**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH1010 UNIVERSITY MATHEMATICS 2023-2024 Term 1**
**Suggested Solutions of WeBWork Coursework 7**

If you find any errors or typos, please email us at
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1. (1 point) Fine the maximum area of a triangle formed in the first quadrant by the $x$-axis, $y$-axis and a tangent line to the graph of $f = (x+6)^{-2}$.

   Area =_________

> **Solution:**
> Let $P\left(t, \frac{1}{(t+6)^2}\right)$ be a point on the graph of the curve $y = \frac{1}{(x+6)^2}$ in the first quadrant. The tangent line to the curve at $P$ is
>
> $$L(x) = \frac{1}{(t+6)^2} - \frac{2(x-t)}{(t+6)^3},$$
>
> which has $x$-intercept $a = \frac{3t+6}{2}$ and $y$-intercept $b = \frac{3t+6}{(t+6)^3}$. The area of the triangle in question is
>
> $$A(t) = \frac{1}{2}ab = \frac{(3t+6)^2}{4(t+6)^3}.$$
>
> Solve
>
> $$A'(t) = \frac{(3t+6)(3\cdot 6 - 3t)}{4(t+6)^4} = 0$$
>
> for $0 \le t$ to obtain $t = 6$. Because $A(0) = \frac{1}{4\cdot 6}$, $A(6) = \frac{1}{2\cdot 6}$ and $A(t) \to 0$ as $t \to \infty$, it follows that the maximum area is $A(6) \approx 0.0833333$.

2. (2 points) Find the point $(x, y)$ of $x^2 + 14xy + 49y^2 = 100$ that is closest to the origin and lies in the first quadrant.

   $x =$_________

   $y =$_________

> **Solution:** The distance of point $(x, y)$ to the origin is $dist(x, y) = \sqrt{x^2+y^2}$. Then from the equation we can find that
>
> $$x^2 + 14xy + 49y^2 = 100 = (x+7y)^2.$$
>
> Therefore, $x + 7y = 10$ in the first quadrant, and
>
> $$dist(x,y)^2 = (10-7y)^2+y^2 = 50y^2-140y+100 = 50(y^2-\frac{14}{5}y+\frac{49}{25})+2 = 50(y-\frac{7}{5})^2+2$$

This means that the minimum of distance take value when $y = \frac{7}{5}$, and $x = \frac{1}{5}$.

3. (3 points) Use L'Hôpital's Rule (possibly more than once) to evaluate the following limit
$\lim_{x\to\infty} (\frac{12x^3+13x^2}{9x^3-11}) =$ ________
If the answer equals $\infty$ or $-\infty$, write INF or -INF in the blank.

**Solution:**

$$\begin{aligned}
\lim_{x\to\infty} \frac{12x^3+13x^2}{9x^3-11} &= \lim_{x\to\infty} \frac{(12x^3+13x^2)'}{(9x^3-11)'} = \lim_{x\to\infty} \frac{36x^2+26x}{27x^2} \\
&= \lim_{x\to\infty} \frac{(36x^2+26x)'}{(27x^2)'} = \lim_{x\to\infty} \frac{36x+13}{27x} \\
&= \lim_{x\to\infty} \frac{(36x+13)'}{(27x)'} = \lim_{x\to\infty} \frac{36}{27} \\
&= \frac{4}{3}
\end{aligned}$$

4. (4 points) Compute
$\lim_{x\to 0} \dfrac{e^x - e^{-x}}{2\sin x} =$ ____________

**Solution:**

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{2\sin x} = \lim_{x\to 0} \frac{(e^x - e^{-x})'}{(2\sin x)'} = \lim_{x\to 0} \frac{e^x + e^{-x}}{2\cos x} = \frac{e^0 + e^0}{2\cos 0} = 1$$

5. (5 points) Apply L'Hôpital's Rule to evaluate the following limit. It may be necessary to apply it more than once.
$\lim_{x\to 0^+} (\tan x)^{\sin x} =$ ________

**Solution:** We first consider the limit $\lim_{x\to 0^+} \sin x \ln \tan x$

$$\begin{aligned}
\lim_{x\to 0^+} \sin x \ln \tan x &= \lim_{x\to 0^+} \frac{\ln \tan x}{\frac{1}{\sin x}} = \lim_{x\to 0^+} \frac{(\ln \tan x)'}{(\frac{1}{\sin x})'} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{\tan x} \cdot \frac{1}{\cos^2 x}}{-\frac{\cos x}{\sin^2 x}} = -\lim_{x\to 0^+} \frac{\sin x}{\cos^2 x} \\
&= 0
\end{aligned}$$

So

$$\lim_{x\to 0^+}(\tan x)^{\sin x} = \lim_{x\to 0^+} e^{\sin x \ln \tan x} = e^{\lim\limits_{x\to 0^+} \sin x \ln \tan x} = e^0 = 1$$

6. (6 points) Evaluate

$$\lim_{x\to 0} \frac{\cos(x) - 1 + \frac{x^2}{2}}{11x^4}.$$

Limit =__________

**Solution:**

$$\begin{aligned}\lim_{x\to 0} \frac{\cos(x) - 1 + \frac{x^2}{2}}{11x^4} &= \lim_{x\to 0} \frac{-\sin(x) + x}{44x^3} \\ &= \lim_{x\to 0} \frac{-\cos(x) + 1}{12 \times 11x^2} \\ &= \lim_{x\to 0} \frac{\sin(x)}{24 \times 11x} \\ &= \frac{1}{264}\end{aligned}$$

