THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2023-2024 Term 1
Suggested Solutions of WeBWork Coursework 10

If you find any errors or typos, please email us at
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1. (1 point)
Use the Fundamental Theorem of Calculus to evaluate (if it exists)

$$\int_{-\pi}^{\pi} f(x)\,dx,$$

where

$$f(x) = \begin{cases} 5x & \text{if } -\pi \leq x \leq 0 \\ 5\sin(x) & \text{if } 0 < x \leq \pi \end{cases}$$

If the integral does not exist, type "DNE" as your answer.

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Solution:
$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{0} 5xdx + \int_0^{\pi} 5\sin(x)dx$
$= \left.\frac{5x^2}{2}\right|_{-\pi}^{0} - 5\cos(x)|_0^{\pi}$
$= 10 - \frac{5\pi^2}{2}$

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2. (1 point)
Evaluate the limit $\lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \frac{7j}{n^2}$.

$\lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \frac{7j}{n^2}$ = __________
Solution:
Let $S_n = \sum\limits_{j=1}^{n} \frac{7j}{n^2} = \frac{7}{n^2}\sum\limits_{j=1}^{n} j = \frac{7}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right) = \frac{7}{2} + \frac{7}{2n}$.
Then $\lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \frac{7j}{n^2} = \lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \frac{7}{2} + \frac{7}{2n} = \frac{7}{2}$.

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3. (1 point) The following sum

$$\sqrt{9 - \left(\frac{3}{n}\right)^2} \cdot \frac{3}{n} + \sqrt{9 - \left(\frac{6}{n}\right)^2} \cdot \frac{3}{n} + \ldots + \sqrt{9 - \left(\frac{3n}{n}\right)^2} \cdot \frac{3}{n}$$

is a right Riemann sum with n subintervals of equal length for the definite integral

$$\int_0^b f(x)\,dx$$

where b = __________
and $f(x)$ = __________

Solution: It is clear

$$\sqrt{9-\left(\frac{3}{n}\right)^2}\cdot\frac{3}{n}+\sqrt{9-\left(\frac{6}{n}\right)^2}\cdot\frac{3}{n}+\ldots+\sqrt{9-\left(\frac{3n}{n}\right)^2}\cdot\frac{3}{n}=\frac{3}{n}\sum_{i=1}^{n}\sqrt{9-(\frac{3i}{n})^2}$$

thus if one compare the form of Riemann sum, we know the interval $[0,3]$ is equally divided into n subintervals and the intergrand function is $\sqrt{9-x^2}$

thus $b=3, f(x)=\sqrt{9-x^2}$

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4. (1 point) Compute the following limit. Use INF to denote ∞ and MINF to denote $-\infty$.

$\lim_{x\to 0}\frac{x}{\int_x^{x^2}\sqrt[3]{27-6t^3}\,dt}=$ __________

Solution: Let $f(x)=\int_x^{x^2}\sqrt[3]{27-6t^3}\,dt$ then $f'(x)=2x\sqrt[3]{27-6x^6}-\sqrt[3]{27-6x^3}$

thus it is clear $\lim_{x\to 0}f'(x)=-3$.By using L'Hopital's rule, we know

$$\lim_{x\to 0}\frac{x}{\int_x^{x^2}\sqrt[3]{27-6t^3}\,dt}=\lim_{x\to 0}\frac{x}{f(x)}=\lim_{x\to 0}\frac{1}{f'(x)}=-\frac{1}{3}$$

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5. (1 point)

Evaluate the integral

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}}10t^3\cos(t^2)\,dt$$

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Solution: We use change of variable $t^2=x$ and integration by part

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}}10t^3\cos(t^2)\,dt=\int_{\sqrt{\pi/2}}^{\sqrt{\pi}}5t^2\cos(t^2)\,dt^2=\int_{\pi/2}^{\pi}5x\cos(x)dx$$

$$=5x\sin(x)|_{\pi/2}^{\pi}-\int_{\pi/2}^{\pi}5\sin(x)dx=-\frac{5\pi}{2}+5\cos(x)|_{\pi/2}^{\pi}=-5-\frac{5\pi}{2}$$

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6. (1 point)

Evaluate the integral

$$\int_0^4\left|\sqrt{x+2}-x\right|dx$$

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Solution: Note that by simple computation we know $\sqrt{x+2}\geq x$ if $x\in(0,2)$
and $\sqrt{x+2}\leq x$ if $\in(2,4)$ thus we have

$$\int_0^4\left|\sqrt{x+2}-x\right|dx=\int_0^2\sqrt{x+2}-x\,dx+\int_2^4 x-\sqrt{x+2}\,dx$$

$$=\frac{2}{3}(x+2)^{\frac{3}{2}}-\frac{x^2}{2}\Big|_0^2+\left(\frac{x^2}{2}-\frac{2}{3}(x+2)^{\frac{3}{2}}\Big|_2^4\right)=\frac{44}{3}-\frac{4\sqrt{2}}{3}-4\sqrt{6}$$

7. (1 point)

The interval $[0,4]$ is partitioned into n equal subintervals, and a number x_i is arbitrarily chosen in the i^{th} subinterval for each i. Then:

$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6x_i+7}{n}=$ ______________

Solution:

Solution:

Let's interpret the sum as a Riemann sum.

Recall that the Riemann sum for a function $f(x)$ on the interval $[0,4]$ has the form $\sum_{i=1}^{n} f(x_i)\frac{4}{n}$ since the length of each subinterval is $\Delta x=\frac{4}{n}$.

