GEOMETRY OF THE MAURER-CARTAN EQUATION NEAR DEGENERATE CALABI-YAU VARIETIES

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Abstract. In this paper, we construct a differential graded Batalin-Vilkovisky (dgBV) algebra $PV^{*,*}(X)$ associated to a possibly degenerate Calabi-Yau variety $X$ equipped with local deformation data. This gives a singular version of the (extended) Kodaira-Spencer dgLa in the Calabi-Yau setting. We work out an abstract algebraic framework and use a local-to-global Čech-de Rham–type gluing construction. Under the Hodge-to-de Rham degeneracy assumption as well as a local assumption that guarantees freeness of the Hodge bundle and applying standard techniques in BV algebras [35, 32, 51], we prove an unobstructedness theorem, which can be regarded as a singular version of the famous Bogomolov-Tian-Todorov theorem [3, 52, 53], in the spirit of the work of Katzarkov-Kontsevich-Pantev [35, 32]. In particular, we recover the existence of smoothing for both log smooth Calabi-Yau varieties (studied by Friedman [16] and Kawamata-Namikawa [34]) and maximally degenerate Calabi-Yau varieties (studied by Kontsevich-Soibelman [37] and Gross-Siebert [24]). We also demonstrate how our construction can be applied to produce a logarithmic Frobenius manifold structure on a formal neighborhood of the extended moduli space using the technique of Barannikov-Kontsevich [2, 1].

1. Introduction

1.1. Background. Deformation theory of Calabi-Yau manifolds $X$ plays an important role in algebraic geometry, mirror symmetry and mathematical physics. Two major approaches to deformation theory are the Čech approach [48] and the Kodaira-Spencer approach [44].

In the Calabi-Yau case, the advantage of the Kodaira-Spencer differential graded Lie algebra (abbrev. dgLa) $\Omega^{0,*}(X,T_X^{1,0})$ is that it can be upgraded to a differential graded Batalin-Vilkovisky (abbrev. dgBV) algebra $\Omega^{0,*}(X,\wedge^*T_X^{1,0})$ which is equipped with a BV operator $\Delta$ constructed from the holomorphic volume form $\omega$ that satisfies the famous Bogomolov-Tian-Todorov Lemma [3, 52, 53]:

\[
(-1)^{|v|}[v, w] := \Delta(v \wedge w) - \Delta(v) \wedge w - (-1)^{|v|}v \wedge \Delta(w).
\]

This naturally gives unobstructedness of deformations, or equivalently, local smoothness of the moduli space of compact Calabi-Yau manifolds. The holomorphic volume form not only yields the unobstructedness result but also gives a local structure of Frobenius manifold on the extended moduli space by the work of Barannikov-Kontsevich [2, 1].

By degenerating a family of Calabi-Yau manifolds $\{X_q\}$ to limit points in the compactified moduli space, one would obtain a degenerate Calabi-Yau variety which is usually equipped with a natural log structure in the sense of Fontaine-Illusie and K. Kato [30]. Smoothing of degenerate log Calabi-Yau varieties via log deformations is thus fundamental in understanding the global moduli space of Calabi-Yau manifolds. Two celebrated results in this direction are, the unobstructedness of smoothing of log smooth Calabi-Yau varieties studied by Friedman [16] and Kawamata-Namikawa [34] and that of maximally degenerate Calabi-Yau varieties studied by Kontsevich-Soibelman [37] and Gross-Siebert [24], where two entirely different methods, namely, the $T^1$-lifting technique in the former case and the scattering diagram technique in the latter case, are used respectively.
The philosophy that any deformation problem should be governed by the Maurer-Cartan equation in a dgLa (or $L_\infty$-algebra) is well-known, after pioneering works of Quillen, Deligne and Drinfeld (see e.g. [36]). A lot has been done in this direction; see e.g. [13, 12, 14, 15, 26, 42, 41, 43]. Employing this framework, Bogomolov-Tian-Todorov–type theorems have been formulated and proven in various settings, e.g. in [35, 32, 33, 29, 27, 28]. In many cases, the existence of an underlying dgLa and the associated Maurer-Cartan equation provides a tool to solve the geometric deformation problem using algebraic techniques.

On the other hand, our earlier work [5] and [6], where we attempted to realize Fukaya’s asymptotic approach [18], showed how asymptotic expansions of Maurer-Cartan solutions for the Kodaira-Spencer dgLa $\Omega^0(X, \wedge^* T_X^1, 0)$, where $X$ is a torus bundle over a base $B$, give rise to scattering diagrams and tropical disk counts as the torus fibers shrink. Scattering diagrams are combinatorial structures used by Kontsevich-Soibelman [37] and Gross-Siebert [24] as quantum-corrected gluing data to solve the important reconstruction problem in mirror symmetry. What we showed in [5, 6] is that scattering diagrams are indeed hidden in and encoded by Maurer-Cartan solutions. Therefore, it is natural to ask whether a dgLa (or better, a dgBV algebra) which governs the smoothing of (maximally) degenerate Calabi-Yau varieties can be constructed. One major difficulty arises from non-trivial topology change in such a degeneration, which is dictated by (the residue of) the Gauss-Manin connection. Simply taking the trivial product $\Omega^0(X, \wedge^* T_X^1, 0)[[q]]$ cannot record such changes in topology and hence does not lead to smoothing, in sharp contrast to deformations of smooth manifolds where the topology is unchanged.

In this paper, we construct the desired dgBV algebra $PV^{*,*}(X)$ associated to a degenerate Calabi-Yau variety $X$, which plays the same role as the Kodaira-Spencer dgBV algebra $\Omega^0(X, \wedge^* T_X^1, 0)[[q]]$ in the smooth case. We also prove that the classical part of the Maurer-Cartan equation (see Definition 5.10) indeed governs geometric smoothings of $X$. More precisely, motivated by the divisorial log deformation theory studied by Gross-Siebert in [22, 23], we develop an abstract algebraic framework and build a dgBV algebra $PV^{*,*}(X)$ from local deformation (or thickening) data attached to $X$ via a local-to-global Čech-de Rham–gluing construction. Such a simplicial construction of $PV^{*,*}(X)$ encodes the aforementioned nontrivial topology change because the local thickenings are not locally trivial, and also allows us to directly link the smoothing of degenerate Calabi-Yau varieties with the Hodge theory developed in [2, 1, 35, 32, 40]. This will be important in subsequent study of degenerate Calabi-Yau varieties and the higher genus $B$-model [7] via dgBV algebras.

We give two main applications, namely, we prove an unobstructedness or Bogomolov-Tian-Todorov–type theorem using the purely algebraic techniques in [35, 32, 51] and, under certain assumptions, construct a logarithmic Frobenius manifold structure on a formal neighborhood in the extended moduli space using the technique of Barannikov-Kontsevich [2, 1]. We also demonstrate how our framework can be applied to the cases studied by Friedman [16] and Kawamata-Namikawa [34] as well as Kontsevich-Soibelman [37] and Gross-Siebert [22, 23, 24].

1.2. Main results.

1.2.1. A singular version of the Kodaira-Spencer dgBV algebra. To begin with, let $Q$ be a monoid and $\mathbb{C}[Q]$ be the universal coefficient ring equipped with the monomial ideal $m = \langle Q \setminus \{0\} \rangle$. Consider a complex analytic space $(X, \mathcal{O}_X)$ and a covering $\mathcal{V} = \{V_\alpha\}$ by Stein open subsets, together with local deformation (or thickening) data on each $V_\alpha$, which consist of a $k$th-order coherent sheaf of BV algebras $(k\mathcal{C}_\alpha^*, \wedge, k\Delta_\alpha)$ over $kR := \mathbb{C}[Q]/m^{k+1}$ acting on a $k$th-order coherent sheaf of de Rham modules $(k\mathcal{K}_\alpha^*, \wedge, k\partial_\alpha)$ (see Definitions 2.13 & 2.17 in [2]) for each $k \in \mathbb{Z}_{\geq 0}$. 
Also fix another covering $\mathcal{U} = \{U_i\}_{i \in \mathbb{N}}$ by Stein open subsets together with higher order local patching data, which are isomorphisms $k^\psi_{\alpha, i} : kG^\ast_{\alpha}|_{U_i} \to kG^\ast_{\beta}|_{U_i}$ (Definition 2.15). In geometric situations, these patching isomorphisms come from local uniqueness of the local thickening data, and they are not compatible on the nose but rather differ by a collection of local sections from $kG^\ast_{\alpha}$ (see Definition 2.15).

The ordinary Čech approach to deformation theory is done by solving for compatible gluings $k^g_{\alpha, \beta} : kG^\ast_{\alpha} \to kG^\ast_{\beta}$ and understanding the obstructions in doing so. In our approach, instead of gluing directly, we first take a dg resolution of the sheaf $kG^\ast_{\alpha}$, given as a sheaf of dgBV algebras $kPV^\ast_{\alpha}$ defined by the Thom-Whitney construction 55 10. Then we solve for compatible gluings $k^g_{\alpha, \beta} : kPV^\ast_{\alpha} \to kPV^\ast_{\beta}$ which satisfy the cocycle condition and are compatible for different orders $k$. This is plausible since the local sheaves of dgBV algebras $kPV^\ast_{\alpha}$ are more topological than the sheaves $kG^\ast_{\alpha}$ themselves.

The key point is that we only need to construct an almost dgBV algebra because then known algebraic techniques can be applied to prove unobstructedness. Our first main result is the following:

**Theorem 1.1** (=Theorem 3.18 + Proposition 3.24 + Theorem 3.34). There exists an almost differential graded Batalin-Vilkovisky (abbrev. dgBV) algebra of polyvector fields 1

$$\langle PV^{\ast, \ast}(X), \bar{\partial}, \Delta, \wedge \rangle$$

over $\mathbb{C}[[Q]]$ meaning that it satisfies all the dgBV algebra identities except that the equations $\partial^2 = \bar{\partial}\Delta + \Delta\bar{\partial} = 0$ hold only in the 0-th order, i.e. when restricted to $0PV^{\ast, \ast}(X) = PV^{\ast, \ast}(X) \otimes \mathbb{C}[[Q]] / (\mathbb{C}[[Q]] / (Q \setminus \{0\}))$.

In geometric situations such as the log smooth case considered by Friedman 16 and Kawamata-Namikawa 34 and the maximally degenerate case considered by Kontsevich-Soibelman 37 and Gross-Siebert 29, the above theorem produces a singular version of the Kodaira-Spencer dgBV algebra governing the smoothing of the singular Calabi-Yau varieties.

1.2.2. Unobstructedness. Now consider the extended Maurer-Cartan equation

$$\bar{\partial} + t\Delta + [\varphi, \cdot] \] = 0 \tag{1.2}$$

for $\varphi \in PV^{\ast, \ast}(X)[[t]]$ Using standard techniques in the theory of BV algebras 35 32 51, we prove an unobstructedness theorem under the Hodge-to-de Rham degeneracy Assumption 5.4 that $H^*(0PV(X)[[t]], \bar{\partial} + t\Delta)$ is a free $\mathbb{C}[[t]]$ module and a local Assumption 4.15 that guarantees freeness of the Hodge bundle (Lemma 4.17).

**Theorem 1.2** (=Theorem 5.5 + Lemma 5.11 + Proposition 5.13). Under Assumptions 4.15 and 5.4, the extended Maurer-Cartan equation (1.2) can be solved order by order for $\varphi \in PV^{\ast, \ast}(X) \otimes \mathbb{C}[[t]]$. In particular, under the same assumptions, geometric deformations (i.e. smoothing) of $X$ over $Spf(\mathbb{C}[[Q]])$ are unobstructed.

This can be regarded as a singular version of the famous Bogomolov-Tian-Todorov (BTT) theorem 3 52 53 for smoothing degenerate Calabi-Yau varieties, formulated in the spirit of the framework set up by Katzarkov-Kontsevich-Pantev 35 32.

Assumption 4.15 depends on how good the models of the local smoothing are. So in the Calabi-Yau setting, Theorem 1.2 essentially reduces smoothability of $X$ to the Hodge-to-de Rham degeneracy (i.e. Assumption 5.4).

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1 Note that here $\bar{\partial}$ is not the Dolbeault operator on $X$, while it plays the role of $\bar{\partial}$ in the smooth case and so we write it as $\bar{\partial}$.

2 Here $t$ is the descendant parameter as in 11.
1.2.3. Logarithmic Frobenius structures. Another advantage of having a dgBV algebra of polyvector fields $PV^{*,*}(X)$ associated to $X$ is that it made possible a direct construction of a semi-infinite log variation of Hodge structures (abbrev. $\frac{\partial}{\partial t}$-LVHS; see [6.1] for its definition) by adapting the techniques developed by Barannikov-Kontsevich [2] combined with the extra data of a mixed Hodge structure on $\mathbb{H}^*(X,\Omega^*)$.

From the log structure on $X$, one can construct the residue action $N_\nu$ of the Gauss-Manin connection acting on the cohomology $\mathbb{H}^*(X,\Omega^*)$ for each constant vector field on $\text{Spec}(\mathbb{C}[Q])$ given by $\nu \in (Q^op)^{\vee} \otimes_{\mathbb{Z}} \mathbb{R}$. We assume that there is a weight filtration of the form
\[
\{0\} \subset W_{\leq 0} \subset \cdots \subset W_{\leq r} \subset \cdots \subset W_d = \mathbb{H}^*(X,\Omega^*)
\]
indexed by half-integer weights $r \in \frac{1}{2}\mathbb{Z}$ which is opposite to the Hodge filtration $\mathcal{F}^*(\mathbb{H}^*(X,\Omega^*))$ (introduced in Definition 2.10) in the sense of Assumption 6.12. We further assume the existence of a trace map $\text{tr} : \mathbb{H}^*(X,\Omega^*) \rightarrow \mathbb{C}$ (playing the role of integration in the smooth case) such that the associated pairing $0^p(\alpha,\beta) := \text{tr}(\alpha \wedge \beta)$ is non-degenerate as described in Assumption 6.17.

Our last main result shows that the above data can be incorporated with our dgBV algebra $PV^{*,*}(X)$ to obtain a logarithmic Frobenius manifold structure on a formal neighborhood of the extended complex moduli space near $X$. To do that we suitably enlarge our coefficient ring $\mathbb{C}[Q]$ to include all the extended moduli parameters (which is again denoted as $\mathbb{C}[Q]$ by abuse of notations).

Given a versal solution $\varphi$ to the Maurer-Cartan equation (1.2), we define the semi-infinite Hodge bundle $\mathcal{H}_+$ over $\text{Spf}(\mathbb{C}[Q][[t]])$ to be the $\mathbb{C}[Q][[t]]$ module $\mathcal{H}_+ := \lim_{\rightarrow} H^*(\mathbb{C}[Q][[t]],\bar{\partial} + t \Delta + [\varphi,\cdot])$. Over the central fiber $\text{Spec}(\mathbb{C}) \hookrightarrow \text{Spf}(\mathbb{C}[[Q]])$, the above weight filtration can be used to define a $\mathbb{C}[t^{-1}]$-submodule of the form $0^\mathcal{H}_- := \bigoplus_r W_{\leq r} \mathbb{C}[t^{-1}]t^{-r+d-2} \subset \mathbb{H}^*(X,\Omega^*)[[t^\pm 1]]$ of $\mathbb{H}^*(X,\Omega^*)[[t^\pm 1]]$, which is opposite to the Hodge bundle over $\text{Spec}(\mathbb{C})$. Flatness of the Gauss-Manin connection then allows us to extend the opposite filtration $0^\mathcal{H}_-$ to the whole moduli $\text{Spf}(\mathbb{C}[[Q]])$ as a $\mathbb{C}[Q][[t^{-1}]]$-submodule preserved by it. Furthermore, the pairing $0^p$ can be extended to a flat pairing on $\langle \cdot,\cdot \rangle : \mathcal{H}_+ \times \mathcal{H}_+ \rightarrow \mathbb{C}[[Q][t]]$ with respect to the Gauss-Manin connection.

**Theorem 1.3** (=Theorem 6.28). The triple $(\mathcal{H}_+,\nabla,\langle \cdot,\cdot \rangle)$ is a semi-infinite log variation of Hodge structures in the sense of Definition 6.2. Under Assumption 6.12, we can construct an opposite filtration $\mathcal{H}_-$ to the Hodge bundle $\mathcal{H}_+$ in the sense of Definition 6.5. Furthermore, there exists a versal solution to the Maurer-Cartan equation (1.2) such that $e^{v/t}$ gives a miniversal section of the Hodge bundle in the sense of 6.26. As a consequence, there is a structure of logarithmic Frobenius manifold on the formal neighborhood $\text{Spf}(\mathbb{C}[[Q]])$ of $X$ in the extended moduli space constructed from these data under Assumption 6.17.

1.2.4. Geometric applications. In the final two sections, we apply our results to the geometric settings studied by Friedman [16] and Kawamata-Namikawa [34] (the log smooth case) and Kontsevich-Soibelman [37] and Gross-Siebert [24] (the maximally degenerate case). In both cases, there is a covering $V_\alpha \subset X$ together with a local thickening $V_\alpha$ (which is toric in both cases) of each $V_\alpha$ over $\text{Spec}(\mathbb{C}[Q])$ as
\[ V_\alpha \subset V_\alpha \quad \text{Spec}(\mathbb{C}) \hookrightarrow \text{Spec}(\mathbb{C}[Q]) \]
which serves local models for smoothing of $X$.

Let $Z \subset X$ be the codimension 2 singular locus of the log-structure of $X$ and write the inclusion of the smooth locus as $j : X \setminus Z \rightarrow X$. We take $kG_\alpha^*$ to be the push-forward of the sheaf of relative log
polyvector fields from $X \setminus Z$ to $X$ and $kK^*_\alpha$ as the push-forward of the sheaf of total log holomorphic de Rham complex from $X \setminus Z$ to $X$. The higher order patching data $k\psi_{\alpha\beta,i}$ comes from uniqueness of the local model near a point in $X$. These data fit into our framework. Also, both freeness of the Hodge bundle (or Assumption 4.15) and the Hodge-to-de Rham degeneracy (i.e. Assumption 5.4) have been proven in \cite[Lemma 4.1]{31} in the log smooth case and in \cite[Theorems 3.26 & 4.1]{23} in the maximally degenerate case. Therefore, we recover the following smoothing result in both cases as a corollary of our main results.

**Corollary 1.4** (see Corollaries 7.4 and 8.8). In both the log smooth and maximally degenerate cases, the complex analytic space $(X,\mathcal{O}_X)$ is smoothable, i.e. there exists a $k^{th}$-order thickening $(k^X,k\mathcal{O})$ over $kS^\dagger$ locally modeled on $k\mathcal{V}_\alpha$ for each $k \in \mathbb{Z}_{\geq 0}$, and these thickenings are compatible.

Furthermore, under suitable assumptions in the two settings, Theorem 1.3 produces a structure of logarithmic Frobenius manifold on the formal neighborhood of $X$ in the extended moduli space (see Corollaries 7.4 and 8.10).

In a very recent work \cite{11}, S. Felten, M. Filip and H. Ruddat proved that Assumptions 4.15 and 5.4 hold for so-called toroidal crossing spaces – a very general class of spaces which includes both log smooth and maximally degenerate Calabi-Yau varieties. Applying our framework and Theorem 1.2 then proves the existence of smoothing for such spaces (see \cite{11} for more details); furthermore, if both Assumptions 6.12 and 6.17 hold, then Theorem 1.3 would yield a logarithmic Frobenius manifold structure in a formal neighborhood of their extended moduli spaces.

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**Notation Summary**

**Notation 1.5.** We fix a rank $s$ lattice $K$ together with a strictly convex $s$-dimensional rational polyhedral cone $Q_\mathbb{R} \subset K_\mathbb{R} := K \otimes_\mathbb{Z} \mathbb{R}$. We let $Q := Q_\mathbb{R} \cap K$ and call it the universal monoid. We consider the ring $R := \mathbb{C}[Q]$ and write a monomial element as $q^m \in R$ for $m \in Q$, and consider the maximal ideal given by $m := \mathbb{C}[Q \setminus \{0\}]$. We let $kR := R/m^{k+1}$ be the Artinian ring, and $\hat{R} := \lim_{k} kR$ be the completion of $R$. We further equip $R$, $kR$ and $\hat{R}$ with the natural monoid homomorphism $Q \to R, m \mapsto q^m$, giving them the structure of a log ring (see \cite[Definition 2.11]{24}); the corresponding log spaces will be denoted as $S^\dagger$, $kS^\dagger$ and $\hat{S}^\dagger$ respectively.

Furthermore, we let $\Omega^s_\dagger := R \otimes_\mathbb{C} \wedge^s(K_\mathbb{C})$, $k\Omega^s_\dagger := kR \otimes_\mathbb{C} \wedge^s(K_\mathbb{C})$ and $\hat{\Omega}^s_\dagger := \hat{R} \otimes_\mathbb{C} \wedge^s(K_\mathbb{C})$ (here $K_\mathbb{C} = K \otimes_\mathbb{Z} \mathbb{C}$) be the spaces of log de Rham differentials on $S^\dagger$, $kS^\dagger$ and $\hat{S}^\dagger$ respectively, where we write $1 \otimes m = d \log q^m$ for $m \in K$; these are equipped with the de Rham differential $\partial$ satisfying
\[ \partial(q^m) = q^m d \log q^m. \] We also denote by \( \Theta_{S^1} := R \otimes_{\mathbb{C}} K^0_{\mathbb{C}}, \Theta_{S^1} \) and \( \Theta_{S^1} \), respectively, the spaces of log derivations, which are equipped with a natural Lie bracket \([\cdot, \cdot]\). We write \( \partial_n \) for the element \( 1 \otimes n \) with action \( \partial_n(q^m) = (m,n)q^m \), where \((m,n)\) is the natural pairing between \( K_{\mathbb{C}} \) and \( K_{\mathbb{C}}^0 \).

For a \( \mathbb{Z}^2 \)-graded vector space \( V^* = \bigoplus_{p,q} V^{p,q} \), we write \( V^k = \bigoplus_{p+q=k} V^{p,q} \), and \( V^* = \bigoplus_{k} V^k \) if we only care about the total degree. We also simply write \( V^k \) if we do not need the grading.

Throughout this paper, we are dealing with two Čech covers \( V = (V_n)_n \) and \( U = (U_j)_{j \in \mathbb{Z}_+} \) at the same time and also \( k \)-th-order thickenings, so we will use the following (rather unusual) notations: The top left hand corner in a notation \( ^k \) refers to the order of \( \bullet \). The bottom left hand corner in a notation \( \bullet \) will stand for something constructed from the Koszul filtration on \( \mathcal{K}_{\alpha} \)'s (as in Definitions \([2,9] \) \) and \([2,17] \), where \( \bullet \) can be \( r, r_1 : r_2 \) or \( \parallel \) (meaning relative forms). The bottom right hand corner is reserved for the Čech indices. We write \( \bullet_{a_0 \cdots a_t} \) for the Čech indices of \( V \) and \( \bullet_{i_0 \cdots i_t} \) for the Čech indices of \( U \), and if they appear at the same time, we write \( \bullet_{a_0 \cdots a_t,i_0 \cdots i_t} \).

2. The abstract setup

2.1. BV algebras and modules.

**Definition 2.1.** A graded Batalin-Vilkovisky (abbrev. BV) algebra is a unital \( \mathbb{Z} \)-graded \( \mathbb{C} \)-algebra \((V^*, \wedge)\) together with a degree 1 operator \( \Delta \) such that \( \Delta(1) = 0, \Delta^2 = 0 \) and the operator \( \delta_{\nu} : V^* \to V^*+|v|+1 \) defined by \( \delta_{\nu}(w) := \Delta(v \wedge w) - \Delta(v) \wedge w - (-1)^{|v|} v \wedge \Delta(w) \) is a derivation of degree \(|v|+1 \) for any homogeneous element \( v \in V^* \) (here \(|v| \) denotes the degree of \( v \)).

**Definition 2.2.** A differential graded Batalin-Vilkovisky (abbrev. dgBV) algebra is a graded BV algebra \((V^*, \wedge, \Delta)\) together with a degree 1 operator \( \bar{\partial} \) satisfying
\[
\bar{\partial}(\alpha \wedge \beta) = (\bar{\partial}(\alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge (\bar{\partial}(\beta)), \quad \bar{\partial}^2 = \bar{\partial} \Delta + \Delta \bar{\partial} = 0.
\]

**Definition 2.3.** A differential graded Lie algebra (abbrev. dgLa) is a triple \((L^*, d, [\cdot, \cdot])\), where \( L = \bigoplus_{l \in \mathbb{Z}} L^l \), \([\cdot, \cdot] : L^* \otimes L^* \to L^* \) is a graded skew-symmetric pairing satisfying the Jacobi identity
\[
[a, [b, c]] + (-1)^{|a||b|+|a||c|}[b, [c, a]] + (-1)^{|a||c|+|b||c|}[c, [a, b]] = 0
\]
for homogeneous elements \( a, b, c \in L^* \), and \( d : L^* \to L^{*+1} \) is a degree 1 differential satisfying \( d^2 = 0 \) and the Leibniz rule \( d[a, b] = [da, b] + (-1)^{|a|} [a, db] \) for homogeneous elements \( a, b \in L^* \).

Given a BV algebra \((V^*, \wedge, \Delta)\), the map \([\cdot, \cdot] : V \otimes V \to V \) defined by \([v, w] = (-1)^{|v|} \delta_{\nu}(w) \) is called the associated Lie bracket\(^3\) Using this bracket, the triple \((V^*[-1] \Delta, [\cdot, \cdot])\) forms a dgLa.

**Notation 2.4.** Given a nilpotent graded Lie algebra \( L^* \), we define a product \( \circ \) by the Baker-Campbell-Hausdorff formula: \( v \circ w := v + w + \frac{1}{2} [v, w] + \cdots \) for \( v, w \in V^* \). The pair \((L^*, \circ)\) is called the exponential group of \( L^* \) and is denoted by \( \exp(L^*) \).

**Lemma 2.5** (See e.g. \([12] \)). For a dgLa \((L^*, d, [\cdot, \cdot])\), we consider the endomorphism \( ad_{\vartheta} := [\vartheta, \cdot] \) for an element \( \vartheta \in L^0 \) such that \( ad_{\vartheta} \) is nilpotent. Then we have the formula
\[
e^{ad_{\vartheta}}(d + [\xi, -])e^{-ad_{\vartheta}} = d + \left[e^{ad_{\vartheta}}(\xi), \cdot\right] = \left[\frac{e^{ad_{\vartheta}} - 1}{ad_{\vartheta}}(d\vartheta), \cdot\right]
\]
for \( \xi \in L^* \). For a nilpotent element \( \vartheta \in L^0 \), we define the gauge action
\[
\exp(\vartheta) \circ \xi := e^{ad_{\vartheta}}(\xi) - \frac{e^{ad_{\vartheta}} - 1}{ad_{\vartheta}}(d\vartheta)
\]
\(^3\)For polyvector fields on a Calabi-Yau manifold, we have \([\cdot, \cdot] = -[\cdot, \cdot]_{\text{sn}} \), where \([\cdot, \cdot]_{\text{sn}} \) is the Schouten-Nijenhuis bracket; see e.g. \([25, \S 6.A] \).
for $\xi \in L^*$. Then we have $\exp(\vartheta_1) \ast (\exp(\vartheta_2) \ast \xi) = \exp(\vartheta_1 \circ \vartheta_2) \ast \xi$, where $\circ$ is the Baker-Campbell-Hausdorff product as in Notation 2.4.

**Definition 2.6** (see e.g. [38]). A BV module $(M^*, \partial)$ over a BV algebra $(V^*, \wedge, \Delta)$ is a complex of $\mathbb{C}$-vector spaces equipped with a degree 1 differential $\partial$ and a graded action by $(V^*, \wedge)$, which will be denoted as $v = v \cdot : M^* \to M^{*+|v|}$ (for a homogeneous element $v \in V^*$) and called the interior multiplication or contraction by $v$, such that if we let $(-1)^{|v|} \mathcal{L}_v := \partial \circ (v \cdot) - (-1)^{|v|} (v \cdot) \circ \partial$, where $[\cdot, \cdot]$ is the graded commutator for operators, then $[\mathcal{L}_{v_1}, \mathcal{L}_{v_2}] = [v_1, v_2]_\partial$.

