# Suggested Solutions of WeBWork Coursework 6

If you find any errors or typos, please email us at
math1010@math.cuhk.edu.hk

**1.** (1 point)

$$f(x) = \frac{x}{x^2+8x+12}$$

a) Give the domain of $f$ (in interval notation) ______

b) Determine the intervals on which $f$ is increasing and decreasing.
Your answer should either be a single interval, such as "(0,1)", a comma separated list of intervals, such as "(-inf, 2), (3,4)" , or the word "none".
$f$ is increasing on: ______
$f$ is decreasing on: ______

For each interval, do take care to consider whether end points should be included.

**Solution:**

Since $f = \frac{x}{x^2+8x+12}$ is a rational function, its domain is all real numbers, excluding those at which the denominator is zero. The denominator factors:

$$x^2+8x+12 = (x+6)(x+2),$$

so the domain is $(-\infty,-6)\cup(-6,-2)\cup(-2,\infty)$.

$f'(x) = \frac{-x^2+12}{(x^2+8x+12)^2}$. Setting equal to zero and solving, there are two critical numbers, $x = \pm\sqrt{12}$.

Use the first derivative test, choosing sample points in each interval. Note, the intervals are determined by both critical numbers and the points excluded from the domain.

| Interval | Sign of $f'$ at sample | Conclusion |
|---|---|---|
| $(-\infty,-6)$ | negative | decreasing |
| $(-6,-\sqrt{12})$ | negative | decreasing |
| $(-\sqrt{12},-2)$ | positive | increasing |
| $(-2,\sqrt{12})$ | positive | increasing |
| $(\sqrt{12},\infty)$ | negative | decreasing |

Based on the signs in each interval there is a relative maximum at $x = \sqrt{12}$ and a relative minimum at $x = -\sqrt{12}$.

*Correct Answers:*

- `(-infinity,-6) U (-6,-2) U (-2,infinity)`
- `[-3.4641,-2), (-2,3.4641]`
- `(-infinity,-6), (-6,-3.4641], [3.4641,infinity)`

**2.** (1 point) Let $f(x) = 8\sqrt{x} - 8x$ for $x > 0$. Find the intervals on which $f$ is increasing (decreasing). Pay attention to endpoints!

1. $f$ is increasing on the intervals ____________
2. $f$ is decreasing on the intervals ____________

**Notes:** Your answer should either be a single interval, such as (0,1), a comma separated list of intervals, such as (-inf, 2), (3,4), or the word "none".
**Solution:**

$$f' = \frac{4}{\sqrt{x}} - 8$$

Let $f' > 0$,

$$x \in (0, 0.25]$$

Let $f' < 0$,

$$x \in [0.25, infinity)$$

*Correct Answers:*

- `(0,0.25]`
- `[0.25,infinity)`

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**3.** (1 point)

Find the critical point and the interval on which the given function is increasing or decreasing, and apply the First Derivative Test to the critical point. Let
$f(x) = 3x - 9\ln(10x), x > 0$

Critical Point = ________

Is $f$ a maximum or minumum at the critical point? [?]

The **open** interval on the left of the critical point is ________.
On this interval, $f$ is [?] while $f'$ is [?].

The **open** interval on the right of the critical point is ________.
On this interval, $f$ is [?] while $f'$ is [?].

**Solution:** First we need to calculate $f'$, thus

$$f' = 3 - 9 \cdot 10 \frac{1}{10x}$$

Setting this equal to zero and solving for $x$ leads to the critical point 3.

We know that $f$ is the same sign on the intervals defined by the critical points. This evaluating $f'$ and some point in each of these intervals, determines if $f'$ is positive, which implies $f$ is increasing or negative, which implies $f$ is decreasing on that interval.

From the sign change of $f'$ at a critical point, we can determine if it is a local maximum/minimum. From $+$ to $-$, a local maximum. From $-$ to $+$, a local minimum.

