THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2023-2024 Term 1
Suggested Solutions of WeBWorK Coursework 2

If you find any errors or typos, please email us at
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1. Find the domain of the function:
$$f(x) = \frac{\sqrt{7+x}}{x^2-36}$$
and write your answer in the interval notation.

Solution 1.
$7+x \geq 0$ which implies $x \geq -7$ and $x^2-36 \neq 0$ which implies $x \neq \pm 6$.
In this case, the domain in the interval notation is $[-7,-6) \cup (-6,6) \cup (6,+\infty)$.

2. The domain of the function $g(x) = log_a(x^2-9)$ is $(-\infty,)$ and $(, +\infty)$

Solution 2.
For a log function to be defined, $x^2-9>0$ which implies $x<-3$ or $x>3$. In this case: $(-\infty,-3) \cup (3,+\infty)$.

3. Given that $f(x) = \frac{1}{x}$ and $g(x) = 4x-7$, calculate
(a) $(f \circ g)(x) =$, its domain is all real numbers except
(b) $(g \circ f)(x) =$, its domain is all real numbers except
(c) $(f \circ f)(x) =$, its domain is all real numbers except
(d) $(g \circ g)(x) =$, its domain is $(,)$
Note: If needed enter ∞ as inf and $-\infty$ as -inf .

Solution 3.
(a) $(f \circ g)(x) = \dfrac{1}{4x-7}$; except $x = \dfrac{7}{4}$.
(b) $(g \circ f)(x) = \dfrac{4}{x} - 7$; except $x = 0$.
(c) $(f \circ f)(x) = \dfrac{1}{\frac{1}{x}}$; except $x = 0$.
(d) $(g \circ g)(x) = 4(4x-7)-7 = 16x-35$; domain is $(-\infty,\infty)$.

4. Given the functions $f(x) = \dfrac{x-8}{x-6}$ and $g(x) = \sqrt{x+1}$, find the following domains. Use interval notation.
(a) Domain of f;
(b) Domain of g;
(c) Domain of $f+g$;
(d) Domain of $\dfrac{f}{g}$;
(e) Domain of $\dfrac{g}{f}$;
(f) Domain of $f(g(x))$;
(g) Domain of $g(f(x))$.

Solution 4.
(a) $(-\infty, 6) \cup (6, \infty)$;
(b) $[-1, \infty)$;
(c) $(f+g)(x) = \frac{x-8}{x-6} + \sqrt{x+1}$, *the domain of* $(f+g)(x)$ *is* $[-1, 6) \cup (6, \infty)$;
(d) $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{\frac{x-8}{x-6}}{\sqrt{x+1}}$, *x should lie in domain of f,g and g cannot be 0. Thus the domain of* $\frac{f}{g}$ *is* $(-1, 6) \cup (6, \infty)$;
(e) $\frac{g}{f}(x) = \frac{\sqrt{x+1}}{\frac{x-8}{x-6}}$, *x should lie in domain of f,g and f cannot be 0. Thus the domain is* $[-1, 6) \cup (6, 8) \cup (8, \infty)$;
(f) $f(g(x)) = f(\sqrt{x+1})$, *x should lie in domain of g and* $g - 6$ *cannot be 0. Thus the domain is* $[-1, 35) \cup (35, \infty)$;
(g) $g(f(x)) = g\left(\frac{x-8}{x-6}\right)$, *x should lie in domain of f and* $f+1 \geq 0$*the domain is* $(-\infty, 6) \cup [7, \infty)$.

5. Suppose $f(x) = 6x - 7$ and $g(y) = \frac{y}{6} + \frac{7}{6}$.
(a) Find the composition $g(f(x))$;
(b) Find the composition $f(g(x))$.

