Supplementary note 3: Haar and Walsh function

<u>Definition 3.1</u> : (Haar functions)

The Haar functions are defined recursively as follows:

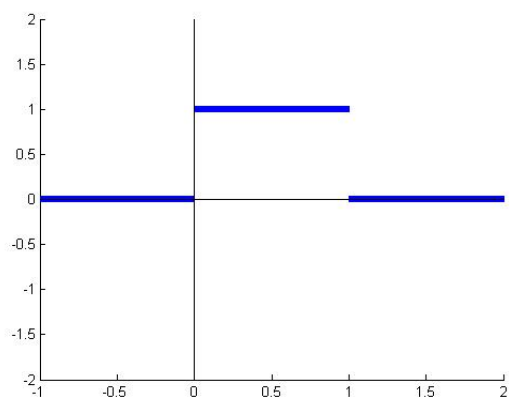
$$H_0(t) \equiv 1 \quad \text{for } 0 \le t < 1$$

$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t < 1 \end{cases}$$

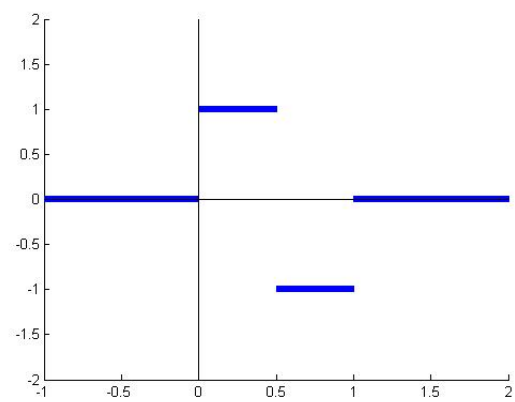
$$H_{2^p+n}(t) \equiv \begin{cases} \sqrt{2}^p & \text{if } \dfrac{n}{2^p} \le t < \dfrac{n+0.5}{2^p} \\ -\sqrt{2}^p & \text{if } \dfrac{n+0.5}{2^p} \le t < \dfrac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p = 1, 2, \cdots; n = 0, 1, 2, \cdots, 2^p - 1$
($H_{2^p+n}(t)$ is compactly supported a smaller region if p is bigger)

H_0



H_1



<u>Definition 3.2</u> : (Walsh functions)

The Walsh functions are defined recursively as follows:

$$W_{2j+q}(t) \equiv (-1)^{\lfloor \frac{j}{2} \rfloor + q} \left\{ W_j(2t) + (-1)^{j+q} W_j(2t-1) \right\}$$

where $\lfloor \frac{j}{2} \rfloor$ = smallest integer smaller or equal to $\frac{j}{2}$; $q = 0$ or 1; $j = 0, 1, 2, \cdots$ and

