THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2023-2024 Term 1
Suggested Solutions of WeBWorK Coursework 4

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Problem 1.

Let

$$f(x) = \begin{cases} -3x, & x < 7, \\ 1, & x = 7, \\ 3x, & x > 7. \end{cases}$$

$\lim_{x \to 7^-} f(x) = -21$;

$\lim_{x \to 7^+} f(x) = 21$;

$\lim_{x \to 7} f(x) = \text{DNE}$;

$f(7)$=1.

Is f continuous at $x = 7$? NO!

Solution :

$\lim_{x \to 7^-} f(x) = \lim_{x \to 7^-} -3x = -21$;

$\lim_{x \to 7^+} f(x) = \lim_{x \to 7^+} 3x = 21$;

Since $\lim_{x \to 7^-} f(x) \neq \lim_{x \to 7^+} f(x)$, we get $\lim_{x \to 7} f(x) = \text{DNE}$;

$f(7) = 1$.

f is not continuous at $x = 7$ since the limit of f at 7 does not exist.

Problem 2.

Let $f(x) = |x - 8|$. Evaluate the following limits.

$\lim_{x \to 8^-} \frac{f(x) - f(8)}{x - 8}$= -1;

$\lim_{x \to 8^+} \frac{f(x) - f(8)}{x - 8}$= 1;

Thus the function $f(x)$ is not differentiable at 8.

Solution :

$\lim_{x\to 8^-} \dfrac{f(x)-f(8)}{x-8} = \lim_{x\to 8^-} \dfrac{(8-x)-(8-8)}{x-8} =$ -1;
$\lim_{x\to 8^+} \dfrac{f(x)-f(8)}{x-8} = \lim_{x\to 8^+} \dfrac{(x-8)-(8-8)}{x-8} = 1;$
Since $\lim_{x\to 8^-} \dfrac{f(x)-f(8)}{x-8} \neq \lim_{x\to 8^+} \dfrac{f(x)-f(8)}{x-8}$, we know that $f(x)$ is not differentiable at 8.

Problem 3.

Find $f^{'}(x)$ and $f^{'}(0)$, where:

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(a) Find the derivative of $f(x)$ for x not equal 0.

$f^{'}(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$;

(b) Find the derivative of $f(x)$ for x equal 0.

$f^{'}(0) = 0.$

Solution :

(a) For $x \neq 0, f^{'}(x) = (x^2)^{'}\sin(\frac{1}{x}) + x^2(\sin(\frac{1}{x}))^{'} = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x});$

(b) For $x = 0, f^{'}(x) = \lim_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \dfrac{x^2\sin(\frac{1}{x})}{x-0} = \lim_{x\to 0} x\sin\left(\dfrac{1}{x}\right) = 0$ by the Squeeze Theorem.

Problem 4.

Let

$$f(x) = \begin{cases} -9x^2 + 5x & \text{for } x < 0, \\ 5x^2 - 3 & \text{for } x \geq 0. \end{cases}$$

According to the definition of the derivative, to compute $f^{'}(0)$, we need to compute the left-hand limit

$\lim_{h\to 0^-} \dfrac{(-9h^2+5h+3)}{h}$, which is $-\infty$,

and the right-hand limit

$\lim_{h\to 0^+} 5h$, which is 0.

We conclude that $f'(0)$ is DNE.

Solution :

The left-hand limit: $Lf'(0) = \lim_{h \to 0^-} \dfrac{f(h) - f(0)}{h - 0} = \dfrac{-9h^2 + 5h - (-3)}{h}$

The right-hand limit: $Rf'(0) = \lim_{h \to 0^+} \dfrac{f(h) - f(0)}{h - 0} = \dfrac{5h^2 - 3 - (-3)}{h} = \dfrac{5h^2}{h}$

Since $\lim_{h \to 0^-} \dfrac{f(h) - f(0)}{h - 0} \neq \lim_{h \to 0^+} \dfrac{f(h) - f(0)}{h - 0}$, we know that $f(x)$ is not differentiable at 0.

Problem 5.

Evaluate the following limits. If needed, enter 'INF' for ∞ and '-INF' for $-\infty$.