Solution 5.
$g(f(x)) = g(6x-7) = x$ *and* $f(g(x)) = f\left(\frac{x}{6} + \frac{7}{6}\right) = x$

6. Find the inverse function of $y = f(x) = \frac{8-6x}{6-8x}$.

Solution 6.
As $y = f(x) = \frac{8-6x}{6-8x}$, *we have* $y = \frac{3}{4}\left(1 + \frac{7}{3}\frac{1}{3-4x}\right)$. *In this case,* $4x = 3 - \frac{7}{4y-3}$. *Finally,* $x = \frac{3y-4}{4y-3}$.

7. Evaluate the following limit by simplifying the expression (first answer box) and then evaluating the limit (second answer box). $\lim_{x \to 4} \frac{x^2 + 2x - 24}{x - 4}$

Solution 7.
$x^2 + 2x - 24 = (x+6)(x-4)$. *Thus for* $x \neq 4$, $\frac{x^2 + 2x - 24}{x-4} = x + 6$.
$\lim_{x \to 4} \frac{x^2 + 2x - 24}{x - 4} = \lim_{x \to 4}(x+6) = 10$.

8. Evaluate the limit

$$\lim_{s \to 3} = \frac{\frac{1}{s} - \frac{1}{3}}{s - 3}$$

Solution 8.
For $s \neq 3$, $\frac{\frac{1}{s} - \frac{1}{3}}{s-3} = -\frac{1}{3s}$. *Thus if* $s \to 3$, $\lim_{s \to 3} = \frac{\frac{1}{s} - \frac{1}{3}}{s-3} = -\frac{1}{9}$.

9. Let a be a positive real number. Evaluate the limit:

$$\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{2(x-a)} =$$

Solution 9.

$x-a=(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})$, thus for $x \neq a$: $\dfrac{\sqrt{x}-\sqrt{a}}{2(x-a)} = \dfrac{1}{2(\sqrt{x}+\sqrt{a})}$.

In this case, $\lim\limits_{x\to a} \dfrac{\sqrt{x}-\sqrt{a}}{2(x-a)} = \lim\limits_{x\to a} \dfrac{1}{2(\sqrt{x}+\sqrt{a})} = \dfrac{1}{4\sqrt{a}}$.

10. Determine whether the sequences are increasing, decreasing, or not monotonic. If increasing, enter I as your answer. If decreasing, enter D as your answer. If not monotonic, enter N as your answer.

1. $a_n = \dfrac{1}{4n+9}$;
2. $a_n = \dfrac{cosn}{4^n}$;
3. $a_n = \dfrac{n-4}{n+4}$;
4. $a_n = \dfrac{\sqrt{n+4}}{9n+4}$.

Solution 10.

1. As n increases, $4n+9$ increases. $a_{n+1} < a_n$, thus D;

2. As cosn can lager than 0 and smaller than 0 as n increasing, so a_n can be positive or negative, thus N;

3. $a_n = 1 - \frac{8}{n+4}, \frac{8}{n+4} > \frac{8}{n+5}, a_n < a_{n+1}$, thus I;

4. Let $m = n+4$, then $\dfrac{\sqrt{n+4}}{9n+4} = \sqrt{\dfrac{1}{81m - 576 + \frac{1024}{m}}}$ which decreases as m increases. $a_{n+1} < a_n$, thus D.

11. Determine whether the sequence $a_n = \dfrac{1^3}{n^4} + ... + \dfrac{n^3}{n^4}$ converges or diverges. If it converges, find the limit.

Solution 11.

Notice that $\sum i^3 = \dfrac{k^2(k+1)^2}{4}$, thus $a_n = \dfrac{1^3+2^3+...+n^3}{n^4} = \dfrac{1}{n^4}\left(\dfrac{n^2(n+1)^2}{4}\right) = \dfrac{(n+1)^2}{4n^2} = \dfrac{n^2+2n+1}{4n^2} = \dfrac{1}{4} + \dfrac{1}{2n} + \dfrac{1}{4n^2}$. In this case, $\lim\limits_{n\to\infty} a_n = \dfrac{1}{4}$.