Given a BV module $(M^*, \partial)$ over $(V^*, \wedge, \Delta)$, we have $[\partial, \mathcal{L}_v] = 0$, $\mathcal{L}_{[v_1,v_2]} = \{\mathcal{L}_{v_1}, \mathcal{L}_{v_2}\}$ and $\mathcal{L}_{v_1 \wedge v_2} = (-1)^{|v_2|} \mathcal{L}_{v_1} \circ (v_2 \cdot) + (v_1 \cdot) \circ \mathcal{L}_{v_2}$.

**Definition 2.7.** A BV module $(M^*, \partial)$ over $(V^*, \wedge, \Delta)$ is called a de Rham module if there is a unital differential graded algebra (abbrv. dga) structure $(M^*, \wedge, \partial)$ such that for $v \in V^{-1}$, $v \cdot$ acts as a derivation, i.e. $v \cdot (w_1 \wedge w_2) = (v \cdot w_1) \wedge w_2 + (-1)^{|w_1|} w_1 \wedge (v \cdot w_2)$. If in addition there is a finite decreasing filtration of BV submodules $\{0\} = M^{-1} \cdots \subset M^0 \subset \cdots M^* = M^*$, then we call it a filtered de Rham module.\(^4\)

Given a de Rham module $(M^*, \partial)$ over $(V^*, \wedge, \Delta)$, it is easy to check that for $v \in V^{-1}$, $\mathcal{L}_v$ acts as a derivation, i.e. $\mathcal{L}_v(w_1 \wedge w_2) = (\mathcal{L}_v w_1) \wedge w_2 + w_1 \wedge (\mathcal{L}_v w_2)$.

**Lemma 2.8.** Given a BV algebra $(V^*, \wedge, \Delta)$ acting on a BV module $(M^*, \partial)$ both with bounded degree, together with an element $v \in V^{-1}$ such that the operator $v \wedge$ is nilpotent and element $\omega$ such that $\partial \omega = 0$ satisfying $\Delta(\alpha) \wedge \omega = \partial(\alpha \wedge \omega)$, we have the following identities

$$
\exp([\Delta, v \wedge]) (1) = \exp \left( \sum_{k=0}^{\infty} \frac{\delta^k_v}{(k+1)!} (\Delta v) \right), \quad \exp([\partial, v \cdot]) \omega = \exp \left( \sum_{k=0}^{\infty} \frac{\delta^k_v}{(k+1)!} (\Delta v) \right) \wedge \omega
$$

where $\delta_v$ is the operator defined in Definition 2.7.

**Proof.** To prove the first identity, notice that $[\Delta, v \wedge] = \delta_v + (\Delta v) \wedge$ and

$$
\exp \left( \sum_{k=0}^{\infty} \frac{\delta^k_v}{(k+1)!} (\Delta v) \right) = 1 + \sum_{m \geq 1; s_1, \ldots, s_m > 0} \sum_{0 \leq k_1 < s_1, \ldots, s_{m+1} = 0} \frac{\delta^{s_1}_{k_1}}{(k_1+1)!} (\Delta v)^{s_1} \cdots \frac{\delta^{s_m}_{k_{m+1}}}{(k_{m+1}+1)!} (\Delta v)^{s_m} (s_1! \cdots s_m!)
$$

So it suffices to establish the equality

$$
\frac{(\delta_v + (\Delta v) \wedge)^L}{L!} (1) = \sum_{0 \leq k_1 < \ldots < k_m; s_1, \ldots, s_m > 0; \ (k_1+1)s_1 + \ldots + (k_{m+1}+1)s_m = L} \frac{\delta^{s_1}_{k_1}}{(k_1+1)!} (\Delta v)^{s_1} \cdots \frac{\delta^{s_m}_{k_{m+1}}}{(k_{m+1}+1)!} (\Delta v)^{s_m} (s_1! \cdots s_m!),
$$

which can be proven by induction on $L$. Essentially the same proof gives the second identity. \(\square\)

### 2.2. The 0th-order data.
Let $(X, \mathcal{O}_X)$ be a $d$-dimensional compact complex analytic space.

**Definition 2.9.** A 0th-order datum over $X$ consists of:

- a coherent sheaf of graded BV algebras $^{(0)}G^*, [\cdot, \cdot], \wedge, ^{(0)}\Delta$ over $X$ (with $-d \leq * \leq 0$), called the 0th-order complex of polyvector fields, such that $^{(0)}G^0 = \mathcal{O}_X$ and the natural Lie algebra morphism $^{(0)}G^{-1} \to \text{Der}(\mathcal{O}_X)$, $v \mapsto [v, \cdot]$ is injective,

\(^4\)This is motivated by the de Rham complex equipped with the Koszul filtration associated to a family of varieties; see e.g. [25] Chapter 10.4.
• a coherent sheaf of dga’s \( (0\mathcal{K}^*, \wedge, 0\partial) \) over \( X \) (with \( 0 \leq \ast \leq d + s \)) endowed with a dg module structure over the dga \( 0\Omega^*_{S^1} \), called the 0th-order de Rham complex, and equipped with the natural filtration \( 0\mathcal{K}^* \) defined by \( 0\mathcal{K}^* := 0\Omega^*_{S^1} \wedge 0\mathcal{K}^* \) (here \( \wedge \) denotes the dga action),

• a de Rham module structure on \( 0\mathcal{K}^* \) over \( 0\mathcal{G}^* \) such that \( [\varphi, \alpha \wedge] = 0 \) for any \( \varphi \in 0\mathcal{G}^* \) and \( \alpha \in 0\Omega^*_{S^1} \), and

• an element \( 0\omega \in \Gamma(X, 0\mathcal{K}^d / 0\mathcal{K}^d) \) with \( 0\partial(0\omega) = 0 \), called the 0th-order volume element, such that

\[
\begin{align*}
(1) & \text{ the map } 0\omega : (0\mathcal{G}^*[d], 0\Delta) \to (0\mathcal{K}^*/0\mathcal{K}^*, 0\partial) \text{ is an isomorphism, and} \\
(2) & \text{ the map } 0\sigma^{-1} : 0\Omega^*_{S^1} \otimes (0\mathcal{K}^*/0\mathcal{K}^*[d]) \to 0\mathcal{K}^*/0\mathcal{K}^*[d], \text{ given by taking wedge product by } \\
& \quad \Omega^*(0\mathcal{S}^1) \text{ (here } [d] \text{ is the upshift of the complex by degree } d), \text{ is also an isomorphism.}
\end{align*}
\]

Note that \( (0\mathcal{K}^*, \wedge, 0\partial) \) is a filtered de Rham module over \( 0\mathcal{G}^* \) using the filtration \( 0\mathcal{K}^* \), and the map \( 0\sigma^{-1} \) is an isomorphism of BV modules. We write \( (0\mathcal{K}^*, 0\partial) := (0\mathcal{K}^*/0\mathcal{K}^*, 0\partial) \) and \( 0\sigma := (0\sigma^{-1})^{-1} \).

We consider the hypercohomology \( \mathbb{H}^*(0\mathcal{K}^*, 0\partial) \) of the complex of sheaves \( (0\mathcal{K}^*, 0\partial) \).

**Definition 2.10.** For each \( r \in \frac{1}{2}\mathbb{Z} \), let \( \mathcal{F}^{\geq r}\mathbb{H}^l \) be the image of the linear map \( \mathbb{H}^l(0\mathcal{K}^r, 0\partial) \to \mathbb{H}^l(0\mathcal{K}^*, 0\partial) \), where \( p \) is the smallest integer such that \( 2p \geq 2r + l - d \). Then

\[
0 \subset \mathcal{F}^{\geq d} \subset \mathcal{F}^{\geq d-1} \subset \cdots \subset \mathcal{F}^{\geq r} \subset \cdots \subset \mathcal{F}^{\geq 0} = \mathbb{H}^*(0\mathcal{K}^*, 0\partial)
\]

is called the Hodge filtration.

We usually write \( \mathcal{F}^{\geq r} \) instead of \( \mathcal{F}^{\geq r}\mathbb{H}^l \) when there is no confusion.

We have the following exact sequence of sheaves from Definition 2.9:

\[
0 \to 0\Omega^1_{S^1} \otimes_{0\mathcal{K}^*} [-1] \cong 0\mathcal{K}^*/2\mathcal{K}^* \to 0\mathcal{K}^*/\mathcal{K}^* \to 0\mathcal{K}^*/0\mathcal{K}^* \cong 0\mathcal{K}^*/0\mathcal{K}^* \to 0
\]

**Definition 2.11.** Suppose we take the long exact sequence associated to the hypercohomology of (2.1), we obtain the 0th-order Gauss-Manin (abbrev. GM) connection:

\[
0\nabla : \mathbb{H}^*(0\mathcal{K}^*, 0\partial) \to 0\Omega^1_{S^1} \otimes \mathbb{H}^*(0\mathcal{K}^*, 0\partial).
\]

Note that the 0th-order GM connection is indeed the residue of the usual GM connection.

**Proposition 2.12.** Griffith’s transversality holds for \( 0\nabla \), i.e. \( 0\nabla(\mathcal{F}^{\geq r}) \subset 0\Omega^1_{S^1} \otimes \mathcal{F}^{\geq r-1} \).

The proof of this is the same as [15] Corollary 10.31. With \( [0\omega] \in \mathcal{F}^{\geq d}\mathbb{H}^0 \) and Griffith’s transversality, we then define the 0th-order Kodaira-Spencer map as \( 0\nabla([0\omega]) : 0\Omega^1_{S^1} \to \mathcal{F}^{d-1}\mathbb{H}^0 \).

**2.3. The higher order data.** We fix an open cover \( \mathcal{V} \) of \( X \) which consists of Stein open subsets \( V_\alpha \subset X \).

**Definition 2.13.** A local thickening datum of the complex of polyvector fields (with respect to \( \mathcal{V} \)) consists of, for each \( k \in \mathbb{Z}_{\geq 0} \) and \( V_\alpha \in \mathcal{V} \),

• a coherent sheaf of BV algebras \( (k\mathcal{G}^*_\alpha, \{\cdot, \cdot\}, \Delta_\alpha) \) over \( V_\alpha \) such that \( k\mathcal{G}^*_\alpha \) is also a sheaf of algebras over \( k\mathcal{R} \) so that \( \{\cdot, \cdot\}, \Delta_\alpha \) are \( k\mathcal{R} \)-bilinear and \( \Delta_\alpha \) is \( k\mathcal{R} \)-linear, and

---

\(^5\)We follow Barannikov [1] for the convention on the index \( r \) of the Hodge filtration, which differs from the usual one by a shift.
a surjective morphism of sheaves of BV algebras $k^{1,1}g^*_\alpha : k^{1,1}G^*_\alpha \to kG^*_\alpha$ which is $k^{1,1}R$-linear and induces a sheaf isomorphism upon tensoring with $kR$ satisfying the following conditions:

1. $(0G^*_\alpha, [\cdot, \cdot, \cdot, \cdot, \cdot]) = (0G^*_\alpha, [\cdot, \cdot, \cdot, \cdot, \cdot])|_{V_\alpha}$.
2. $kG^*_\alpha$ is flat over $kR$, i.e., the stalk $(kG^*_\alpha)_x$ is flat over $kR$ for any $x \in V_\alpha$, and
3. the natural Lie algebra morphism $kG^*_\alpha \to \text{Der}(kG^0_\alpha)$ is injective.

We write $k_l b_\alpha := l^{1,1}b_\alpha \cdots o k^1 b_\alpha : kG^*_\alpha \to lG^*_\alpha$ for every $k > l$, and $k^k b_\alpha \equiv \text{id}$. We also introduce the following notation: Given two elements $a \in k^1 G^*_\alpha$, $b \in k^2 G^*_\alpha$ and $l \leq \min\{k_1, k_2\}$, we say that $a = b \mod (m^{l+1})$ if and only if $k_1, l b_\alpha(a) = k_2, l b_\alpha(b)$.

**Notation 2.14.** We also fix, once and for all, a cover $U$ of $X$ which consists of a countable collection of Stein open subsets $U = \{U_i\}_{i \in \mathbb{Z}^+}$ forming a basis of topology. We refer readers to [9, Chapter IX Theorem 2.13] for the existence of such a cover. Note that an arbitrary finite intersection of Stein open subsets remains Stein.

**Definition 2.15.** A patching datum of the complex of polyvector fields (with respect to $U, V$) consists of, for each $k \in \mathbb{Z}_{\geq 0}$ and triple $(U_i; V_\alpha, V_\beta)$ with $U_i \subset V_{\alpha \beta} := V_\alpha \cap V_\beta$, a sheaf isomorphism $k\psi_{\alpha \beta, i} : kG^*_\alpha|_{U_i} \to kG^*_\beta|_{U_i}$ over $kR$ fitting into the diagram

\[
\begin{array}{ccc}
kG^*_\alpha|_{U_i} & \xrightarrow{k\psi_{\alpha \beta, i}} & kG^*_\beta|_{U_i} \\
\downarrow{k_0 b_\alpha} & & \downarrow{k_0 b_\alpha} \\
0G^*|_{U_i} & \equiv & 0G^*|_{U_i},
\end{array}
\]

and an element $k\varpi_{\alpha \beta, i} \in kG^0_\alpha(U_i)$ with $k\varpi_{\alpha \beta, i} = 0 \mod(m)$ such that

\[
k\psi_{\alpha \beta, i} \circ k\Delta_{\beta} \circ k\psi_{\alpha \beta, i} - k\Delta_{\alpha} = [k\varpi_{\alpha \beta, i}, \cdot]
\]

satisfying the following conditions:

1. $k\psi_{\alpha \beta, i} = k\psi_{\alpha \beta, i}^{-1}, 0\psi_{\alpha \beta, i} \equiv \text{id}$;
2. for $k > l$ and $U_i \subset V_{\alpha \beta}$, there exists $k_l b_{\alpha \beta, i} \in kg^{-1}_\alpha(U_i)$ with $k_l b_{\alpha \beta, i} = 0 \mod(m)$ such that

\[
l\psi_{\alpha \beta, i} \circ k_l b_{\beta \alpha, i} = \exp\left(k_l b_{\alpha \beta, i}, \cdot \right) \circ k_l b_{\alpha \beta, i};
\]
3. for $k \in \mathbb{Z}_{\geq 0}$ and $U_i, U_j \subset V_{\alpha \beta}$, there exists $k_p_{\alpha \beta, ij} \in kg^{-1}_\alpha(U_i \cap U_j)$ with $k_p_{\alpha \beta, ij} = 0 \mod(m)$ such that

\[
\left(k\psi_{\alpha \beta, i}|_{U_i \cap U_j}\right) \circ \left(k\psi_{\alpha \beta, i}|_{U_i \cap U_j}\right) = \exp\left(k_p_{\alpha \beta, ij}, \cdot \right); \text{ and}
\]
4. for $k \in \mathbb{Z}_{\geq 0}$ and $U_i \subset V_{\alpha \beta \gamma} := V_\alpha \cap V_\beta \cap V_\gamma$, there exists $k_0_{\alpha \beta \gamma, i} \in kg^{-1}_\alpha(U_i)$ with $k_0_{\alpha \beta \gamma, i} = 0 \mod(m)$ such that

\[
\left(k\psi_{\alpha \beta, i}|_{U_i}\right) \circ \left(k\psi_{\alpha \beta, i}|_{U_i}\right) \circ \left(k\psi_{\alpha \beta, i}|_{U_i}\right) = \exp\left(k_0_{\alpha \beta \gamma, i}, \cdot \right).
\]

**Lemma 2.16.** The elements $k_l b_{\alpha \beta, i}$'s, $k_p_{\alpha \beta, ij}$'s and $k_0_{\alpha \beta \gamma, i}$'s are uniquely determined by the patching isomorphisms $k\psi_{\alpha \beta, i}$'s.

**Proof.** We just prove the statement for the elements $k_p_{\alpha \beta, ij}$'s as the other cases are similar. Suppose we have another set of elements $k_p_{\alpha \beta, ij}$'s satisfying (2.5), then we have $\text{exp}(k_p_{\alpha \beta, ij}, k_p_{\alpha \beta, ij}, \cdot) \equiv \text{id}$ as actions on $kG^0_\alpha(U_{ij})$ where $U_{ij} = U_i \cap U_j$. The result then follows from an order-by-order argument.
using the assumptions that $k^0p_\alpha(kp_{\alpha\beta,ij} - kp_{\alpha\beta,ij}) = 0$ and that the map $kG_\alpha^{-1} \to \text{Der}(kG_\alpha^0)$ is injective.

**Definition 2.17.** A local thickening datum of the de Rham complex (with respect to $\mathcal{V}$) consists of, for each $k \in \mathbb{Z}_{\geq 0}$ and $V_\alpha \in \mathcal{V}$,

- a coherent sheaf of dgas $(^kK_\alpha^*\wedge, ^k\partial_\alpha)$ with a dg module structure over $^k\Omega^*_G$ equipped with the natural filtration $^kK_\alpha^* := ^k\Omega_{S}^{\geq s} \wedge ^kK_\alpha^*[s]$,
- a de Rham module structure on $^kK_\alpha^*$ over $^kG_\alpha^*$ such that $[\varphi \wedge, \alpha \wedge] = 0$ for any $\varphi \in ^kG_\alpha^*$ and $\alpha \in ^k\Omega_{S}^*$,
- a surjective $^kR^{1}$-linear morphism $^k1_{\alpha} : ^kK_\alpha^* \to ^kK_\alpha^*$ inducing an isomorphism upon tensoring with $^kR$ which is compatible with both $^k1_{\alpha}$ and $^kG_\alpha^*$ and $^k\Omega^*_G \to ^k\Omega^*_G$ under the contraction and dg actions respectively, and
- an element $^k\omega_\alpha \in \Gamma(V_\alpha, 0^kG_\alpha^0/1^kK_\alpha^0)$ satisfying $^k\partial_\alpha(^k\omega_\alpha) = 0$ called the local $^k$-th order volume element

such that

1. $^kK_\alpha^*$ is flat over $^kR$ for $0 \leq s$;
2. $^k1_{\alpha}$ is an isomorphism over $0^k\Omega^*_G$;
3. $(0^kK_\alpha^*/0^k\Omega^*_G, \wedge, 0^k\partial_\alpha) = (0^kK_\alpha^*/0^k\Omega^*_G, \wedge, 0^k\partial_\alpha)|_{V_\alpha}$ and $0^k\omega_\alpha = 0\omega|_{V_\alpha}$;
4. the map $^k\omega_\alpha : (G_\alpha^*/[d], ^k\Delta_\alpha) \to (0^kK_\alpha^*/[k\alpha], ^k\partial_\alpha)$ is an isomorphism, and
5. the map $^k\sigma_{\alpha}^{-1} : ^k\Omega_{G}^* \otimes _R (0^kK_\alpha^*/[k\alpha] \wedge [r]) \to ^kK_\alpha^*/[k\alpha] \wedge [r]$, given by taking wedge product by $^k\Omega_{G}^*$, is also an isomorphism.

Note that $^kK_\alpha^*$ is a filtered de Rham module over $^kG_\alpha^*$ using the filtration $^kK_\alpha^*$. We write $^kK_\alpha^* := 0^kK_\alpha^*/[k\alpha]$ and $^k\sigma_{\alpha} = (^k\sigma_{\alpha}^{-1})^{-1}$.

We also write $^k1_{\alpha} := 1 + 1_{\alpha} \circ \cdots \circ 1_{\alpha}$ for every $k > l$ and $^k1_{\alpha} \equiv \text{id}$, and introduce the following notation: Given two elements $a \in ^k\Omega_{G}^* \oplus ^k\Omega_{G}^*$, $b \in ^k\Omega_{G}^* \oplus ^k\Omega_{G}^*$ and $l \leq \min\{k_1, k_2\}$, we say that $a = b$ (mod $m^{l+1}$) if and only if $^k1_{\alpha}(a) = ^k1_{\alpha}(b)$.

From Definition 2.17, we have the following diagram of BV modules

(2.7) \[ \begin{array}{c}
0 \to ^kK \otimes _Z ^k1_{\alpha} \to ^kK \otimes _Z ^k1_{\alpha} \otimes _R ^k\Omega_{G}^* \to ^kK \otimes _Z ^k\Omega_{G}^*/[k\alpha] \to ^kK \otimes _Z ^k\Omega_{G}^*/[k\alpha] \to 0
\end{array} \]

and the natural filtration $^k\Omega_{G}^*/[k\alpha] \to ^kK \otimes _Z ^k\Omega_{G}^*/[k\alpha] \to ^kK \otimes _Z ^k\Omega_{G}^*/[k\alpha] \to 0$.

**Definition 2.18.** A patching datum of the de Rham complex (with respect to $\mathcal{U}, \mathcal{V}$) consists of, for each $k \in \mathbb{Z}_{\geq 0}$ and triple $(U_\alpha, V_\alpha, V_\beta)$ with $U_\alpha \subset V_\alpha$, a sheaf isomorphism $^k\psi_{\alpha\beta, i}$ of dg modules over $^k\Omega_{S}^*$ such that it fits into the diagram

\[ \begin{array}{c}
^kK_\alpha^*|_{U_\alpha} \to ^kK_\beta^*|_{U_\alpha} \\
\downarrow ^k1_{\alpha} \quad \downarrow ^k1_{\beta} \\
^k\Omega_{G}^*|_{U_\alpha} \to ^k\Omega_{G}^*|_{U_\alpha}
\end{array} \]

and satisfying the following conditions:

\[ \text{Here we abuse notations and use } ^k1_{\alpha} \text{ for both } ^k1_{\alpha} \text{ and } ^k1_{\alpha}. \]
Remark 2.19. We can deduce (2.4) from (2.9) as follows: From (2.9), we have \( k \hat{\psi}_{\alpha,i} \circ (\Delta_k \omega_\beta | U_i) \circ k \hat{\psi}_{\alpha,i}(\gamma) = (\gamma \wedge \exp(k \omega_{\alpha,i} | U_i)) \Delta_k (k \omega_{\alpha} | U_i) \), so
\[
(k \hat{\psi}_{\alpha,i} \circ \Delta_k \circ k \hat{\psi}_{\alpha,i})(\gamma) = k \Delta_k (\gamma \wedge \exp(k \omega_{\alpha,i} | U_i)) = (k \Delta_k (\gamma) + [k \omega_{\alpha,i}, \gamma] \wedge \exp(k \omega_{\alpha,i}))
\]
for any \( \gamma \in kG_\alpha^* (U_i) \), which gives \( k \Delta_k (\gamma) + [k \omega_{\alpha,i}, \gamma] = (k \hat{\psi}_{\alpha,i} \circ \Delta_k \circ k \hat{\psi}_{\alpha,i})(\gamma) \).

3. Abstract construction of the Čech-Thom-Whitney complex

3.1. The simplicial set \( A^* (\Delta) \). In this subsection, we recall some notations and facts about the simplicial sets \( A^* (\Delta) \) of polynomial differential forms with coefficient \( k = \mathbb{Q}, \mathbb{R}, \mathbb{C} \); we will simply write \( A^* (\Delta) \) when \( k = \mathbb{C} \), which will be the case for all other parts of this paper.
Notation 3.1. We let $\text{Mon}$ (resp. $s\text{Mon}$) be the category of finite ordinals $[n] = \{0, 1, \ldots, n\}$ in which morphisms are non-decreasing maps (resp. strictly increasing maps). We denote by $d_{i,n}: [n-1] \to [n]$ the unique strictly increasing map which skips the $i$-th element, and by $e_{i,n}: [n+1] \to [n]$ the unique non-decreasing map sending both $i$ and $i+1$ to the same element $i$.

Note that every morphism in $\text{Mon}$ can be decomposed as a composition of the maps $d_{i,n}$’s and $e_{i,n}$’s, and any morphism in $s\text{Mon}$ can be decomposed as a composition of the maps $d_{i,n}$’s.

Definition 3.2 ([19]). Let $\mathcal{C}$ be a category. A (semi-)simplicial object in $\mathcal{C}$ is a contravariant functor $A(\bullet): \text{Mon} \to \mathcal{C}$ (resp. $sA(\bullet): \text{sMon} \to \mathcal{C}$), and a (semi-)cosimplicial object in $\mathcal{C}$ is a covariant function $A(\bullet): \text{Mon} \to \mathcal{C}$ (resp. $sA(\bullet): \text{sMon} \to \mathcal{C}$).

Definition 3.3 ([19]). Let $k$ be a field which is either $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Consider the dga

$$A^*_k(\Delta_n) := \mathbb{k}[x_0, \ldots, x_n, dx_0, \ldots, dx_n] \left(\sum_{i=0}^{n} x_i - 1, \sum_{i=0}^{n} dx_i\right),$$

with $\deg(x_i) = 0$, $\deg(dx_i) = 1$, and equipped with the degree 1 differential $d$ defined by $d(x_i) = dx_i$ and the Leibniz rule. Given $a: [n] \to [m]$ in $\text{Mon}$, we let $a^* := A_k(a): A^*_k(\Delta_m) \to A^*_k(\Delta_n)$ be the unique dga morphism satisfying $a^*(x_j) = \sum_{i \in [n]: a(i) = j} x_i$ for any $j \in [n]$. From this we obtain a simplicial object in the category of dga’s, which we denote as $A^*_k(\bullet)$.

Notation 3.4. We denote by $\partial_n$ the boundary of $\Delta_n$, and let

$$\partial^*_k(\Delta_n) := \{ (\alpha_0, \ldots, \alpha_n) \mid \alpha_i \in A^*_k(\Delta_{n-1}), d^*_{n-1}(\alpha_{i}) = d^*_{j-1,n-1}(\alpha_{i}) \text{ for } 0 \leq i < j \leq n \}$$

be the space of polynomial differential forms on $\Delta_n$. There is a natural restriction map defined by $\beta|_{\partial_n} := (d^*_{0,n}(\beta), \ldots, d^*_{n,n}(\beta))$ for $\beta \in A^*_k(\Delta_n)$.

The following extension lemma will be frequently used in subsequent constructions:

Lemma 3.5 (Lemma 9.4 in [19]). For any $\alpha = (\alpha_0, \ldots, \alpha_n) \in A^*_k(\Delta_n)$, there exists $\beta \in A^*_k(\bullet)$ such that $\beta|_{\partial_n} = \alpha$.

Notation 3.6. We let $\square_n := \Delta_1 \times \Delta_1$, where $\Delta_1 := \{(t_0, t_1) \mid 0 \leq t_i \leq 1, t_0 + t_1 = 1\}$, and

$$A^*_k(\square_n) := A^*_k(\Delta_1) \otimes_k A^*_k(\Delta_1) = \mathbb{k}[x_0, \ldots, x_n, dx_0, \ldots, dx_n, t_0, t_1, dt_0, dt_1] \left(\sum_{i=0}^{n} x_i - 1, \sum_{i=0}^{n} dx_i, t_0 + t_1 - 1, dt_0 + dt_1\right),$$

Besides the restriction maps $d^*_{j,n}: A^*_k(\square_n) \to A^*_k(\Delta_{n-1})$ induced from that on $\Delta_n$, we also have the maps $r^*_j: A^*_k(\square_n) \to A^*_k(\Delta_n)$ defined by putting $t_j = 1$ (and $t_{1-j} = 0$).

Notation 3.7. We denote by $\square_n$ the boundary of $\square_n$, and let

$$A^*_k(\square_n) := \left\{ (\alpha_0, \ldots, \alpha_n, \beta_0, \beta_1) \mid \begin{array}{l}
\alpha_i \in A^*_k(\square_{n-1}), \beta_i \in A^*(\Delta_n), \\
d^*_{n-1}(\alpha_j) = d^*_{j-1,n-1}(\alpha_i) \text{ for } 0 \leq i < j \leq n, \\
r^*_i(\alpha_j) = d^*_{n,n}(\beta_i) \text{ for } i = 0, 1 \text{ and } 0 \leq j \leq n
\end{array} \right\}$$

be the space of polynomial differential forms on $\square_n$. There is a natural restriction map defined by $\gamma|_{\square_n} := (d^*_{0,n}(\gamma), \ldots, d^*_{n,n}(\gamma), r^*_0(\gamma), r^*_1(\gamma))$ for $\gamma \in A^*_k(\square_n)$.

Lemma 3.8. For any $(\alpha_0, \ldots, \alpha_n, \beta_0, \beta_1) \in A^*_k(\square_n)$, there exists $\gamma \in A^*_k(\square_n)$ such that $\gamma|_{\square_n} = (\alpha_0, \ldots, \alpha_n, \beta_0, \beta_1)$.