*Correct Answers:*

- `3`
- `Local Min`

- (0,3)
- Decreasing
- Negative
- (3,infinity)
- Increasing
- Positive

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**4.** (1 point)

Determine the intervals on which the given function is concave up or down and find the point of inflection. Let

$$f(x) = x\left(x - 9\sqrt{x}\right)$$

The x-coordinate of the point of inflection is ______

The **open** interval on the left of the inflection point is ______, and on this interval $f$ is [?].
The **open** interval on the right is ______, and on this interval $f$ is [?].

**Solution:**
One can compute $f'$ using the product rule, but it is easier to re-write
$f(x) = x(x - 9\sqrt{x}) = x^2 - 9x^{3/2}$
Then $f'(x) = 2x - 9 \cdot \frac{3}{2}x^{1/2}$ and $f''(x) = 2 - 9 \cdot \frac{3}{4}x^{-1/2} = 2 - \frac{9 \cdot 3}{4\sqrt{x}}$.

Now, $f$ is Concave Down for $0 < x < 11.3906$ since $f''(x) < 0$ there. Moreover, $f$ is Concave Up for $x > 11.3906$ since $f''(x) > 0$ there.

Finally, because $f''(x)$ changes sign at $x = 11.3906$, $f(x)$ has a point of inflection at $x = 9$.

*Correct Answers:*

- 729/64
- (0,11.3906)
- Concave Down
- (11.3906,infinity)
- Concave Up

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**5.** (1 point) Suppose that

$$f(x) = \frac{6e^x}{6e^x + 6}.$$

(A) Find all critical values of $f$. If there are no critical values, enter *None* . If there are more than one, enter them separated by commas.

Critical value(s) = ______

(B) Use **<u>interval notation</u>** to indicate where $f(x)$ is concave up.

Concave up: ______

(C) Use **interval notation** to indicate where $f(x)$ is concave down.

Concave down: ______

(D) Find all inflection points of $f$. If there are no inflection points, enter *None* . If there are more than one, enter them separated by commas.
Inflection point(s) at $x =$ ______

**Solution:**

$$f(x) = \frac{e^x}{e^x+1}$$

so $f$ is increasing on $R$ and there is no critical value.

$$f'(x) = \frac{e^x}{(e^x+1)^2}$$

$$f^{(2)}(x) = \frac{e^x(1-e^x)}{(e^x+1)^3}$$

So $f$ is concave up when $x < 0$ and concave down when $x > 0$. The inflection point is $x = 0$.

*Correct Answers:*

- `None`
- `(-infinity,0)`
- `(0,infinity)`
- `0`

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**6.** (1 point) Find the extreme values of the function $f$ on the interval $[0.5, 6]$. If an extreme value does not exist, enter **DNE** .

$$f(x) = x^6 + \frac{6}{x}$$

Absolute minimum value: ______

Absolute maximum value: ______

**Solution:**

Set the derivative equal to zero to locate all critical numbers.

$$\begin{aligned} f'(x) = 6x^5 - \frac{6}{x^2} &= 0 \\ x^5 &= \frac{1}{x^2} \\ x^7 &= 1 \\ x &= 1 \end{aligned}$$

The only critical numbers is $x = 1$ Find the value of $f$ at this critical number and the endpoints:

$$\begin{aligned} f(0.5) &= 12.015625 \\ f(1) &= 7 \\ f(6) &= 46657 \end{aligned}$$

The absolute minimum value is 7, and the absolute maximum value is 46657.

- `7`
- `46657`

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**7.** (1 point)

Let $f(x) = \frac{(x+5)^2}{(x-5)^2}$.

Answer the following questions (for multiple answers enter each separated by commas e.g (a) 0,2 or (c) (-2,3),(0,-4) if no value enter "none".

(a) Vertical Asymptotes $x =$ ____________

(b) Horizontal Asymptotes $y =$ ____________

(c) Points where the graph crosses a horizontal asymptote $(x,y) =$ ____________

(d) Critical Points $(x,y) =$ ____________

(e) Inflection Points $(x,y) =$ ____________

SOLUTION

(a) $f(x) = \frac{(x+5)^2}{(x-5)^2}$ has zero denominator and hence a vertical asymptote when $x = 5$ .

