THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2023-2024 Term 1
Suggested Solutions of WeBWorK Coursework 1

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math1010@math.cuhk.edu.hk

(1) In each part, find a formula for the general term of the sequence, starting with $n = 1$.

(a) $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

(b) $\frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \ldots$

(c) $\frac{3}{4}, \frac{15}{16}, \frac{63}{64}, \frac{255}{256} \ldots$

(d) $0, \frac{1}{\sqrt{\pi}}, \frac{4}{\sqrt[3]{\pi}}, \frac{9}{\sqrt[4]{\pi}}, \ldots$

Solution:

(a) The general term of the sequence is $a_n = \frac{1}{2^{n+1}}$.

(b) The general term of the sequence is $a_n = (-1)^{n+1}\frac{1}{2^{n+1}}$.

(c) The general term of the sequence is $a_n = 1 - \frac{1}{4^n}$.

(d) The general term of the sequence is $a_n = \frac{(n-1)^2}{\sqrt[n]{\pi}}$.

(2) Determine whether the sequence $a_n = \frac{n^{13} + \sin(14n+6)}{n^{14}+6}$ converge or diverge. If it converges, find the limit.

Solution: Consider the sequence $b_n = \frac{n^{13}+1}{n^{14}+6}$ and $c_n = \frac{n^{13}-1}{n^{14}+6}$. Note that $c_n \leq a_n \leq b_n$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$. By Sandwich theorem, we have $\lim_{n\to\infty} a_n = 0$.

(3) Use algebra to simplify the expression before evaluating the limit. In particular, factor the highest power of n from the numerator and denominator, then cancel as many factors of n as possible.

Solution:

$$\lim_{n\to\infty} \frac{2n}{(3n^3+5)^{1/3}} = \lim_{n\to\infty} \frac{2n}{n(3+5/n^3)^{1/3}} = \lim_{n\to\infty} \frac{2}{(3+5/n^3)^{1/3}} = \frac{2}{3^{1/3}}$$

(4) Part 1: Evaluating a series

Consider the sequence $\{a_n\} = \left\{\frac{2}{n^2+2n}\right\}$.

(a) Find $\lim_{n\to\infty} a_n$ if it exists.

(b) Find $\sum_{n=1}^{\infty} a_n$ the sum of all terms in this sequence, which is defined as the limit of the partial sums, if it exists.

Solution:

(a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{n^2+2n} = \lim_{n\to\infty} \frac{2/n^2}{1+2/n} = \frac{0}{1+0} = 0.$

(b) Note that, for $n \geq 1$,

$$a_n = \frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}.$$

Hence, for $N \geq 2$,

$$\begin{aligned}
\sum_{n=1}^{N} a_n &= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+2}\right) \\
&= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+2}\right) \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N}\right) - \frac{1}{N+1} - \frac{1}{N+2} \\
&= \frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}.
\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} \left(\frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right) = \frac{3}{2}.$$

Part 2: Evaluating another series

Consider the sequence $\{b_n\} = \left\{\ln\left(\frac{n+1}{n}\right)\right\}$.

(a) Find $\lim_{n\to\infty} b_n$ if it exists.

(b) Find $\sum_{n=1}^{\infty} b_n$ if it exists.

Solution:

(a) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right) = \ln(1+0) = 0.$

(b) Note that, for $n \geq 1$,

$$b_n = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n.$$

Hence, for $N \geq 1$,

$$\begin{aligned}
\sum_{n=1}^{N} b_n &= \sum_{n=1}^{N} (\ln(n+1) - \ln n) \\
&= \sum_{n=1}^{N} \ln(n+1) - \sum_{n=1}^{N} \ln n \\
&= \sum_{n=2}^{N+1} \ln n - \sum_{n=1}^{N} \ln n \\
&= \ln(N+1) - \ln 1 \\
&= \ln(N+1)
\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} b_n = \lim_{N\to\infty} \sum_{n=1}^{N} b_n = \lim_{N\to\infty} \ln(N+1) = +\infty.$$

Part 3: Developing conceptual understanding

Suppose $\{c_n\}$ is a sequence.