This variation of Lemma 3.5 can be proven by the same technique as in [19] Lemma 9.4.\footnote{In the case $k = \mathbb{R}$, this can be thought of as the space of polynomial differential forms on $\mathbb{R}^{n+1}$ restricted to the $n$-simplex $\Delta_n$.}
3.2. Local Thom-Whitney complexes. Consider a sheaf of BV algebras \((\mathcal{G}^*, \wedge, \Delta)\) on a topological space \(V\) together with an acyclic cover \(\mathcal{U} = \{U_i\}_{i \in \mathbb{Z}_+}\) of \(V\) such that \(H^0(U_{i_0 \cdots i_l}, \mathcal{G}^j) = 0\) for all \(j\) and all finite intersections \(U_{i_0 \cdots i_l} := U_{i_0} \cap \cdots \cap U_{i_l}\). In particular, this allows us to compute the sheaf cohomology \(H^*(V, \mathcal{G}^j)\) and the hypercohomology \(H^*(\mathcal{U}, \mathcal{G}^j)\) using the Čech complex \(\check{C}^*(\mathcal{U}, \mathcal{G}^j)\) and the total complex of \(\check{C}^*(\mathcal{U}, \mathcal{G}^j)\) respectively.

Let \(\mathcal{I} = \{(i_0, \ldots, i_l) \mid i_j \in \mathbb{Z}_+, i_0 < i_1 < \cdots < i_l\}\) be the index set. Let \(\Delta_i\) be the standard \(l\)-simplex in \(\mathbb{R}^{l+1}\) and \(A^q(\Delta_i)\) be the space of \(\mathbb{C}\)-valued polynomial differential \(q\)-forms on \(\Delta_i\). Also let \(d_{j,l} : \Delta_{l-1} \rightarrow \Delta_i\) be the inclusion of the \(j\)th-facet in \(\Delta_i\) and let \(d^*_j\) be the pullback map. See Definition 3.3 and Notation 3.1 in [3.1] for details.

**Definition 3.9** (see e.g. [55] 10, [12]). The Thom-Whitney complex is defined as \(TW^{*,*}(\mathcal{G}) := \bigoplus_{p,q} TW^{p,q}(\mathcal{G})\) where

\[ TW^{p,q}(\mathcal{G}) := \left\{ (\varphi_{i_0 \cdots i_l})_{(i_0, \ldots, i_l) \in \mathcal{I}} \mid \varphi_{i_0 \cdots i_l} \in A^q(\Delta_i) \otimes \mathcal{G}^p(U_{i_0 \cdots i_l}), \ d^*_j(\varphi_{i_0 \cdots i_l}) = \varphi_{i_0 \cdots \hat{i}_j \cdots i_l}|_{U_{i_0 \cdots i_l}} \right\}. \]

It is equipped with the structures \((\wedge, \partial, \Delta, \wedge)\) defined componentwise by

\[ (\alpha_I \otimes v_I) \wedge (\beta_I \otimes w_I) := (-1)^{|v_I||\beta_I|} (\alpha_I \wedge \beta_I) \otimes (v_I \wedge w_I), \]
\[ \partial(\alpha_I \otimes v_I) := (d\alpha_I) \otimes v_I, \quad \Delta(\alpha_I \otimes v_I) := (-1)^{|\alpha_I|} \alpha_I \otimes (\Delta v_I), \]

for \(\alpha_I, \beta_I \in A^*(\Delta_i)\) and \(v_I, w_I \in \mathcal{G}^*(U_I)\), where \(I = (i_0, \ldots, i_l) \in \mathcal{I}\) and \(l = |I| - 1\).

**Remark 3.10.** We use the notation \(\partial\) since it plays the role of the Dolbeault operator in the classical deformation theory of smooth Calabi-Yau manifolds.

\((TW^{*,*}(\mathcal{G}), \partial, \Delta, \wedge)\) forms a dgBV algebra in the sense of Definition 2.2 From Definitions 2.2 and 3.9 the Lie bracket on the Thom-Whitney complex is determined componentwise by the formula

\[ [\alpha_I \otimes v_I, \beta_I \otimes w_I] := (-1)^{|v_I|+1|\beta_I|} (\alpha_I \wedge \beta_I) \otimes [v_I, w_I], \]

for \(\alpha_I, \beta_I \in A^*(\Delta_i)\) and \(v_I, w_I \in \mathcal{G}^*(U_I)\) where \(l = |I| - 1\).

We consider the integration map \(\mathcal{I} : TW^{p,q}(\mathcal{G}) \rightarrow \check{C}^q(\mathcal{U}, \mathcal{G}^p)\) defined by

\[ \mathcal{I}(\alpha_{i_0 \cdots i_l}) := \left( \int_{\Delta_i} \otimes \text{id} \right) (\alpha_{i_0 \cdots i_l}) \]

for each component \(\alpha_{i_0 \cdots i_l} \in A^q(\Delta_i) \otimes \mathcal{G}^p(U_{i_0 \cdots i_l})\) of \((\alpha_{i_0 \cdots i_l})_{(i_0, \ldots, i_l) \in \mathcal{I}} \in TW^{p,q}(\mathcal{G})\). Notice that \(\mathcal{I}\) is a chain morphism from \((TW^{*,*}(\mathcal{G}), \partial)\) to \((\check{C}^q(\mathcal{U}, \mathcal{G}^p), \delta)\), where \(\delta\) is the Čech differential. Taking the total complexes gives a chain morphism from \((TW^{*,*}(\mathcal{G}), \partial \pm \Delta)\) to \(\check{C}^*(\mathcal{U}, \mathcal{G}^*)\), which is equipped with the total Čech differential \(\partial \pm \Delta\).

**Lemma 3.11** ([55]). The maps \(\mathcal{I} : TW^{p,*}(\mathcal{G}) \rightarrow \check{C}^q(\mathcal{U}, \mathcal{G}^p)\) and \(\mathcal{I} : TW^{*,*}(\mathcal{G}) \rightarrow \check{C}^*(\mathcal{U}, \mathcal{G}^*)\) are quasi-isomorphisms.

**Remark 3.12.** Comparing to the standard construction of the Thom-Whitney complex in e.g. [12] where one considers \((\varphi_{i_0 \cdots i_l})_{(i_0, \ldots, i_l) \in \mathcal{I}} \in \prod_{I \geq 0} \left( A^*(\Delta_i) \otimes \mathcal{G}^p(U_{i_0 \cdots i_l}) \right)\), we are taking a bigger complex in Definition 3.3 for the purpose of later constructions. However, the original proof of Lemma 3.11 works in exactly the same way for this bigger complex, and hence \(TW^{p,*}(\mathcal{G})\) also serves as a resolution of \(\mathcal{G}^p\).

\(^8\) Readers may assume that \(\mathcal{G}^*\) is a bounded complex for the purpose of this paper.
Definition 3.13. Given the $0^\text{th}$-order complex of polyvector fields $(0\mathcal{G}^*, \wedge, 0\Delta)$ over $X$ (Definition 2.9), we use the cover $\mathcal{U}$ in Notation 2.14 to define the $0^\text{th}$-order Thom-Whitney complex $(TW^{*,*}(0\mathcal{G}), \bar{\partial}, 0\Delta, \wedge)$. To simplify notations, we write $0TW^{*,*}$ to stand for $TW^{*,*}(0\mathcal{G})$.

Given a finite intersection of open subsets $V_{a_0\ldots a_\ell} := V_{a_0} \cap \cdots \cap V_{a_\ell}$ of the cover $\mathcal{V}$, and local thickenings of the complex of polyvector fields $(0\mathcal{G}^*_\alpha, \wedge, k\Delta_\alpha)$ over $V_{a_i}$ for each $k \in \mathbb{Z}_{\geq 0}$ (Definition 2.13), we use the cover $U_{a_0\ldots a_\ell} := \{ U \in \mathcal{U} \mid U \subset V_{a_0\ldots a_\ell} \}$ to define the local Thom-Whitney complex $(TW^{*,*}(k\mathcal{G}^*_\alpha|_{V_{a_0\ldots a_\ell}}), \bar{\partial}, k\Delta_\alpha, \wedge)$ over $V_{a_0\ldots a_\ell}$. To simplify notations, we write $kTW^{*,*}_{\alpha\beta}$ to stand for $TW^{*,*}(k\mathcal{G}^*_\alpha|_{V_{a_0\ldots a_\ell}})$.

The covers $\mathcal{U}$ and $U_{a_0\ldots a_\ell}$ satisfy the acyclic assumption at the beginning of this section because $0\mathcal{G}^*_\alpha$ and $k\mathcal{G}^*_\alpha$ are coherent sheaves and all the open sets in these covers are Stein:

Theorem 3.14 (Cartan’s Theorem B [4]; see e.g. Chapter IX Corollary 4.11 in [9]). For a coherent sheaf $\mathcal{F}$ over a Stein space $U$, we have $H^2(0(U, \mathcal{F}) = 0$.

3.3. The gluing morphisms.

3.3.1. Existence of a set of compatible gluing morphisms. The aim of this subsection is to construct, for each $k \in \mathbb{Z}_{\geq 0}$ and any pair $V_\alpha, V_\beta \in \mathcal{V}$, an isomorphism

\[ (k \mathcal{G}^*_\alpha \wedge, k\Delta_\alpha) \rightarrow (k \mathcal{G}^*_\beta \wedge, k\Delta_\beta) \]

as a collection of maps $(k \mathcal{G}^*_\alpha|_{U_i}) \rightarrow (k \mathcal{G}^*_\beta|_{U_i})$ so that for each $\varphi = (\varphi_I)_{I \in I} \in kTW^{*,*}_{\alpha\beta}$ with $\varphi_I \in \mathcal{A}^*(\mathcal{U}_I) \otimes k\mathcal{G}^*_\alpha(U_I)$ we have $\left( k \mathcal{G}^*_\beta(\varphi) \right)_I = k \mathcal{G}^*_\alpha(\varphi_I)$, which preserves the algebraic structures $[\cdot, \cdot], \wedge$ and satisfies the following condition:

Condition 3.15. (1) for $U_i \subset V_\alpha \cap V_\beta$, we have

\[ k \mathcal{g}_{\alpha, \beta, i} = \exp((k \mathcal{a}_{\alpha, \beta, i} \wedge) \circ k \mathcal{g}_{\alpha, \beta, i} \]

for some element $k \mathcal{a}_{\alpha, \beta, i} \in k \mathcal{G}^*_\beta(U_i)$ with $k \mathcal{a}_{\alpha, \beta, i} = 0\ (mod\ m)$;

(2) for $U_{i_0}, \ldots, U_{i_l} \subset V_\alpha \cap V_\beta$, we have

\[ k \mathcal{g}_{\alpha, \beta, i_0 \cdots i_l} = \exp((k \mathcal{g}_{\alpha, \beta, i_0 \cdots i_l, \varphi}) \circ k \mathcal{g}_{\alpha, \beta, i_0 \cdots i_l}) \]

for some element $k \mathcal{g}_{\alpha, \beta, i_0 \cdots i_l} \in k \mathcal{G}^*_\beta(U_{i_0} \cdots U_{i_l})$ with $k \mathcal{g}_{\alpha, \beta, i_0 \cdots i_l} = 0\ (mod\ m)$; and

(3) the elements $k \mathcal{g}_{\alpha, \beta, i_0 \cdots i_l}$ satisfy the relation:

\[ d_{j, l}^* (k \mathcal{g}_{\alpha, \beta, i_0 \cdots i_l}) = \left\{ \begin{array}{ll} k \mathcal{g}_{\alpha, \beta, i_0 \cdots i_l} & \text{for } j > 0, \\
k \mathcal{g}_{\alpha, \beta, i_0 \cdots i_l} \circ k \mathcal{g}_{\alpha, \beta, i_0} & \text{for } j = 0, \end{array} \right. \]

where $k \mathcal{g}_{\alpha, \beta, i_0 \cdots i_l} \in k \mathcal{G}^*_\beta(U_{i_0} \cdots U_{i_l})$ is the unique element such that

\[ \exp((k \mathcal{g}_{\alpha, \beta, i_0}) \circ k \mathcal{g}_{\alpha, \beta, i_0}) = k \mathcal{g}_{\alpha, \beta, i_1}. \]

Lemma 3.16. Suppose that the morphisms $k \mathcal{g}_{\alpha, \beta}$’s, each of which is a collection of maps $(k \mathcal{g}_{\alpha, \beta}|_{\mathcal{U}_I})$, all satisfy Condition 3.15. For any $\varphi = (\varphi_I)_{I \in I} \in kTW^{*,*}_{\alpha\beta}$, we have $k \mathcal{g}_{\alpha, \beta}(\varphi_I)_I \in kTW^{*,*}_{\alpha\beta}$.

\[ 9 \text{Here } d_{j, l}^* \text{ is induced by the corresponding map } d_{j, l}^* : \mathcal{A}^*(\mathcal{U}_I) \rightarrow \mathcal{A}^*(\mathcal{U}_{I-1}) \text{ on the simplicial set } \mathcal{A}^*(\mathcal{U}_I) \text{ introduced in Definition 3.3 and } \circ \text{ is the Baker-Campbell-Hausdorff product in Notation 2.4.} \]
Proof. Suppose that we have \((\varphi_l)_{l \in I} \in kT^{\ast,\ast}_{\alpha l} \beta \gamma\) such that \(\varphi_{i_0 \cdots i_l} \in A(U_{i_0 \cdots i_l})\) and \(\varphi_{i_0 \cdots i_j \cdots i_l} = d_{j, l}^* (\varphi_{i_0 \cdots i_l})\). Letting \((k g_{\alpha \beta l})_{i_0 \cdots i_l} := (exp([k g_{\alpha \beta}, g_{\alpha \beta l}]) (\varphi_{i_0 \cdots i_l}))\), we have

\[
(k g_{\alpha \beta l})_{i_0 \cdots i_l} = (exp([k g_{\alpha \beta}, g_{\alpha \beta l}])) (\varphi_{i_0 \cdots i_l})
\]

and

\[
(k g_{\alpha \beta l})_{i_0 \cdots i_l} = (exp([k g_{\alpha \beta}, g_{\alpha \beta l}])) (\varphi_{i_0 \cdots i_l})
\]

which are the required conditions for \(k g_{\alpha \beta l} \in kT^{\ast,\ast}_{\alpha l} \beta \gamma\).

Given a multi-index \((\alpha_0 \cdots \alpha_\ell)\), we have, for each \(j = 0, \ldots, \ell\), a natural restriction map

\[
\tau_{\alpha_j} : kT^{\ast,\ast}_{\alpha_l \alpha_0 \cdots \alpha_{j-1} \alpha_j} \rightarrow kT^{\ast,\ast}_{\alpha_l \alpha_0 \cdots \alpha_j}
\]

defined componentwise by

\[
\tau_{\alpha_j} (\varphi_{i \in I}) = (\varphi_l)_{l \in I}
\]

for \((\varphi_l)_{l \in I} \in kT^{\ast,\ast}_{\alpha_l \alpha_0 \cdots \alpha_{j-1} \alpha_j} \beta \gamma\), where \(I' = \{(i_0, \ldots, i_l) \in I \mid U_{i_l} \subset V_{\alpha_0 \cdots \alpha_j}\}\). The map \(\tau_{\alpha_j}\) is a morphism of dgBV algebras.

Now for a triple \(V_\alpha, V_\beta, V_\gamma \in \mathcal{V}\), we define the restriction of \(k g_{\alpha \beta l}\) to \(kT^{\ast,\ast}_{\alpha_l \alpha_0 \cdots \alpha_j \alpha_k} l \gamma\) as the unique map \(k g_{\alpha \beta} : kT^{\ast,\ast}_{\alpha_l \alpha_0 \cdots \alpha_j \alpha_k} l \gamma\) that fits into the diagram

\[
kT^{\ast,\ast}_{\alpha_l \alpha_0 \cdots \alpha_j \alpha_k} l \gamma \xrightarrow{k g_{\alpha \beta}} kT^{\ast,\ast}_{\alpha_l \alpha_0 \cdots \alpha_j \alpha_k} l \gamma
\]

\[
kT^{\ast,\ast}_{\alpha_l \alpha_0 \cdots \alpha_j \alpha_k} l \gamma \xrightarrow{k g_{\alpha \beta}} kT^{\ast,\ast}_{\alpha_l \alpha_0 \cdots \alpha_j \alpha_k} l \gamma
\]

\[
kT^{\ast,\ast}_{\alpha_l \alpha_0 \cdots \alpha_j \alpha_k} l \gamma \xrightarrow{k g_{\alpha \beta}} kT^{\ast,\ast}_{\alpha_l \alpha_0 \cdots \alpha_j \alpha_k} l \gamma
\]

\[
kT^{\ast,\ast}_{\alpha_l \alpha_0 \cdots \alpha_j \alpha_k} l \gamma \xrightarrow{k g_{\alpha \beta}} kT^{\ast,\ast}_{\alpha_l \alpha_0 \cdots \alpha_j \alpha_k} l \gamma
\]

**Definition 3.17.** The morphisms \(\{k g_{\alpha \beta}\}_{\alpha \beta l}\) satisfying Condition 3.15 are said to form a set of compatible gluing morphisms if in addition the following conditions are satisfied:

1. \(0 g_{\alpha \beta} = id\) for all \(\alpha, \beta\);
2. (compatibility between different orders) for each \(k \in \mathbb{Z}_{\geq 0}\) and any pair \(V_\alpha, V_\beta \in \mathcal{V}\),

\[
k g_{\alpha \beta} \circ k g_{\alpha \beta} = k g_{\alpha \beta} \circ k g_{\alpha \beta};
\]

3. (cocycle condition) for each \(k \in \mathbb{Z}_{\geq 0}\) and any triple \(V_\alpha, V_\beta, V_\gamma \in \mathcal{V}\),

\[
k g_{\alpha \beta} \circ k g_{\alpha \beta} \circ k g_{\alpha \beta} = id
\]

when \(k g_{\alpha \beta}, k g_{\alpha \beta} \gamma\) and \(k g_{\alpha \beta}\) are restricted to \(kT^{\ast,\ast}_{\alpha_\alpha \beta \gamma}, kT^{\ast,\ast}_{\beta \alpha \beta \gamma}\) and \(kT^{\ast,\ast}_{\gamma \alpha \beta \gamma}\) respectively.

**Theorem 3.18.** There exists a set of compatible gluing morphisms \(\{k g_{\alpha \beta}\}\).
Lemma 3.19. Fixing $U_{i_0}, \ldots, U_{i_1} \in \mathcal{U}$ and $-d \leq j \leq 0$, we consider the index set $I_{i_0 \cdots i_1} := \{\alpha \mid U_{i_r} \subset V_{\alpha} \text{ for all } 0 \leq r \leq 1\}$ and the following Čech complex $\check{C}^*(I_{i_0 \cdots i_1}, 0^G) \otimes \mathbb{V}$ of vector spaces

$$
0 \rightarrow \prod_{\alpha \in I_{i_0 \cdots i_1}} 0^G(U_{i_0 \cdots i_1} \cap V_\alpha) \rightarrow \prod_{\alpha, \beta \in I_{i_0 \cdots i_1}} 0^G(U_{i_0 \cdots i_1} \cap V_{\alpha \beta}) \rightarrow \prod_{\alpha, \beta, \gamma \in I_{i_0 \cdots i_1}} 0^G(U_{i_0 \cdots i_1} \cap V_{\alpha \beta \gamma}) \cdots ,
$$

where each arrow is the Čech differential associated to the index set $I_{i_0 \cdots i_1}$. Then we have

$$H^{>0}(\check{C}(I_{i_0 \cdots i_1}, 0^G)) = 0 \text{ and } H^0(\check{C}(I_{i_0 \cdots i_1}, 0^G)) = 0^G(U_{i_0 \cdots i_1});$$

the same holds for $0^G \otimes \mathbb{V}$ for any vector space $\mathbb{V}$.

Proof. We consider the topological space $pt$ consisting of a single point and an indexed cover $(V_\alpha)_{\alpha \in I_{i_0 \cdots i_1}}$ such that $V_\alpha = pt$ for each $\alpha$. Then we take a constant sheaf $\mathcal{F}$ over $pt$ with $\mathcal{F}(pt) = 0^G(U_{i_0 \cdots i_1})$. Since $0^G(U_{i_0 \cdots i_1} \cap V_{\alpha_0 \cdots \alpha_t}) = 0^G(U_{i_0 \cdots i_1}) = \mathcal{F}(V_{\alpha_0 \cdots \alpha_t})$ for any $\alpha_0, \ldots, \alpha_t \in I_{i_0 \cdots i_1}$, we have a natural isomorphism $\check{C}^*(I_{i_0 \cdots i_1}, 0^G) \cong \check{C}^*(I_{i_0 \cdots i_1}, \mathcal{F})$. The result then follows by considering the Čech cohomology of $pt$.

Lemma 3.20 (Lifting Lemma). Let $b : \mathcal{F} \rightarrow \mathcal{H}$ be a surjective morphism of sheaves over $V := V_{\alpha_0 \cdots \alpha_t}$. For a Stein open subset $U := U_{i_0 \cdots i_1} \subset V$, let $\mathcal{V} \subset \mathcal{A}^q(\mathbf{A}) \otimes \mathcal{H}(U)$ and $\partial(\mathcal{V}) \subset \mathcal{A}^q(\delta_1) \otimes \mathcal{F}(U)$ such that $b(\partial(\mathcal{V})) = \mathcal{V}|_{\delta_1}$. Then there exists $\mathcal{V} \subset \mathcal{A}^q(\mathbf{A}) \otimes \mathcal{F}(U)$ such that $\mathcal{V}|_{\delta_1} = \partial(\mathcal{V})$ and $b(\mathcal{V}) = \mathcal{V}$. The same holds if $\mathcal{A}^q(\mathbf{A})$ and $\mathcal{A}^q(\delta_1)$ are replaced by $\mathcal{A}^q(\mathbf{A}|_I)$ and $\mathcal{A}^q(\mathbf{A}|_I)$ respectively.

Proof. By Lemma 3.3 there is a lifting $\mathcal{V} \subset \mathcal{A}^q(\mathbf{A}) \otimes \mathcal{F}(U)$ such that $\mathcal{V}|_{\delta_1} = \partial(\mathcal{V})$. Let $\mathcal{W} := \mathcal{W} - b(\mathcal{V}) \subset \mathcal{A}^q(\mathbf{A}) \otimes \mathcal{H}(U)$, where $\mathcal{A}^q(\mathbf{A}) \otimes \mathcal{H}(U)$ is the space of differential 0-forms whose restriction to $\delta_1$ is 0. Since $U$ is Stein, the map $b : \mathcal{F}(U) \rightarrow \mathcal{H}(U)$ is surjective. So we have a lifting $\tilde{u}$ to $\mathcal{A}^q(\mathbf{A}) \otimes \mathcal{F}(U)$.

Now the element $\mathcal{V} := \mathcal{V} + \tilde{u}$ satisfies the desired properties. The same proof applies to the case involving $\mathbf{A}|_I$ and $\mathbf{A}|_I$.

Lemma 3.21 (Key Lemma). Suppose we are given a set of gluing morphisms $\{k_{g_{\alpha \beta}}\}$ for some $k \geq 0$ satisfying Condition 3.15 and the cocycle condition (3.10). Then there exists a set of $\{k_{g_{\alpha \beta}}\}$ satisfying Condition 3.15, the compatibility condition (3.9) as well as the cocycle condition (3.10).

Proof. We will prove by induction on $l$ where $l = |I| - 1$ for a multi-index $I = (i_0, \ldots, i_l) \in \mathcal{I}$.

For $l = 0$, we fix $i = i_0$. From (3.5) in Condition 3.15, we have $k_{g_{\alpha \beta}, i} = \exp([-k_{\psi_{\alpha \beta}, i}, \cdot]) \circ k_{\psi_{\alpha \beta}, i}$ for some $k_{\alpha \beta, i} \in k_\mathcal{G}^{-1}(U_i)$. Also, from (2.4) in Definition 2.15, there exist elements $k_{b_{\alpha \beta}, i} \in k_\mathcal{G}^{-1}(U_i)$ such that

$$k_{b_{\alpha \beta}, i} \circ k_{\psi_{\alpha \beta}, i} = k_{\psi_{\alpha \beta}, i} \circ \exp((k_{b_{\alpha \beta}, i}(k_{b_{\alpha \beta}, i}, \cdot)) \circ k_{\psi_{\alpha \beta}, i}) \text{ where we use the fact that } k_{\psi_{\alpha \beta}, i} \text{ is an isomorphism preserving the Lie bracket } [\cdot, \cdot].$$

We have $k_{g_{\alpha \beta}, i} \circ k_{b_{\alpha \beta}, i} = \exp([-k_{\psi_{\alpha \beta}, i}, \cdot]) \circ \exp([-k_{\psi_{\alpha \beta}, i}(k_{b_{\alpha \beta}, i}, \cdot), \cdot]) \circ k_{b_{\alpha \beta}, i}$ by taking a lifting $\mathbf{Y}_{\alpha \beta, i}$ of the term $k_{\alpha \beta, i} \circ (k_{\psi_{\alpha \beta}, i}(k_{b_{\alpha \beta}, i}, \cdot))$ from $k_\mathcal{G}^{-1}(U_i)$ to $k_\mathcal{G}^{-1}(U_i)$ in the above equation (using the surjectivity of the map $k_{g_{\alpha \beta}, i} : k_\mathcal{G} \rightarrow k_{\mathcal{G}}$), we define a lifting of $k_{g_{\alpha \beta}, i}$:

$$k_{g_{\alpha \beta}, i} := \exp([\mathbf{Y}_{\alpha \beta, i}, \cdot]) \circ k_{\psi_{\alpha \beta}, i} : k_\mathcal{G}_{\alpha, i} \rightarrow k_\mathcal{G}_{\beta, i}.$$

As endomorphisms of $k_\mathcal{G}_{\alpha, i}$, we have $k_{g_{\alpha \beta}, i} \circ k_{g_{\alpha \beta}, i} \circ k_{g_{\alpha \beta}, i} = \exp([-k_{O_{\alpha \beta, i}, \cdot}, \cdot])$ for some $k_{O_{\alpha \beta, i}} \in k_{\mathcal{G}}^{-1}(U_i)$. From the fact that $k_{g_{\alpha \beta}, i} \circ k_{g_{\alpha \beta}, i} \circ k_{g_{\alpha \beta}, i} = \text{id}$ (mod $m^{k+1}$), we have $\exp([k_{g_{\alpha \beta}, i}, \cdot]) = k_{g_{\alpha \beta}, i} = 0$ (mod $m^{k+1}$) as the map $k_{\mathcal{G}}^{-1} \rightarrow \text{Der}(k_{\mathcal{G}})$ is injective (see Definition 2.13). Since every stalk $(k_{\mathcal{G}}, x)$ is a free
Hence we have \( k+1O_{\alpha\beta;\gamma,i} \in (m^{k+1}/m^{k+2}) \otimes C^0G^{-1}(U_i) \).

Now we consider the Čech complex \( \check{C}^*(U_i; C^0G^{-1}) \otimes C^0G^{k+1}(m^{k+1}/m^{k+2}) \) as in Lemma 3.19. The collection \( (k+1O_{\alpha\beta;\gamma,i})_{\alpha\beta;\gamma,i \in U_i} \) is a 2-cocycle in \( \check{C}^2(U_i; C^0G^{-1}) \otimes C^0G^{k+1}(m^{k+1}/m^{k+2}) \). By Lemma 3.19 for the case \( l = 0 \), there exists \( (k+1c_{\alpha\beta;i})_{\alpha\beta \in C^l(U_i; C^0G^{-1}) \otimes C^0G^{k+1}(m^{k+1}/m^{k+2}) \) whose image under the Čech differential is precisely \( (k+1O_{\alpha\beta;\gamma,i})_{\alpha\beta;\gamma} \). By the identification (3.11), we can regard \( k+1c_{\alpha\beta;i} \) as an element in \( k+1G_{\beta;\gamma}(U_i) \) such that \( k+1c_{\alpha\beta;i} = 0 \) (mod \( m^{k+1} \)). Therefore letting \( k+1g_{\alpha\beta;i} := \exp(k+1c_{\alpha\beta;i}) \) we have the cocycle condition \( k+1g_{\alpha\beta;\gamma,i} \circ k+1g_{\beta\gamma;\alpha,i} \circ k+1g_{\alpha\beta;i} = \text{id} \).