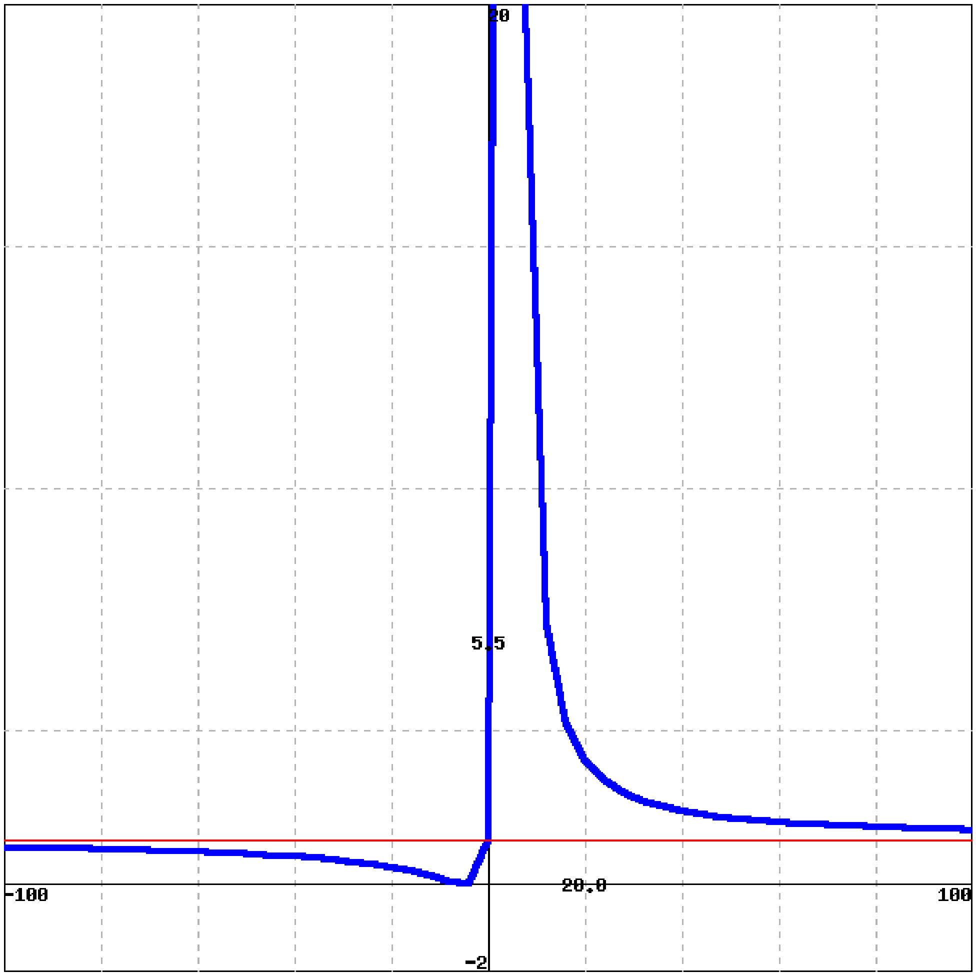
(b) $\lim\limits_{x\to\pm\infty} \dfrac{(x+5)^2}{(x-5)^2} = 1$ so there is a horizontal asymptote at $y = 1$.

(c) $\frac{(x+5)^2}{(x-5)^2} = 1$ only when $x = 0$ so that $(0,1)$ is the only point where the graph crosses the horizontal asymptote.

(d) $f'(x) = -\frac{20(x+5)}{(x-5)^3} = 0$ when $x = -5$ gives the only Stationary Point $(-5,0)$.

(e) $f''(x) = \frac{40(x+10)}{(x-5)^4} = 0$ when $x = -10$ so $(-10, \frac{1}{9})$ is also an Inflection Point.

(click on image to enlarge)




$y = f(x)$

*Correct Answers:*

- 5
- 1
- (0,1)
- (-5,0)
- (-10,1/9)

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**8.** (1 point)

Consider the functions $f(x) = e^{x-1} - 1$ and $g(x) = x - 1$. These are continuous and differentiable for $x > 0$. In this problem we use the Racetrack Principle to show that one of these functions is greater than the other, except at one point where they are equal.