(a) If $\lim\limits_{n\to\infty} c_n = 0$, then the series $\sum\limits_{n=1}^{\infty} c_n$

- must
- may or may not
- cannot

converge.

(b) If $\lim\limits_{n\to\infty} c_n \neq 0$, then the series $\sum\limits_{n=1}^{\infty} c_n$

- must
- may or may not
- cannot

converge.

(c) If the series $\sum\limits_{n=1}^{\infty} c_n$ converges, then $\lim\limits_{n\to\infty} c_n$

- must
- may or may not
- cannot

be equal to 0.

Solution:

(a) If $\lim\limits_{n\to\infty} c_n = 0$, then the series $\sum\limits_{n=1}^{\infty} c_n$ **may or may not** converge. Just look back at parts 1 and 2.

(b) If $\lim\limits_{n\to\infty} c_n \neq 0$, then the series $\sum\limits_{n=1}^{\infty} c_n$ **cannot** converge.

(c) If the series $\sum\limits_{n=1}^{\infty} c_n$ converges, then $\lim\limits_{n\to\infty} c_n$ **must** be equal to 0.

Explanation. (a) Just see Part 1 $\lim\limits_{n\to\infty} a_n = 0$, $\sum\limits_{n=1}^{\infty} a_n$ converges while for Part 2, $\lim\limits_{n\to\infty} b_n = 0$ but $\sum\limits_{n=1}^{\infty} b_n$ diverges.

(b) This is a result of (c) by the argument of contradiction: first claim that statement (c) holds, now suppose (b) is false, i.e. the series $\sum_{n=1}^{\infty} c_n$ converges, then according to (c) c_n must tend to zero, which contradicts the initial setting $\lim\limits_{n\to\infty} c_n \neq 0$.

(c) Finally, we prove statement (c): Just set the partial sum

$$S_n = \sum_{k=1}^{n} c_k.$$

Then the series $\sum_{n=1}^{\infty} c_n$ converges is equivalent to $\lim_{n\to\infty} S_n = C < \infty$. Note that $\lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} S_n = C$ by setting $S_0 = 0$, therefore

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = C - C = 0.$$

□

(5) Consider the recursively defined sequence:

$$a_1 = 7$$
$$a_{n+1} = \frac{n+1}{n^2} a_n, \quad \text{for } n \geq 1$$

(a) The sequence is

- Eventually monotone increasing
- Eventually monotone decreasing
- Neither

(b) The sequence is bounded below by

(c) The sequence is bounded above by

(d) The limit of the sequence is

Solution:

(a) The sequence is eventually monotone decreasing since $a_2 = 14 > 7 = a_1$ while

$$a_{n+1} = \frac{n+1}{n^2} a_n \leq a_n \quad \text{for } n \geq 2.$$

(b) Clearly $a_n \geq 0$ for all $n \geq 1$. So the sequence is bounded below by 0.

(c) From (a) we see that $a_n \leq a_2 = 14$ for $n \geq 1$. So the sequence is bounded above by 14.

(d) By Monotone Convergence Theorem, $\{a_n\}$ converges to some limit ℓ. Thus

$$\begin{aligned}\ell = \lim_{n\to\infty} a_{n+1} &= \left(\lim_{n\to\infty} \frac{n+1}{n^2}\right)\left(\lim_{n\to\infty} a_n\right) \\ &= \left(\lim_{n\to\infty} (\frac{1}{n} + \frac{1}{n^2})\right)\ell \\ &= 0 \cdot \ell = 0.\end{aligned}$$

Therefore the limit of the sequence $\{a_n\}$ is 0.

