THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2023-2024 Term 1
Suggested Solutions of WeBWork Coursework 8

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1. (1 point)
Using an upper-case ”C” for any arbitrary constants, find the general indefinite integral:

$$\int\left(-8x^2+10+\frac{1}{x^2+1}\right)dx$$

Integral = ____________________

Solution:

$$\int(-8x^2+10+\frac{1}{x^2+1})dx=\frac{-8}{3}x^3+10x+\arctan(x)+C$$

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2. (1 point)
Evaluate the integral

$$\int\frac{10t^5}{\sqrt{t^2+2}}\,dt$$

Note: Use an upper-case ”C” for the constant of integration.

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Solution:
Sub $u=t^2+2$, then $du=2t\,dt$

$$\begin{aligned}&\int\frac{10t^5}{\sqrt{t^2+2}}\,dt\\ &=\int\frac{5t^4}{\sqrt{u}}\,du\\ &=\int\frac{5(u-2)^2}{\sqrt{u}}\,du\\ &=\int 5u^{3/2}-20u^{1/2}+20u^{-1/2}\,du\\ &=5\cdot\frac{2}{5}u^{5/2}-20\cdot\frac{2}{3}u^{3/2}+20\cdot 2u^{1/2}+C\\ &=2(t^2+2)^{5/2}-\frac{40}{3}(t^2+2)^{3/2}+40(t^2+2)^{1/2}+C\end{aligned}$$

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3. (1 point)
Given that $f''(x)=\cos(x)$, $f'(\pi/2)=2$ and $f(\pi/2)=8$ find:

$f'(x)=$ __________
$f(x)=$ __________

Solution:
We begin by finding $f'(x)$.
The general solution is:
$f'(x)=\int\cos(x)dx=\sin(x)+C$

We choose C so that the condition $f'(\pi/2) = 2$ is satisfied. Since $f'(\pi/2) = \sin(\pi/2) + C = 1 + C = 2$, we have $C = 1$ and thus:

$$f'(x) = \sin(x) + 1.$$

Now, to find $f(x)$, we solve the integral $\int \sin(x) + 1dx$ to find the following general solution:

$$f(x) = \int \sin(x) + 1dx = -\cos(x) + 1x + D$$

We choose D so that the initial solution $f(\pi/2) = 8$ is satisfied. Since

$$f(\pi/2) = -\cos(\pi/2) + 1\frac{\pi}{2} + D = 0 + 1\frac{\pi}{2} + D = 8,$$

we have $D = 8 - 1\frac{\pi}{2} = 8 - 1 \cdot \pi/2$ and the particular solution is:

$$f(x) = -\cos(x) + 1x + 8 - 1 \cdot \pi/2$$

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4. (1 point) Consider the function $f(x)$ whose second derivative is $f''(x) = 10x + 8\sin(x)$. If $f(0) = 2$ and $f'(0) = 3$, what is $f(x)$?

Answer: ____________________

Solution:

To solve this formula we will use the standard rule for the antiderivatives of a polynomial and the fact that the antiderivative of sin is $-\cos$ while the antiderivative of cos is sin. Furthermore, we will use the information given to us about $f'(0)$ and $f(0)$ to eliminate any arbitrary constants.

Finding the general antiderivative of $f''(x)$ gives a formula for $f'(x)$ as shown below:

$$f'(x) = \frac{10}{2}x^2 - 8\cos(x) + C_1$$

We are told that $f'(0) = 3$ so that we can solve for C_1 in the expression above as shown.

$$f'(0) = \frac{10}{2}(0)^2 - 8\cos(x) + C_1$$

$$3 = -8 + C_1$$
$$C_1 = 11$$

Therefore, the formula for $f'(x)$ is:

$$f'(x) = 5x^2 - 8\cos(x) + 11$$

Now, we take the antiderivative of $f'(x)$ to find the general formula for the function $f(x)$.

$$f(x) = \frac{5}{3}x^3 - 8\sin(x) + 11x + C_2$$

We are told that $f(0) = 2$ so that we can solve for C_2 as we did for C_1 above.