For the induction step, we assume that maps \( k+1g_{\alpha\beta,io...ij} \) satisfying all the required conditions have been constructed for each multi-index \( (i_0 \ldots i_j) \) with \( j \leq l-1 \). We shall construct \( k+1g_{\alpha\beta,io...ij} \) for any multi-index \( (i_0, \ldots, i_l) \). In view of Condition 3.15 what we need are elements \( k+1d_{\alpha\beta,io...ij} \in A^0(\Delta_i) \otimes k+1G^{-1}_\beta(U_{io...ij}) \) satisfying (3.7) and the cocycle condition (3.10), the latter of which can be written explicitly as

\[
\exp(k+1d_{\alpha\beta,io...ij}) \circ k+1g_{\alpha\beta,i_0} \circ \exp(k+1d_{\beta\gamma,io...ij}) \circ k+1g_{\beta\gamma,i_0} \circ \exp(k+1d_{\alpha\beta,io...ij}) \circ k+1g_{\alpha\beta,i_0} = \exp(k+1d_{\alpha\beta,io...ij}) \circ \exp(k+1g_{\alpha\beta,i_0}(k+1d_{\beta\gamma,io...ij})) \circ \exp(k+1g_{\alpha\beta,i_0}(k+1d_{\alpha\beta,io...ij}))) = \text{id}.
\]

Using the \( k+1d_{\alpha\beta,io...ij} \)'s that were defined previously, we let

\[
\partial_{k+1d_{\alpha\beta,io...ij}} := (k+1d_{\alpha\beta,io...ij} + k+1\phi_{\alpha\beta,i_0} + k+1\phi_{\alpha\beta,i_0} + \cdots + k+1\phi_{\alpha\beta,i_0} + k+1\phi_{\alpha\beta,i_0}),
\]

where \( k+1\phi_{\alpha\beta,i_0} \) is defined in Condition 3.15. For \( 0 \leq r_1 < r_2 \leq l \), we have

\[
\begin{align*}
\partial(k+1d_{\alpha\beta,io...ij}) &:= (k+1d_{\alpha\beta,io...ij} + k+1\phi_{\alpha\beta,i_0} + k+1\phi_{\alpha\beta,i_0} + \cdots + k+1\phi_{\alpha\beta,i_0} + k+1\phi_{\alpha\beta,i_0}) \\
&= \phi_{\alpha\beta,i_0} + \phi_{\alpha\beta,i_0} + \cdots + \phi_{\alpha\beta,i_0} + \phi_{\alpha\beta,i_0} = \phi_{\alpha\beta,i_0}
\end{align*}
\]

where the last case follows from the identity \( k+1d_{\alpha\beta,io...ij} + k+1\phi_{\alpha\beta,i_0} + k+1\phi_{\alpha\beta,i_0} + \cdots + k+1\phi_{\alpha\beta,i_0} \), which in turn follows from the definition of \( k+1\phi_{\alpha\beta,i_0} \) in Condition 3.15. Therefore we have \( \partial(k+1d_{\alpha\beta,io...ij}) \in A^0(\Delta_i) \otimes k+1G^{-1}_\beta(U_{io...ij}) \). By Lemma 3.20 we obtain \( k+1d_{\alpha\beta,io...ij} \in A^0(\Delta_i) \otimes k+1G^{-1}_\beta(U_{io...ij}) \) satisfying

\[
\partial(k+1d_{\alpha\beta,io...ij})|_{\Delta_i} = \partial(k+1d_{\alpha\beta,io...ij})|_{\Delta_i} = \partial(k+1d_{\alpha\beta,io...ij}) \quad (\mod m^{k+1}).
\]

Therefore, we have an obstruction term \( k+1O_{\alpha\beta,io...ij} \in A^0(\Delta_i) \otimes k+1G^{-1}_\beta(U_{io...ij}) \) given by

\[
k+1O_{\alpha\beta,io...ij} = k+1\phi_{\alpha\beta,io...ij} + k+1\phi_{\alpha\beta,io...ij} + \cdots + k+1\phi_{\alpha\beta,io...ij} + k+1\phi_{\alpha\beta,io...ij}
\]

which satisfies \( k+1O_{\alpha\beta,io...ij} = 0 \) (mod \( m^{k+1} \)). Direct computation gives \( \exp(k+1O_{\alpha\beta,io...ij}) = \text{id} \) for all \( r = 0, \ldots, l \). Using injectivity of \( k+1G_{\alpha;\gamma}^{-1} \rightarrow \text{Der}(k+1G_{\alpha}) \) we deduce \( (k+1O_{\alpha\beta,io...ij})|_{\Delta_i} = 0 \).
Via (3.11) again, we may regard the term $k+1\mathcal{O}_{\alpha\beta, i_0 \cdots i_i}$ as lying in $A^0_{\alpha\beta}(\mathbf{A}_I) \otimes G^{-1}(U_{i_0 \cdots i_i}) \otimes \mathfrak{m}^{k+1}/\mathfrak{m}^{k+2}$. By a similar argument as in the $l = 0$ case, we obtain an element $(k+1\mathcal{O}_{\alpha\beta, i_0 \cdots i_i})_{\alpha\beta}$ whose image under the Čech differential is precisely $(k+1\mathcal{O}_{\alpha\beta, i_0 \cdots i_i})_{\alpha\beta}$, and such that $(k+1\mathcal{O}_{\alpha\beta, i_0 \cdots i_i})_{\alpha\beta} = 0$. Therefore setting $k+1\mathcal{D}_{\alpha\beta, i_0 \cdots i_i} := k+1\mathcal{O}_{\alpha\beta, i_0 \cdots i_i} \otimes k+1\mathcal{D}_{\alpha\beta, i_0 \cdots i_i}$ solves the required cocycle condition (3.10). We also have $k+1\mathcal{D}_{\alpha\beta, i_0 \cdots i_i} = k^j\mathcal{D}_{\alpha\beta, i_0 \cdots i_i} = 0 (\text{mod } \mathfrak{m}^{k+1})$ by our construction, and $(k+1\mathcal{D}_{\alpha\beta, i_0 \cdots i_i})_{\alpha\beta} = 0 (\text{mod } \mathfrak{m}^{k+1})$ which is the required compatibility condition (3.7). This completes the proof of the lemma.

**Proof of Theorem 3.18** We prove by induction on the order $k$. For the initial case $k = 0$, as $\mathfrak{m}G^*$ is globally defined on $X$ with $\mathfrak{m}G^* = G^*|_{\mathbb{V}}$ (see Definition 2.13), we can (and have to) set $g_{\alpha\beta,i} = 0 \in \mathbb{V}$ and $\partial_{\alpha\beta,i} = 0$. The induction step is proven in Lemma 3.21.

### 3.3.2. Homotopy between two sets of gluing morphisms.

The set of compatible gluing morphisms $\{^k g_{\alpha\beta}\}$ constructed in Theorem 3.18 is not unique (except for $k = 0$). To understand the relation between two sets of such data, say, $\{^k g_{\alpha\beta}(0)\}$ and $\{^k g_{\alpha\beta}(1)\}$, we need, for each $k \in \mathbb{Z}_{\geq 0}$ and any pair $V_{\alpha}, V_{\beta} \in \mathbb{V}$, an isomorphism

$$k^h_{\alpha\beta} : kTW_{\alpha\beta}(\mathbf{A}_I) \to kTW_{\beta\alpha}(\mathbf{A}_I),$$

as a collection of maps $(k^h_{\alpha\beta})_{I} \in \mathbb{I}$ so that for each $\varphi = (\varphi_I)_{I} \in kTW_{\alpha\beta}(\mathbf{A}_I) \otimes A^*(\mathbf{A}_I)$ we have $(k^h_{\alpha\beta}(\varphi))_{I} = (k^h_{\alpha\beta}(\varphi))_{I}$, which preserves the algebraic structures $[\cdot, \cdot], \wedge$ obtained via tensoring with the dga $A^*(\mathbf{A}_I)$ and fits into the following commutative diagram

$$\begin{array}{ccc}
TW_{\alpha\beta}(\mathbf{A}_I) & \xrightarrow{r_0} & TW_{\alpha\beta}(\mathbf{A}_I) \\
\downarrow{k^h_{\alpha\beta}(0)} & & \downarrow{k^h_{\alpha\beta}(1)} \\
TW_{\beta\alpha}(\mathbf{A}_I) & \xrightarrow{r_1} & TW_{\beta\alpha}(\mathbf{A}_I)
\end{array}$$

Here $kTW_{\alpha\beta}(\mathbf{A}_I)$ is the Thom-Whitney complex constructed from the sheaf $A^*(\mathbf{A}_I) \otimes G_{\alpha\beta}^*|_{\mathbb{V}}$ where the degree $*$ in $kTW_{\alpha\beta}(\mathbf{A}_I)$ refers to the total degree on $A^*(\mathbf{A}_I) = A^*(\mathbf{A}_I) \otimes A^*(\mathbf{A}_I)$, and $r_j : A^*(\mathbf{A}_I) \to A^*(\mathbf{A}_I)$ is induced by the evaluation $A^*(\mathbf{A}_I) \to \mathbb{C}$ at $t_j = 1$ for $j = 0, 1$ as in Notation 3.7. The isomorphisms $k^h_{\alpha\beta}$'s are said to constitute a homotopy from $\{^k g_{\alpha\beta}(0)\}$ to $\{^k g_{\alpha\beta}(1)\}$ if they further satisfy the following condition (cf. Condition 3.15):

**Condition 3.22.**

1. For $U_i \subset V_\alpha \cap V_\beta$, we have

$$k^h_{\alpha\beta,i} = \exp(\{^k a_{\alpha\beta,i} \cdot \cdot \cdot \}) \circ \psi_{\alpha\beta,i},$$

for some element $k\mathcal{O}_{\alpha\beta,i} \in A^0(\mathbf{A}_I) \otimes G_{\beta}^{-1}(U_i)$ with $k\mathcal{O}_{\alpha\beta,i} = 0 \pmod{\mathfrak{m}}$;

2. For $U_{i_0}, \ldots, U_{i_l} \subset V_\alpha \cap V_\beta$, we have

$$k^h_{\alpha\beta,i_0 \cdots i_l} = \exp(\{^k x_{\alpha\beta,i_0 \cdots i_l} \cdot \cdot \cdot \}) \circ \left( k^h_{\alpha\beta,i_0 \cap U_{i_0 \cdots i_l}} \right),$$

for some element $k\mathcal{O}_{\alpha\beta,i_0 \cdots i_l} \in A^0(\mathbf{A}_I) \otimes A^0(\mathbf{A}_I) \otimes G_{\beta}^{-1}(U_{i_0 \cdots i_l})$ with $k\mathcal{O}_{\alpha\beta,i_0 \cdots i_l} = 0 \pmod{\mathfrak{m}}$;

3. The elements $k\mathcal{O}_{\alpha\beta,i}$'s satisfy the relation

$$r_j^*(k\mathcal{O}_{\alpha\beta,i}) = \begin{cases} k\mathcal{O}_{\alpha\beta,i}(0) & \text{for } j = 0, \\
k\mathcal{O}_{\alpha\beta,i}(1) & \text{for } j = 1, \end{cases}$$

where $k\mathcal{O}_{\alpha\beta,i}(j)$ is the element associated to $k\mathcal{O}_{\alpha\beta,i}(j)$ as in (3.5).
(4) the elements $k_{\alpha\beta,i_0\cdots i_l}$ satisfy the relation (cf. (3.7)):

\[ d_j^*\left(k_{\alpha\beta,i_0\cdots i_l}\right) = \begin{cases} k_{\alpha\beta,i_0\cdots i_l} & \text{for } j > 0, \\ k_{\alpha\beta,i_0\cdots i_l} + k_{o\beta,i_0i_l} & \text{for } j = 0, \end{cases} \]

where $k_{o\beta,i_0i_l} \in A^0(\Delta_1) \otimes G^{-1}_\beta(U_{i_0i_l})$ is the unique element such that $exp\left(k_{o\beta,i_0i_l}ight) \circ k_{o\beta,i_0i_l} = k_{o\beta,i_0i_l}$, and the relation

\[ r_j^*\left(k_{\alpha\beta,i_0\cdots i_l}\right) = \begin{cases} k_{\partial\alpha\beta,i_0\cdots i_l}(0) & \text{for } j = 0, \\ k_{\partial\alpha\beta,i_0\cdots i_l}(1) & \text{for } j = 1, \end{cases} \]

where $k_{\partial\alpha\beta,i_0\cdots i_l}(j) \in A^0(\Delta_1) \otimes G^{-1}_\beta(U_{i_0i_l})$ is the element associated to $k_{\alpha\beta,i_0\cdots i_l}(j)$ as in (3.6) for $j = 0, 1$.

**Definition 3.23.** A homotopy $\{k_{\alpha\beta}(0)\}$ to $\{k_{\alpha\beta}(1)\}$ is said to be compatible if in addition the following conditions are satisfied:

1. $0_{h_{\alpha\beta}} = id$ for all $\alpha, \beta$;
2. (compatibility between different orders) for each $k \in \mathbb{Z}_{\geq 0}$ and any pair $V_\alpha, V_\beta \in \mathcal{V}$,

\[ k_{h_{\alpha\beta}} \circ k_{\partial\alpha\beta} = k_{\partial\alpha\beta} \circ k_{h_{\alpha\beta}}; \]

3. (cocycle condition) for each $k \in \mathbb{Z}_{\geq 0}$ and any triple $V_\alpha, V_\beta, V_\gamma \in \mathcal{V}$,

\[ k_{h_{\alpha\beta}} \circ k_{h_{\beta\gamma}} \circ k_{h_{\alpha\gamma}} = id \]

when $k_{h_{\alpha\beta}}, k_{h_{\beta\gamma}}, k_{h_{\alpha\gamma}}$ are restricted to $kTW^*_{\alpha\beta\gamma}(1), kTW^*_{\beta\alpha\gamma}(1), kTW^*_{\gamma\alpha\beta}(1)$.

The same induction argument as in Theorem 3.18 proves the following:

**Proposition 3.24.** Given any two sets of compatible gluing morphisms $\{k_{\alpha\beta}(0)\}$ and $\{k_{\alpha\beta}(1)\}$, there exists a compatible homotopy $\{h_{\alpha\beta}\}$ from $\{k_{\alpha\beta}(0)\}$ to $\{k_{\alpha\beta}(1)\}$.

### 3.4. The Čech-Thom-Whitney complex.

The goal of this subsection is to construct a Čech-Thom-Whitney complex $\hat{\mathcal{C}}^*(TW, g)$ for each $k \in \mathbb{Z}_{\geq 0}$ from a given set $g = \{k_{\alpha\beta}\}$ of compatible gluing morphisms.

**Definition 3.25.** For $\ell \in \mathbb{Z}_{\geq 0}$, we let $kTW^*_{\alpha_0\cdots \alpha_\ell}(g) := \bigoplus_{i=0}^{\ell} kTW^*_{\alpha_{i_0}\cdots \alpha_\ell}(g)$ be the set of elements $(\varphi_0, \cdots, \varphi_{\ell})$ such that $\varphi_j = k_{\alpha_{ij}}(\varphi_i)$. Then the $k^{th}$-order Čech-Thom-Whitney complex over $X$, $\hat{\mathcal{C}}^*(TW, g)$ is defined by setting $\hat{\mathcal{C}}^0(TW^0g, g) := \prod_{\alpha_0\cdots \alpha_{\ell}}^\mathcal{C}^0_{\alpha_{i_0}\cdots \alpha_\ell}(g)$ and $\hat{\mathcal{C}}^0(TW, g) := \bigoplus_{p,q}^\mathcal{C}^0(TW_{p,q}, g)$ for each $k \in \mathbb{Z}_{\geq 0}$.

This is equipped with the Čech differential $k\delta_{\ell} := \sum_{j=0}^{\ell+1}(-1)^j r_{j,\ell+1} : \hat{\mathcal{C}}^j(TW, g) \to \hat{\mathcal{C}}^{j+1}(TW, g)$, where $r_{j,\ell} : \hat{\mathcal{C}}^{j-1}(TW, g) \to \hat{\mathcal{C}}^j(TW, g)$ is the natural restriction map defined componentwise by the map $r_{j,\ell} : kTW^*_{\alpha_{i_0}\cdots \alpha_\ell}(g) \to kTW^*_{\alpha_{i_0}\cdots \alpha_\ell}(g)$ which in turn comes from (3.8).

We define the $k^{th}$-order complex of polyvector fields over $X$ by $PV^*, g) := \text{Ker}(k\delta_{\ell})$ and denote the natural inclusion $kPV^*, g) \to \hat{\mathcal{C}}^0(TW, g)$ by $k\delta_{\ell-1}$, so we have the following sequence of maps

\[ 0 \to kPV^0g, g) \to k\hat{\mathcal{C}}^0(TW^0g, g) \to k\hat{\mathcal{C}}^1(TW^0g, g) \to \cdots \to k\hat{\mathcal{C}}^\ell(TW^0g, g) \to \cdots \]

For $\ell \in \mathbb{Z}_{\geq 0}$ and $k \geq l$, there is a natural map $k_{l,\ell} : k\hat{\mathcal{C}}^l(TW^0g, g) \to k\hat{\mathcal{C}}^{l+1}(TW^0g, g)$ defined componentwise by the map $k_{l,\ell} : k_TW^0_{\alpha_j\cdots \alpha_\ell} \to k_TW^0_{\alpha_j\cdots \alpha_\ell}$ obtained from $k_{l,\ell} : kG^*_{\alpha_j} \to kG^*_{\alpha_j}$ (see Definition 2.13). Similarly, we have the natural maps $k_{l,0} : kPV_{p,q} \to kPV_{p,q}$. 
**Definition 3.26.** The Čech-Thom-Whitney complex $\tilde{C}^k(TW, g) = \bigoplus_{p,q} \tilde{C}^k(TW_{p,q}, g)$ is defined by taking the inverse limit $\tilde{C}^k(TW_{p,q}, g) := \varprojlim_k \tilde{C}^k(TW_{p,q}, g)$ along the maps $k+1, k_0 : k+1 \tilde{C}^k(TW_{p,q}, g) \to k \tilde{C}^k(TW_{p,q}, g)$.

The complex of polyvector fields $PV^{*,*}(g) = \bigoplus_{p,q} PV^{p,q}(g)$ is defined by taking the inverse limit $PV^{p,q}(g) := \varprojlim_k PV^{p,q}(g)$ along the maps $k+1, k_0 : k+1 PV^{p,q}(g) \to k PV^{p,q}(g)$

The maps $k_0$'s commute with the Čech differentials $k \delta^k's$ and $\delta^l's$, so we have the following sequence of maps

\[ 0 \to PV^{p,q}(g) \to \tilde{C}^0(TW_{p,q}, g) \to \tilde{C}^1(TW_{p,q}, g) \to \cdots \to \tilde{C}^k(TW_{p,q}, g) \to \cdots \]

**Lemma 3.27.** Given $k+1 \mathbf{w} \in k+1 \tilde{C}^k(TW, g)$ with $k+1 \delta_{k+1}(k+1 \mathbf{w}) = 0$ and $k \mathbf{v} \in k \tilde{C}^k(TW, g)$ satisfying $k \delta^k(k \mathbf{v}) = k+1 \mathbf{w}$ (mod $k^{k+1}$), there exists a lifting $k+1 \mathbf{v} \in k+1 \tilde{C}^k(TW, g)$ such that $k+1 \delta_{k+1}(k+1 \mathbf{v}) = k+1 \mathbf{w}$. As a consequence, both (3.22) and (3.23) are exact sequences.

**Proof.** We only need to prove the first statement of the lemma because the second statement follows by induction on $k$ (note that the initial case for this induction is $k = -1$ where we take the trivial sequence whose terms are all zero).

Without loss of generality, we can assume that $k+1 \mathbf{w} \in k+1 \tilde{C}^k(TW_{p,q}, g)$ and $k \mathbf{v} \in k \tilde{C}^k(TW_{p,q}, g)$ for some fixed $p$ and $q$. We need to construct $k+1 \mathbf{v}_{\alpha_0, \ldots, \alpha_{\ell}} \in k+1 \tilde{C}^{k+1}(TW_{p,q}, g)$ for every multi-index $(\alpha_0, \ldots, \alpha_{\ell})$ which, by Definition 3.25, can be written as

\[ k+1 \mathbf{v}_{\alpha_0, \ldots, \alpha_{\ell}} = \left( k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}, \ldots, k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}} \right) \]

satisfying $k+1 \mathbf{v}_{\alpha_0, \ldots, \alpha_{\ell}} = k+1 g_{\alpha_0, \alpha_0} (k+1 \mathbf{v}_{\alpha_0, \ldots, \alpha_{\ell}})$, and each component $k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}$ is of the form

\[ k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}} = \left( k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}(i_0, \ldots, i_\ell) \right), \quad \text{where} \quad U_{i_0, \ldots, i_\ell} \subseteq V_{\alpha_0, \ldots, \alpha_{\ell}} \quad \text{and} \quad k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}(i_0, \ldots, i_\ell) \in \mathcal{A}^q(A_q) \otimes k+1 \mathcal{G}^p \left( U_{i_0, \ldots, i_\ell} \right) \]

and $k \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}(i_0, \ldots, i_\ell)$, such that $d^j(k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}(i_0, \ldots, i_\ell)) = k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}(i_0, \ldots, i_\ell, U_{i_0, \ldots, i_\ell})$ (see Definition 3.9).

We will use induction on $l$ to prove the existence of such an element.

The initial case is $l = 0$. We fix $U_{i_0, \ldots, i_0}$ and consider all the multi-indices $(\alpha_0, \ldots, \alpha_{\ell})$ such that $U_{i_0, \ldots, i_0} \subseteq V_{\alpha_0, \ldots, \alpha_{\ell}}$ for $r = 0, \ldots, q$. Using the fact that $k+1 \mathcal{G}^p$ is free over $k+1 R$, we can take a lifting $k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}(i_0, \ldots, i_\ell)$ of $k \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}(i_0, \ldots, i_\ell)$. Then we let $k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}(i_0, \ldots, i_\ell) := k+1 g_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}(k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}(i_0, \ldots, i_\ell))$ for $j = 1, \ldots, \ell$ and set

\[ k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}(i_0, \ldots, i_\ell) = \left( k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}(i_0, \ldots, i_\ell), \ldots, k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell}}(i_0, \ldots, i_\ell) \right). \]

Now the element

\[ k+1 \mathbf{w}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell+1}; i_0, \ldots, i_\ell} := k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell+1}; i_0, \ldots, i_\ell} - k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell+1}; i_0, \ldots, i_\ell}, \]

\[ + \cdots + (-1)^j k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell+1}; i_0, \ldots, i_\ell} = k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell+1}; i_0, \ldots, i_\ell}, \]

satisfies the condition that $k+1 \mathbf{w}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell+1}; i_0, \ldots, i_\ell} = 0$ (mod $k^{k+1}$). Under the identification (3.11), we can treat $k+1 \mathbf{w}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell+1}; i_0, \ldots, i_\ell}$ as an $(\ell+1)$-cocycle in the Čech complex $\tilde{C}^{k+1}(I_{i_0, \ldots, i_\ell}, \mathcal{G}^p) \otimes \mathcal{A}^q(A_q) \otimes (k^{k+1}/m^{k+2})$. By Lemma 3.19 there exists $(k+1 \mathbf{c}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell+1}; i_0, \ldots, i_\ell})_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell+1}} \subseteq \tilde{C}^{k+1}(I_{i_0, \ldots, i_\ell}, \mathcal{G}^p) \otimes \mathcal{A}^q(A_q) \otimes (k^{k+1}/m^{k+2})$ whose image under the Čech differential is precisely $(k+1 \mathbf{w}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell+1}; i_0, \ldots, i_\ell})_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell+1}}$. Therefore if we let

\[ k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell+1}; i_0, \ldots, i_\ell} := k+1 \mathbf{v}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell+1}; i_0, \ldots, i_\ell} - k+1 \mathbf{c}_{\alpha_0, \alpha_0, \ldots, \alpha_{\ell+1}; i_0, \ldots, i_\ell}, \]
then its image under the Čech differential is $(k+1) v_{α_0...α_{l+1};i_0...i_l}^{0...α_l}$ as desired.

Next we suppose that we are given $k+1 v_{α_0...α_{l+1};i_0...i_l}$ for some $l ≥ q$. Then we need to construct $k+1 v_{α_0...α_{l+1};i_0...i_l}$ for any $U_{i_0...i_l}$ and $V_{α_0...α_l}$ such that $U_{i_r} ⊂ V_{α_0...α_r}$ for $r = 1, ..., l+1$. We fixed $U_{i_0...i_l}$ and consider one such $V_{α_0...α_l}$.

Letting

$$∂(k+1 v_{α_0...α_{l+1};i_0...i_l}) := (k+1 v_{α_0...α_{l+1};i_0...i_l}, ..., k+1 v_{α_0...α_{l+1};i_0...i_l})$$

gives an element in $A^q(Δ_{l+1}) ⊕ k+1 g^{p}_{α_0}(U_{i_0...i_l})$. Using Lemma 3.20 we can construct an element $k+1 v_{α_0...α_{l+1};i_0...i_l} ∈ A^q(Δ_{l+1}) ⊕ k+1 g^{p}_{α_0}(U_{i_0...i_l})$ such that

$$k+1 v_{α_0...α_{l+1};i_0...i_l} = k v_{α_0...α_{l+1};i_0...i_l} (mod m^{k+1}),$$

$$k+1 v_{α_0...α_{l+1};i_0...i_l} = ∂(k+1 v_{α_0...α_{l+1};i_0...i_l}).$$

We then let $k+1 v_{α_j;α_0...α_{l+1};i_0...i_l} := k+1 v_{α_j;α_0...α_{l+1};i_0...i_l}^{k+1} v_{α_0...α_{l+1};i_0...i_l}$ for $j = 1, ..., l$ and set

$$k+1 v_{α_0...α_{l+1};i_0...i_l} := (k+1 v_{α_0...α_{l+1};i_0...i_l}, ..., k+1 v_{α_0...α_{l+1};i_0...i_l}).$$

Now the elements $k+1 δ_j(k+1 v_{α_0...α_{l+1};i_0...i_l})$ and $k+1 w_{α_0...α_{l+1};i_0...i_l}$ agree modulo $m^{k+1}$ and on the boundary $Δ_{l+1}$ of the simplex $Δ_{l+1}$, so the rest of the proof of this induction step would be the same as the initial case $l = q$. □

**Corollary 3.28.** For all $k, l ∈ Z_0$, the map $k+1 : k+1 C_ℓ(TW^{p,q}, g) → k C_ℓ(TW^{p,q}, g)$ and hence the induced map $k+1 : C_ℓ(TW^{p,q}, g) → C_ℓ(TW^{p,q}, g)$ are surjective; in particular, both $k+1 PV^{p,q}(g) → k PV^{p,q}(g)$ and $k+1 PV^{p,q}(g) → k PV^{p,q}(g)$ are surjective.

**Proof.** It suffices to show that for any $kv ∈ k C_ℓ(TW, g)$, there exists $k+1 v ∈ k+1 C_ℓ(TW, g)$ such that $k+1 k(y) = k v$. If $k+1 k(y) = 0$, then applying Lemma 3.27 with $k+1 v = 0$ gives the desired $k+1 v$. For a general $k v$, we let $k w = k δ_j(k v)$. Since $k+1 k(y) = 0$, we can find a lifting $k+1 k(y)$ such that $k+1 k(y)(k+1 v) = k v$. Applying Lemma 3.27 again, we obtain $k+1 v$ satisfying $k+1 k(y)(k+1 v) = k v$. □

**Definition 3.29.** Let $g(0) = \{g_{αβ}(0)\}$ and $g(1) = \{g_{αβ}(1)\}$ be two sets of compatible gluing morphisms, and $h = \{h_{αβ}\}$ be a compatible homotopy from $g(0)$ to $g(1)$. For $ℓ ≥ 0$, we let $k TW_{α_0...α_{l+1};i_0...i_l}^{p,q}(h) ⊂ T^k W_{α_0...α_{l+1};i_0...i_l}(h)$ be the set of elements $(ϕ_0, ..., ϕ_l)$ such that $ϕ_j = h_{α_0...α_{l+1};i_0...i_l}$. Then, for each $k ∈ Z_0$, the $k$-th order homotopy Čech-Thom-Whiteley complex is defined by setting $k C_ℓ(TW^{p,q}, h) := ∏_{α_0...α_{l+1}} k TW^{p,q}_{α_0...α_{l+1};i_0...i_l}(h)$ and $k C_ℓ(TW, h) = ∏_{p,q} k C_ℓ(TW^{p,q}, h)$. We have the natural restriction map $r_{j,ℓ} : k C_ℓ-1(TW, h) → k C_ℓ(TW, h)$ as in Definition 3.25.