$$W_0(t) \equiv \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

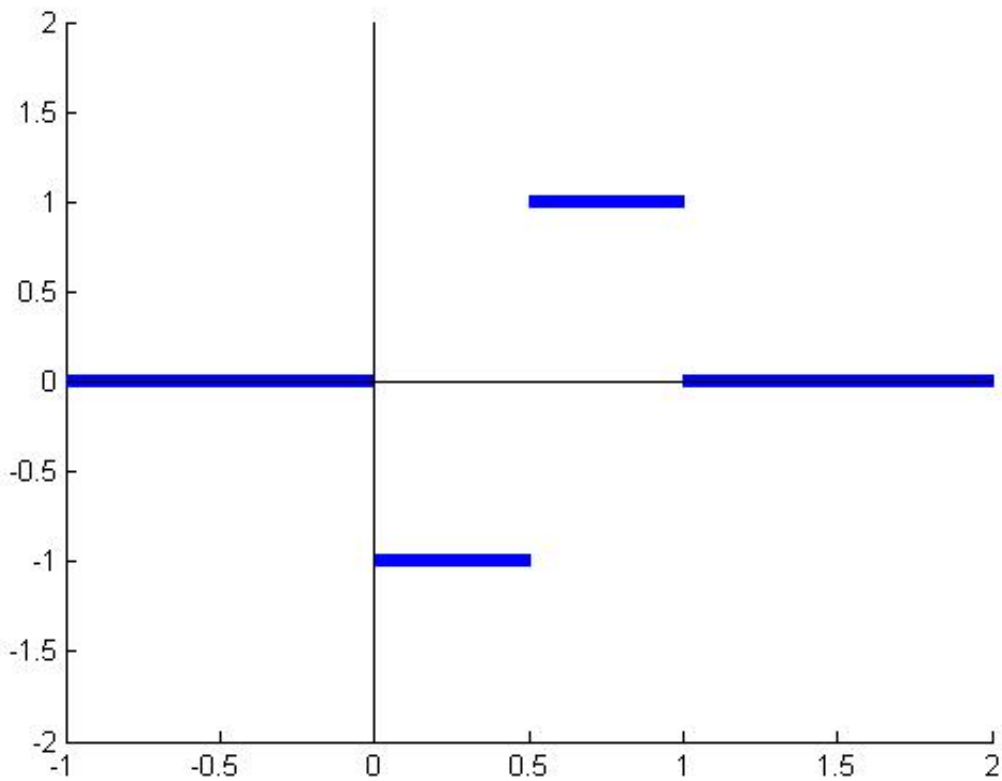
<u>Example 3.1</u>: Compute $W_1(t)$.

Put $j = 0, q = 1$. Then:

$$\begin{aligned} W_1(t) &\equiv (-1)^{\lfloor 0 \rfloor + 1} \left\{ W_0(2t) + (-1)^1 W_0(2t-1) \right\} \\ &= (-1) \left\{ W_0(2t) + (-1)^1 W_0(2t-1) \right\} \end{aligned}$$

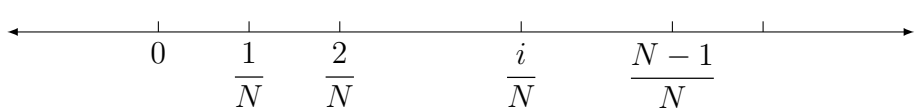
Check that for $0 \leq t < \frac{1}{2},\ W_0(2t) = 1,\ W_0(2t-1) = 0 \Rightarrow W_1(t) = -1$

For $\frac{1}{2} \leq t < 1,\ W_0(2t) = 0,\ W_0(2t-1) = 1 \Rightarrow W_1(t) = 1$



<u>Definition 3.3</u>: (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is performed as follows. Divide t variable by the size of matrix $= N$. That's:



Let $H(k,i) \equiv H_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \cdots, N-1$

We obtain the Haar Transform matrix:

$$\tilde{H} \equiv \frac{1}{\sqrt{N}} H \qquad \text{where } H \equiv (H(k,i))_{0 \leq k, 1 \leq N-1}$$

(Then $\tilde{H}^T \tilde{H} = I$)

The Haar Transform of $f \in M_{n \times n}$ is defined as:

$$g = \tilde{H} f \tilde{H}^T$$

Definition 3.4: (Discrete Walsh Transform)

The Walsh Transform of a $N \times N$ image is defined similarly as Haar Transform.

Define $W(k,i) \equiv W_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \cdots, N-1$

Then, the Walsh Transform matrix:

$$\tilde{W} \equiv \frac{1}{\sqrt{N}} W \qquad \text{where } W \equiv (W(k,i))_{0 \le k, 1 \le N-1}$$

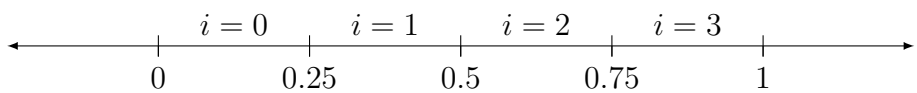
(Then $\tilde{W}^T \tilde{W} = I$)