$$f(0) = \frac{5}{3}(0)^3 - 8\sin(0) + 11(0) + C_2$$
$$2 = C_2$$

Therefore, the specific value for the function $f(x)$ is given by:

$$f(x) = \frac{5}{3}x^3 - 8\sin x + 11x + 2$$

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5. (1 point) The form of the partial fraction decomposition of a rational function is given below.

$$\frac{20x - 10x^2 - 45}{(x-4)(x^2+9)} = \frac{A}{x-4} + \frac{Bx+C}{x^2+9}$$

$A =$ _____ $B =$ _____ $C =$ _____

Now evaluate the indefinite integral.

$\int \frac{20x - 10x^2 - 45}{(x-4)(x^2+9)}dx =$ ________________________ $+D$, where D is an arbitrary constant.

Solution:

Multiplying by the least common denominator gives

$$20x - 10x^2 - 45 = A(x^2+9) + (Bx+C)(x-4)$$

Rearranging terms on the right hand side, yields

$$20x - 10x^2 - 45 = (A+B)x^2 + (-4B+C)x + 9A - 4C$$

Now we equate the coefficients:

$$\begin{aligned} A + B &= -10 \\ -4B + C &= 20 \\ 9A - 4C &= -45 \end{aligned}$$

Solving the system gives $A = -5$, $B = -5$ and $C = 0$ so the partial fraction decomposition is

$$\frac{20x - 10x^2 - 45}{(x-4)(x^2+9)} = \frac{-5}{x-4} + \frac{-5x}{x^2+9}$$

The definite integral is then

$$\int \frac{20x - 10x^2 - 45}{(x-4)(x^2+9)}dx = -5\ln(|x-4|) + (-2.5)\ln\left(x^2+9\right) + C$$

6. (1 point) Evaluate the integral using an appropriate substitution.

$\int \frac{\sec^2(3\sqrt{x})}{\sqrt{x}}\,dx =$ ____________________ $+C$

Solution:

For $u = 3\sqrt{x}$ we have $\frac{2}{3}\,du = \frac{dx}{\sqrt{x}}$ and hence;

$$\int \frac{\sec^2(3\sqrt{x})}{\sqrt{x}}\,dx = \tfrac{2}{3}\int \sec^2(u)\;du = \tfrac{2}{3}\;\tan(u) + C = \tfrac{2}{3}\tan\big(3\sqrt{x}\big) + C$$

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7. (1 point) Match each indefinite integral on the left with the corresponding functions on the right.

Remember: to verify $\int f(x)\,dx = F(x) + C$, all you have to do is verify that $\frac{d}{dx}[F(x)] = f(x)$.

Because this is a matching problem, WeBWorK will not tell you which answers are right and which are wrong. You must get all of the answers correct to get credit for the problem. There is no partial credit.

____1. $\int \frac{1}{\sqrt{8x-x^2}}\,dx$

____2. $\int x\sqrt{x^2-16}\,dx$

____3. $\int \frac{1}{x^2+16}\,dx$

____4. $\int x\sqrt{x^2+16}\,dx$

A. $\arcsin\left(\frac{x-4}{4}\right) + C$

B. $\frac{1}{3}(x^2-16)^{3/2} + C$

C. $\frac{1}{3}(x^2+16)^{3/2} + C$

D. $\frac{1}{4}\arctan\left(\frac{x}{4}\right) + C$

Solution:

1.

$$\int \frac{1}{\sqrt{8x-x^2}}\,dx = \int \frac{1}{\sqrt{16-(x-4)^2}}\,dx = \arcsin\left(\frac{x-4}{4}\right) + C$$

2.

$$\int x\sqrt{x^2-16}\,dx = \int \frac{1}{2}\sqrt{x^2-16}\,dx^2 = \frac{1}{3}(x^2-16)^{3/2} + C$$

3.

$$\begin{aligned}\int \frac{1}{x^2+16}\,dx &= \int \frac{1}{(4y)^2+16}\,d(4y)\\ &= \frac{1}{4}\arctan(y) + C\\ &= \frac{1}{4}\arctan\left(\frac{x}{4}\right) + C\end{aligned}$$