(a) $\lim_{x \to \infty} \dfrac{\sqrt{9 + 2x^2}}{3 + 11x} = \dfrac{\sqrt{2}}{11}$.

(b) $\lim_{x \to -\infty} \dfrac{\sqrt{9 + 2x^2}}{3 + 11x} = -\dfrac{\sqrt{2}}{11}$.

Solution :

(a) $\lim_{x \to \infty} \dfrac{\sqrt{9 + 2x^2}}{3 + 11x} = \lim_{x \to \infty} \dfrac{\sqrt{\frac{9}{x^2} + 2}}{\frac{3}{x} + 11} = \dfrac{\sqrt{2}}{11}$.

(b) $\lim_{x \to -\infty} \dfrac{\sqrt{9 + 2x^2}}{3 + 11x} = \lim_{x \to -\infty} \dfrac{\sqrt{\frac{9}{x^2} + 2}}{-\frac{3}{x} - 11} = -\dfrac{\sqrt{2}}{11}$.

Problem 6.

Find a and b so that the function

$$f(x) = \begin{cases} 2x^3 - 4x^2 + 6, & x < -2, \\ ax + b, & x \geq -2 \end{cases}$$

is both continuous and differentiable.

$a = 40$

$b = 54$

Solution :

To make $f(x)$ continuous, we must have $\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x)$.

Hence, $2 \cdot (-2)^3 - 4 \cdot (-2)^2 + 6 = -2a + b$, i.e. $-26 = -2a + b$.

And we also have $f(-2) = -26 = -2a + b$.

To make $f(x)$ differentiable, we must have $Lf^{'}(-2) = Rf^{'}(-2)$, i.e.

$$\lim_{x \to -2^-} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \to -2^+} \frac{f(x) - f(-2)}{x - (-2)},$$

that is

$$\lim_{x \to -2^-} \frac{(2x^3 - 4x^2 + 6) - (-26)}{x - (-2)} = \lim_{x \to -2^+} \frac{(ax + b) - (-2a + b)}{x - (-2)},$$

which is reduced to

$$\lim_{x \to -2^-} \frac{(x+2)(2x^2 - 8x + 16)}{x + 2} = \lim_{x \to -2^+} a.$$

Hence $a = 2(-2)^2 - 8(-2) + 16 = 40$, and $b = 2a - 26 = 54$.
Remark: $Lf^{'}(-2)$ can also be calculated using

$$\lim_{x \to -2^-} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \to -2^-} (2x^3 - 4x^2 + 6)^{'} = \lim_{x \to -2^-} 3 \cdot 2x^2 - 2 \cdot 4x.$$

Problem 7.

Suppose $f^{'}(x)$ exists for all x in (a, b).

Mark all true items with a check. There may be more than one correct answer.

A. $f(x)$ is continuous on (a, b).

Comment: This is true. The continuity of $f(x)$ can be deduced from the fact that $f^{'}(x)$ exists for all x in (a,b).

B. $f(x)$ is continuous at $x = a$.

Comment: This is not always be true. Since (a, b) is an open interval, and $f'(x)$ is defined locally in this interval, we can't know the value of $f(x)$ at the endpoint x =a.

C. $f(x)$ is defined for all x in (a, b).

Comment: This is true.

D. $f^{'}(x)$ is differential on (a, b).

Comment: This is false. Maybe $f^{'}(x)$ is not continuous.

Problem 8.

If $f^{'}(a)$ exists, then $\lim_{x \to a} f(x)$:

A. must exist, but there is not enough information to determine its value.

B. is equal to $f(a)$.

Comment: This is correct. Since $f(x)$ is continuous at $x = a$.

C. is equal to $f^{'}(a)$.

D. might not exist.

E. does not exist.

Solution :

• A is incorrect. Since $f^{'}(a)$ exists, then we deduce that $f(x)$ is continuous at $x = a$, hence $\lim_{x \to a} f(x)$ is equal to $f(a)$.

• B is correct. Since $f^{'}(a)$ exists, then we deduce that $f(x)$ is continuous at $x = a$, hence $\lim_{x \to a} f(x)$ is equal to $f(a)$.