The Walsh Transform of $f \in M_{n \times n}$ is defined as:

$$g = \tilde{W} f \tilde{W}^T$$

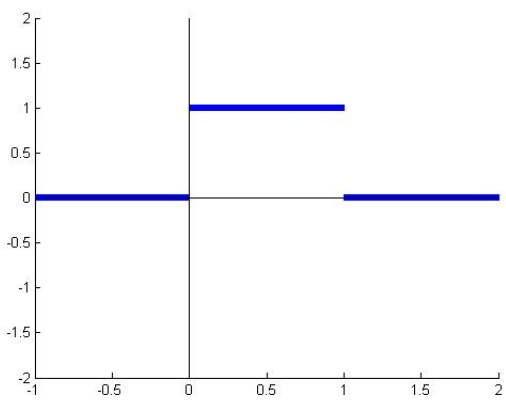
Example 3.2: Compute the Haar Transform matrix for a 4×4 image.

Solution: Divide $[0, 1]$ into 4 portions:

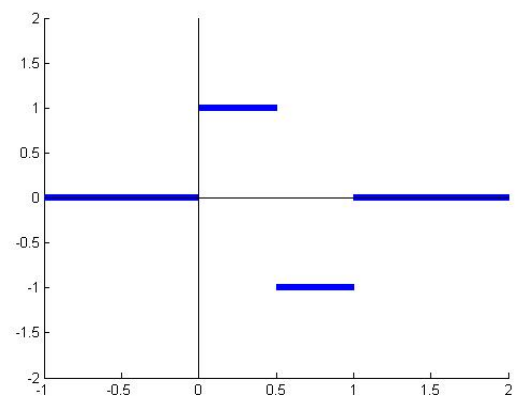


Check that:

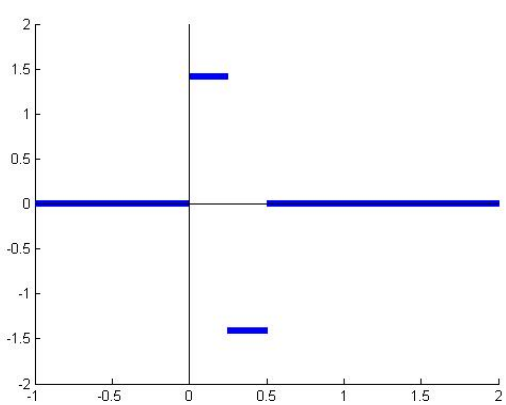
H_0



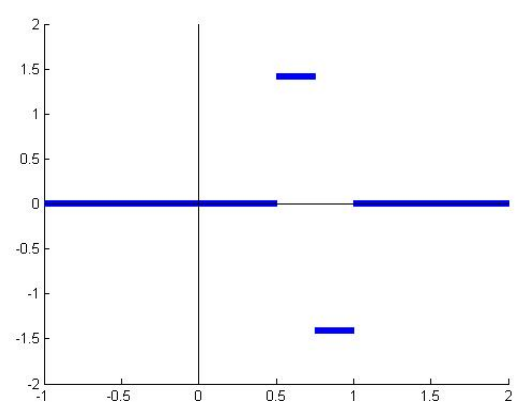
H_1



H_3



H_2



We get that:

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \quad \text{and} \quad \tilde{H} = \frac{1}{\sqrt{4}} H = \frac{1}{2} H$$

Easy to check that $\tilde{H}^T \tilde{H} = I$.

Example 3.3: Compute the Haar Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

$$g = \tilde{H} f \tilde{H}^T = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Example 3.4: Suppose g in Example 3.3 is changed to:

$$g = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Reconstruct the original image.

Solution:

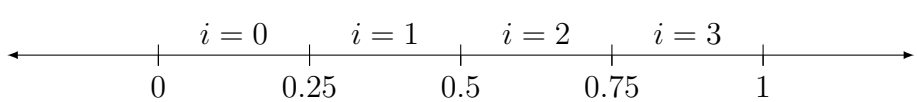
$$f = \tilde{H}^T f \tilde{H} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Remark:

- Haar Transform usually produces coefficient matrix with more zeroes (compression)
- Error in the coefficient matrix causes localized error in the reconstructed image (Assign detail of accuracy in compression)