$\sum_{i=1}^{n}\frac{6x_i+7}{n}=\sum_{i=1}^{n}\frac{6x_i+7}{4}\cdot\frac{4}{n}$, therefore the given sum is the Riemann sum for $f(x)=\frac{6x+7}{4}$.

The limit of the Riemann sum as n approaches infinity is the integral of the function $f(x)$ from 0 to 4, thus

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{6x+7}{4}\cdot\frac{4}{n}=\int_0^4\frac{6x+7}{4}dx=\frac{1}{4}\int_0^4(6x+7)dx=\frac{1}{4}\left(3x^2+7x\right)\Big|_0^4=\frac{1}{4}\left(3\cdot4^2+7\cdot4\right)=19$$

8. (1 point)

(a) Consider the integral $\int_0^{\pi}\sin(5x)\,dx$. Which of the following expressions represents the integral as a limit of Riemann sums?

- A. $\lim_{n\to\infty}\sum_{i=1}^{n}\sin\left(\frac{5\pi i}{n}\right)$
- B. $\lim_{n\to\infty}\sum_{i=1}^{n}\sin\left(\frac{\pi i}{n}\right)$
- C. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin\left(\frac{5\pi i}{n}\right)$
- D. $\lim_{n\to\infty}\sum_{i=1}^{n}\sin\left(\pi+\frac{5\pi i}{n}\right)$
- E. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin\left(\frac{\pi i}{n}\right)$
- F. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin\left(\pi+\frac{5\pi i}{n}\right)$

(b) Limit in the correct answer to (a) = ______________

Solution: (a)let $f(x)=\sin(5x)$ then devide interval $[0,\pi]$ into n equal size subintervals, and use the language of Riemann sum, we have

$$\int_0^{\pi}\sin(5x)dx=\int_0^{\pi}f(x)dx=\frac{\pi}{n}\lim_{n\to\infty}\sum_{i=1}^{n}f(\frac{\pi i}{n})=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin(\frac{5\pi i}{n})$$

thus C is the right expression.

(b)

$$\int_0^{\pi}\sin(5x)dx=-\frac{\cos(5x)}{5}\Big|_0^{\pi}=\frac{2}{5}$$

9. (1 point)

Consider the integral $\int_2^6 \frac{x}{1+x^5}\,dx$. Which of the following expressions represents the integral as a limit of Riemann sums?

- A. $\lim_{n\to\infty}\sum_{i=1}^{n} \frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)^5}$
- B. $\lim_{n\to\infty}\sum_{i=1}^{n} \frac{4}{n}\frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)^5}$
- C. $\lim_{n\to\infty}\sum_{i=1}^{n} \frac{4}{n}\frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)}$
- D. $\lim_{n\to\infty}\sum_{i=1}^{n} \frac{2+\frac{6i}{n}}{1+\left(2+\frac{6i}{n}\right)^5}$
- E. $\lim_{n\to\infty}\sum_{i=1}^{n} \frac{6}{n}\frac{2+\frac{6i}{n}}{1+\left(2+\frac{6i}{n}\right)^5}$
- F. $\lim_{n\to\infty}\sum_{i=1}^{n} \frac{6}{n}\frac{2+\frac{6i}{n}}{1+\left(2+\frac{6i}{n}\right)}$

Solution: By dividing interval $[0,4]$ into n equal-size subintervals,we have

$$\int_2^6 \frac{x}{1+x^5}\,dx = \int_0^4 \frac{x+2}{1+(x+2)^5}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} \frac{4}{n}\frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)^5}$$

Thus B is the right expression.

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10. (1 point)

Let $F(x) = \int_4^x \frac{9}{\ln(2t)}\,dt$, for $x \geq 4$.

A. $F'(x) =$ ________________

B. On what interval or intervals is F increasing?

$x \in$ ________________

(Give your answer as an interval or a list of intervals, e.g., **(-infinity,8]** *or* **(1,5),(7,10)**, *or enter* **none***for no intervals.)*

C. On what interval or intervals is the graph of F concave up?

$x \in$ ________________

(Give your answer as an interval or a list of intervals, e.g., **(-infinity,8]** *or* **(1,5),(7,10)**, *or enter* **none***for no intervals.)*

Solution:

SOLUTION

A. $F'(x) = \frac{9}{\ln(2x)}$ by the Construction Theorem.

B. For $x \geq 4$, $F'(x) > 0$, so $F(x)$ is increasing for all $x \in [4,\infty)$.

C. $F''(x) = -9\frac{1}{x\ln(2x)^2} < 0$ for $x \geq 4$, so the graph of $F(x)$ is concave down for all $x \in [4,\infty)$ (and is concave up for no intervals).

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11. (1 point)

Suppose that $F(x) = \int_1^x f(t)\,dt$, where

$$f(t) = \int_1^{t^4} \frac{\sqrt{6+u^6}}{u}\,du.$$

Find $F''(2)$.

$F''(2)$ = ______________

Solution: since $F(x) = \int_1^x f(t)\,dt$ and

$$f(x) = \int_1^{x^4} \frac{\sqrt{6+u^6}}{u}\,du.$$

we have $F'(x) = f(x)$ and

$$f'(x) = 4x^3 \cdot \frac{\sqrt{6+x^{24}}}{x^4} = \frac{4\sqrt{6+x^{24}}}{x}$$

thus

$$F''(x) = f'(x) = \frac{4\sqrt{6+x^{24}}}{x}$$

$$F''(2) = f'(x) = \frac{4\sqrt{6+2^{24}}}{2} \approxeq 8192.00146484362$$