THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2023-2024 Term 1
Suggested Solutions of WeBWork Coursework 5

If you find any errors or typos, please email us at
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(1) (1 point)

Differentiate the following function:

$$f(t) = \sqrt[6]{t} - \frac{1}{\sqrt[6]{t}}$$

$f'(t) =$ ________________

Solution:

$$\begin{aligned} f'(t) &= \left(t^{\frac{1}{6}}\right)' - \left(t^{-\frac{1}{6}}\right)' \\ &= \frac{1}{6}t^{-\frac{5}{6}} - \left(-\frac{1}{6}t^{-\frac{7}{6}}\right) \\ &= \frac{1}{6}t^{-\frac{5}{6}} + \frac{1}{6}t^{-\frac{7}{6}} \\ &= \frac{1}{6}\left(t^{-\frac{5}{6}} + t^{-\frac{7}{6}}\right). \end{aligned}$$

Correct Answers:

$$\frac{1}{6}\left(t^{-\frac{5}{6}} + t^{-\frac{7}{6}}\right)$$

(2) (1 point)

Calculate the derivative of the following function.

$$f(x) = \frac{e^x}{(e^x+3)(x+4)}$$

$f'(x) =$ ________________

Solution:

To compute $f'(x)$ we begin with quotient rule

$$f'(x) = \frac{[e^x]' \cdot (e^x+3)(x+4) - e^x \cdot [(e^x+3)(x+4)]'}{[(e^x+3)(x+4)]^2}.$$

Next, recall that $[e^x]' = e^x$, and use the product rule to compute

$$[(e^x+3)(x+4)]' = [e^x+3]' \cdot (x+4) + (e^x+3) \cdot [x+4]'$$

which equals

$$e^x \cdot (x+4) + (e^x+3) \cdot 1.$$

Therefore

$$f'(x) = \frac{e^x \cdot (e^x+3)(x+4) - e^x \cdot [e^x(x+4) + (e^x+3)]}{[(e^x+3)(x+4)]^2}$$

and after factoring out e^x in the numerator, expanding

$$(e^x+3)(x+4) = xe^x + 4e^x + 3x + 12,$$

and distributing the minus sign, we get

$$f'(x) = \frac{e^x(xe^x + 4e^x + 3x + 12 - xe^x - 4e^x - e^x - 3)}{[(e^x + 3)(x + 4)]^2}$$

which simplifies to

$$f'(x) = \frac{e^x(3x - e^x + 9)}{[(e^x + 3)(x + 4)]^2}.$$

Correct Answers:

$$\frac{e^x(3x - e^x + 9)}{[(e^x + 3)(x + 4)]^2}$$

(3) (1 point)

Differentiate $g(x) = \ln\left(\frac{3 - x}{3 + x}\right)$.

Solution:

$$\begin{aligned} g'(x) &= \left(\frac{3 - x}{3 + x}\right)^{-1} \cdot \left(\frac{3 - x}{3 + x}\right)' \\ &= \frac{3 + x}{3 - x} \cdot \frac{(-1) \cdot (3 + x) - (3 - x) \cdot 1}{(3 + x)^2} \\ &= \frac{1}{3 - x} \cdot \frac{-6}{3 + x} \\ &= \frac{6}{x^2 - 9}. \end{aligned}$$

Correct Answers:

$$\frac{6}{x^2 - 9}$$

(4) (1 point)

Let $f(x) = |x| \ln(2 - x)$. Find $f'(x)$.

$$f'(x) = \begin{cases} \\ \end{cases}$$

__________ if $x < c$
__________ if $x = c$
__________ if $c < x < d$

where $c =$ ___ and $d =$ ___.

Enter 'DNE' if the derivative does not exist.

Solution: One can find that $c = 0$, $d = 2$.

If $x < 0$,

$$f(x) = -x \ln(2 - x),$$

so

$$f'(x) = -\ln(2 - x) + (-x) \cdot \frac{-1}{2 - x} = -\ln(2 - x) + \frac{x}{2 - x}.$$

If $0 < x < 2$,

$$f(x) = x \ln(2 - x),$$

so

$$f'(x) = \ln(2 - x) - \frac{x}{2 - x}.$$

But if $x = 0$, we must use the definition of $f'(0)$. Let's consider the left and right derivatives of f at $x = 0$.