**(a)** Find a point $c$ such that $f(c) = g(c)$. $c =$ ______

**(b)** Find the equation of the tangent line to $f(x) = e^{x-1} - 1$ at $x = c$ for the value of $c$ that you found in (a).
$y =$ ____________

**(c)** Based on your work in (a) and (b), what can you say about the derivatives of $f$ and $g$?
$f'(x)$ [?/</=/>] $g'(x)$ for $0 < x < c$, and
$f'(x)$ [?/</=/>] $g'(x)$ for $c < x < \infty$.

**(d)** Therefore, the Racetrack Principle gives
$f(x)$ [?/<=/=/>=] $g(x)$ for $x \leq c$, and
$f(x)$ [?/<=/=/>=] $g(x)$ for $x \geq c$.

**Solution:**

Note that at $c = 1$ we have $f(c) = g(c) = 0$. Then, at $x = 1$ $f'(x) = e^{x-1}$, so that $f'(1) = 1$, and the equation of the tangent line is $y = x - 1$.

Then we note that for $x \leq 1$ we have $f'(x) \leq g'(x) = 1$ and for $x \geq 1$, that $f'(x) \geq g'(x) = 1$. Therefore, by the Racetrack Principle, we know that $f(x) \geq g(x)$ at every $x$.

*Correct Answers:*

- `1`
- `x-1`
- `<`
- `>`
- `>=`
- `>=`

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**9.** (1 point)

Consider the function $f(x) = x^2 - 4x + 2$ on the interval $[0,4]$. Verify that this function satisfies the three hypotheses of Rolle's Theorem on the inverval.

$f(x)$ is __________ on $[0,4]$;
$f(x)$ is __________ on $(0,4)$;
$f(0) = f(4) =$ __________.

Then by Rolle's theorem, there exists a $c$ such that $f'(c) = 0$.
Find the value $c$.
$c =$ _____

**Solution:**

$$f'(x) = 2x - 4$$

So $c = 2$.

*Correct Answers:*

- `continuous`
- `differentiable`
- `2`
- `2`

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**10.** (1 point) Find the absolute maximum and absolute minimum values of $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ on the interval $[-5,5]$.

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**1.** Find the absolute maximum of $f$ on the interval.
Answer: __________

**2.** Find the absolute minimum of $f$ on the interval.
Answer: __________

**Solution:**

$$f'(x) = \frac{4x}{(x^2+1)^2}$$

So $f'(x) < 0, when x < 0$ and $f'(x) > 0, when x > 0$. $f(-5) = f(5) = \frac{24}{26}$, $f(0) = -1$. Then the absolute maximum of f is $\frac{24}{26}$ and the absolute minimum is $-1$.

*Correct Answers:*

- `24/26`
- `-1`

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**11.** (1 point)

Answer the following True-False quiz. (Enter "T" or "F".)

___1. $(f(x)+g(x))' = f'(x)+g'(x)$.
___2. If $f(x) = e^2$, then $f'(x) = 2e$.
___3. If $f'(c) = 0$, then $c$ is either a local maximum or a local minimum.
___4. If $f(x)$ and $g(x)$ are increasing on an interval $I$, then $f(x)g(x)$ is increasing on $I$.
___5. If a function has a local maximum at $c$, then $f'(c)$ exists and is equal to 0.
___6. Continuous functions are always differentiable.
___7. If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at $c$.

**Solution:** $2.f'(x) = 0$.

3.c may be a saddle point.

4 Suppose the interval is $[-1,0]$ and $f = g = x$, then$f * g = x^2$is decreasing on $[-1,0]$.

5 Suppose $f = -|x|$, it has a local maximum at$x = 0$, but $f'(0)$ does not exist.

6 Suppose a simple example $f = |x|$ and a complicated example Weierstrass function.

*Correct Answers:*

- `T`
- `F`
- `F`
- `F`
- `F`
- `F`
- `T`