Let $k δ_ℓ := ∑_{j=0}^l (-1)^j k r_{j,ℓ+1} : k C_ℓ(TW, h) → k C_ℓ+1(TW, h)$ be the Čech differential acting on $k C_ℓ(TW, h)$. Then the $k$-th order homotopy polyvector field on $X$ is defined as $k PV^{*,*}(h) := Ker(k δ_0)$. So we have the following sequences

(3.24) $0 → k PV^{p,q}(h) → k C_ℓ(TW^{p,q}, h) → ... → k C_ℓ(TW^{p,q}, h) → ...$,

(3.25) $0 → PV^{p,q}(h) → C_ℓ(TW^{p,q}, h) → ... → C_ℓ(TW^{p,q}, h) → ...$,

where (3.25) is obtained from (3.24) by taking the inverse limit. We also write $C_ℓ(TW, h) := ∑_{p,q} C_ℓ(TW^{p,q}, h)$.

We further let $k r_j^* : k C_ℓ(TW^{p,q}, g(j)) → k C_ℓ(TW^{p,q}, g(j))$ and $k r_j^* : k PV^{p,q}(h) → k PV^{p,q}(g(j))$ be the maps induced by $r_j^* : A^*(Δ_1) → A^*(Δ_1)$ for $j = 0, 1$, and let $r_j^* := lim_k r_j^*$. Then we have the
Lemma 3.30. Given $k+1 \omega \in k+1 \check{c}^2(TW, h)$ with $k+1 \delta_{\ell+1}(k+1 \omega) = 0$, $k+1 \alpha_j \in k+1 \check{c}^2(TW, g(j))$ satisfying $k+1 \delta_{\ell}(k+1 \alpha_j) = k+1 \check{r}_j(k+1 \omega)$ and $k \nu \in k \check{c}^2(TW, h)$ such that $k \delta_{\ell}(k \nu) = k+1 \nu \mod m^{k+1}$ and $k+1 \check{r}_j(k \nu) = k+1 \alpha_j \mod m^{k+1}$, there exists $k+1 \nu \in k+1 \check{c}^2(TW, h)$ such that $k+1 \delta_{\ell}(k+1 \nu) = k \nu$, $k+1 \check{r}_j(k+1 \nu) = k+1 \alpha_j$ and $k+1 \delta_{\ell}(k+1 \nu) = k+1 \omega$. As a consequence, both (3.24) and (3.25) are exact sequences.

Furthermore, the maps $\alpha \beta : k \check{c}^2(TWP^q, h) \to k \check{c}^2(TWP^q, h)$ and $\alpha \beta : PV^p, q(h) \to PV^p, q(h)$, as well as $k \check{r}_j : PV^p + (h) \to PV^p + (g(j))$ and $k \check{r}_j : PV^p + (h) \to PV^p + (g(j))$ are all surjective.

3.5. The dgBV algebra structure. The complex $PV^p, q(g)$ (as well as $PV^p, q(h)$) constructed in [3.4] is only a graded vector space. In this subsection, we equip it with two differential operators $\check{\partial}$ and $\check{\Delta}$, turning it into a dgBV algebra.

We fix a set of compatible gluing morphisms $g = \{k \gamma_{\alpha \beta}\}$ consisting of isomorphisms $k \gamma_{\alpha \beta} : kTW^p, q_{\alpha, \alpha} \to kTW^p, q_{\beta, \beta}$ for each $k \in Z_{\geq 0}$ and pair $V_{\alpha}, V_{\beta} \subset V$. Both $kTW^p, q_{\alpha, \alpha}$ and $kTW^p, q_{\beta, \beta}$ are dgBV algebras with differentials and BV operators given by $k \check{\partial}_{\alpha}$, $k \check{\partial}_{\beta}$ and $k \check{\Delta}_{\alpha}$, $k \check{\Delta}_{\beta}$ respectively.

Lemma 3.31. For each $k \in Z_{\geq 0}$ and pair $V_{\alpha}, V_{\beta} \subset V$, there exists $k \omega_{\alpha, \beta} \in kTW^\alpha_{\alpha, \beta}$ such that $k \omega_{\alpha, \beta} = 0 \mod m$, $k+1 \omega_{\alpha, \beta} = k \omega_{\alpha, \beta} \mod m^{k+1}$ and $k g_{\alpha \beta} \circ k \check{\partial}_{\beta} \circ k g_{\alpha \beta} = [k \omega_{\alpha, \beta}, \gamma_{\alpha \beta}]$. Furthermore, if we let $k \omega_{\alpha, \beta} : (k \gamma_{\alpha \beta}, k \gamma_{\alpha \beta}(k \omega_{\alpha, \beta}))$, then $(k \omega_{\alpha, \beta})_{\alpha \beta}$ is a Čech 1-cocycle in $k \check{c}^1(TW^{p, q}, g)$.

Proof. By Lemma 2.7, we have

$$\exp(-[k \partial_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu] \circ d \circ \exp([k \partial_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu]) = d - \frac{1}{[k \partial_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu]} d[k \partial_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu}],} $$

where $d$ is the de Rham differential acting on $A^*(\mathbf{a})$ (recall that $k \check{\partial}_{\beta}$ is induced by the de Rham differential on $A^*(\mathbf{a})$). Then using (3.6) in Condition 3.15 (i.e. $k g_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu = \exp([k \partial_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu]) \circ (k g_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu})$, we obtain

$$k g_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu} \circ d \circ k g_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu} = d - \frac{1}{[k \partial_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu]} d[k \partial_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu}],} $$

Now we put

$$k \omega_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu} := -k g_{\alpha \beta, i_0 \partial_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu}] - 1 d[k \partial_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu}],} $$

Then $k+1 \omega_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu} = k \omega_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu} \mod m^{k+1}$. To check that it is well-defined as an element in $kTW^\alpha_{\alpha, \beta}$, we compute using Lemma 2.5 to get $d + [d_{\gamma_{\check{r}_j}, (k \omega_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu}), \gamma_{\alpha \beta, i_0 \cdots i_{\ell-1}, \nu}],}$. By
injectivity of $k \mathcal{G}_a^{-1} \hookrightarrow \text{Der}(k \mathcal{G}_a^T)$, this implies that $d^*_a(k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}) = k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}$. To see that it is a Čech cocycle, we deduce from its definition that $[k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}] = k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}$. Thus, by direct computation, we have $[k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}] = 0$ and can conclude that

$$k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l} = 0.$$ 

□

We have a similar result concerning the difference between the BV operators $k \Delta_a$ and $k \Delta_\beta$:

**Lemma 3.32.** For each $k \in \mathbb{Z}_{\geq 0}$ and pair $V_\alpha, V_\beta \subset V$, there exists $k \mathcal{F}_{\alpha;\beta} \in k \mathcal{W}_{a;\alpha}^0$ such that $k \mathcal{F}_{\alpha;\beta} = 0 \pmod{m}$, $k+1 \mathcal{F}_{\alpha;\beta} = k \mathcal{F}_{\alpha;\beta} \pmod{m+1}$, and $k g_\beta \circ k \Delta_\beta \circ k g_\beta - k \Delta_a = [k \mathcal{F}_{\alpha;\beta}]$. Furthermore, if we let $k \mathcal{F}_{\alpha \beta} := (k \mathcal{F}_{\alpha;\beta}, k g_\beta(k \mathcal{F}_{\alpha;\beta}))$, then $(k \mathcal{F}_{\alpha \beta})_{\alpha \beta}$ is a Čech 1-cocycle in $k \mathcal{C}^1(T \mathcal{W}^0, g)$.

**Proof.** To simplify notations, we introduce the power series

$$T(x) := e^{-x} - 1 = \sum_{k=0}^{\infty} (-1)^k x^k (k+1)!$$

Similar to the previous proof, we have

$$\exp(-[k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}]) \circ \exp(-[k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}]) \circ k \Delta_\beta \circ \exp([k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}]) \circ \exp([k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}])$$

$$= k \Delta_\beta - ([\exp(-[k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}]) \circ T([k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}]) \circ k \Delta_\beta)(k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}),]$$

$$- [(T(k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l})) \circ k \Delta_\beta)(k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l},]$$

using Lemma 2.5. By (3.5) in Condition 3.15 we can write $k g_\beta \circ k \Delta_\beta \circ k g_\beta - k \Delta_a = [k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}]$, so

$$k g_\beta \circ k \Delta_\beta \circ k g_\beta - k \Delta_a = -[(k g_\beta \circ T([k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}]) \circ k \Delta_\beta)(k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l},)]$$

$$- [k \psi_{\alpha \beta, i_0 \circ T([k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}]) \circ k \Delta_\beta)(k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l},]$$

Now we put

$$k \mathcal{F}_{\alpha;\beta, i_0 \cdots i_l} := - (k g_\beta \circ T([k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}]) \circ k \Delta_\beta)(k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l})$$

$$- (k \psi_{\alpha \beta, i_0 \circ T([k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}]) \circ k \Delta_\beta)(k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l},]$$

$$= - (k g_\beta \circ T([k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}]) \circ k \Delta_\beta)(k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}) + 1 \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}$$

where $1 \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}$ denotes the constant function with value 1 on $\mathcal{W}_{\alpha \beta, i_0 \cdots i_l}$. We need to check the following conditions for the elements $k \mathcal{F}_{\alpha;\beta, i_0 \cdots i_l}$’s:

1. $k+1 \mathcal{F}_{\alpha;\beta, i_0 \cdots i_l} = k \mathcal{F}_{\alpha;\beta, i_0 \cdots i_l} \pmod{m+1}$;

2. $k \mathcal{F}_{\alpha;\beta} := (k \mathcal{F}_{\alpha;\beta, i_0 \cdots i_l})_{i_0 \cdots i_l} \in k \mathcal{W}_{a;\alpha}^0$ (see Definition 3.9);

3. letting $k \mathcal{F}_{\alpha;\beta} := (k \mathcal{F}_{\alpha;\beta, i_0 \cdots i_l})_{i_0 \cdots i_l} \in k \mathcal{W}_{a;\alpha}^0$, we have

$$k g_\beta \circ T([k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}]) \circ k \Delta_\beta)(k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}) + 1 \mathcal{W}_{\alpha \beta, i_0 \cdots i_l} \circ k \mathcal{F}_{\alpha;\beta, i_0 \cdots i_l}$$

$$= - (k g_\beta \circ T([k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}]) \circ k \Delta_\beta)(k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}) + 1 \mathcal{W}_{\alpha \beta, i_0 \cdots i_l} \circ k \mathcal{F}_{\alpha;\beta, i_0 \cdots i_l}$$

where $k \mathcal{F}_{\alpha;\beta, i_0 \cdots i_l} = (k \psi_{\alpha;\beta, i_0 \circ T([k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l}]) \circ k \Delta_\beta)(k \mathcal{W}_{\alpha \beta, i_0 \cdots i_l})$ and

4. that $(k \mathcal{F}_{\alpha;\beta})_{\alpha \beta}$ is a Čech 1-cocycle in $k \mathcal{C}^1(T \mathcal{W}^0, g)$.

The properties (1)-(4) are proven by applying the comparison (2.9) of the volume forms in Definition 2.18 (which can be regarded as a more refined piece of information than the comparison of BV operators in (2.3)) and Lemma 2.8 in the same manner, together with some rather tedious (at least notationally) calculations. For simplicity, we shall only present the proof of (1) here.
To prove (1), first notice that the term \( (k g_{\alpha}, i_0) \circ \text{Der}(k G, [-, -]) \) already satisfies the equality, so we only need to consider the case for \( l = 0 \). In the rest of this proof, we shall work (mod \( \mathbf{m}^{k+1} \)), meaning that all equalities hold (mod \( \mathbf{m}^{k+1} \)). First of all, the equation \( k+1, k \gamma_{\beta} \circ k+1, g_{\alpha, i_0} = k g_{\alpha, i_0} \circ k+1, k \gamma_{\alpha} \) (mod \( \mathbf{m}^{k+1} \)) can be rewritten as

\[
\exp(-k a_{\alpha, i_0}; \cdot) \circ \exp(k_{\alpha, i_0; \cdot}) \circ k+1, k \gamma_{\beta} = k \psi_{\alpha, i_0} \circ k+1, k \gamma_{\alpha} \circ k+1, k \psi_{\alpha, i_0} = \exp(k+1, b_{\alpha, i_0}; \cdot) \circ k+1, k \gamma_{\beta}
\]

using \([3, 5]\) and \([2, 4]\), so we have

\[
-k+1 a_{\alpha, i_0} = (-k+1, b_{\beta, i_0}) \circ (-k a_{\beta, i_0})
\]

by the injectivity of \( k G^{-1} \rightarrow \text{Der}(k G) \).

Applying Lemma \([2, 5]\) to the dgLa \((k G^{*}, [-, -], k \Delta_{\beta})\), we get \((\exp(-k+1, b_{\beta, i_0}) \circ \exp(-k a_{\beta, i_0}) \circ 0 = \exp(-k+1 a_{\beta, i_0}) \circ 0, \) which can be expanded as

\[
0 = -\exp(-k+1, b_{\beta, i_0}) \circ T([-k a_{\beta, i_0}; \cdot]) \circ (k \Delta_{\beta})\)

\[-\exp(k+1, k b_{\beta, i_0}) \circ (k \Delta_{\beta})\]

\[+ \exp([-k+1 a_{\beta, i_0}; \cdot]) \circ (k \Delta_{\beta})\]

\[= k_{\beta, i_0} - k_{\alpha, i_0} \circ T([-k+1, k b_{\beta, i_0}; \cdot]) \circ (k \Delta_{\beta})
\]

From \([2, 9]\), we learn that \( k_{\beta, i_0} = -k_{\alpha, i_0} \). Hence it remains to show that

\[
k_{\beta, i_0} + k_{\alpha, i_0} \circ (k+1, k b_{\beta, i_0}) = (T([-k+1, k b_{\beta, i_0}; \cdot]) \circ (k+1, k b_{\beta, i_0})
\]

which follows from the relation

\[
\exp(k_{\beta, i_0} + k_{\alpha, i_0} \circ (k+1, k b_{\beta, i_0})) \circ (k \omega_{\beta}) = (\psi_{\alpha, i_0} \circ k+1, k \gamma_{\alpha} \circ \psi_{\alpha, i_0}) \circ (k+1, k \omega_{\beta})
\]

\[= \exp(L_{k+1, k b_{\beta, i_0}}) (k \omega_{\beta}) = \exp([k \partial_{\beta}, (k+1, k b_{\beta, i_0})]) (k \omega_{\beta})
\]

coming from Definition \([2, 18]\) and using Lemma \([2, 8]\).}

The same results hold with exactly the same proofs for the homotopy Čech-Thom-Whitney complex with gluing morphisms \( k h_{\alpha, i_0} : k TW_{\alpha, i_0}(\mathbf{1}) \rightarrow k TW_{\beta, i_0}(\mathbf{1}) \), where \( k TW_{\alpha, i_0}(\mathbf{1}) \) and \( k TW_{\beta, i_0}(\mathbf{1}) \) are equipped with the differentials \( k D_{\alpha} := d_{\mathbf{1}} \otimes 1 + 1 \otimes k \partial_{\alpha} \) and \( k D_{\beta} := d_{\mathbf{1}} \otimes 1 + 1 \otimes k \partial_{\beta} \) and BV operators \( k \Delta_{\alpha} \) and \( k \Delta_{\beta} \) respectively. Such results are summarized in the following Lemma.

**Lemma 3.33.** There exist \( k w_{\alpha, i_0} \in k TW_{\alpha, i_0}(\mathbf{1}) \) and \( k F_{\alpha, i_0} \in k TW_{\alpha, i_0}(\mathbf{1}) \) such that \( k w_{\alpha, i_0} = 0 \) (mod \( \mathbf{m} \)) and \( k F_{\alpha, i_0} = 0 \) (mod \( \mathbf{m} \)), and

\[
k_{h_{\alpha, i_0}} \circ k D_{\beta} \circ k h_{\alpha, i_0} = [k w_{\alpha, i_0}; \cdot, \cdot], \quad k h_{\alpha, i_0} \circ k \Delta_{\beta} \circ k h_{\alpha, i_0} = [k D_{\alpha}; \cdot, \cdot, \cdot].
\]

Furthermore, if we let

\[
k_{w_{\alpha, i_0}} := \langle k w_{\alpha, i_0}, k h_{\alpha, i_0} \rangle, \quad k F_{\alpha, i_0} := \langle k F_{\alpha, i_0}, k h_{\alpha, i_0} \rangle,
\]

then \( k w_{\alpha, i_0} \) and \( k F_{\alpha, i_0} \) are Čech 1-cocycles in the complex \( k \tilde{C}(TW, h) \).

We conclude this subsection by the following theorem:
Theorem 3.34. There exist elements $\mathbf{d} = (d_\alpha)_\alpha = \lim_k (k^{d}_\alpha)_\alpha \in \mathcal{C}^0(TW^{-1,1}, g)$ and $f = (f_\alpha)_\alpha = \lim_k (k^{f}_\alpha)_\alpha \in \mathcal{C}^0(TW^{0,0}, g)$ such that
\[
g_{\beta \alpha} \circ (\partial_\beta + [d_\beta, \cdot]) \circ g_{\alpha \beta} = \partial_\alpha + [d_\alpha, \cdot], \quad g_{\beta \alpha} \circ (\Delta_\beta + [f_\beta, \cdot]) \circ g_{\alpha \beta} = \Delta_\alpha + [f_\alpha, \cdot].
\]
Also, $(\partial_\alpha + [d_\alpha, \cdot])_\alpha$ and $(\Delta_\alpha + [f_\alpha, \cdot])_\alpha$ glue to give operators $\partial$ and $\Delta$ on $PV^{*,*}(g)$ such that
(1) $\partial$ is a derivation of $[\cdot, \cdot]$ and $\wedge$ in the sense that
\[
\partial [u, v] = [\partial u, v] + (-1)^{|u|+1} [u, \partial v], \quad \partial (u \wedge v) = (\partial u) \wedge v + (-1)^{|u|} u \wedge (\partial v),
\]
where $|u|$ and $|v|$ denote respectively the total degrees (i.e. $|u| = p + q$ if $u \in PV^{p,q}(g)$) of the homogeneous elements $u, v \in PV^*(g)$;
(2) the BV operator $\Delta$ satisfies the BV equation and is a derivation for the bracket $[\cdot, \cdot]$, i.e.
\[
\Delta [u, v] = [\Delta u, v] + (-1)^{|u|+1} [u, \Delta v], \quad \Delta (u \wedge v) = (\Delta u) \wedge v + (-1)^{|u|} u \wedge (\Delta v) + (-1)^{|u|} [u, v],
\]
for homogeneous elements $u, v \in PV^*(g)$; and
(3) we have $\Delta^2 = 0$ and
\[
\bar{\partial}^2 = 0 = \bar{\partial} \Delta + \Delta \bar{\partial} \mod m,
\]
so $(PV^{*,*}, \wedge, \partial, \Delta) \mod m$ forms a dgBV algebra.

Moreover, if $(\mathbf{d}', f')$ is another pair of such elements defining operators $\partial'$ and $\Delta'$, then we have
\[
\mathbf{d}' - \partial = [v_1, \cdot], \quad \Delta' - \Delta = [v_2, \cdot],
\]
for some $v_1 \in PV^{-1,1}(g)$ and $v_2 \in PV^{0,0}(g)$.

Proof. In view of Lemmas 3.31 and 3.32 we have a Čech 1-cocycle $w = (w_{\alpha \beta})_{\alpha \beta} = \lim_k (k^{w}_{\alpha \beta})_{\alpha \beta}$ and $f = (f_{\alpha \beta})_{\alpha \beta} = \lim_k (k^{f}_{\alpha \beta})_{\alpha \beta}$. Using the exactness of the Čech-Thom-Whitney complex in Lemma 3.27, we obtain $\mathbf{d} \in \mathcal{C}^0(TW^{-1,1}, g)$ and $f \in \mathcal{C}^0(TW^{0,0}, g)$ such that the images of $-\mathbf{d}$ and $-f$ under the Čech differential $d_0$ are $w$ and $f$ respectively, and also $\mathbf{d} = 0 \mod m$ and $f = 0 \mod m$. Therefore we obtain the identities
\[
g_{\beta \alpha} \circ (\partial_\beta + [d_\beta, \cdot]) \circ g_{\alpha \beta} = \partial_\alpha + [d_\alpha, \cdot], \quad g_{\beta \alpha} \circ (\Delta_\beta + [f_\beta, \cdot]) \circ g_{\alpha \beta} = \Delta_\alpha + [f_\alpha, \cdot].
\]
Also notice that if we have another choice of $\mathbf{d}'$ and $f'$ such that the images of $-\mathbf{d}'$ and $-f'$ under the Čech differential $d_0$ are $w$ and $f$ respectively, then we must have $\mathbf{d}' - \partial = \partial - \mathbf{d}' = [v_1, \cdot]$ and $f' - \partial = f - f' = [v_2, \cdot]$ for some elements $v_1 \in PV^{-1,1}(g)$ and $v_2 \in PV^{0,0}(g)$.

It remains to show that the operators $\partial$ and $\Delta$ defined by $\mathbf{d}$ and $f$ satisfying the desired properties. First note that we have an injection $\delta_{-1} : PV^{p,q}(g) \hookrightarrow \mathcal{C}^0(TW^{p,q}, g) = \prod_k TW^{p,q}$, where we write $TW^{p,q}_\alpha := \lim_k TW^{p,q}_\alpha$. Also the product $\wedge$ and the Lie bracket $[\cdot, \cdot]$ on $PV^{*,*}(g)$ are induced by those on each $TW^{*,*}_\alpha$. Since $\partial$ and $\Delta$ are defined by gluing the operators $(\partial_\alpha + [d_\alpha, \cdot])_\alpha$ and $(\Delta_\alpha + [f_\alpha, \cdot])_\alpha$, we only have to check the required identities on each $TW^{*,*}_\alpha$, which hold because both $[d_\alpha, \cdot]$'s and $[f_\alpha, \cdot]$ are derivations of degree 1 and $\mathbf{d} = 0 \mod m$. Also, $(\Delta_\alpha + [f_\alpha, \cdot])^2 = \Delta_\alpha^2 + [\Delta_\alpha(f_\alpha), \cdot] = 0$ ($\Delta_\alpha(f_\alpha) = 0$ for degree reason), so we have $\Delta^2 = 0$. \qed

For the homotopy Čech-Thom-Whitney complex we have the following proposition which is parallel to Theorem 3.34.
Proposition 3.35. There exist elements $\mathbf{D} = (D_\alpha)_\alpha = \lim_k (k^{D}_\alpha)_\alpha \in \mathcal{C}^0(TW^{-1,1}, h)$ and $\mathbf{F} = (F_\alpha)_\alpha = \lim_k (k^{F}_\alpha)_\alpha \in \mathcal{C}^0(TW^{0,0}, h)$ such that
\[
h_{\beta \alpha} \circ (D_\beta + [D_\beta, \cdot]) \circ h_{\alpha \beta} = D_\alpha + [D_\alpha, \cdot], \quad h_{\beta \alpha} \circ (D_\beta + [F_\beta, \cdot]) \circ h_{\alpha \beta} = F_\alpha + [F_\alpha, \cdot].
\]
Furthermore, \( (D_\alpha + [D_\alpha, \cdot])_\alpha \) and \((\Delta_\alpha + [\Delta_\alpha, \cdot])_\alpha\) glue to give operators \( D \) and \( \Delta \) on \( PV^* \) so that \((PV^*)^*(h)\) satisfies (1) – (3) of Theorem 3.34 (with \( D \) playing the role of \( \bar{\partial} \)).

4. Abstract construction of the de Rham differential complex

4.1. The de Rham complex. Given a set of compatible gluing morphisms \( g = \{k g_{\alpha \beta}\} \), the goal of this subsection is to glue the local filtered de Rham modules \( kK_\alpha \) over \( V_\alpha \) to form a global differential graded algebra over \( X \). Similar to \([3,2]\), we consider a sheaf of filtered de Rham modules \((\mathcal{K}_*, \mathcal{K}_*, \land, \partial)\) over a sheaf of BV algebras \((\mathcal{G}, \land, \Delta)\) on \( V \) and a countable acyclic cover \( U = \{ U_i \}_{i \in \mathbb{Z}_+} \) of \( V \) which satisfies the condition that \( H^{>0}(U_{i_0 \cdots i_r}, \mathcal{K}_I) = 0 \) for all \( j, r \) and all finite intersections \( U_{i_0 \cdots i_r} := U_{i_0} \cap \cdots \cap U_{i_r} \).

**Definition 4.1.** We let

\[ TW^{p,q}(\mathcal{K}) := \{ (\eta_{i_0 \cdots i_n})_{(i_0, \ldots, i_n) \in I} \mid \eta_{i_0 \cdots i_n} \in A^q(\mathfrak{A}) \otimes_{\mathbb{C}} \mathcal{K}_p(U_{i_0 \cdots i_n}), \quad d^*_J(\eta_{i_0 \cdots i_n}) = \eta_{i_0 \cdots i_n \mid U_{i_0 \cdots i_n}} \}, \]

and \( TW^{*,*}(\mathcal{K}) := \bigoplus_{p,q} TW^{p,q}(\mathcal{K}) \). It is equipped with a natural filtration \( TW^{*,*}(\mathcal{K}) \) inherited from \( \mathcal{K} \) and the structures \((\land, \partial, \bar{\partial})\) defined componentwise by

\[
(\alpha_I \otimes u_I) \land (\beta_I \otimes w_I) := (-1)^{|u_I||\beta_I|}(\alpha_I \land \beta_I) \otimes (u_I \land w_I),
\]

\[
\bar{\partial}(\alpha_I \otimes u_I) := (d\alpha_I) \otimes u_I, \quad \partial(\alpha_I \otimes u_I) := (-1)^{|\alpha_I|}\alpha_I \otimes (\partial u_I),
\]

for \( \alpha_I, \beta_I \in A^*(\mathfrak{A}) \) and \( u_I, w_I \in \mathcal{K}(U_I) \) where \( l = |I| - 1 \). Furthermore, there is an action \( \iota_{\varphi} : TW^* (\mathcal{K}) \to TW^{*+1}(\mathcal{K}) \) defined componentwise by

\[
(\alpha_I \otimes v_I) \iota_{\beta_I \otimes u_I} := (-1)^{|\beta_I||v_I|}(\alpha_I \land \beta_I) \otimes (v_I \land u_I),
\]

for \( \alpha_I, \beta_I \in A^*(\mathfrak{A}), \quad v_I \in \mathcal{G}(U_I) \) and \( u_I \in \mathcal{K}(U_I) \), where \( |\varphi| = p + q \) for \( \varphi \in TW^{p,q}(\mathcal{K}) \) and \( l = |I| - 1 \).

Direct computation shows that \( (TW^*(\mathcal{K}), \land, \partial) \) is a filtered de Rham module over the BV algebra \( (TW^*(\mathcal{G}), \land, \Delta) \) with the identity \( L_{\alpha_I \otimes v_I} (\beta_I \otimes u_I) = (-1)^{|v_I+1||\beta_I|} (\alpha_I \land \beta_I) \otimes (L_{v_I} u_I) \) for \( \alpha_I, \beta_I \in A^*(\mathfrak{A}), v_I \in \mathcal{G}(U_I) \) and \( u_I \in \mathcal{K}(U_I) \) where \( l = |I| - 1 \). Also, \((TW^*(\mathcal{K}), \land, \bar{\partial})\) is a dga with the relation \( \overline{\partial}(\varphi \cdot \eta) = \overline{\partial}(\varphi) \cdot \eta + (-1)^{|\varphi|} \varphi \cdot (\bar{\partial} \eta) \) for \( \varphi \in TW^*(\mathcal{G}) \) and \( \eta \in TW^*(\mathcal{K}) \).