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h\ln(2-h) - 0}{h} = \lim_{h\to 0^+} \ln(2-h) = \ln 2.$$

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-h\ln(2-h) - 0}{h} = \lim_{h\to 0^-} -\ln(2-h) = -\ln 2.$$

Since $\ln 2 \neq -\ln 2$, the derivative doesn't exist at $x = 0$.

Correct Answers:
- $x/(2-x) - ln(2-x)$
- DNE
- $ln(2-x) - x/(2-x)$
- 0
- 2

(5) (1 point)

Find $\frac{dy}{dx}$ if

$$6x^3y^2 - 4x^2y = 3.$$

Express your answer in terms of x, y if necessary.

$\frac{dy}{dx} =$ ____________

Solution: Taking the derivative with respect to x we get

$$0 = 18x^2y^2 + 12x^3y\frac{dy}{dx} - 8xy - 4x^2\frac{dy}{dx},$$

so

$$8xy - 18x^2y^2 = \left(12x^3y - 4x^2\right)\frac{dy}{dx}.$$

Therefore,

$$\frac{dy}{dx} = \frac{8xy - 18x^2y^2}{12x^3y - 4x^2}.$$

Correct Answers:

$$\frac{dy}{dx} = \frac{8xy - 18x^2y^2}{12x^3y - 4x^2}$$

(6) (1 point)

Consider the following function: $y = x^{x^2}$.

$\frac{dy}{dx} =$ ________________

(you will lose 25% of your points if you do)

Solution:

After taking the log of both sides, you should get: $\ln y = x^2 \ln x$.

Taking the derivative: $\frac{1}{y}\frac{dy}{dx} = 2x\ln x + x$.

Therefore, $\frac{dy}{dx} = y(2x\ln x + x) = x^{x^2+1}(2\ln x + 1)$.

Correct Answers:

$$\frac{dy}{dx} = x^{x^2+1}(2\ln x + 1)$$

(7) (1 point)

Let $f(x) = \dfrac{4x^3}{(3-2x)^5}$.

Find the equation of the line tangent to the graph of f at $x = 1$.

Tangent line: $y =$ ______________________

Solution: Differentiating gives

$$\begin{aligned} f'(x) &= \frac{12x^2(3-2x)^5 - 4x^3 \cdot 5(3-2x)^4(-2)}{(3-2x)^{10}} \\ &= \frac{12x^2(3-2x) - 4x^3 \cdot 5 \cdot (-2)}{(3-2x)^6} \\ &= \frac{36x^2 + 16x^3}{(3-2x)^6}. \end{aligned}$$

And hence the slope of the tangent line of the graph at $x = 1$ is $f'(1) = 52$. Since $f(1) = 4$ and the point $(1, 4)$ is also on this line, we know the tangent line $y - 4 = 52(x-1)$, that is, $y = 52x - 48$.

Correct Answers:

- `52x-48`

(8) (1 point)

If the equation of motion of a particle is given by $s(t) = A\cos(wt + d)$, the particle is said to undergo *simple harmonic motion* . Assume $0 \leq d < \pi$.

(a) Find the velocity of the particle at time t.
(b) What is the smallest positive value of t for which the velocity is 0? Assume that w and d are positive.

(a) $v(t) =$ ______________
(b) $t =$ ______________

Solution:
(a) Differentiating with respect to t gives: $v(t) = s'(t) = -Aw\sin(wt + d)$.
(b) By (a), $v(t) = 0$ implies $\sin(wt + d) = 0$, then $wt + d = n\pi$, where n is any integer. So $t = \dfrac{n\pi - d}{w}$. Since $0 \leq d < \pi$ and $w > 0$, the smallest positive vale of t is given by taking $n = 1$ and we get

$$t = \frac{\pi - d}{w}.$$

Correct Answers:

- `-Aw sin(wt+d)`
- `(pi-d)/w`

(9) (1 point)

A parabola is defined by the equation

$$x^2 - 2xy + y^2 + 8x - 12y + 36 = 0$$

The parabola has horizontal tangent lines at the **point(s)** _______.