7. (7 points) Evaluate

$$\lim_{x\to 0} \frac{\ln(1-x) + x + \frac{x^2}{2}}{14x^3}.$$

Limit =__________

**Solution:**

$$\begin{aligned}\lim_{x\to 0} \frac{\ln(1-x) + x + \frac{x^2}{2}}{14x^3} &= \lim_{x\to 0} \frac{-1/(1-x) + 1 + x}{14 \times 3x^2} \\ &= \lim_{x\to 0} \frac{-1/(1-x)^2 + 1}{14 \times 6x} \\ &= \lim_{x\to 0} \frac{-2/(1-x)^3}{14 \times 6} = -\frac{1}{42}\end{aligned}$$

8. (8 points) Find the first three **nonzero** terms of the Taylor series for the function $f(x) = \sqrt{4x - x^2}$ about the point $a = 2$.

**Solution:** $f(2) = 2$

$f'(x) = \dfrac{4-2x}{2\sqrt{4x-x^2}} = \dfrac{2-x}{\sqrt{4x-x^2}}$

$f'(2) = 0$

$f''(x) = -\dfrac{4x-x^2}{(4x-x^2)^{\frac{3}{2}}} - \dfrac{(2-x)(4-2x)}{2(4x-x^2)^{\frac{3}{2}}} = -\dfrac{4}{(4x-x^2)^{\frac{3}{2}}}$

$f''(2) = -\dfrac{1}{2}$

$f^{(3)}(x) = \dfrac{12(2-x)}{(4x-x^2)^{\frac{5}{2}}}$

$f^{(3)}(2) = 0$

$f^{(4)}(x) = -\dfrac{12(4x^2-16x+20)}{(4x-x^2)^{\frac{7}{2}}}$

$f^{(4)}(5) = -\dfrac{3}{8}$

Then

$$\begin{aligned}\sqrt{4x-x^2} &= 2 + \frac{1}{2!}(-\frac{1}{2})(x-2)^2 + \frac{1}{4!}(-\frac{3}{8})(x-2)^4 + ... \\ &= 5 - \frac{1}{4}(x-2)^2 - \frac{1}{64}(x-2)^4 + ...\end{aligned}$$

9. (9 points) Compute $T_2(x)$ at $x = 1$ for $y = e^x$ and use a calculator to compute the error $|e^x - T_2(x)|$ at $x = 0.3$ .

$T_2(x) =$ ____________

$|e^x - T_2(x)| =$ ____________

**Solution:**

$$y^{(n)} = e^x,\ n \in \mathbf{Z}$$

Therefore, $T_2(x) = e + e(x-1) + \frac{e}{2}(x-1)^2$, and $|e^x - T_2(x)| \approx 0.131605$ .

10. (10 points) Write the Taylor series for $f(x) = \ln(\sec(x))$ at $x = 0$ as $\sum_{n=0}^{\infty} c_n x^n$. Find the first five coefficients.

**Solution:** $f(x) = -\ln(\cos(x))$ , Then $f'(x) = \frac{\sin(x)}{\cos(x)} = \tan(x)$, $f^{(2)}(x) = (\tan(x))' = \sec^2(x)$,
$f^{(3)}(x) = 2\sec(x)\tan(x)$, $f^{(4)}(x) = 2\sec^3(x) + 2\sec(x)\tan^2(x)$. Taking value at $x = 0$, we get the first five coefficients: $c_0 = 0,\ c_1 = 0,\ c_2 = \frac{1}{2},\ c_3 = 0,\ c_4 = \frac{1}{12}$.

11. (11 points) Write the Taylor series for $f(x) = \sin(x)$ at $x = \frac{\pi}{2}$ as $\sum_{n=0}^{\infty} c_n(x - \frac{\pi}{2})^n$

**Solution:** Since $f'(x) = \cos(x), f^{(2)}(x) = -\sin(x)$, it's easy to find that

$$f^{(4n)} = \sin(x),\ f^{(4n+1)} = \cos(x),\ f^{(4n+2)} = -\sin(x),\ f^{(4n+3)} = -\cos(x)$$

Therefore the Taylor series take value at $x = \frac{\pi}{2}$ is $1 + \sum_{n=1}^{\infty} \frac{1}{4n!}(x - \frac{\pi}{2})^{4n} - \sum_{n=0}^{\infty} \frac{1}{(4n+2)!}(x - \frac{\pi}{2})^{4n+2}$.

12. (12 points) Suppose that $f(x)$ and $g(x)$ are given by the power series

$$f(x) = 3 + 6x + 2x^2 + 5x^3 + \cdots$$

and

$$g(x) = 6 + 6x + 5x^2 + 4x^3 + \cdots$$

Find the first few terms of the series for

$$h(x) = f(x) \cdot g(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

**Solution:** Denote the coefficients of series $f$ and $g$ by $a_n$ and $b_n$. Then their product have coefficients $c_k = \sum_{i+j=k,i,j\geq 0} a_i b_j$. Therefore $c_0 = 18,\ c_1 = 18+36 = 54,\ c_2 = 15 + 12 + 36 = 63,\ c_3 = 12 + 30 + 30 + 12 = 84$.