**Proposition 4.2.** There is an exact sequence

\[ 0 \to TW^{p,q}(r+1 \mathcal{K}) \to TW^{p,q}(r \mathcal{K}) \to TW^{p,q}(r \mathcal{K}/r+1 \mathcal{K}) \to 0 \]

induced naturally by the exact sequence \[ 0 \to r+1 \mathcal{K} \to r \mathcal{K} \to \mathcal{K}/r+1 \mathcal{K} \to 0. \]

**Proof.** The only nontrivial part is the surjectivity of the map \( p : TW^{p,q}(r \mathcal{K}) \to TW^{p,q}(r \mathcal{K}/r+1 \mathcal{K}) \), which is induced from surjective maps \( p : r \mathcal{K}_p(U_{i_0 \cdots i_l}) \to (r \mathcal{K}_p/r+1 \mathcal{K}_p)(U_{i_0 \cdots i_l}) \). We fix \( \eta = (\eta_{i_0 \cdots i_l})_{(i_0, \ldots, i_l) \in I} \in TW^{p,q}(r \mathcal{K}/r+1 \mathcal{K}) \) with \( \eta_{i_0 \cdots i_l} \in A^q(\mathfrak{A}_l) \otimes (r \mathcal{K}_p/r+1 \mathcal{K}_p)(U_{i_0 \cdots i_l}) \), and show by induction on \( l \) that there exists a lifting \( \tilde{\eta}_{i_0 \cdots i_l} \in A^q(\mathfrak{A}_l) \otimes r \mathcal{K}_p(U_{i_0 \cdots i_l}) \) for all \( 0 \leq l \leq l' \) satisfying \( p(\tilde{\eta}_{i_0 \cdots i_l}) = \eta_{i_0 \cdots i_l} \) and \( d^*_J(\tilde{\eta}_{i_0 \cdots i_l}) = \tilde{\eta}_{i_0 \cdots i_l \mid U_{i_0 \cdots i_l}} \) for all \( 0 \leq j \leq l' \). The initial case \( l = q \) follows from the surjectivity of the map \( p : r \mathcal{K}_p(U_{i_0 \cdots i_q}) \to (r \mathcal{K}_p/r+1 \mathcal{K}_p)(U_{i_0 \cdots i_q}) \) over the Stein open subset \( U_{i_0 \cdots i_q} \). For the induction step, let \( d(\tilde{\eta}_{i_0 \cdots i_l}) := (\tilde{\eta}_{i_0 \cdots i_l}, \ldots, \tilde{\eta}_{i_0 \cdots i_l}) \in A^q(\Delta_l) \otimes r \mathcal{K}_p(U_{i_0 \cdots i_l}) \). Then the Lifting Lemma 3.20 gives an extension \( \tilde{\eta}_{i_0 \cdots i_l} \in A^q(\mathfrak{A}_l) \otimes r \mathcal{K}_p(U_{i_0 \cdots i_l}) \) such that \( \tilde{\eta}_{i_0 \cdots i_l} \mid \Delta_l = \bar{\partial}(\tilde{\eta}_{i_0 \cdots i_l}) \) and \( \tilde{p}_{i_0 \cdots i_l}(\tilde{\eta}_{i_0 \cdots i_l}) = \eta_{i_0 \cdots i_l} \), as desired.

**Notation 4.3.** We will write \( r_1 kK_\alpha := \frac{kK_\alpha}{r_1 kK_\alpha} \) for any \( r_1 \leq r_2 \), and extend the notation \( \frac{kK_\alpha}{0} \) as before.
Given a set of compatible gluing morphisms $g = \{ k g_{\alpha \beta} \}$ as in 3.4 we can extend them to gluing morphisms acting on $k^{*} \mathcal{A}_{\alpha \beta} := TW_{*}^{*}(k \mathcal{K}_{\alpha \beta} | \nu_{\alpha \beta})$.

**Definition 4.4.** For each pair $V_{\alpha}, V_{\beta} \subset V$, the morphism $k^{\gamma} g_{\alpha \beta} = (k^{\gamma} g_{\alpha \beta, i_{0} \ldots i_{l}})_{(i_{0}, \ldots, i_{l})} \in \mathcal{I} : k^{*} \mathcal{A}_{\alpha \beta} \rightarrow k^{*} \mathcal{A}_{\alpha \beta}$ is defined componentwise by

$$k^{\gamma} g_{\alpha \beta, i_{0} \ldots i_{l}}(\eta_{i_{0} \ldots i_{l}}) := (\exp(\mathcal{L}_{k^{\gamma} g_{\alpha \beta, i_{0} \ldots i_{l}}} \circ \exp(\mathcal{L}_{k^{\gamma} g_{\alpha \beta, i_{0}}} \circ k^{\gamma} \hat{\psi}_{\alpha \beta, i_{l}})) \eta_{i_{0} \ldots i_{l}}$$

for any multi-index $(i_{0}, \ldots, i_{l}) \in \mathcal{I}$ such that $U_{i_{j}} \subset V_{\alpha \beta}$ for every $0 \leq j \leq l$.

From Definition 2.18 we see that the differences between the morphisms $k^{\gamma} \hat{\psi}_{\alpha \beta, i}$’s are captured by taking Lie derivatives of the same elements $k^{\gamma} \hat{b}_{\alpha \beta, i}$’s, $k^{\gamma} \hat{p}_{\alpha \beta, i}$’s and $k^{\gamma} \hat{c}_{\alpha \beta, i}$’s as for the morphisms $k^{\gamma} \psi_{\alpha \beta, i}$’s. So the morphisms $k^{\gamma} g_{\alpha \beta} : k^{*} \mathcal{A}_{\alpha \beta} \rightarrow k^{*} \mathcal{A}_{\alpha \beta}$ are well-defined and satisfying $k^{\gamma+1} g_{\alpha \beta} = k^{\gamma} g_{\alpha \beta}$ (mod $m^{k+1}$), $k^{\gamma} \hat{g}_{\alpha \beta} \circ k^{\gamma} \hat{g}_{\alpha \gamma} \circ k^{\gamma} \hat{g}_{\alpha \beta} = \text{id}$. As a result, we can define the Čech-Thom-Whitney complex $\check{C}^{*}(\Lambda, \hat{g})$ as in 3.4

**Definition 4.5.** For $\ell \geq 0$, we let $k^{*} \mathcal{A}_{\alpha \beta} = \bigoplus_{i=0}^{\ell} k^{*} \mathcal{A}_{\alpha \beta}^{*}$ be the set of elements $(\eta_{0}, \ldots, \eta_{\ell})$ such that $\eta_{j} = k^{\gamma} g_{\alpha, \alpha}(\eta_{j})$. Then we set $k^{\gamma} \check{C}^{*}(\Lambda, \hat{g}) := \prod_{i=0}^{\ell} k^{*} \mathcal{A}_{\alpha \beta}^{*}$ for each $k \in \mathbb{Z}_{\geq 0}$

and $k^{\gamma} \check{C}^{*}(\Lambda, \hat{g}) := \bigoplus_{p, q} k^{\gamma} \check{C}^{*}(\Lambda, \hat{g})$, which is equipped with the natural restriction maps $r_{j, i}: k^{\gamma} \check{C}^{*-1}(\Lambda, \hat{g}) \rightarrow k^{\gamma} \check{C}^{*-1}(\Lambda, \hat{g})$.

We let $k^{\gamma} \check{C}^{*} := \bigoplus_{j=0}^{\ell+1} k^{\gamma} \check{C}^{*}(\Lambda, \hat{g}) \rightarrow k^{\gamma} \check{C}^{*+1}(\Lambda, \hat{g})$ be the Čech differential. The $k^{\gamma}$th order total de Rham complex over $X$ is then defined to be $k^{\gamma} \check{C}^{*}(\hat{g}) := \text{Ker}(k^{\gamma} \delta_{0})$. Denoting the natural inclusion $k^{\gamma} \check{C}^{*}(\hat{g}) \rightarrow k^{\gamma} \check{C}^{0}(\Lambda, \hat{g})$ by $k^{\gamma} \delta_{-1}$, we obtain the following sequence of maps

$$0 \rightarrow k^{\gamma} \check{C}^{0}(\Lambda, \hat{g}) \rightarrow k^{\gamma} \check{C}^{1}(\Lambda, \hat{g}) \rightarrow k^{\gamma} \check{C}^{2}(\Lambda, \hat{g}) \rightarrow \cdots \rightarrow k^{\gamma} \check{C}^{\ell}(\Lambda, \hat{g}) \rightarrow \cdots ,$$

$$0 \rightarrow k^{\gamma} \check{C}^{\ell}(\Lambda, \hat{g}) \rightarrow k^{\gamma} \check{C}^{\ell+1}(\Lambda, \hat{g}) \rightarrow k^{\gamma} \check{C}^{\ell+2}(\Lambda, \hat{g}) \rightarrow \cdots \rightarrow k^{\gamma} \check{C}^{0}(\Lambda, \hat{g}) \rightarrow \cdots ,$$

where the second sequence is obtained by taking inverse limits $\lim_{\ell \rightarrow k} k^{\gamma} \check{C}$ of the first sequence.

Furthermore, we let $k^{\gamma} \check{\partial}$ and $k^{\gamma} \partial$ be the operators on $k^{\gamma} \check{C}^{*}(\hat{g})$ obtained by gluing of the operators $(k^{\gamma} \partial_{\alpha} + \mathcal{L}_{k^{\gamma} b_{\alpha}})_{\alpha}$’s (where $(k^{\gamma} b_{\alpha})_{\alpha} \in k^{\gamma} \check{C}^{1}(TW^{-1, 1}, g)$ is the element obtained from Theorem 3.34) and $k^{\gamma} \partial_{\alpha}$’s on $k^{*} \mathcal{A}_{\alpha \alpha}$, and $\partial := \lim_{\ell \rightarrow k} k^{\gamma} \partial$ and $\partial := \lim_{\ell \rightarrow k} k^{\gamma} \partial$ be the corresponding inverse limits.

**Proposition 4.6.** Let $(\Lambda^{*}(\hat{g}), \wedge, \partial)$ be a filtered de Rham complex over the BV algebra $(PV^{*}^{*}(\Lambda), \wedge, \Delta)$, and we have $\Delta^{2} = 0 = \partial \partial + \partial \partial \Delta (\text{mod } m)$ as well as the relations

$$\bar{\partial}(\eta \wedge \mu) = \bar{\partial}(\eta) \wedge \mu + (-1)^{|\eta|} \eta \wedge (\partial \Delta), \quad \bar{\partial}(\varphi \varphi) = (-1)^{|\varphi|} \varphi \varphi \bar{\partial}(\eta),$$

for $\varphi \in PV^{*}^{*}(\Lambda)$ and $\eta, \mu \in \Lambda^{*}(\hat{g})$. Furthermore, the filtration $(\Lambda^{*}(\hat{g}), \wedge, \Delta)$ satisfies the relation

$$r_{k^{\gamma}}^{\gamma}(\hat{g})/ r_{k^{\gamma}+1}^{\gamma}(\hat{g}) = r_{k^{\gamma}+1}^{\gamma}(\hat{g}).$$

**Proof.** Since $\bar{\partial}$ and $\partial$ are constructed from the operators $(k^{\gamma} \partial_{\alpha} + \mathcal{L}_{k^{\gamma} b_{\alpha}})_{\alpha}$’s and $k^{\gamma} \partial_{\alpha}$’s on $k^{*} \mathcal{A}_{\alpha \alpha}$, we only have to check the relations for each $(k^{3} \mathcal{A}_{\alpha \alpha}^{*}, \wedge, \partial)$, which is a filtered de Rham module over the BV algebra $(k^{3} TW^{*}^{*} \mathcal{A}_{\alpha \alpha}^{*}, \wedge, \Delta)$. To see the last relation, note that there is an exact sequence of Čech-Thom-Whitney complexes $0 \rightarrow k^{\gamma} \check{C}^{*}(\Lambda^{*}(\hat{g})) \rightarrow k^{\gamma} \check{C}^{*}(\Lambda^{*}(\hat{g})) \rightarrow k^{\gamma} \check{C}^{*}(\Lambda^{*}(\hat{g})) \rightarrow 0$ using Proposition 4.2. Taking the kernel and inverse limits then gives the exact sequence

$$0 \rightarrow r_{k^{\gamma}+1}^{\gamma}(\hat{g}) \rightarrow r_{k^{\gamma}}^{\gamma}(\hat{g}) \rightarrow r_{k^{\gamma}+1}^{\gamma}(\hat{g}) \rightarrow 0.$$
The result follows. 

**Notation 4.7.** We will simplify notations by writing $\ast \mathcal{A}^{\ast \ast} = \mathcal{A}(\hat{g})$ and $PV^{\ast \ast} = PV^{\ast \ast}(g)$ if the gluing morphisms $\hat{g} = \{k \alpha_\beta\}$ and $g = \{k \alpha_\beta\}$ are clear from the context. We will further denote the relative de Rham complex (over $\operatorname{spf}(R)$) as $\ast \mathcal{A}^{\ast \ast} :=_0 \mathcal{A}^{\ast \ast} /_1 \mathcal{A}^{\ast \ast}$. 

**Proposition 4.8.** Using the element $(k f_{1\alpha}) \in \mathcal{C}_0(TW^{0,0}, g)$ obtained from Theorem 3.34, we obtain the element $(\exp(k f_{1\alpha})) \omega_{\alpha} \in \mathcal{C}_0(0,1) \mathcal{A}^{d,0}, \hat{g})$ whose components glue to form a global element $k \omega \in \mathcal{A}^{d,0}$, i.e. we have $(k \hat{g}_{\alpha_\beta} \circ \exp(k f_{1\alpha}))(k \omega_{\alpha}) = \exp(k f_{1\alpha})(k \omega_{\beta})$. Furthermore, we have $k^{+1} \omega = k \omega \pmod{m^{k+1}}$. In view of this, we define the relative volume form to be $\omega := \lim_{k \to \infty} k \omega$. 

**Proof.** Similar to the proof of Lemma 3.32, we use the power series $T(x)$ in (3.26) to simplify notations. We fix $V_{\alpha_\beta}$ and $U_{i_0\ldots i_l}$ such that $U_{ij} \subset V_{\alpha_\beta}$ for all $0 \leq j \leq l$. We need to show that 

$$(\exp(\mathcal{L}_{k \partial_{\alpha_\beta},i_0\ldots i_l}))(k \omega_{\alpha}) \circ (k \alpha_\beta) = (k \omega_{\alpha}) \exp(k f_{1\alpha,i_0\ldots i_l})(k \omega_{\alpha}) = (k \omega_{\alpha}) \exp(k f_{1\alpha,i_0\ldots i_l})(k \omega_{\alpha}).$$ 

We begin with the case $l = 0$. Making use of the identities 

$$\exp([\partial, \partial])(u \wedge v) = (\exp([\partial, \partial])(u)) \wedge (\exp([\partial, \partial])(v)),$$

$$\exp(\mathcal{L}_{\partial})(v \partial w) = (\exp(\mathcal{L}_{\partial})(v)) \partial (\exp(\mathcal{L}_{\partial})(w))$$

for $\partial \in k \mathcal{G}_\beta^{-1}(U_{i_0\ldots i_l})$, $u, v, w \in k \mathcal{G}_\beta^0(U_{i_0\ldots i_l})$ and $w \in 0, k \mathcal{G}_\beta^d(U_{i_0\ldots i_l})$, we have 

$$(k \hat{g}_{\alpha_\beta} \circ \exp(k f_{1\alpha,i_0\ldots i_l}))(k \omega_{\alpha}) = \exp(k f_{1\alpha,i_0\ldots i_l})(k \omega_{\alpha}) \exp(k f_{1\alpha,i_0\ldots i_l})(k \omega_{\alpha}) = (k \omega_{\alpha}) \exp(k f_{1\alpha,i_0\ldots i_l})(k \omega_{\alpha}) = (k \omega_{\alpha}) \exp(k f_{1\alpha,i_0\ldots i_l})(k \omega_{\alpha}).$$

where $k f_{1\alpha,i_0\ldots i_l} = k g_{\alpha_\beta}(k f_{1\alpha,i_0\ldots i_l}) = k g_{\alpha_\beta}(-k \psi_{\alpha_\beta} \circ \mathcal{T}(k f_{1\alpha,i_0\ldots i_l})) \circ k \Delta(\alpha_\beta)(k f_{1\alpha,i_0\ldots i_l}) + k w_{\alpha_\beta}$ is the component of the term $k f_{1\alpha,i_0\ldots i_l}$ obtained in Lemma 3.32.

The general case $l \geq 0$ is similar, as we have 

$$(k \hat{g}_{\alpha_\beta} \circ \exp(k f_{1\alpha,i_0\ldots i_l}))(k \omega_{\alpha}) = (k \omega_{\alpha}) \exp(k f_{1\alpha,i_0\ldots i_l})(k \omega_{\alpha}) \exp(k f_{1\alpha,i_0\ldots i_l})(k \omega_{\alpha}) = (k \omega_{\alpha}) \exp(k f_{1\alpha,i_0\ldots i_l})(k \omega_{\alpha}).$$

For the element $k l_{\alpha}$, we have 

$$(k \hat{\partial}_{\alpha} + [k \omega_{\alpha}])^2 = [k l_{\alpha},]$$

on $\ast \mathcal{W}^{\ast \ast}$ and since $k g_{\alpha_\beta} \circ (k \hat{\partial}_{\alpha} + [k \omega_{\alpha}]) \circ k g_{\alpha_\beta} = k \hat{\partial}_{\alpha} + [k \omega_{\alpha}]$, we deduce that $[k g_{\alpha_\beta}(k l_{\alpha}),] = [k l_{\beta},]$ and hence $k g_{\alpha_\beta}(k l_{\alpha}) = k l_{\beta}$ by the injectivity of $k \mathcal{G}^{-1} \hookrightarrow \operatorname{Der}(k \mathcal{G})$. 

**Lemma 4.9.** The elements $k l_{\alpha} := k \hat{\partial}_{\alpha}(k \omega_{\alpha}) + \frac{1}{2}[k \hat{\partial}_{\alpha}, k \omega_{\alpha}]$ and $k l_{\alpha} := k \Delta(\alpha_\beta)(k \omega_{\alpha}) + k \partial(\alpha_\beta)(k l_{\alpha}) + [k \partial_{\alpha}, k l_{\alpha}]$ glue to give global elements $k l_{\alpha} = (k l_{\alpha})_{\alpha} \in k PV^{-1,2}$ and $k \eta_{\alpha} = (k \eta_{\alpha})_{\alpha} \in k PV^{0,1}$ respectively. 

**Proof.** For the element $k l_{\alpha}$, we have 

$$(k \hat{\partial}_{\alpha} + [k \omega_{\alpha}])^2 = [k l_{\alpha},]$$

on $\ast \mathcal{W}^{\ast \ast}$ and since $k g_{\alpha_\beta} \circ (k \hat{\partial}_{\alpha} + [k \omega_{\alpha}]) \circ k g_{\alpha_\beta} = k \hat{\partial}_{\alpha} + [k \omega_{\alpha}]$, we deduce that $[k g_{\alpha_\beta}(k l_{\alpha}),] = [k l_{\beta},]$ and hence $k g_{\alpha_\beta}(k l_{\alpha}) = k l_{\beta}$ by the injectivity of $k \mathcal{G}^{-1} \hookrightarrow \operatorname{Der}(k \mathcal{G})$. 


For the element $k\eta$, we have $kg_{\alpha\beta}(k\partial_{\alpha}) = k\partial_{\beta} + k\nu_{\beta,\alpha\beta}$ and $kg_{\alpha\beta}k\partial_{\alpha} = k\partial_{\beta} + k\nu_{\beta,\alpha\beta}$ from the constructions in Theorem 3.34. Making use of the relations $kg_{\alpha\beta}k\partial_{\alpha} = k\partial_{\alpha} + [k\nu_{\beta,\alpha\beta}, \cdot] \circ kg_{\alpha\beta}$ and $k\Delta_{\beta} \circ kg_{\alpha\beta} = kg_{\alpha\beta} \circ k\Delta_{\beta} + [k\nu_{\beta,\alpha\beta}, \cdot] \circ kg_{\alpha\beta}$ from Lemmas 3.31 and 3.32 we have

\[
kg_{\alpha\beta}(k\Delta_{\beta}(k\partial_{\alpha})) = kg_{\alpha\beta}(k\Delta_{\beta}(k\partial_{\alpha})) + [k\nu_{\beta,\alpha\beta}, \cdot] \circ kg_{\alpha\beta} = kg_{\alpha\beta}(k\Delta_{\beta}(k\partial_{\alpha}) + [k\nu_{\beta,\alpha\beta}, \cdot] \circ kg_{\alpha\beta}) = kg_{\alpha\beta}(k\Delta_{\beta}(k\partial_{\alpha}) + [k\nu_{\beta,\alpha\beta}, \cdot] \circ kg_{\alpha\beta}).
\]

Hence it remains to show that $k\Delta_{\beta}(k\nu_{\beta,\alpha\beta}) + k\partial_{\beta}(k\nu_{\beta,\alpha\beta}) - [k\nu_{\beta,\alpha\beta}, k\nu_{\beta,\alpha\beta}] = 0$ in $kTW_{\beta}^{\ast,*}$.

We fix a multi-index $(i_0, \ldots, i_l) \in \Lambda$, and recall from the proofs of Lemmas 3.31 and 3.32 the formulas:

\[
k\partial_{\beta}(k\nu_{\beta,\alpha\beta}) - (k\nu_{\beta,\alpha\beta})_{i_0} = 0 \quad \text{and} \quad (k\partial_{\beta}(k\nu_{\beta,\alpha\beta}))_{i_0} = 0.
\]

Since $A$ is gauge equivalent to 0 in the above dgLa, we have the equation $(k\partial_{\beta} + k\partial_{\beta})A + 2[A, A] = 0$ whose component in $A^1(\Lambda_1) \otimes kG^0_\beta(U_{a \cdots i})$ can be extracted as

\[
k\partial_{\beta}(k\partial_{\alpha\beta}) = (k\partial_{\beta}(k\partial_{\alpha\beta}))_{i_0} = 0.
\]

Therefore, the $(i_0, \ldots, i_l)$-component of the term $k\Delta_{\beta}(k\nu_{\beta,\alpha\beta}) + k\partial_{\beta}(k\nu_{\beta,\alpha\beta}) - [k\nu_{\beta,\alpha\beta}, k\nu_{\beta,\alpha\beta}]$ is given by

\[
k\Delta_{\beta}(k\nu_{\beta,\alpha\beta}) + k\partial_{\beta}(k\nu_{\beta,\alpha\beta}) - [k\nu_{\beta,\alpha\beta}, k\nu_{\beta,\alpha\beta}] = kg_{\alpha\beta}(k\partial_{\alpha\beta}) - k\nu_{\beta,\alpha\beta}(k\partial_{\alpha\beta}) = 0.
\]

\begin{definition}
We let $I := \lim k \in PV^{-1,2}$ and $\eta := \lim k \in PV^{0,1}$. The operator $d = \partial + \partial + L_1$, which acts on $A_1^{\ast,*}$ preserves the filtration, is called the total de Rham differential. We also denote the pull back of $d$ to $PV^{\ast,*}$ under the isomorphism $\omega : PV^{\ast,*} \to \mathbb{A}^{d+\ast,*}$ by $d$.
\end{definition}

\begin{proposition}
The pair $(A_1^{\ast,*}, d)$ forms a filtered complex, i.e. $d^2 = 0$ and $d$ preserves the filtration. We also have $d = \partial + \Delta + (1 + \eta)\wedge$ on $PV^{\ast,*}$.
\end{proposition}

\begin{proof}
From the discussion right before Proposition 4.12 we compute $d^2 = (\partial + \partial + L_1)^2 = (\partial + \partial)^2 - L_1$. If we compute $(\partial + \partial)^2$ locally on $A_1^{\ast,*}$, we obtain $(\partial_\alpha + L_{\partial_\alpha} + \partial_\alpha)^2 = L_{\partial_\alpha}(2\alpha) + L_{\partial_\alpha}^2 = L_{\partial_\alpha} + \frac{1}{2}[\partial_\alpha, \partial_\alpha] = L_{L_1}$. So we get $(\partial + \partial)^2 = L_1$ and hence $d^2 = 0$. As for $\partial$, we compute locally on $TW^{\ast,*}_{\alpha}$. Taking $\gamma \in TW^{\ast,*}_{\alpha}$, we have

\[
d(\gamma \wedge \exp(\partial)(\omega_\alpha)) = (\partial_\alpha + L_{\partial_\alpha} + \partial_\alpha + L_\omega) ((\gamma \wedge \exp(\partial))(\partial)(\omega_\alpha)) = (\partial_\alpha(\gamma) + [\partial_\alpha, \gamma] + \Delta_\alpha(\gamma) + [f_\alpha, \gamma] + \omega_\alpha \wedge \gamma + \eta_\alpha \wedge \gamma) \wedge \exp(\partial)(\omega_\alpha),
\]

which gives the identity $d = \partial + \Delta + (1 + \eta)\wedge$.
\end{proof}
4.2. The Gauss-Manin connection. Using the natural isomorphisms \( k\sigma : (1, k\mathcal{K}, k\partial) \cong K \otimes \mathbb{Z} \)
\((0, 1)\mathcal{K}[−1], k\partial\) from Definition 2.17, we obtain isomorphisms
\[
 k\sigma : 1, 2, k\mathcal{A}_{\alpha, \omega} \to K \otimes \mathbb{Z} k\mathcal{A}_{\alpha, \omega}[-1],
\]
which can be patched together to give an isomorphism of complexes \( k\sigma : 1, 2, k\mathcal{A}_{\alpha, \omega} \to K \otimes \mathbb{Z} k\mathcal{A}_{\alpha, \omega}[-1] \)
which is equipped with the differential \( k\mathcal{d} \). This produces an exact sequence of complexes:
\[
(4.3)\quad 0 \to K \otimes \mathbb{Z} k\mathcal{A}_{\alpha, \omega}[-1] \to k\mathcal{A}_{\alpha, \omega}/k^2\mathcal{A}_{\alpha, \omega} \to k\mathcal{A}_{\alpha, \omega} \to 0,
\]
which we use to define the Gauss-Manin connection (cf. Definition 2.11).

**Definition 4.12.** Taking the long exact sequence associated to (4.3), we get the map
\[
(4.4)\quad k\nabla : H^*(\mathbb{A}, k\mathcal{d}) \to K \otimes \mathbb{Z} H^*(\mathbb{A}, k\mathcal{d}),
\]
which is called the \( k^{th} \)-order Gauss-Manin (abbrev. GM) connection over \( kR \). Taking inverse limit over \( k \) gives the formal Gauss-Manin connection over \( \hat{R} \):
\[
\nabla : H^*\mathbb{A}, \mathcal{d}) \to K \otimes \mathbb{Z} H^*(\mathbb{A}, \mathcal{d}).
\]

The modules \( H^*(\mathbb{A}, k\mathcal{d}) \) and \( H^*(\mathbb{A}, \mathcal{d}) \) over \( kR \) and \( \hat{R} \) are respectively called the \( k^{th} \)-order Hodge bundle and the formal Hodge bundle.

**Remark 4.13.** By its construction, the complex \( (0, 1, 0, 0) \) serves as a resolution of the complex \( (0, 1, 0, 0) \), and the cohomology \( H^*(0, 1, 0, 0) \) computes the hypercohomology \( H^*(0, 1, 0, 0) \). So the \( 0^{th} \)-order Gauss-Manin connection \( 0\nabla \) agrees with the one introduced in Definition 2.11.

**Proposition 4.14.** The Gauss-Manin connection \( \nabla \) defined in Definition 4.12 is a flat connection, i.e. the map \( \nabla^2 : \hat{H}^*(\mathbb{A}, \mathcal{d}) \to \wedge^2 (K\mathcal{C}) \otimes H^*(\mathbb{A}, \mathcal{d}) \) is a zero map.

**Proof.** It suffices to show the \( k^{th} \)-order Gauss-Manin connection \( k\nabla \) is flat for every \( k \). Consider the short exact sequence (4.3), and take a cohomology class \([\eta] \in H^*(\mathbb{A}, k\mathcal{d})\) represented by an element \( \eta \in \mathbb{A} \). Then we take a lifting \( \tilde{\eta} \in \mathbb{A} \) so that \( k\nabla([\eta]) \) is represented by the element \( k\mathcal{d}(\tilde{\eta}) \in \mathbb{A} \). We write \( k\nabla([\eta]) = \sum_\alpha \alpha_i \otimes [\xi_i] \) for \( \alpha_i \in \mathcal{O}_{X} \) and \( [\xi_i] \in H^*(\mathbb{A}, k\mathcal{d}) \). Once again we take a representative \( \xi_i \in \mathbb{A} \) for \([\xi_i]\) and by our construction we have an element \( \mathcal{e} \in \mathbb{A} \) such that \( \sum_\alpha \alpha_i \otimes [\xi_i] = k\mathcal{d}(\tilde{\eta}) + \mathcal{e} \). Therefore if we consider the exact sequence of complexes
\[
0 \to k\mathbb{A}/k^2\mathbb{A} \to k\mathbb{A}/k^3\mathbb{A} \to k\mathbb{A}/k^4\mathbb{A} \to 0,
\]
we have \( k\mathcal{d}(\sum_\alpha \alpha_i \otimes [\xi_i]) = k\mathcal{d}(\mathcal{e}) \in \mathbb{A}/k^2\mathbb{A} \). Note that \( (k\nabla)^2([\eta]) \) is represented by the cohomology class of the element \( k\mathcal{d}(\sum_\alpha \alpha_i \otimes [\xi_i]) \in \mathbb{A}/k^3\mathbb{A} \) by the isomorphism induced by \( k\sigma_\alpha \)'s from Definition 2.17. Hence we have \( k\mathcal{d}(\sum_\alpha \alpha_i \otimes [\xi_i]) = k\mathcal{d}(\mathcal{e}) = 0 \in k\mathcal{O}_{X} \otimes (k\mathcal{R}) \mathbb{A}[-2] \).