• C is incorrect. The reason is as same as before.

• D is incorrect. The reason is as same as before.

• E is incorrect. The reason is as same as before.

Problem 9.

Give the interval(s) on which the function is continuous.

$$h(k) = \sqrt{9-k} + \sqrt{k+7}$$

Solution :

$[-7, 9]$

Since the square root function is continuous on its domain, we just need to calculate the domain of this function. That is, $\begin{cases} 9-k \geq 0 \\ k+7 \geq 0 \end{cases}$ which implies $\begin{cases} k \leq 9 \\ k \geq -7 \end{cases}$.

Problem 10.

Shown below are six statements about functions. Match each statement to one of the functions shown below which BEST matches that statement.

1. $\lim_{x \to 8^+} f(x)$ and $\lim_{x \to 8^-} f(x)$ both exist and are finite, but they are not equal.

2.The graph of $y = f(x)$ has vertical tangent line at $(8, f(8))$.

3. $\lim_{x\to 8^-} f(x) = -\infty$.

4. $\lim_{x\to 8^+} f(x)$ exists but $\lim_{x\to 8^-} f(x)$ doesn't.

5. $\lim_{x\to 8} f(x) = \infty$.

6. $\lim_{x\to 8} f(x)$ exists but f is not continuous at 8.

A. $f(x) = \dfrac{1}{x-8}$

B. $f(x) = \begin{cases} \cos(\frac{1}{x-8}) & x < 8 \\ 0 & x = 8 \\ 3x+48 & x > 8 \end{cases}$

C. $f(x) = \begin{cases} 3x & x < 8 \\ 0 & x = 8 \\ 48-3x & x > 8 \end{cases}$

D. $f(x) = \sqrt[3]{x-8}$

E. $f(x) = \begin{cases} 3x & x < 8 \\ 0 & x = 8 \\ 3x-48 & x > 8 \end{cases}$

F. $f(x) = \dfrac{1}{(x-8)^2}$

Solution :

- For A, $\lim_{x\to 8^-} f(x) = -\infty$ and $\lim_{x\to 8^+} f(x) = +\infty$.

- For B, $\lim_{x\to 8^-} f(x)$ doesn't exist , $\lim_{x\to 8^+} f(x) = 72$, hence $\lim_{x\to 8^+} f(x)$ exists but $\lim_{x\to 8^-} f(x)$ doesn't.

- For C, $\lim_{x\to 8^-} f(x) = 24$, $\lim_{x\to 8^+} f(x) = 24$, hence $\lim_{x\to 8} f(x)$ exists but $f(8) = 0 \neq 24$, so f is not continuous at 8.

- For D, since the derivative of $f(x)$ is $\frac{1}{3}(x-8)^{-\frac{2}{3}}$, and $f'(8) = \infty$, the graph of $y = f(x)$ has vertical tangent line at $(8, f(8))$.

- For E, $\lim_{x\to 8^-} f(x) = 24$, $\lim_{x\to 8^+} f(x) = -24$, hence $\lim_{x\to 8^+} f(x)$ and $\lim_{x\to 8^-} f(x)$ both exist and are finite, but they are not equal.

- For F, $\lim_{x\to 8^-} f(x) = \infty$, $\lim_{x\to 8^+} f(x) = \infty$, hence $\lim_{x\to 8} f(x) = \infty$.

Problem 11.

Why is the following function discontinuous at $x = 0$?

$$f(x) = \begin{cases} e^x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

(a) $f(0)$ does not exist.

(b) $\lim_{x \to 0} f(x)$ does not exist (or is infinite).

(c) both (a) and (b).

(d) $f(0)$ and $\lim_{x \to 0} f(x)$ exist, but they are not equal.

Solution :

- (a) is incorrect. Since $f(0) = 0^2 = 0$;

- (b) is correct. The function is discontinuous at $x = 0$ because $\lim_{x \to 0^+} f(x) = 0^2 = 0$ is not equal to $\lim_{x \to 0^-} f(x) = e^0 = 1$.

- (c) is incorrect.

- (d) is incorrect. This is because $\lim_{x \to 0} f(x)$ does not exist .