4.

$$\int x\sqrt{x^2+16}dx = \int \frac{1}{2}\sqrt{x^2+16}dx^2 = \frac{1}{3}(x^2+16)^{3/2}+C$$

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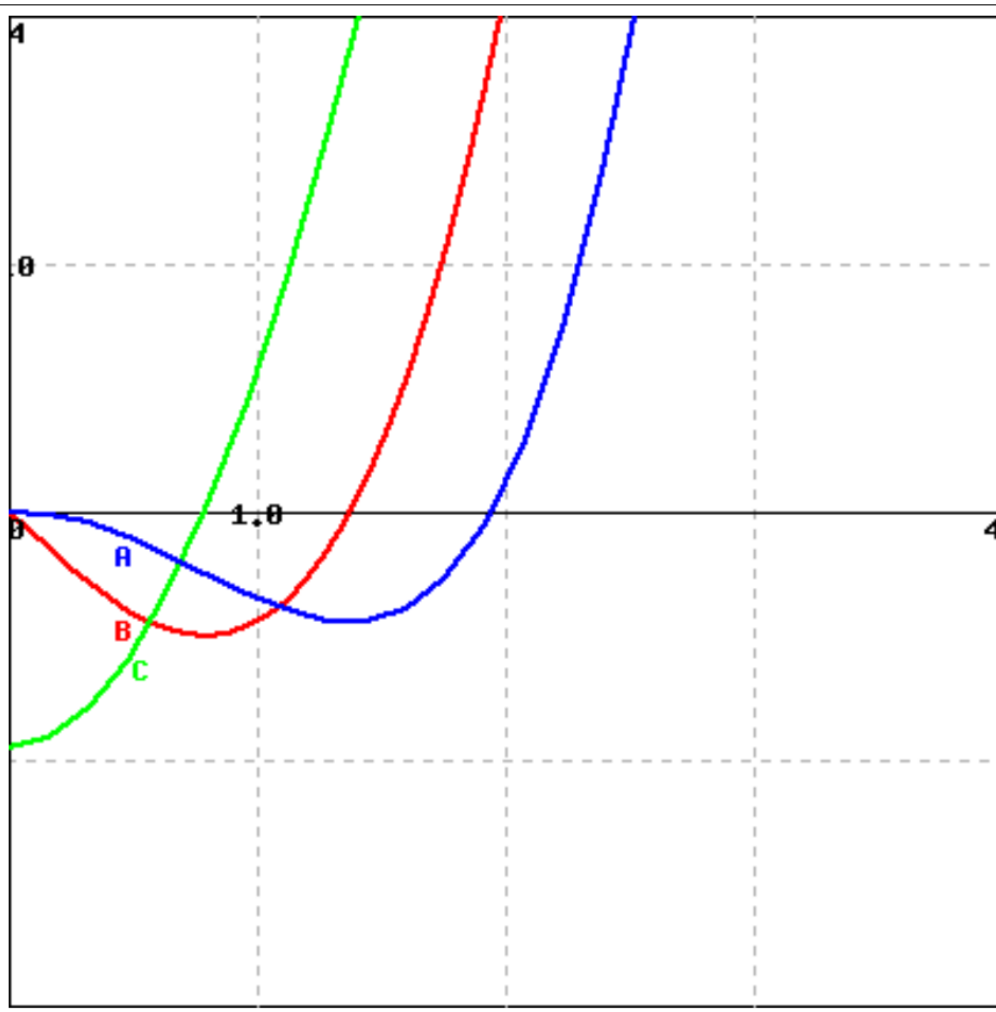
8. (1 point)
Evaluate the integral

$$\int \frac{7}{\sqrt{x+1}+\sqrt{x}}dx$$

Note: Use an upper-case "C" for the constant of integration.