4.3. Freeness of the Hodge bundle from a local criterion. To prove the desired unobstructedness result, we need freeness of the Hodge bundle; in geometric situations, this has been established in various cases \[31, 50, 34, 23\]. In this subsection, we generalize the techniques in \[23, 34, 50\] to prove the freeness of the \( k^{th} \)-order Hodge bundle \( H^*(\mathbb{A}, k\mathcal{d}) \) over \( kR \) in our abstract setting (Lemma 4.17) under a local criterion (Assumption 4.15).
4.3.1. A local criterion. Recall from Notation 1.15 that we have a strictly convex polyhedral cone $Q_{\mathbb{R}} \subset K_{\mathbb{R}}$, the coefficient ring $R = \mathbb{C}[Q]$, and the log space $S^t$ (or the formal log space $\hat{S}^t$) parametrizing the moduli space near the degenerate Calabi-Yau variety $(X, O_X)$. For every primitive element $n \in \text{int}(Q_{\mathbb{R}}) \cap K_{\mathbb{R}}$, we have a natural ring homomorphism $i_n : \mathbb{C}[Q] \rightarrow \mathbb{C}[q]$, $q^m \mapsto q^{(m,n)}$, where $(\cdot, \cdot)$ denotes the natural pairing between $K$ and $K_{\mathbb{R}}$, and then taking spectra gives a map $i_n : A^{1,1} \rightarrow S^t$ (or $k_i : k A^{1,1} \rightarrow k S^t$ for each $k \in \mathbb{Z}_{\geq 0}$), where $A^{1,1}$ is the log space associated to the log ring $\mathbb{C}[q]^*$ which is equipped with the monoid homomorphism $\mathbb{N} \rightarrow \mathbb{C}[q]$, $k \mapsto q^k$.

Geometrically, taking base change with the map $A^{1,1} \rightarrow S^t$ should be viewed as restricting the family to the 1-dimensional family determined by $n$. In our abstract setting, we consider the tensor product $k^*G_{n,\alpha} := k^*G_{\alpha} \otimes_{k^*} \mathbb{C}[q]/(q^{k+1})$. Then tensoring the maps $k^*\psi_{\alpha,\beta,i}$'s with $\mathbb{C}[q]/(q^{k+1})$ give patching morphisms for the $k^*G_{n,\alpha}$'s which will be denoted as $k^*\psi_{n;\alpha,\beta,i}$. Similarly we use $k^*b_{n;\alpha,\beta,i}$'s, $k^*b_{n;\alpha,\beta,ij}$'s, $k^*a_{n;\alpha,\beta,i}$'s and $k^*b_{n;\alpha,\beta,ij}$'s to denote the tensor products of the corresponding terms appearing in Definition 2.15 with $\mathbb{C}[q]/(q^{k+1})$. Note that all the relations in Definition 2.15 still hold after taking the tensor products.

In view of the isomorphism $\omega^{k} : k PV^{*,*} \rightarrow k A^{d++,*}$ in Definition 4.10 and the fact that the complex $(k PV^{*,*}, k \mathfrak{d})$ is free over $k R$ (meaning that the differential is $k R$-linear), we see that $(k A^{d++,*}, k \mathfrak{d})$ is also free over $k R$. Then taking tensor product with $\mathbb{C}[q]/(q^{k+1})$ (for a fixed $n$), we obtain the relative de Rham complex $(A^{*} \otimes_{k^*} \mathbb{C}[q]/(q^{k+1}), \mathfrak{d})$ over $\mathbb{C}[q]/(q^{k+1})$.

Now the filtered de Rham module $k^*K_{n,\alpha}$ plays the role of the sheaf of holomorphic de Rham complex on the thickening of $\mathcal{V}_\alpha$. We need to consider restrictions of these holomorphic differential forms to the 1-dimensional family $\text{Spec}(\mathbb{C}[q]/(q^{k+1}))$, but naively taking tensor product with $\mathbb{C}[q]/(q^{k+1})$ does not give the desired answer. In our abstract setting, the existence of such restrictions can be formulated as the following assumption (which is motivated by the proof of 23 Theorem 4.1):

**Assumption 4.15.** For each $n \in \text{int}(Q_{\mathbb{R}}) \cap K_{\mathbb{R}}$, $k \in \mathbb{Z}_{\geq 0}$ and $\mathcal{V}_\alpha \in \mathcal{V}$, we assume there exists a coherent sheaf of dga's

$$(k^*K_{n,\alpha}^{\ast}, \wedge, k \partial_{n,\alpha})$$

equipped with a d module structure over $k^*\Omega^{1,1}_{\alpha}$, the natural filtration

$$k^*K_{n,\alpha}^{\ast} = k^*K_{n,\alpha}^{0} \supset k^*K_{n,\alpha}^{1} \supset \ldots \supset k^*K_{n,\alpha}^{l} = \{0\}$$

where $k^*K_{n,\alpha}^{0} = d \log(q) \ast k^*K_{n,\alpha}^{1}$, and a de Rham module structure over $(k^*G_{n,\alpha}^{\ast}, \wedge, k \Delta_{n,\alpha})$ satisfying all the conditions in Definitions 2.17 and 2.18 (in particular, we have surjective morphisms $k^*\omega_{n,\alpha} : k^*K_{n,\alpha}^{0} \rightarrow k^*K_{n,\alpha}^{0}$ for $k \geq 1$, a volume element $k^*\omega_{n,\alpha} \in k^*K_{n,\alpha}^{l}/k^*K_{n,\alpha}^{l-1}$, an isomorphism $k^*\sigma_{n,\alpha} : (k^*K_{n,\alpha}^{l}/k^*K_{n,\alpha}^{l-1}, k \partial_{n,\alpha}) \cong (k^*K_{n,\alpha}^{l}/k^*K_{n,\alpha}^{l-1}, k \partial_{n,\alpha})$, and patching isomorphisms $k^* \psi_{n;\alpha,\beta,i} : k^*K_{n,\alpha}^{l} \mid_{U_i} \rightarrow k^*K_{n,\beta}^{l} \mid_{U_i}$ for triples $(U_i; V_\alpha, V_\beta)$ with $U_i \subset V_\alpha$ fulfilling all the required conditions). We further assume that the complex $(k^*K_{n,\alpha}^{*}[u], k \partial_{n,\alpha})$, where

$$k \partial_{n,\alpha} \left( \sum_{s=0}^{l} \nu_s u^s \right) := \sum_{s} (k \partial_{n,\alpha} \nu_s) u^s + s d \log(q) \ast \nu_s u^{s-1},$$

satisfies the holomorphic Poincaré Lemma in the sense that for each Stein open subset $U$ and any $\sum \nu_s u^s \ast k^*K_{n,\alpha}^{*}[U]$, with $k \partial_{n,\alpha} \nu_s = 0$, we have $\sum \eta_s u^s \ast k^*K_{n,\alpha}^{*}[U]$ satisfying $k \partial_{n,\alpha} (\sum \eta_s u^s) = \sum \nu_s u^s$ on $U$, and if in addition $k^*\omega_{n,\alpha} (\nu_s) = 0$ in $(0_{k^*\Omega^{1,1}_{\alpha}}/k^*K_{n,\alpha}^{l})(U)$, then $\sum \eta_s u^s$ can be chosen so that $k^*\omega_{n,\alpha} (\nu_s) = 0$ in $(0_{k^*\Omega^{1,1}_{\alpha}}/k^*K_{n,\alpha}^{l})(U)$.
Assumption 4.15 allows us to construct the total de Rham complex \( \mathcal{A}^{\bullet}_{n,1} \) such that \( \| I \mathcal{A} \otimes_R (\mathbb{C}[q]/(q^{k+1})) = k \| I \mathcal{A}^{\bullet}/k \| I \mathcal{A}^\bullet =: k \| I \mathcal{A}^\bullet. \)

4.3.2. Freeness of the Hodge bundle. We consider a general monomial ideal \( I \) of \( R \) such that \( m^k \subset I \) for some integer \( k \in \mathbb{Z}_+ \), and we let \( \| I \mathcal{A} := \| I \mathcal{A} \otimes_R (R/I) \) equipped with the differential by taking tensor product which is also denoted by \( \| d \). We consider two such ideals \( I \subset J \subset I \) and the following exact sequence of complexes

\[
0 \rightarrow \| I \mathcal{A} \otimes_C (J/I) \rightarrow \| I \mathcal{A} \rightarrow \| J \mathcal{A} \rightarrow 0.
\]

Then we consider the long exact sequence associated to (4.5) and let

\[
\| I, J \delta : H^\ast(\| I \mathcal{A}, J \mathcal{A}) \rightarrow H^\ast(\| I \mathcal{A}, 0 \mathcal{A})[1] \otimes_C (J/I)
\]

as in the proof of [34, Lemma 4.1].

**Lemma 4.16.** Suppose we have a filtration \( I = I_l \subset I_{l-1} \subset \cdots \subset I_0 = J \) of monomial ideals. Then the connecting homomorphism \( \| I, J \delta \) in (4.6) is zero if and only if the corresponding connecting homomorphism \( \| I_{j+1}, J \delta \) (abbrev. by \( j+1, J \delta \)) is zero for each \( j = 0, \ldots, l - 1 \).

**Proof.** First assume that the homomorphism \( \| I, J \delta \) is non-zero. Let \( m \) be the minimum of those \( j = 1, \ldots, l \) such that the composition

\[
H^\ast(\| I \mathcal{A}, J \mathcal{A}) \rightarrow H^\ast(\| I \mathcal{A}, J \mathcal{A}) \rightarrow H^\ast(\| I \mathcal{A}, 0 \mathcal{A}) \otimes_C (J/I)
\]

is non-zero. Then we consider the commutative diagram

\[
\begin{array}{ccc}
H^\ast(\| I \mathcal{A}, J \mathcal{A}) & \rightarrow & H^\ast(\| I \mathcal{A}, J \mathcal{A}) \\
\downarrow & & \downarrow \\
H^\ast(\| I \mathcal{A}, J \mathcal{A}) & \rightarrow & H^\ast(\| I \mathcal{A}, J \mathcal{A}) \\
\downarrow & & \downarrow \\
H^\ast(\| I \mathcal{A}, J \mathcal{A}) & \rightarrow & H^\ast(\| I \mathcal{A}, J \mathcal{A}) \\
\downarrow & & \downarrow \\
\end{array}
\]

Notice that the connecting homomorphism \( \| m, J \delta \) is non-zero with its image lying in the subspace \( H^\ast(\% I \mathcal{A}, 0 \mathcal{A}) \otimes_C (I_m/I_m) \). So \( m, J \delta \) is non-zero.

Conversely, using the above commutative diagram, we observe that if \( m, J \delta \) is non-zero for some \( 0 \leq m \leq l - 1 \) then the connecting homomorphism \( \| I, J \delta \) cannot be zero. \( \Box \)

To prove triviality of the Hodge bundle, we need a decreasing tower of ideals \( m = J_1 \supset J_2 \supset \cdots \) such that \( m \cdot J_i \subset J_{i+1} \) for each \( i \), \( J_i/J_{i+1} \) is at most one-dimensional and \( \mathcal{R} = \lim_{\leftarrow} (R/J_i) \). We should show that the connecting homomorphism

\[
J_{i+1, J_i} : H^\ast(\| J_i \mathcal{A}, J_i \mathcal{A}) \rightarrow H^\ast+1(\% J_i \mathcal{A}, 0 \mathcal{A}) \otimes_C (J_i/J_{i+1})
\]

is zero for \( i = 1, 2, \ldots \).

To construct such a tower, we take an element \( n_0 \in \text{int}(Q^\vee_R) \cap K^\vee \) and define the monomial ideal \( J_i := (q^m \mid m \in Q, \ (m, n_0) \geq i \) \), giving a sequence \( J_1 \supset J_2 \supset \cdots \), which should be further refined. For each \( i \), notice that the finite dimensional vector space \( J_i/J_{i+1} \) has a basis \( q^m \) given by the lattice points \( m \in Q \) with \( (m, n_0) = i \). We take a generic element \( e \in \text{int}(Q^\vee_R) \cap K^\vee \) such that
that \((m_1, e) \neq (m_2, e)\) for all such \(m \in Q\) with \((m, n_0) = i\) (this can be done since there are finitely many such \(m\)’s). We further take \(L\) large enough so that if we let \(n = Ln_0 + e\), there is an integer \(l\) such that \((m, n) \geq l\) for \(m \in Q\) with \((m, n_0) \geq i + 1\) and \((m', n) \leq l - 1\) for \(m' \in Q\) with \((m', n_0) = i\).

We can therefore define the refined filtration \(J_{i+1} = I_{j, l} \subset I_{j, l-1} \subset \cdots \subset I_{j, 0} = \hat{J}_i\) such that \(I_{j, s}\) is the monomial ideal generated by those \(q^m\) with \(m \in Q\), \((m, n_0) \geq i\) and \((m, n) \geq s\). Such a choice ensures that there is at most one \(m \in Q\) such that \((m, n_0) = i\) and \((m, n) = s\) for a fixed \(s\), and hence \(I_{j, s-1}/I_{j, s}\) is at most one-dimensional. Making such a refinement for each \(\hat{J}_i \supset \hat{J}_{i+1}\) and possibly renumbering the sequence, we obtain the desired sequence \(\hat{J}_i \supset \hat{J}_{i+1} \supset \cdots\). For each pair \(J_j/J_{j+1} \cong \mathbb{C}\), we notice that there is an \(n\) together with \(i_n : R \to \mathbb{C}[q]\) and some \(k \in \mathbb{Z}_+\) such that \(\hat{i}_n^{-1}(q^k) = J_j\) and \(\hat{i}_n^{-1}(q^{k+1}) = J_{j+1} + i_n : J_j/J_{j+1} \cong \mathbb{C} \cdot q^k\).

We have the following commutative diagram of complexes:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & A^* \otimes \mathbb{C} (J_j/J_{j+1}) & \rightarrow & J_{j+1}^* & \rightarrow & J_j^* & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & A^* \otimes \mathbb{C} \cdot q^k & \rightarrow & k^1_{\alpha} A_n^* & \rightarrow & h_{\alpha} A_n^* & \rightarrow & 0 \\
\end{array}
\]

such that the induced map \(i^*_n : H^*(0||A^*) \otimes \mathbb{C} (J_j/J_{j+1}) \rightarrow H^*(0||A^*) \otimes \mathbb{C} \cdot q^k\) is an isomorphism. Therefore it remains to show that \(H^*(k||A^*_n)\) is a free \(\mathbb{C}[q] /(q^{k+1})\) module for each \(k\).

**Lemma 4.17.** Under Assumption \[4.15\], \(H^*(k||A, K d)\) is a free \(R/K\) module for any ideal \(K \subset m^L \subset K\) for some \(L\).

**Proof.** We first consider the case \(K = J_i\) for some \(i\). Similar to [31, p. 404] and the proof of [23, Theorem 4.1], it suffices to show that the map \(k,0 d_n : H^*(k||A^*_n, k d_n) \rightarrow H^*(0||A^*_n, 0 d_n)\), which is induced by the maps \(k,0 d_n\)'s in Assumption \[4.15\] is surjective for all \(k \in \mathbb{Z}_{\geq 0}\). Following the proof of [23, Theorem 4.1], we consider the complex \((k^* A_n^*, k d_n)\) constructed from the complexes \((k^0 \mathcal{K}_n^* [u], k^0 \mathcal{K}_n^* [u])\)'s as in Definition \[4.10\]. There is a natural restriction map \(k,0 d_n : k^* A_n^* \rightarrow 0\| A_n^*\) defined by \(k,0 d_n(\sum_{n=0}^{\infty} \eta_n u^n) = k,0 d_n(\eta_0)\) on \(k^* \mathcal{K}_n^*\). Since \(k,0 d_n\) (and hence the induced map on cohomology) factors through \(k,0 d_n\), we only need to show that the map \(k,0 d_n : H^*(k||A_n^*, k d_n) \rightarrow H^*(0||A_n^*, 0 d_n)\) is an isomorphism.

By gluing the sheaves \(k^0 \mathcal{K}_n^* [u]\)'s in Definition \[4.5\], we can construct the Čech-Thom-Whitney complexes \(\tilde{k} \mathcal{C}^*(A_n^*, \hat{g}_n)\) (resp. \(\tilde{k} \mathcal{C}^*(B_n^*, \hat{g}_n)\)) and obtain the exact sequences:

\[
\begin{align*}
0 & \rightarrow \tilde{k} \mathcal{A}_n^* (\hat{g}_n) \rightarrow \tilde{k} \mathcal{C}^0 (\tilde{k} \mathcal{A}_n^* , \hat{g}_n) \rightarrow \tilde{k} \mathcal{C}^1 (\tilde{k} \mathcal{A}_n^* , \hat{g}_n) \rightarrow \cdots \rightarrow \tilde{k} \mathcal{C}^\ell (\tilde{k} \mathcal{A}_n^* , \hat{g}_n) \rightarrow \cdots , \\
0 & \rightarrow \tilde{k} \mathcal{B}_n^* (\hat{g}_n) \rightarrow \tilde{k} \mathcal{C}^0 (\tilde{k} \mathcal{B}_n^* , \hat{g}_n) \rightarrow \tilde{k} \mathcal{C}^1 (\tilde{k} \mathcal{B}_n^* , \hat{g}_n) \rightarrow \cdots \rightarrow \tilde{k} \mathcal{C}^\ell (\tilde{k} \mathcal{B}_n^* , \hat{g}_n) \rightarrow \cdots .
\end{align*}
\]

Now we have a commutative diagram:

\[
\begin{array}{cccc}
\tilde{k} \mathcal{A}_n^* & \rightarrow & k \mathcal{A}_n^* (\hat{g}_n) & \rightarrow & \tilde{k} \mathcal{C}^*(B_n^*, \hat{g}_n) \\
\downarrow & & \downarrow & & \downarrow \\
k,0 d_n & \rightarrow & k,0 d_n & \rightarrow & \mathcal{C}^*(0||A_n^*, \hat{g}_n)
\end{array}
\]
where the horizontal arrows are quasi-isomorphisms. So what we need is to show that \( k,0 \_{\hat{g}_n} : k C^*(B^*_{\hat{A}}, \hat{g}_n) \rightarrow 0 C^*(B^*_n, \hat{g}_n) \) is a quasi-isomorphism.

The decreasing filtrations
\[
F^\geq l \left( k C^*(B^*_{\hat{A}}, \hat{g}_n) \right) := k C^*(B^*_n, \hat{g}_n), \quad F^\geq l \left( 0 C^*(B^*_n, \hat{g}_n) \right) := 0 C^*(B^*_n, \hat{g}_n)
\]
induce spectral sequences with
\[
E_0^{pq} \left( k C^*(B^*_{\hat{A}}, \hat{g}_n) \right) = \bigoplus_{p+\ell=r} k C^\ell(B^p_{\hat{A}}, \hat{g}_n), \quad E_0^{pq} \left( 0 C^*(B^*_n, \hat{g}_n) \right) = \bigoplus_{p+\ell=r} 0 C^\ell(B^p_{\hat{A}}, \hat{g}_n)
\]
respectively converging to their cohomologies. Therefore it remains to prove that the map
\[
k,0_n : \bigoplus_{p+\ell=r} k C^\ell(B^p_{\hat{A}}, \hat{g}_n) \rightarrow \bigoplus_{p+\ell=r} 0 C^\ell(B^p_{\hat{A}}, \hat{g}_n)
\]
induces an isomorphism on the cohomology of the \( E_0 \)-page for each fixed \( q \).

If we further consider the filtrations \( \bigoplus_{p+\ell=r} k C^\ell(B^p_{\hat{A}}, \hat{g}_n) \) and \( \bigoplus_{p+\ell=r} 0 C^\ell(B^p_{\hat{A}}, \hat{g}_n) \), then we only need to show that the induced map
\[
k,0 \rightarrow 0 \Rightarrow \bigoplus_{p+\ell=r} k C^\ell(B^p_{\hat{A}}, \hat{g}_n) \rightarrow \bigoplus_{p+\ell=r} 0 C^\ell(B^p_{\hat{A}}, \hat{g}_n)
\]
on the corresponding \( E_0 \)-page is a quasi-isomorphism for any fixed \( c \) and \( q \), where \( k B^{p,q}_{\alpha_0-\ldots-\alpha_\ell}(\hat{g}_n) \) is constructed by gluing together the sheaves \( k K^*_{\alpha_0-\ldots-\alpha_\ell} \)'s as in Definition 4.5.

Note that the differential on \( \bigoplus_{p+\ell=r} \prod_{\alpha_0-\ldots-\alpha_\ell} k B^{p,q}_{\alpha_0-\ldots-\alpha_\ell}(\hat{g}_n) \) is given componentwise by the differential \( \hat{\partial}_{\hat{\alpha}_0-\ldots-\hat{\alpha}_\ell} : k B^{p,q}_{\alpha_0-\ldots-\alpha_\ell}(\hat{g}_n) \rightarrow k B^{p+1,q}_{\alpha_0-\ldots-\alpha_\ell}(\hat{g}_n) \) (where the term \( k B^{p,q}_{\alpha_0-\ldots-\alpha_\ell}(\hat{g}_n) \subset \bigoplus_{\ell=0}^\ell k B^{p,q}_{\alpha_0-\ldots-\alpha_\ell} \) is defined as in Definition 4.10 using \( k K^* \)'s). Using similar argument as in Lemma 3.27, we can see that the bottom horizontal map \( k,0 \rightarrow 0 \) in the above diagram is surjective. Finally, the kernel complex of this map is acyclic by the holomorphic Poincaré Lemma in Assumption 4.15 and arguments similar to Lemma 3.27.

For a general ideal \( K \subset m \), one can argue that \( H^*_{\hat{A}^*_{\hat{K}}}(K^*, K d) \) is a free \( R/K \) module as follows. We consider the sequence of ideals \( m = J_1 + K \supset J_2 + K \supset \ldots \supset J_l + K = K \) for some \( l \). Then one can prove that \( H^*_{\hat{A}^*_{\hat{J}_{j+1}+K}}(K^*, J_{j+1}+K d) \rightarrow H^*_{\hat{A}^*_{\hat{J}_{j}+K}}(K^*, J_j+K d) \) is surjective by induction on \( j \). Details are left to the readers.

5. An abstract unobstructedness theorem

Theorem 3.34 produces an almost differential graded Batalin-Vilkovisky (abbrev. dgBV) algebra \( (PV^{*,*}, \hat{\partial}, \Delta, \wedge) \) (where “almost” means \( (\hat{\partial}+\Delta)^2 \) is zero only at 0th-order), together with an almost de Rham module \( (\hat{A}^{*,*}, \hat{\partial}, \vartheta, \Delta, \wedge) \) (where “almost” means \( (\hat{\partial}+\vartheta)^2 \) is zero only at 0th-order) and the volume element \( \omega \in \hat{A}^{0,0} \). From these we can prove an unobstructedness theorem, using the techniques from [1] [33, 32, 51].
5.1. **Solving the Maurer-Cartan equation from dgBV structures.** We first introduce some notations, following Barannikov [1]:

**Notation 5.1.** Let \( t \) be a formal variable. We consider the spaces of formal power series or Laurent series in \( t \) or \( t^{\frac{1}{2}} \) with values in polynomial fields

\[
\begin{align*}
&k PV^p_0[[t]], \ k PV^p_q[[t^{\frac{1}{2}}]], \ k PV^p_q[[t^{\frac{1}{2}}]][[t^{-\frac{1}{2}}]],
\end{align*}
\]

together with a scaling morphism \( l_t : k PV^p_q[[t^{\frac{1}{2}}]][[t^{-\frac{1}{2}}]] \to k PV^p_q[[t^{\frac{1}{2}}]][[t^{-\frac{1}{2}}]] \) induced by \( l_t(\varphi) = t^{\frac{d-n-2}{2}} \varphi \) for \( \varphi \in k PV^p_q[[t^{\frac{1}{2}}]][[t^{-\frac{1}{2}}]] \). We have the identification \( k \tilde{\partial} := t^{\frac{1}{2}} l_t^{-1} \circ k \partial \circ l_t = k \tilde{\partial} + t(k \Delta) + t^{-1}(k I + t(k \eta)) \wedge \). We also consider spaces of formal power series or Laurent series in \( t \) or \( t^{\frac{1}{2}} \) with values in the relative de Rham module

\[
\begin{align*}
&k \| A^p_q[[t^{\frac{1}{2}}]], \ k \| A^p_q[[t^{\frac{1}{2}}]][[t^{-\frac{1}{2}}]],
\end{align*}
\]

together with the rescaling \( l_t : k \| A^p_q[[t^{\frac{1}{2}}]][[t^{-\frac{1}{2}}]] \to k \| A^p_q[[t^{\frac{1}{2}}]][[t^{-\frac{1}{2}}]] \) given by \( l_t(\alpha) = t^{\frac{d-n+2}{2}} \alpha \) which preserves the filtration on \( \| A \), and gives \( l_t(\varphi, \omega) = l_t(\varphi \cdot k \omega) \) and \( k \tilde{\partial} := t^{\frac{1}{2}} l_t^{-1} \circ k \partial \circ l_t = k \tilde{\partial} + t(k \Delta) + t^{-1}(k L_t) \).

For the purpose of constructing log Frobenius structures in the next section, we consider a finite-dimensional graded vector space \( \mathbb{V}^\circ \) and the associated graded symmetric algebra \( T := \text{Sym}(\mathbb{V}^\circ) \), equipped with the maximal ideal \( \mathbb{I} \) generated by \( \mathbb{V}^\circ \). We will abuse notations by using \( \mathbb{m} \) and \( \mathbb{I} \) again to denote the respective ideals of \( R_T := R \otimes T \), where \( R \) is the coefficient ring introduced in Notation 1.5. We also let \( \mathbb{I} : = \mathbb{m} + \mathbb{I} \) be the ideal generated by \( \mathbb{m} \otimes T + R \otimes \mathbb{I} \) and write \( k R_T := (R_T/\mathbb{I}^{k+1}) \). We write \( k PV_T := k PV \otimes T \otimes R_T \) and \( k \| A_T := k \| A \otimes T \otimes R_T \). Then \( k PV_T[[t^{\frac{1}{2}}]][[t^{-\frac{1}{2}}]] \) and \( k \| A_T[[t^{\frac{1}{2}}]][[t^{-\frac{1}{2}}]] \) be the complexes of formal series or Laurent series in \( t^{\frac{1}{2}} \) or \( t \) with values in those coefficient rings.

**Remark 5.2.** We can also define the Hodge bundle \( H^* (\| A^*, d) \otimes T \) over the formal power series ring \( \hat{R}_T := \lim_{\leftarrow k} k R_T \), which is equipped with the Gauss-Manin connection \( \nabla \) defined as in Definition 4.12. Then Lemma 4.17 implies that the Hodge bundle \( H^*(\| A^*, d) \otimes T \) is free over \( \hat{R}_T \), or equivalently, \( H^*(k PV_T, k \tilde{d}) \) is free over \( k R_T \) for each \( k \in \mathbb{Z}_{\geq 0} \).

**Definition 5.3.** An element \( k \varphi \in k PV^0_0[[t]] \) with \( k \varphi = 0 \mod \mathbb{m} + \mathbb{I} \) is called a Maurer-Cartan element over \( R_T/\mathbb{I}^{k+1} \) if it satisfies the Maurer-Cartan equation

\[
( k \tilde{\partial} + t(k \Delta) )^k \varphi + \frac{1}{2} [ k \varphi, k \varphi ] + ( k I + t(k \eta) ) = 0,
\]

or equivalently, \( ( k \tilde{\partial} + t(k \Delta) )^k \varphi + [ k \varphi, \cdot ]^2 = 0 \).

