THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2024-2025 Term 1
Homework Assignment 1
Due Date: October 7, 2024 (Monday)

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website https://www.cuhk.edu.hk/policy/academichonesty/

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

____________________ ____________________

Signature Date

General Regulations

- All assignments will be submitted and graded on Gradescope. You can view your grades and submit regrade requests here as well. For submitting your PDF homework on Gradescope, here are a few tips.
- Late assignments will receive a grade of 0.
- For the declaration sheet:

  Either

  Use the attached file, sign and date the statement of Academic Honesty, convert it into a PDF, and submit it with your homework assignments via Gradescope.

  Or

  Write your name on the first page of your submitted homework, and simply write out the sentence "I have read the university regulations."
- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.

- Write your solutions on A4 white paper. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Failure to comply with these instructions will result in a 10-point deduction).

- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

1. Determine the limit of each of the following sequences, or show that the sequence diverges. You may make use of the limit laws and theorems covered in class.

   (a) $a_n = \dfrac{3n-5}{n+1} - \left(\dfrac{3}{5}\right)^n$ for $n \geq 1$.

   (b) $a_n = \sqrt{n}(\sqrt{n+5} - \sqrt{n})$ for $n \geq 1$.

   (c) $a_n = \dfrac{3^n}{n!}$ for $n \geq 1$.

   (d) $a_n = \dfrac{\sin(n^2)}{n}$ for $n \geq 1$.

   (e) $a_n = \dfrac{n}{n + n^{1/n}}$ for $n \geq 1$.

   (f) $a_n = \left(5 + \dfrac{4}{n^2}\right)^{1/3}$ for $n \geq 1$.

2. Consider the following bounded and increasing sequence:

$$\begin{cases} a_1 = \sqrt{3} \\ a_2 = \sqrt{3+\sqrt{3}} \\ a_3 = \sqrt{3+\sqrt{3+\sqrt{3}}} \\ \vdots \\ a_{n+1} = \sqrt{3+a_n} \\ \vdots \end{cases}$$

   Answer the following questions:

   (a) Show that the sequence converges and find its limit.

   (b) Answer the same question when 3 is replaced by an arbitrary integer $k \geq 2$.

3. For this problem, you may make use of the following mathematical result:

   Fact. Let a, r be real numbers, with $r \neq 1$. Let $\{S_n\}$ be the geometric series defined as follows:

$$S_n = \sum_{k=0}^{n} ar^k = a + ar + ar^2 + \cdots + ar^n, \quad n = 0, 1, 2, \ldots.$$

   Then, $S_n = a\left(\dfrac{1-r^{n+1}}{1-r}\right)$.

   (a) Verify that $\{S_n\}$ converges to $\dfrac{a}{1-r}$, whenever $|r| < 1$.

   (b) Use the result of Part (a) to find the limit of the sequence $\{a_n\}$, where

$$a_n = 1 + \frac{3}{4} + \frac{3}{4^2} + \cdots + \frac{3}{4^n}.$$

(c) Use the result of Part (a) to verify that the repeating decimal $1.777\cdots$, often written as $1.\dot{7}$, is equal to $\dfrac{16}{9}$.

4. A sequence $\{a_n\}$ is defined recursively by the following equations:

$$\begin{cases} a_1 = 1, \\ a_{n+1} = \sqrt{7 + 2a_n} \quad \text{for } n \geq 1. \end{cases}$$

Answer the following questions:

(a) Show that $\{a_n\}$ is bounded and monotonic and hence convergent.

(b) Find the limit of $\{a_n\}$.

5. Let $k > 0$ and a_1 be a positive number. Define a sequence $\{a_n\}$ by the relation:

$$a_{n+1} = \sqrt{k + a_n} \quad \text{for } n \geq 1.$$

Let α be the positive root of the equation:

$$x^2 - x - k = 0.$$

(a) Suppose $0 < a_1 < \alpha$. Show that the sequence $\{a_n\}$ is monotonic increasing and converges to α.

(b) Suppose $a_1 > \alpha$. Show that the sequence $\{a_n\}$ is monotonic decreasing and converges to α.

6. Given a sequence $\{a_n\}$ such that $a_1 > a_2 > 0$, and

$$a_{n+2} = \frac{1}{2}(a_{n+1} + a_n), \quad \text{for} \quad n = 1, 2, \cdots.$$

Answer the following questions:

(a) Show that for $n \geq 1$,

$$a_{n+2} - a_n = \frac{(-1)^n}{2^n}(a_1 - a_2)$$

and hence show that the sequence $\{a_1, a_3, a_5, \cdots\}$ is strictly decreasing and that the sequence $\{a_2, a_4, a_6, \cdots\}$ is strictly increasing.

(b) For any positive integers m and n, show that

$$a_{2m} < a_{2n-1}.$$

(c) Show that the two sequences $\{a_1, a_3, a_5, \cdots\}$ and $\{a_2, a_4, a_6, \cdots\}$ converge to the same limit k, where

$$k = \frac{1}{3}(a_1 + 2a_2).$$

7. For each of the given functions, f, find its natural domain, that is, the largest subset of $\mathbb{R}$ on which the expression defining f may be validly computed. Please express your answer in the form of a single interval, or a union of disjoint intervals. For example: $(-\infty, 2) \cup [5, 11)$.