The parabola has vertical tangent lines at the **point(s)** ______.

Solution: Differentiating implicitly with respect to x gives

$$2x + (-2y - 2x\,\frac{dy}{dx}) + 2y\,\frac{dy}{dx} + 8 - 12\,\frac{dy}{dx} = 0,$$

so

$$(y - x - 6)\,\frac{dy}{dx} = y - x - 4,$$

and so

$$\frac{dy}{dx} = \frac{y - x - 4}{y - x - 6}.$$

The tangent line to the parabola is horizontal where $\frac{dy}{dx} = 0$, i.e., where $x - y = -4$. Observe that the equation of the parabola can be rewritten in the form

$$(x - y)^2 + 8(x - y) + 36 - 4y = 0,$$

and $x - y = -4$ gives $20 = 4y$, so $y = 5$, and $x = 1$. Hence, the tangent line to the parabola is horizontal at the point $(1, 5)$ and nowhere else.

The tangent line the the parabola is vertical where

$$0 = \frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{y - x - 6}{y - x - 4},$$

i.e., where $x - y = -6$. Together with the last displayed equation of the parabola, this gives $24 - 4y = 0$, so $y = 6$, and $x = 0$. Hence, the tangent line to the parabola is vertical at the point $(0, 6)$ and nowhere else.

Correct Answers:

- `(1,5)`
- `(0,6)`

(10) (1 point)

Let $f(x) = x^2 \tan^{-1}(7x)$

$f'(x) =$ ______________

Solution: Using the product and chain rules, we see

$$f'(x) = 2x \cdot \tan^{-1}(7x) + x^2 \cdot \frac{7}{1 + (7x)^2} = 2x \tan^{-1}(7x) + \frac{7x^2}{1 + 49x^2}.$$

(Note: here $\tan^{-1}(7x)$ means that $\arctan(7x)$.)

Correct Answers:

$$2x \tan^{-1}(7x) + \frac{7x^2}{1 + 49x^2}$$

(11) (1 point)

Suppose that

$$f(x) = \frac{4x^2}{\sqrt{2x^2 + 1}}.$$

Find $f'(x)$, and then evaluate f' at $x = 2$ and $x = -1$.

$f'(2) =$ ____________

$f'(-1) =$ ____________

Solution:

$$
\begin{aligned}
f'(x) &= \frac{8x \cdot \sqrt{2x^2+1} - 4x^2 \cdot \frac{1}{2}(2x^2+1)^{-\frac{1}{2}} 4x}{2x^2+1} \\
&= \frac{8x(2x^2+1) - 8x^3}{(2x^2+1)^{\frac{3}{2}}} \\
&= 8x \cdot \frac{x^2+1}{(2x^2+1)^{\frac{3}{2}}}
\end{aligned}
$$

So $f'(2) = \frac{80}{27} = 2.96296296296296$
and $f'(-1) = -\frac{16}{\sqrt{27}} = -3.079201435678$

Correct Answers:

- `2.96296296296296`
- `-3.079201435678`

(12) (1 point)

The equation of the tangent line to the graph of $y = x\cos(3x)$ at $x = \pi$ is given by $y = mx + b$ for

$m =$ _____
and
$b =$ _____

Solution: Differentiating gives

$$\frac{dy}{dx} = \cos(3x) + x \cdot [-\sin(3x)3] = \cos(3x) - 3x\sin(3x).$$

And hence the slope of the tangent line of the graph at $x = \pi$ is

$$m = \cos(3\pi) - 3\pi\sin(3\pi) = -1.$$

Since when $x = \pi$, $y = \pi\cos(3\pi) = -\pi$, the point $(\pi, -\pi)$ is also on this line. Hence we know that $-\pi = y = mx + b = (-1)\pi + b$, so $b = 0$.

Correct Answers:

- `-1`
- `0`