Solution:

$$\int \frac{7}{\sqrt{x+1}+\sqrt{x}}dx = \int \frac{7(\sqrt{x+1}-\sqrt{x})}{(\sqrt{x+1}+\sqrt{x})(\sqrt{x+1}-\sqrt{x})}dx$$
$$= \int 7(\sqrt{x+1}-\sqrt{x})dx$$
$$= \frac{14}{3}(x+1)^{\frac{3}{2}} - \frac{14}{3}x^{\frac{3}{2}} + C$$



9. (1 point)
List corresponding features of the graphs of a function f, its first derivative f', and (an) antiderivative F.

Describe a strategy whereby, given a plot showing the graphs of f, f', and F, you can determine which is which. Apply your strategy to identify the graphs A (blue), B (red) and C (green) as the graphs of a function f, its first derivative f', and (an) antiderivative F:

___ is the graph of the function f.

___ is the graph of the function's first derivative f'.

___ is the graph of an antiderivative of the function F.

Solution:

Call the functions in blue, red, and green f_B, f_R, f_G respectively. Note that at $x = 0$ f_G is negative while f_R decreases fast and f_B starts to decrease, so we may guess f_G is f', the highest order derivative governing the increase rate of the other two. Consequently, f_R will be f and f_B be F, so that $f_R = (f_B)'$. Next choose another set of points to verify our guess, say, the zero of f_G (call it a): at $x = a$, $f_G = 0$ while f_R starts to increase and f_B starts to decrease slower(inflection point), which verifies our guess.

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10. (1 point) Find the following indefinite integrals.

$$\int \frac{x}{\sqrt{x+7}}\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ + C$$

$$\int \frac{\cos(t)}{(7\sin(t)+4)^2}\,dt = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ + C$$

Solution:

$$\begin{aligned}
\int \frac{x}{\sqrt{x+7}}dx &= \int \frac{x+7-7}{\sqrt{x+7}}dx \\
&= \int \frac{x+7}{\sqrt{x+7}}dx - \int \frac{7}{\sqrt{x+7}}dx \\
&= \int (x+7)^{\frac{1}{2}}d(x+7) - 2\times 7\times\sqrt{x+7}+C \\
&= \frac{2}{3}(x+7)^{\frac{3}{2}} - 2\times 7\times\sqrt{x+7}+C \\
&= \frac{2}{3}(x+7)\sqrt{x+7} - 14\sqrt{x+7}+C
\end{aligned}$$

$$\begin{aligned}
\int \frac{\cos(t)}{(7\sin(t)+4)^2}dt &= \int \frac{1}{(7\sin(t)+4)^2}d\sin(t) \\
&= -\frac{1}{7(7\sin(t)+4)}+C
\end{aligned}$$

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11. (1 point)

Evaluate the integral

$$\int \frac{8}{x^2\sqrt{4x+1}}\,dx$$

Note: Use an upper-case "C" for the constant of integration.

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Solution: Sub $u = \sqrt{4x+1}$. Then $x^2 = (\frac{u^2-1}{4})^2$, $dx = \frac{1}{2}udu$.

$$\begin{aligned} &\int \frac{8}{x^2\sqrt{4x+1}}\,dx \\ &= 8\int \frac{1}{(\frac{u^2-1}{4})^2 \cdot u} \cdot \frac{1}{2}u\,du \\ &= 8\int \frac{8}{(u^2-1)^2}\,du = 8\int \frac{8}{(u+1)^2(u-1)^2}\,du \end{aligned}$$

Suppose $\frac{8}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2}$, where A,B,C,D are constants.
Then $A = 2$, $B = 2$, $C = -2$, $D = 2$.

$$\begin{aligned} &8\int \frac{8}{(u+1)^2(u-1)^2}\,du \\ &= 8\int (\frac{2}{u+1} + \frac{2}{(u+1)^2} - \frac{2}{u-1} + \frac{2}{(u-1)^2})\,du \\ &= 8\cdot[2\ln|u+1| - \frac{2}{u+1} - 2\ln|u-1| - \frac{2}{u-1}] + C \\ &= 16\cdot[\ln|\sqrt{4x+1}+1| - \frac{1}{\sqrt{4x+1}+1} - \ln|\sqrt{4x+1}-1| - \frac{1}{\sqrt{4x+1}-1}] + C. \end{aligned}$$