   (a) (Optional) $f(x) = \dfrac{1}{2}\sqrt{4 - x^2}$.

   (b) $f(x) = \sqrt{\dfrac{x-3}{x+3}}$.

   (c) (Optional) $f(x) = \ln(3x^2 - 4x + 5)$.

   (d) $f(x) = \ln(\sqrt{x-4} + \sqrt{6-x})$.

   (e) (Optional) $f(x) = \sin^2 x + \cos^4 x$.

   (f) $f(x) = \dfrac{1}{1 + \cos x}$.

   (g) (Optional) $f(x) = 1 - |x - 1|$.

8. Determine whether the given function, f, is injective, surjective, bijective, or none of these. Explain clearly.

   (a) $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = 2x - 1$.

   (b) $f : \{x |\ x \neq 1\} \to \mathbb{R}$, where $f(x) = \dfrac{x^2 - 1}{x - 1}$.

   (c) $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \sqrt[3]{x}$.

   (d) $f : [-1, 1] \to [0, 4)$, where $f(x) = x^2$.

9. Determine whether the given function, f, is increasing, strictly increasing, decreasing, strictly decreasing, bounded, bounded above, or bounded below.

   (a) $f : [0, +\infty) \to \mathbb{R}$, where $f(x) = \dfrac{x}{x+1}$.

   (b) $f : \mathbb{R}^+ \to \mathbb{R}$, where $f(x) = \dfrac{1}{x}$.

10. Find whether the function is even, odd or neither:

   (a) (Optional) $f(x) = x^2 - |x|$

   (b) $f(x) = \log_2\left(x + \sqrt{x^2 + 1}\right)$

   (c) (Optional) $f(x) = x\left(\dfrac{a^x - 1}{a^x + 1}\right)$

   (d) $f(x) = \sin x + \cos x$

11. Without using l'Hôpital's rule, evaluate the limit, if it exists. If not, determine whether the one-sided limits exist (finite or infinite).

(a) $\lim_{x\to 3} \frac{x^3 - 3x^2 + 5x - 15}{x^2 - x - 12}$.

(b) (Optional) $\lim_{x\to 1/2} \frac{1 - 32x^5}{1 - 8x^3}$.

(c) (Optional) $\lim_{x\to 1} \frac{x - \sqrt{2 - x^2}}{2x - \sqrt{2 + 2x^2}}$.

(d) $\lim_{x\to 1} \frac{\sqrt{x^2 + 8} - \sqrt{10 - x^2}}{\sqrt{x^2 + 3} - \sqrt{5 - x^2}}$.

(e) (Optional) $\lim_{x\to 1} \left(\frac{2}{1 - x^2} + \frac{1}{x - 1} \right)$.

(f) $\lim_{x\to a} \left(\frac{2a}{x^2 - a^2} - \frac{1}{x - a} \right)$.

(g) $\lim_{x\to a} \left(\frac{x^m - a^m}{x^n - a^n} \right)$.

(h) $\lim_{x\to 1} \left(\frac{x - 1}{x^{1/4} - 1} \right)$.

(i) (Optional) $\lim_{x\to 0} \left(\frac{x^{7/10} + 3x^{4/3} + 2x}{x^{1/3} + 4x^{2/3} + 2x^{1/5}} \right)$.

12. Without using l'Hôpital's rule, evaluate the limit, if it exists. If not, determine whether the one-sided limits exist (finite or infinite).

(a) $\lim_{x\to \infty} \frac{\sqrt{x^4 + 1} - \sqrt{x^4 - 1}}{x}$.

(b) $\lim_{x\to \infty} \frac{\sqrt{3x^2 - 1} - \sqrt{2x^2 + 1}}{4x + 3}$.

(c) $\lim_{x\to \pi/2} \left(\frac{1 - \sin^3 x}{1 - \sin^2 x} \right)$.

(d) $\lim_{x\to \pi/4} \left(\frac{\sin 2x - (1 + \cos(2x))}{\cos x - \sin x} \right)$.

(e) $\lim_{x\to \pi/4} \frac{\sqrt{2} - \cos x - \sin x}{(4x - \pi)^2}$.

(f) $\lim_{x\to 0} \frac{\sin 7x - \sin x}{\sin 6x}$.

(g) $\lim_{x\to 0} \left(\frac{1 + x}{1 - x} \right)^{1/x}$.

(h) $\lim_{x\to 0} \left(\frac{\sqrt{x + 1} - 1}{\ln(1 + x)} \right)$.

(i) $\lim_{x\to 0}\left(\dfrac{e^{ax}-e^{a}}{x}\right)$ where a is a constant.

(j) $\lim_{x\to 1}\dfrac{1-x(1+|1-x|)}{|1-x|}\cos\left(\dfrac{1}{1-x}\right)$.

13. Evaluate the following limits.

(a) $\lim_{x\to 0^-} x\left|\sin\frac{1}{x}\right|$

(b) $\lim_{x\to +\infty}\dfrac{\sin(\tan x)+\tan(\sin x)}{x+1}$

14. Evaluate the following limits.

(a) $\lim_{x\to \pi/2}\dfrac{\cot x-\cos x}{(\pi-2x)^3}$

(b) $\lim_{x\to 0}\dfrac{\tan^2 x}{\sin(x^2)}$

(c) $\lim_{x\to 0}\dfrac{1-\cos x\sqrt{\cos 2x}}{x^2}$