(6) Consider the recursively defined sequence:

$$a_1 = 1, \quad a_2 = 1$$
$$a_{n+2} = \frac{a_{n+1} + a_n}{2}, \quad \text{for } n \geq 1$$

Find the limit of the sequence if it exists.

Solution:

From the definition of the sequence,

$$a_3 = \frac{a_2 + a_1}{2} = \frac{1+1}{2} = 1,$$

$$a_4 = \frac{a_3 + a_2}{2} = \frac{1+1}{2} = 1,$$

and so on, we thus have

$$a_{n+2} = \frac{a_{n+1} + a_n}{2} = \frac{1+1}{2} = 1, \qquad \text{for } n \geq 1.$$

Hence the sequence is just a constant sequence of 1's, and clearly $\lim_{n\to\infty} a_n = 1$.

(7) Consider the sequence

$$a_n = \frac{n\cos(n\pi)}{2n-1}.$$

Write the first five terms of a_n, and find $\lim_{n\to\infty} a_n$.

Solution: The first five terms are

$$a_1 = -1,\ a_2 = \frac{2}{3},\ a_3 = -\frac{3}{5},\ a_4 = \frac{4}{7},\ a_5 = -\frac{5}{9}.$$

Note that

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{2n\cos(2n\pi)}{4n-1} = \lim_{n\to\infty} \frac{1}{2-1/2n} = \frac{1}{2},$$

while

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} \frac{(2n+1)\cos((2n+1)\pi)}{4n+1} = \lim_{n\to\infty} -\frac{1+1/2n}{2+1/2n} = -\frac{1}{2}.$$

Since $\lim_{n\to\infty} a_{2n} \neq \lim_{n\to\infty} a_{2n+1}$, $\lim_{n\to\infty} a_n$ does not exist.

(8) The sequence $\{a_n\}$ is defined by $a_1 = 2$, and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right),$$

for $n \geq 1$. Assuming that $\{a_n\}$ converges, find its limit.

Solution: Let $a = \lim_{n\to\infty} a_n$. Since $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, we have

$$a = \frac{1}{2}\left(a + \frac{2}{a}\right)$$

$$2a^2 = a^2 + 2$$

$$a^2 = 2.$$

So $a = \sqrt{2}$ or $a = -\sqrt{2}$, where the latter is rejected since $a_n \geq 0$ (rigorous proof by mathematical induction). Therefore, $\lim_{n\to\infty} a_n = a = \sqrt{2}$.

(9) Determine whether the sequence is divergent or convergent. If it is convergent, evaluate its limit.

$$\lim_{n\to\infty} (-1)^n \sin(13/n)$$

Solution: Note that, for $n \geq 1$,

$$-|\sin(13/n)| \leq (-1)^n \sin(13/n) \leq |\sin(13/n)|.$$

Moreover, $\lim_{n\to\infty} |\sin(13/n)| = |\sin(0)| = 0$, and similarly $\lim_{n\to\infty} -|\sin(13/n)| = 0$. Therefore $\lim_{n\to\infty} (-1)^n \sin(13/n) = 0$.

In fact for $N = \left\lfloor \frac{13}{\pi/2} \right\rfloor + 1$, the tail terms $n \geq N$ satisfy

$$-13/n \leq (-1)^n \sin(13/n) \leq 13/n,$$

this is because when $n \geq N$, we have $0 < 13/n < \pi/2$ and for $0 < x < \pi/2$, the inequality $\sin(x) < x$ holds. By squeeze theorem, $\lim_{n\to\infty} (-1)^n \sin(13/n) = \lim_{n\to\infty} 13/n = 0$.

(10) Consider the sequence $a_n = \left\{ \dfrac{4n+1}{4n} - \dfrac{4n}{4n+1} \right\}$. Graph this sequence and use your graph to help you answer the following questions.

Part 1: Is the sequence bounded?

(a) Is the sequence a_n bounded above by a number?

(b) Is the sequence a_n bounded below by a number?