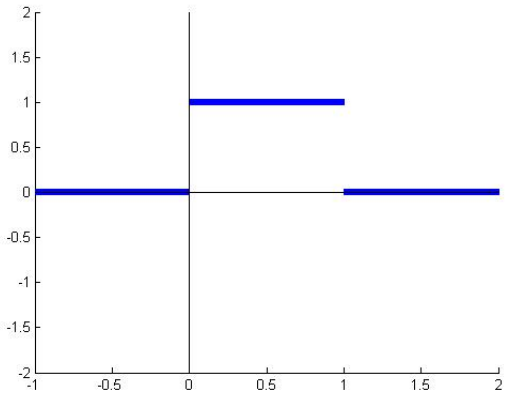
Example 3.5: Compute the Walsh Transform matrix for a 4×4 image.

Solution: Again, divide $[0, 1]$ into 4 portions:

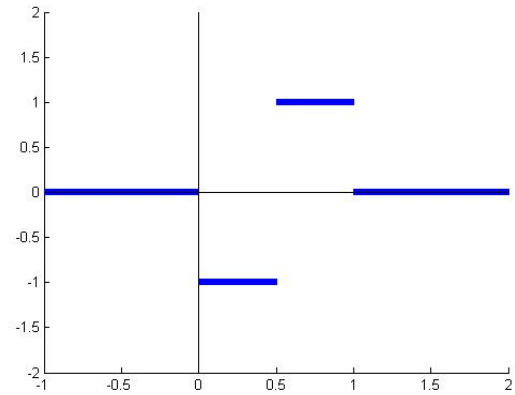


Check that:

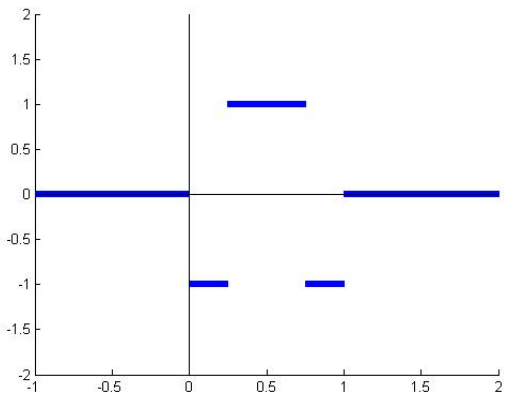
W_0



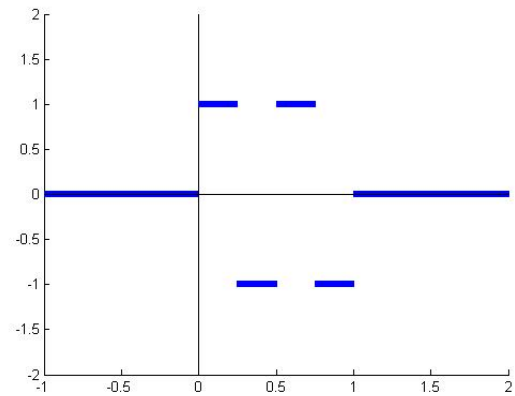
W_1



W_3



W_2



So,

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \tilde{W} = \frac{1}{\sqrt{4}} W = \frac{1}{2} W$$

$(\tilde{H}^T \tilde{H} = I)$

<u>Example 3.6</u>: Compute the Walsh Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

<u>Solution</u>:

$$g = \tilde{W} f \tilde{W}^T = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(Many zeroes $\Rightarrow$ compression)

Remark:

- The idea of Haar / Walsh Transform is to transform an image to "transformed image" with much more zeroes.
- The coefficient in the "transformed" image tells us information of frequency of image intensity changes.

