(1x)\Big|_0^{\pi/4} &= \tfrac{1}{1}\tan^{11}\left(\tfrac{1}{4}\pi\right)\sec\left(\tfrac{1}{4}\pi\right) = \tfrac{1}{1}\left(1\right)^{11}\left(\sqrt{2}\right) \\ &= 1\sqrt{2}, \end{aligned}$$

and from $\sec^3(1x) = \sec^2(1x)\sec(1x) = \left(\tan^2(1x)+1\right)\sec(1x)$, it follows that

$$\begin{aligned} \int_0^{\pi/4}\tan^{10}(1x)\sec^3(1x)\,dx &= \int_0^{\pi/4}\tan^{10}(1x)\left(\tan^2(1x)+1\right)\sec(1x)\,dx \\ &= J+I. \end{aligned}$$

Therefore,

$$J = 1\sqrt{2} - 11J - 11I, \qquad \text{or} \qquad 12J = 1\sqrt{2} - 11I,$$

and so

$$\int_0^{\pi/4}\tan^{12}(x)\sec(x)\;dx = J = \tfrac{1}{12}\sqrt{2} - \tfrac{11}{12}I.$$

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10. (1 point)
Evaluate $\int_{-\pi}^{\pi} f(x)\,dx$, where

$$f(x) = \begin{cases} 4x^2, & -\pi \le x < 0 \\ 5\sin(x), & 0 \le x \le \pi. \end{cases}$$

$\int_{-\pi}^{\pi} f(x)\,dx =$ __________

Solution:

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\,dx &= \textstyle\int_{-\pi}^{0} 4x^2\,dx + \int_0^{\pi} 5\sin(x)\,dx \\ &= \left[\tfrac{4}{3}x^3\right]_{-\pi}^{0} + \left[-5\cos x\right]_0^{\pi} \\ &= \left(\tfrac{4}{3}\cdot 0^3 - \tfrac{4}{3}(-\pi)^3\right) + ((-5\cos\pi + 5\cos 0)) \\ &= \tfrac{4}{3}\pi^3 + 10 \end{aligned}$$

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11. (1 point) Evaluate the integral.
$\int e^x\sqrt{25-e^{2x}}\,dx =$ ______________________ $+C$
Solution:
For $e^x = 5\sin(t)$, $dx = \frac{\cos(t)}{\sin(t)}\;dt$

$$\int e^x\sqrt{25-e^{2x}}\,dx = \int 25\cos^2(t)\;dt = 25\left[\frac{t}{2}+\frac{1}{2}\sin t\cos t\right] + C = \frac{25\sin^{-1}\left(\frac{e^x}{5}\right)}{2} + \frac{e^x\sqrt{25-e^{2x}}}{2} + C$$

12. (1 point) If $f(x) = \int_{x}^{x^2} t^2 dt$

then

$f'(x) =$ ______________

Solution. Using the Fundamental Theorem of Calculus and the chain rule, then for some constant a

$$f(x) = \int_{x}^{x^2} t^2 dt = \int_{a}^{x^2} t^2 dt - \int_{a}^{x} t^2 dt$$

$$\begin{aligned} \frac{df(x)}{dx} &= (x^2)^2 \cdot \frac{d}{dx}(x^2) - x^2 \\ &= (x^2)^2 \cdot (2x) - x^2 \\ &= 2x^5 - x^2 \end{aligned}$$