(c) Select all that apply: The sequence a_n is

A. bounded.
B. bounded below.
C. bounded above.
D. unbounded.

Part 2: Is the sequence monotonic?

The sequence a_n is

A. decreasing.
B. alternating
C. increasing.
D. none of the above

Part 3: Does the sequence converge?

(a) The sequence a_n is

- convergent
- divergent

(b) The limit of the sequence a_n is

Part 4: Conceptual follow up questions

(a) Select all that apply: The sequence $\left\{ (-1)^n \dfrac{10n^2+1}{n^2+n} \right\}$ is

A. monotonic
B. divergent
C. convergent
D. not monotonic
E. unbounded
F. bounded

(b) Select all that apply: The sequence $\left\{ \dfrac{10n^3+1}{n^2+n} \right\}$ is

A. unbounded
B. not monotonic
C. divergent
D. monotonic

E. convergent
F. bounded

(c) If a sequence is bounded, it ....
- must
- may or may not
- cannot

converge.

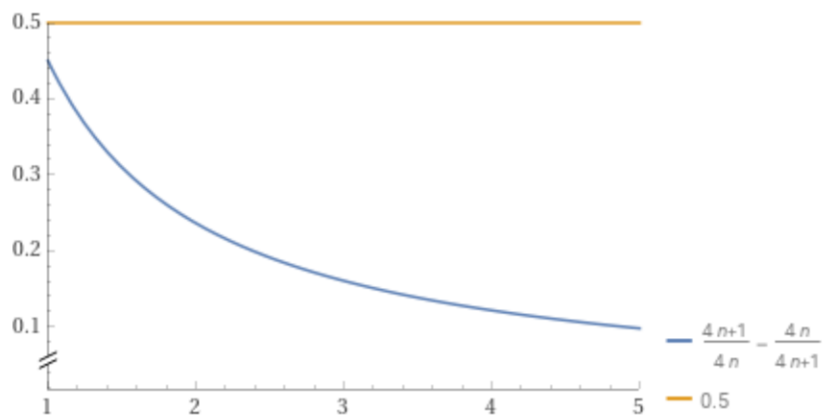
(d) If a sequence is monotonic, it ....
- must
- may or may not
- cannot

converge.

(e) If a sequence is bounded and monotonic, it ....
- must
- may or may not
- cannot

converge.

Solution:



Part 1:

(a) Yes, the sequence is bounded above by $\frac{1}{2}$.
Explanation:
$a_n = \frac{4n+1}{4n} - \frac{4n}{4n+1} = \frac{(4n+1)^2 - (4n)^2}{4n(4n+1)} = \frac{(8n+1)}{4n(4n+1)} < \frac{(8n+2)}{4n(4n+1)} = \frac{2}{4n} \leq \frac{1}{2}$.

(b) Yes, the sequence is bounded below by 0.
Explanation:
$a_n = \frac{4n+1}{4n} - \frac{4n}{4n+1} > \frac{4n}{4n} - \frac{4n}{4n+1} > 0$.

(c) The sequence is bounded, bounded below and bounded above (i.e A, B and C are the correct answers).

Part 2:
The sequence a_n is monotonic decreasing A (or monotonically decreasing).
Explanation:
Compute the terms of this sequence to get

$a_1 = \dfrac{9}{20} \approx 0.45, a_2 = \dfrac{17}{72} \approx 0.24, a_3 = \dfrac{25}{156} \approx 0.16, \ldots$
From this we can see that the sequence is monotonically decreasing.

For the rigorous proof, note that

$$a_n = \frac{(8n+1)}{4n(4n+1)}$$

and then compute $a_{n+1} - a_n$ to find that $a_{n+1} - a_n < 0$.

Part 3:

(a) The sequence a_n is convergent because it's bounded and monotonic.

(b) The limit of the sequence a_n is 0.