Another way to define Walsh function

Definition 3.5 : (Rademacher functions)

A Rademacher function of order n ($n \neq 0$) is defined as:

$$R_n(t) \equiv \text{sign}[\sin(2^n \pi t)] \qquad \text{for } 0 \leq t \leq 1$$

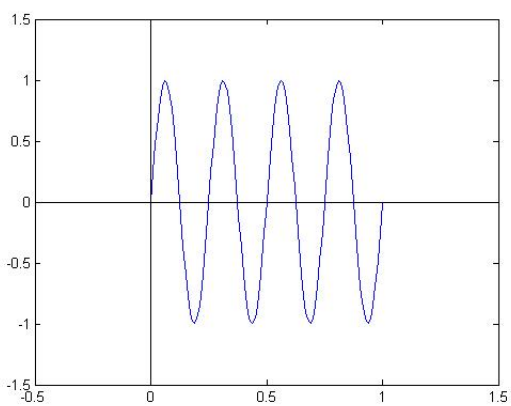
For $n = 0$, $\qquad R_0(t) \equiv 1 \qquad$ for $0 \leq t \leq 1$.
(with the continuation of the function from the left)

Let $N = b_{m+1}2^m + b_m 2^{m-1} + \cdots + b_1 2^0$. Then, the R-Walsh function $\tilde{W}_N$ is given by:

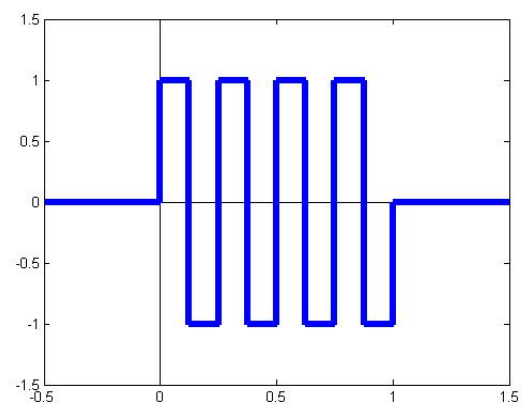
$$\tilde{W}_N = \prod_{i=1, b_i \neq 0}^{m+1} R_i(t)$$

Example 3.7 : Compute R-Walsh function $\tilde{W}_4$ using Rademacher function.

Consider $\sin(8\pi t)$:



Therefore, $R_3(t) =$



As $4 = \underbrace{1}_{b_3} \cdot 2^2 + \underbrace{0}_{b_2} \cdot 2^1 + \underbrace{0}_{b_1} \cdot 2^0$, we have

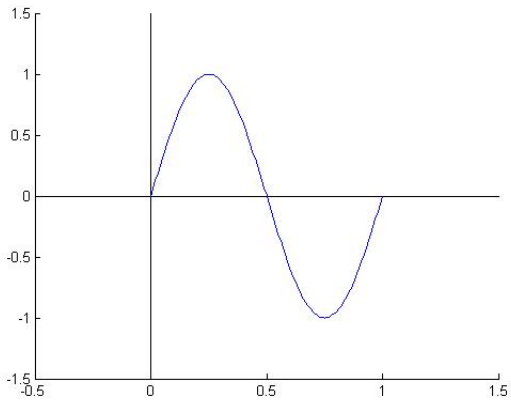
$$\tilde{W}_4 = \prod_{i=1, b_i \neq 0}^{3} R_i(t) = R_3(t)$$

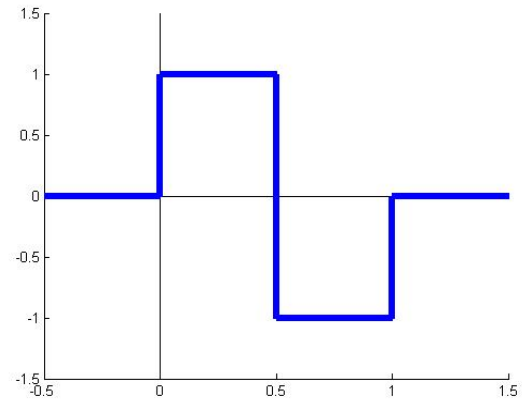
$(W_{2j+q}(t) \equiv (-1)^{\lfloor j/2 \rfloor + q}\{W_j(2t) + (-1)^{j+q} W_j(2t-1)\})$

For $W_3(t)$: As $3 = \underbrace{1}_{b_2} \cdot 2^1 + \underbrace{1}_{b_1} \cdot 2^0$, we have

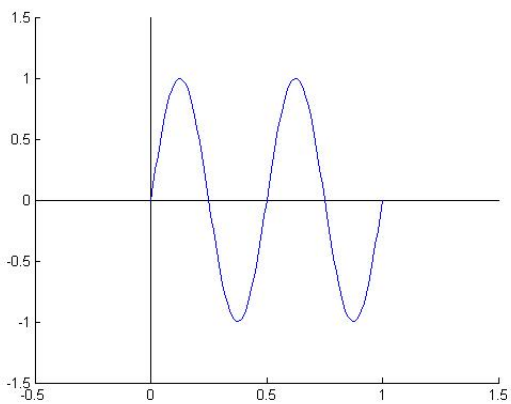
$$\tilde{W}_3(t) = \prod_{i=1, b_i \neq 0}^{2} R_i(t) = R_1(t) R_2(t)$$

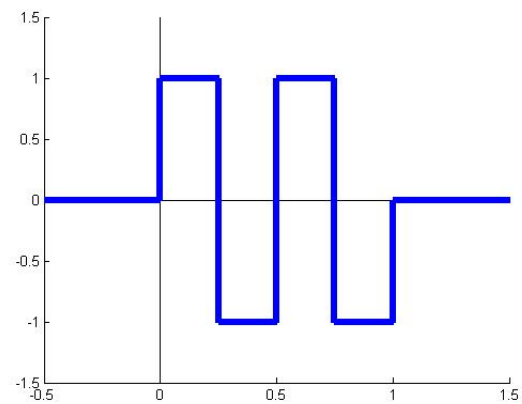
$R_1(t)$:



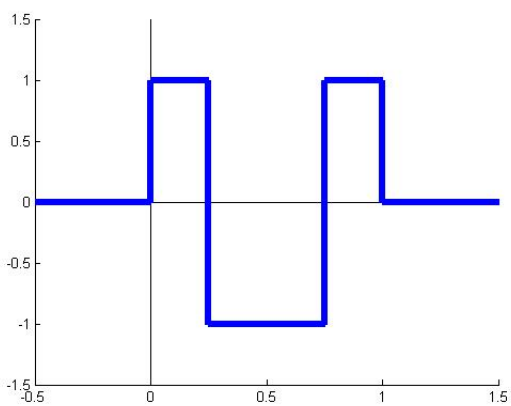


$R_2(t)$:





Therefore, $\tilde{W}_3(t)$:



Relationship between Walsh functions and R-Walsh functions

$W_0(t) = \tilde{W}_0(t), W_1(t) = -\tilde{W}_1(t), W_2(t) = -\tilde{W}_3(t), W_3(t) = \tilde{W}_2(t),$
$W_4(t) = \tilde{W}_6(t), W_5(t) = -\tilde{W}_7(t), W_6(t) = -\tilde{W}_5(t), W_7(t) = \tilde{W}_4(t)$

How to determine R-Walsh $\tilde{W}_i(t)$ associate to Walsh $W_j(t)$

Idea: Write j as:

$$j = b_{m+i}2^m + b_m 2^{m-1} + \cdots + b_1 2^0$$

The binary representation of j is :

$$b_{m+1}b_m \cdots b_1$$

Then, i is given by:

$$c_{m+1}c_m \cdots c_1$$

where

$$c_{m+1} \equiv b_{m+1} \pmod 2, \quad c_k \equiv (b_{k+1} + b_k) \pmod 2$$

The sign is determined from $W_j(0)$.

Example 3.8 Consider $W_7(t)$.

Check that $W_7(t) > 0$.
Now, $7 = 2^2 + 2^1 + 2^0$ so binary representation of 7 is 111.
Therefore, $j = 100$ (binary) $= 4$
Thus, $W_7(t) = \tilde{W}_4(t)$.

What do the elementary images under the Haar and Wash Trasnform look like?

Please refer to the powerpoint Lecture 6 page 7-12.

Properties of each elementary image under the Haar Trasnform

It captures different levels of details with different levels of resolution in the horizontal and vertical directions. Please refer to the powerpoint Lecture 7 p.2-3.