Proof:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{4n+1}{4n} - \frac{4n}{4n+1}\right) = \lim_{n\to\infty}\frac{4n+1}{4n} - \lim_{n\to\infty}\frac{4n}{4n+1} = 1 - 1 = 0.$$

Part 4:

(a) For n is even the sequence becomes $\left\{\dfrac{10n^2+1}{n^2+n}\right\}$ and $\dfrac{10n^2+1}{n^2+n} \leq \dfrac{10n^2+1}{n^2+1} < \dfrac{10n^2+10}{n^2+1} \leq 10$.

For n is odd the sequence becomes $\left\{-\dfrac{10n^2+1}{n^2+n}\right\}$ and $-\dfrac{10n^2+1}{n^2+n} \geq -\dfrac{10n^2+1}{n^2+1} > -\dfrac{10n^2+10}{n^2+1} \geq -10$. Thus the sequence is bounded above by 10 and bounded below by -10.

Therefore, the sequence is bounded but not monotonic because it changes sign.

For even $n = 2k$, $\dfrac{10n^2+1}{n^2+n} = \dfrac{10+1/n^2}{1+1/n}$, we have

$$\lim_{k\to\infty} a_{2k} = \lim_{n\to\infty}\frac{10+1/n^2}{1+1/n} = \frac{\lim_{n\to\infty} 10+1/n^2}{\lim_{n\to\infty} 1+1/n} = 10,$$

while for odd $n = 2k-1$, $-\dfrac{10n^2+1}{n^2+n} = -\dfrac{10+1/n^2}{1+1/n}$, we have

$$\lim_{k\to\infty} a_{2k-1} = \lim_{n\to\infty} -\frac{10+1/n^2}{1+1/n} = -\frac{\lim_{n\to\infty} 10+1/n^2}{\lim_{n\to\infty} 1+1/n} = -10,$$

The limits of even subsequence and odd subsequence do not match, therefore the sequence is divergent.

So the correct answers are B, D, and F.

(b) We denote the sequence by $a_n = \dfrac{10n^3+1}{n^2+n}$. Then for arbitrary n, we have

$$\begin{aligned} a_{n+1} - a_n &= \frac{10(n+1)^3+1}{(n+1)^2+(n+1)} - \frac{10n^3+1}{n^2+n} \\ &= \frac{10(n+1)^3+1}{(n+2)(n+1)} - \frac{10n^3+1}{(n+1)n} \\ &= \frac{[10(n+1)^3+1]n - (10n^3+1)(n+2)}{(n+2)(n+1)n} \\ &= \frac{10n^3+30n^2+10n-2}{(n+2)(n+1)n} \end{aligned}$$

The numerator $10n^3 + 30n^2 + 10n - 2 > 10n - 2 \geq 8 > 0$ for $n \geq 1$, so $a_{n+1} - a_n > 0$ for arbitrary $n \geq 1$, $n \in \mathbb{N}$, hence the sequence is monotonic increasing.

Note that the following inequality holds for $n \geq 1$:

$$\frac{10n^3+1}{n^2+n} > \frac{10n^3}{n^2+n} \geq \frac{10n^3}{n^2+n^2} = 5n$$

so the sequence is unbounded, hence it's divergent.

So the correct answers are A, C, D.

(c) If a sequence is bounded, it may or may not converge.

A bounded sequence may jump up and down indefinitely. Part 4 (a) is an example. The sequence $\left\{(-1)^n \dfrac{10n^2+1}{n^2+n}\right\}$ is bounded but not monotonic and not convergent.

(d) If a sequence is monotonic, it may or may not converge.

A sequence may monotonically tend to $+\infty$ or $-\infty$. Part 4 (b) is an example. The sequence $\left\{\dfrac{10n^3+1}{n^2+n}\right\}$ is monotonically increasing but unbounded, hence it is not convergent.

(e) If a sequence is bounded and monotonic, it must converge. [This is the monotonic convergence theorem.]