Notice that the MC equation (5.1) is also equivalent to \( k d(e_{\tilde{\partial}}(k \varphi) + \omega) = 0 \iff k \tilde{\partial}(e_{\partial}(k \varphi)) = k \tilde{\partial} + k \Delta + (k I + k \eta) \wedge (e_{\partial}(k \varphi)) = 0 \). In order to solve (5.1) using algebraic techniques as in [32], we need Assumption 4.15 which guarantees freeness of the Hodge bundle, as well as a suitable version of the Hodge-to-de Rham degeneracy; recall that these are also the essential conditions to ensure unobstructedness for smoothing of log smooth Calabi-Yau varieties in [33].

Recall from Remark 4.13 that the cohomology \( H^*(\| A^*, 0) \) computes the hypercohomology \( \mathbb{H}^* (\| A^*, 0) \), so the Hodge filtration \( \mathcal{F}^p_{\geq p} \mathbb{H}^* = H^* (\| A^* \geq p, 0) \) (where \( 0 d = \partial + 0 \partial \)) is induced by the filtration \( \mathcal{F}_{\geq p} (\| A^*) := \| A^* \geq p, 0 \) on the complex \( \| A^*, 0 d \).
Assumption 5.4 (Hodge-to-de Rham degeneracy). We assume that the spectral sequence associated to the decreasing filtration $F^{>k}(0, A)$ degenerates at the $E_1$ term.

Assumption 5.4 is equivalent to the condition that $H^*(0 PV[[t]], 0 \tilde{d} = \tilde{\partial} + t(0 \Delta))$ (or equivalently, that $H^*(\frac{\partial}{\partial t}, \tilde{\partial} + t(0 \partial))$) is a finite rank free $\mathbb{C}[[t]]$-module (cf. [32]).

Theorem 5.5. Suppose Assumptions 4.15 and 5.4 hold. Then for any degree 0 element $\psi \in 0 PV[[t]] \otimes \mathbb{C}(I/I^2)$ with $(\tilde{\partial} + t(0 \Delta))\psi = 0$, there exists a Maurer-Cartan element $k \varphi \in k PV^0_{\tilde{\partial}}[[t]]$ over $R_t/I_t^{k+1}$ for each $k \in \mathbb{Z}_{\geq 0}$ such that $k^{+1} \varphi = k \varphi$ (mod $I_t^{k+1}$) and $k \varphi = \psi$ (mod $m + I^2$).

Proof. We will consider the surjective map $k^{+1,k_0}: k^{+1} PV^p_q[[t]] \to k PV^p_q[[t]]$ obtained from Corollary 3.28 and inductively solve for $k \varphi \in k PV^0_{\tilde{\partial}}[[t]]$ for each $k \in \mathbb{Z}_{\geq 0}$ so that $k^{+1,k_0}(k^{+1} \varphi) = k \varphi, k \varphi = \psi$ (mod $m + I^2$) and $k \varphi$ satisfies the Maurer-Cartan equation $\tilde{\partial} + t(0 \partial)$ in $k PV^0_{\tilde{\partial}}[[t]]$.

We begin with $0 \varphi = 0$ and try to solve for $1 \varphi$. As the operator $\tilde{\partial} = \tilde{\partial} + \Delta + (1 + \eta)\wedge$ satisfies $\tilde{\partial}^2 = 0$, we have $\tilde{d}(1 + \eta) = (\tilde{\partial} + \Delta)(1 + \eta) = 0$ (mod $I^2$) (where $0$ (mod $I^2$) means being mapped to zero under $\sim_{1,b}$). Together with the fact that $(1 + \eta) = 0$ (mod $I$), we see that $(1 + \eta)$ represents a cohomology class in $(0 PV^*, 0 \tilde{d} \otimes (I/I^2))$. Since $\tilde{d}(1) = (1 + \eta)$, we deduce that $[1 + \eta] = 0$ in $(1 PV^1, 0 \tilde{d})$. Now applying Lemma 4.17 or Remark 5.2 (freeness of the Hodge bundle) to $(1, A^*, 1 \tilde{d})$ gives the short exact sequence

$$0 \to H^*(0 PV^*) \otimes (I/I^2) \to H^*(0 PV^1) \to H^*(0 PV^*) \to 0$$

under the identification by the volume element $\omega$. We conclude that the class $[1 + \eta]$ is zero in $H^*(0 PV^*) \otimes (I/I^2)$ which means that $(1 + \eta) = (\tilde{\partial} + \Delta)(-\tilde{\zeta})$ (mod $I^2$) for some $\tilde{\zeta} \in 0 PV^0 \otimes (I/I^2)$, and we have $[1 + t \eta] = (\tilde{\partial} + t \Delta)(-\zeta)$ for some $\zeta \in 0 PV^0 [[t]]/t^{-1} \otimes (I/I^2)$.

Applying Assumption 5.4 and using the technique from [51], we can modify $\zeta$ to satisfy $\zeta \in 0 PV^0 [[t]] / \otimes (I/I^2)$ (i.e. removing all the negative powers in $t$), and then we can take $1 \varphi$ to be the image of $\zeta$ in $1 PV^1[[t]]$. Further observe the Maurer-Cartan element $1 \varphi$ can be modified by adding any $\xi \in 1 PV^1[[t]]$ with $\xi = 0$ (mod $I$) and $1 \tilde{d} \xi = 0$ (mod $I^2$). Therefore we can always achieve $1 \varphi + \xi = \psi$ (mod $m + I^2$) by choosing a suitable $\xi$ and letting $1 \varphi + \xi$ be the new $1 \varphi$.

Next suppose $k^{+1} \varphi$ satisfying the Maurer-Cartan equation $\tilde{d}(\exp(k^{+1} \varphi)) = 0$ (mod $I^k$) up to order $k - 1$ has been constructed. Take an arbitrary lifting $k^{+1} \varphi$ in $k PV^1_{\tilde{\partial}}[[t]]$ and let $k^0 := \tilde{d}(\exp(k^{+1} \varphi)) = t^{k+1}l_t(\exp(k^{+1} \varphi/t))$ (mod $I_t^{k+1}$). Then $k^0$ represents a cohomology class in $(0 PV^1 [[t^{1/2}]]/t^{-1/2} \otimes (I/I_t^{k+1}), 0 \tilde{d})$. We again apply Lemma 4.17 to obtain a short exact sequence

$$0 \to H^*(0 PV^* [[t^{1/2}]]/t^{-1/2} \otimes (I/I_t^{k+1}) \to H^*(0 PV^* [[t^{1/2}]]/t^{-1/2}) \to H^*(k^{+1} PV^* [[t^{1/2}]]/t^{-1/2}) \to 0,$$

which forces $k^0 = 0$ as in the initial case. Hence, applying Assumption 5.4 and using the technique from [51] again, we can find $\zeta \in 0 PV^0 [[t]] / \otimes (I/I_t^{k+1})$ such that $\tilde{\partial} + t(0 \Delta)(-\zeta) = l_t^{-1}(k^0)$ and then set $k \varphi := k^{+1} \varphi + \zeta$ to solve the equation.

\[\square\]

5.2. Homotopy between Maurer-Cartan elements for different sets of gluing morphisms. Theorem 5.5 is proven for a fixed set of compatible gluing morphisms $g = \{k g_{\alpha, \beta}\}$. In this subsection, we study how Maurer-Cartan elements for two different sets of compatible gluing morphisms $g(0) = \{k g_{\alpha, \beta}(0)\}$ and $g(1) = \{k g_{\alpha, \beta}(1)\}$ are related through a fixed homotopy $h = \{h_{\alpha, \beta}\}$.
We begin by assuming that the data $\mathcal{D} = (\mathfrak{D}_\alpha)_{\alpha} \in \mathcal{C}_0(TW^{-1,-1}, h)$ and $\mathfrak{F} = (\mathfrak{F}_\alpha)_{\alpha} \in \mathcal{C}_0(TW^{0,0}, h)$ for the construction of the operators $\mathcal{D}$ and $\Delta$ in Proposition 3.35 are related to the data $\mathfrak{d}_j$ and $\bar{f}_j$ for the construction of the operators $\partial_j$ and $\Delta_j$ in Theorem 3.34 by the relations

$$\mathfrak{r}_j^*(\mathcal{D}) = \mathfrak{d}_j, \quad \mathfrak{r}_j^*(\mathfrak{F}) = \bar{f}_j$$

for $j = 0, 1$, where $\mathfrak{r}_j : PV^{*,*}(h) \rightarrow PV^{*,*}(g(j))$ is the map introduced in Definition 3.29.

**Notation 5.6.** Similar to Lemma 4.9, we let $\mathcal{L}_\alpha := \mathfrak{D}_\alpha(\mathfrak{D}_\alpha) + \frac{1}{2}[\mathfrak{D}_\alpha, \mathfrak{D}_\alpha]$ and $\mathfrak{E}_\alpha := \Delta_\alpha(\mathfrak{D}_\alpha) + \mathfrak{D}_\alpha(\mathfrak{F}_\alpha) + [\mathfrak{D}_\alpha, \mathfrak{F}_\alpha]; (\mathfrak{L}_\alpha)_{\alpha}$ and $(\mathfrak{E}_\alpha)_{\alpha}$ glue to give global terms $\mathfrak{L} \in PV^{2,-1}(h)$ and $\mathfrak{E} \in PV^{1,0}(h)$ respectively.

We set $\tilde{\mathcal{D}} := \mathcal{D} + \Delta + (\mathcal{L} + \mathfrak{E})\wedge$, which defines an operator acting on $PV^*(h)$ (and we will use $k\tilde{\mathcal{D}}$ to denote the corresponding operator acting on $kPV^*(h)$). We have $\tilde{\mathcal{D}}^2 = 0$ as in Proposition 4.11.

We also introduce a scaling $l_{\varphi} : kPV^{p,q}(h)[[t^{\frac{1}{2}}]][[t^{-\frac{1}{2}}]] \rightarrow kPV^{p,q}(h)[[t^{\frac{1}{2}}]][[t^{-\frac{1}{2}}]]$ defined by $l_{\varphi}(\varphi) = t^{\frac{1}{2}-\frac{p}{2}} \varphi$ for $\varphi \in kPV^{p,q}(h)$. Then we have the identity $k\tilde{\mathcal{D}} l_{\varphi} := t^{\frac{1}{2}}l_{\varphi}^{-1} \circ k\tilde{\mathcal{D}} \circ l_{\varphi} = k\mathcal{D} + t(k\Delta) + t^{-1}(k\mathcal{L} + t(k\mathfrak{E}))\wedge$ as in Notation 5.1.

**Similar to Notation 5.1, we consider the complex $kPV^*_T(h)[[t]]$ (or formal power series or Laurent series in $t$ or $t^{-1}$) for any graded ring $T = \mathbb{C}[V^*]$.**

**Lemma 5.7.** The natural restriction map $\mathfrak{r}_j^* : (kPV^*(h), k\tilde{\mathcal{D}}) \rightarrow (kPV^*(g(j)), k\mathcal{D})$ is a quasi-isomorphism for $j = 0, 1$ and all $k \in \mathbb{Z}_{\geq 0}$.

**Proof.** We will only give a proof of the case $j = 0$ because the other case is similar. We first consider the following diagram

$$
\begin{array}{cccccccc}
0 & \rightarrow & k^{-1}PV^*(h) \otimes \mathbb{C}(m/m^2) & \rightarrow & kPV^*(h) & \rightarrow & 0PV^*(h) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & k^{-1}PV^*(g(0)) \otimes \mathbb{C}(m/m^2) & \rightarrow & kPV^*(g(0)) & \rightarrow & 0PV^*(g(0)) & \rightarrow & 0
\end{array}
$$

with exact horizontal rows. By passing to the corresponding long exact sequence, we see that it suffices to prove that $\mathfrak{r}_0^* : (0PV^*(h), 0\mathcal{D} + 0\Delta) \rightarrow (0PV^*(g(0)), 0\mathcal{D} + 0\Delta)$ is a quasi-isomorphism. In this case we have $0h = id$ (mod $m$) and $0g(0) = id$ (mod $m$), from which we deduce that $0PV^*(h) = \mathcal{A}^*(\Delta_1) \otimes \mathbb{C}^*0PV^*(g(0))$ in which the operators are related by $0h + 0\Delta = 0\mathcal{D} + 0\Delta + d_s$, where $s$ is the coordinate function on the 1-simplex $\Delta_1$ and $d_s$ is the usual de Rham differential acting on $\mathcal{A}^*(\Delta_1)$. The quasi-isomorphism is then obtained using the homotopy operator constructed by integration $\int_0^s$ along the 1-simplex. Details are left to the readers. 

The following proposition relates Maurer-Cartan elements $\varphi_0$ of $PV^*_T(0)(g(0))[[t]]$ and those of $PV^*_T(h)[[t]]$.

**Proposition 5.8.** Given any Maurer-Cartan element $k\varphi_0 \in kPV_0^*(g(0))[[t]]$ as in Theorem 5.5, there exists a lifting $k\varphi \in kPV_0^*(h)[[t]]$ which is a Maurer-Cartan element for each $k$ such that $k+1\varphi = k\varphi$ (mod $T^{k+1}$) and $\mathfrak{r}_0^*(k\varphi) = k\varphi_0$. If there are two liftings $(k\varphi)_k$ and $(k\psi)_k$ of $(k\varphi)_0$, then there exists a gauge element $k\vartheta \in kPV_1^*(h)[[t]]$ for each $k$ such that $\mathfrak{r}_0^*(k\vartheta) = 0$, $k+1\vartheta = k\vartheta$ (mod $T^{k+1}$) and $e^{k\vartheta} \circ k\varphi = k\psi$.

**Proof.** We construct $k\varphi$ by induction on $k$. Given a Maurer-Cartan element $k^{-1}\mathfrak{r}_0^*(k\varphi \in k^{-1}PV_0^*[h][[t]]$ such that $\mathfrak{r}_0^*(k^{-1}\varphi) = k^{-1}\varphi_0$, our goal is to construct a lifting $k\varphi$ of $k^{-1}\varphi$ with $\mathfrak{r}_0^*(k\varphi) = k\varphi_0$. 


By surjectivity of \( k,k^{-1} : kPV_T(h)[[t]] \to k^{-1}PV_T(h)[[t]] \), we get a lifting \( \widehat{k^{-1} \varphi} \) of \( k^{-1} \varphi \). By surjectivity of \( r_0^* : kPV_T^1(h)[[t]] \to PV_T^0(kg(0))[[t]] \) for any \( k \) from Lemma 3.30, we further obtain a lifting \( \eta \) of \( k \varphi_0 - r_0^*(k^{-1} \varphi) \) such that \( \eta = 0 \) (mod \( \mathcal{I}^k \)) in \( kPV_T^0(h)[[t]] \). Then we set \( \widetilde{k \varphi} := k^{-1} \varphi + \eta \) so that \( r_0^*(\widetilde{k \varphi}) = k \varphi_0 \). Similar to the proof of Theorem 5.5, we define the obstruction class

\[
kO := \widehat{k_D_t(te^{\widetilde{k \varphi}/t}) = (kD + t(k\Delta))\widehat{k \varphi} + \frac{1}{2}[\widetilde{k \varphi}, \widetilde{k \varphi}] + (kE + t(kE))
\]

in \( kPV_T^1[[t]] \) which satisfies \( k,k^{-1} \gamma(kO) = 0 \) and \( kD_t(kO) = 0 \).

Considering the short exact sequences

\[
0 \longrightarrow K^* \longrightarrow (kPV_T^1(h)[[t]], kD_t) \overset{r_0^*}{\longrightarrow} (kPV_T^0(g(0))[[t]], kD_t) \longrightarrow 0,
\]

\[
0 \longrightarrow \widehat{K} \longrightarrow (kPV_T^1(h)[[t]], k\bar{D}) \overset{\bar{r}_0^*}{\longrightarrow} (kPV_T^0(g(0))[[t]], k\bar{D}) \longrightarrow 0
\]

and observing that \( (K^*, kD_t) \) is acyclic, we conclude that \( kO \in K^1 \). Hence we can find \( \zeta \in K^0 \) such that \( kD_t(\zeta) = kO \) and \( \zeta = 0 \) (mod \( \mathcal{I}^k \)). Then \( k \varphi := k \varphi + \zeta \) is the desired lifting of \( k^{-1} \varphi \).

The gauge \( (k\vartheta) \) can be constructed by a similar inductive process. Given \( k^{-1} \vartheta \), we need to construct a lifting \( k \vartheta \in kPV_T^{-1}(h)[[t]] \) which serves as a homotopy from \( k \varphi \) to \( k \psi \). Again we take a lifting \( \widetilde{k \vartheta} \) satisfying \( k,k^{-1} \gamma(k \vartheta) = k^{-1} \vartheta \) and \( r_0^*(k \vartheta) = 0 \), and consider the obstruction

\[
kO := k \psi - \exp \left( [\widehat{k \vartheta}, \cdot] \right)(k \varphi) + \frac{\exp \left( [\widehat{k \vartheta}, \cdot] \right) - 1}{[\widehat{k \vartheta}, \cdot]} ((kD + t(k\Delta)) \widehat{k \vartheta}),
\]

which satisfies \( k,k^{-1} \gamma(kO) = 0 \) and \( \bar{r}_0^*(kO) = 0 \). We can find \( \zeta \in \mathcal{O}PV_T^{-1}[[t]] \otimes (\mathcal{I}^k/\mathcal{I}^{k-1}) \) with \( \bar{r}_0^*(\zeta) = 0 \) such that \( -(0D + t(0\Delta)) \zeta = kO \) and letting \( k \vartheta := k \vartheta + \zeta \) gives the desired gauge element.

Given a homotopy \( h \), we define a map \( F_h \) from the set of Maurer-Cartan elements modulo gauge equivalence with respect to \( g(0) \) to that with respect to \( g(1) \) by \( F_h((k \varphi_0)_k) := (\bar{r}_1^*(k \varphi_0))_k \) with \( k \varphi \in kPV_T^0[[t]] \). Proposition 5.8 says that this map is well-defined, and its inverse \( F_h^{-1} \) is given by reversing the roles of \( g(0) \) and \( g(1) \), so \( F_h \) is a bijection.

Next we consider the situation where we have a fixed set of compatible gluing morphisms \( g = \{g_{\alpha \beta}\} \) but the complex \( kPV^* \) is equipped with two different choices of operators \( k \bar{\partial} \) and \( k \Delta, k \bar{\partial} \) and \( k \Delta' \), whose differences are captured by elements \( v_1 \in PV^{-1,1}(g) \) and \( v_2 \in PV^{0,0}(g) \), as in Theorem 3.34. We write \( v = v_1 + v_2 \) and consider the complex \( A^*(*_A) \otimes_C kPV^* \) equipped with the differential

\[
k\bar{D} := k\bar{d} + d_{*_1} + t_1(v, \cdot) + (t_1(k \bar{\partial} + k \Delta)v + \frac{t_1^2}{2}[v, v]) \land + (dt_1 \land v) \land,
\]

where \( t_1 \) is the coordinate function on the 1-simplex \( *_1 \) and \( d_{*_1} \) is the de Rham differential for \( A^*(*_1) \). We let \( kO_{t_1v} := (t_1(k \bar{\partial} + k \Delta)v + \frac{t_1^2}{2}[v, v]) + (k + k \eta) \) and compute

\[
(k\bar{D})^2 = (k \bar{\partial} + k \Delta + t_1[v, \cdot])^2 - [kO_{t_1v}, \cdot] + dt_1 \land \frac{\partial}{\partial t_1}(kO_{t_1v}) \land - dt_1 \land ((k \bar{\partial} + k \Delta)(v) + t_1[v, v]) \land = [kO_{t_1v}, \cdot] - [kO_{t_1v}, \cdot] + dt_1 \land ((k \bar{\partial} + k \Delta)(v) + t_1[v, v]) \land - dt_1 \land ((k \bar{\partial} + k \Delta)(v) + t_1[v, v]) \land = 0.
\]
Repeating the argument in this subsection but replacing \((kPV^*(h), k\hat{D})\) by \((A^\ast \otimes_C kPV^*, k\hat{D})\) and arguing as in the proof of Proposition 5.8 yields the following:

**Proposition 5.9.** Given any Maurer-Cartan element \(k\varphi_0 \in kPV^0_T[[t]]\) with respect to the operators \(k\hat{D}\) and \(k\Delta\) as in Theorem 5.5, there exists a lifting \(k\varphi \in A^\ast(\Delta_1) \otimes kPV^*_T[[t]]\) which is a Maurer-Cartan element with respect to the operators \((k\hat{D} + d_{\Delta_1} + t_1[v_1, \cdot])\) and \(k\Delta + [v_2, \cdot]\) (meaning that \(\left((k\hat{D} + d_{\Delta_1} + t_1[v_1, \cdot]) + t(k\Delta + [v_2, \cdot]) + [k\varphi, \cdot]\right)^2 = 0\)) for each \(k\) satisfying \(k^{+1}\varphi = k\varphi \pmod{T^{k+1}}\) and \(\mathbf{r}_0(k\varphi) = k\varphi_0\). If there are two liftings \((k\varphi)_k\) and \((k\psi)_k\) of \((k\varphi_0)_k\), then there exists a gauge element \(k\hat{D} \in A^\ast(\Delta_1) \otimes kPV^*_T[[t]]\) for each \(k\) such that \(\mathbf{r}_0(k\varphi) = 0\), \(k^{+1}\varphi = k\varphi \pmod{T^{k+1}}\) and \(e^{k\hat{D}} \ast k\varphi = k\psi\).

Propositions 5.8 and 5.9 together show that the set of gauge equivalence classes of Maurer-Cartan elements is independent of the choice of the gluing morphisms \(g = \{g_{\alpha\beta}\}\) and the choices of the operators \(\hat{D}\) and \(\Delta\) in the construction of \(kPV^*_T[[t]]\).

### 5.3. From a Maurer-Cartan element to geometric Čech gluing

In this subsection, we show that a Maurer-Cartan (MC) element \(\varphi = (k\varphi_k)_{k \in \mathbb{Z} \geq 0}\) as defined in Definition 5.3 contains the data for gluing the sheaves \(kG^\ast\)'s consistently.

We fix a set of gluing morphisms \(g = \{g_{\alpha\beta}\}\) and consider a MC element \(\varphi = (k\varphi_k)_{k \in \mathbb{Z} \geq 0}\) (where we take \(T = \mathbb{C}\)) obtained in Theorem 5.5. Setting \(t = 0\), we have the element \(k\phi := k\varphi|_{t=0}\) which satisfies the following extended MC equation (5.2).

**Definition 5.10.** An element \(k\phi \in kPV^0\) is said to be a Maurer-Cartan element in \(kPV^*\) if it satisfies the extended Maurer-Cartan equation:

\[
(k\hat{D}(k\phi) + \frac{1}{2}[k\phi, k\phi]) + k\mathbf{l} = 0.
\]

Note that \((kPV^{-1}, [-1], k\hat{D}, [\cdot, \cdot])\) forms a dgLa, and an element \(k\psi \in kPV^{-1,1}\) is called a classical Maurer-Cartan element if it satisfies (5.2).

**Lemma 5.11.** In the proof of Theorem 5.5, the Maurer-Cartan element \(k\varphi = k\varphi_0 + k\phi_1 t^1 + \cdots + k\phi_j t^j + \cdots \in kPV^0[[t]],\) where \(k\varphi_0 = k\psi_0 + k\psi_1 + \cdots \) and \(k\psi_1 \in kPV^{-1,1}\), can be constructed so that \(k\psi_0 = 0\). In particular, \(k\psi_1 \in kPV^{-1,1}\) is a classical Maurer-Cartan element.

**Proof.** We prove by induction on \(k\). Recall from the initial step of the inductive proof of Theorem 5.5 that \(1\varphi \in 1PV^0[[t]]\) was constructed so that \(((1\hat{D} + t^1(\Delta))((1\varphi) = 1t + t^1\eta).\) As \(1t \in 1PV^{-1,2}\) and \(1\eta \in 1PV^{0,1},\) we have \((1\hat{D}(1\varphi) = 0).\) Also, we know \(1\Delta(1\psi_0) = 0\) by degree reasons, so we obtain the equation \((1\hat{D} + t^1(\Delta))((1\varphi - 1\psi_0) = 1t + t^1\eta).\) Hence we can replace \(1\varphi\) by \(1\varphi - 1\psi_0\) in the construction so that the desired condition is satisfied.

For the induction step, suppose that \(k^{-1}\varphi = k^{-1}\varphi_0 + k^{-1}\phi_1 t + \cdots \in k^{-1}PV^0\) with \(k^{-1}\psi_0 = 0\) has been constructed. Again recall from the construction in Theorem 5.5 that we have solved the equation

\[
(k\hat{D} + t^1(\Delta))(\hat{\eta}) = (k\hat{D} + t^1(\Delta))(\hat{\eta})
\]

for \(\hat{\eta} \in kPV^0[[t]].\) We are only interested in the coefficient of \(t^0\) of the component lying in \(kPV^0,1\) on the RHS of the above equation, which we denote as \([k\hat{D} + t^1(\Delta)](\hat{\eta})_1.\) By writing \(k^{-1}\phi_0 = \cdots\)
\( k^{-1} \psi_1 + \cdots + k^{-1} \psi_d \) using the induction hypothesis, we have
\[
\left[ k \delta_t \left( t e^{-k^{-1} \varphi / t} \right) \right]_0 = \left[ \left( k \Delta (k^{-1} \varphi) + t(k \Delta (k^{-1} \varphi)) + \frac{1}{2} [k^{-1} \varphi, k^{-1} \varphi] + kI + t(k \eta) \right) \right]_0 \wedge \left( \exp(k^{-1} \varphi / t) \right)_0 = 0.
\]

Therefore by writing \( \hat{\eta} = \zeta_0 + \zeta_0 t^1 + \cdots, \) and \( \zeta_0 = \xi_0 + \cdots + \xi_d \) with \( \xi_i \in kPV^{-i} \), we conclude that \( k \delta_t (\zeta_0) = 0 \) and hence \( (k \delta_t + t(k \Delta))(\zeta_0) = 0 \). As a result, if we replace \( \hat{\eta} \) by \( \hat{\eta} - \zeta_0 \) in the construction, we get the desired element \( k \varphi \) for the induction step.

The second statement follows from the first because \( k \varphi_0 = k \psi_1 + \cdots + k \psi_d \) satisfies the extended MC equation (5.2). Then by degree reasons, we conclude that \( k \delta_t (k \psi_1) + 1/2 [k \psi_1, k \psi_1] + kI = 0. \) □

In view of Lemma 5.11, we restrict our attention to the dgLa \( kPV^{-1} \alpha^* [-1] \) and a classical Maurer-Cartan element \( k \varphi \in kPV^{-1} \). We write \( k \varphi = (k \psi_1)_\alpha \) where \( k \psi_1 \in kTW_{\alpha,1} \) with regard to the Čech-Thom-Whitney complexes in Definition 3.25.

Since \( V_\alpha \) is Stein and \( k \mathcal{G}_\alpha^* \) is a coherent sheaf over \( V_\alpha \), we have \( H^0(kTW_{\alpha,0}^p, [p], k \delta_\alpha) = 0 \) for any \( p \) (here \([p]\) is the degree shift so that \( kTW_{\alpha,0}^p \) is at degree 0). In particular, the operator \( k \delta_\alpha + [k \delta_\alpha, \cdot] + [k \psi_\alpha, \cdot] \) is gauge equivalent to \( k \delta_\alpha \) via a gauge element \( k \psi_\alpha \in kTW_{\alpha,0} \). As \( k+1 \psi_\alpha = k \psi_\alpha \) (mod \( m^{k+1} \)), we can further construct \( k \delta_\alpha \) via induction on \( k \) so that \( k+1 \delta_\alpha = k \delta_\alpha \) (mod \( m^{k+1} \)).

Given any open subset \( W \subset V_{\alpha \beta} \), we use the restrictions \( k \delta_\alpha |_W \in kTW_{\alpha,0}^{-1,0}(k \mathcal{G}_\alpha |_W) \) for \( k \delta_\beta \in kTW_{\beta,0}^{-1,0}(k \mathcal{G}_\beta |_W) \) to define an isomorphism \( k \mathcal{g}_{\alpha \beta} : kTW_{\alpha,0}^{-1,0}(k \mathcal{G}_\alpha |_W) \to kTW_{\beta,0}^{-1,0}(k \mathcal{G}_\beta |_W) \) which fits into the following commutative diagram
\[
\begin{array}{ccc}
  kTW_{\alpha,0}^{-1,0}(k \mathcal{G}_\alpha |_W) & \xrightarrow{k \mathcal{g}_{\alpha \beta}} & kTW_{\beta,0}^{-1,0}(k \mathcal{G}_\beta |_W) \\
  \exp([k \delta_\alpha, \cdot]) & \downarrow & \exp([k \delta_\beta, \cdot]) \\
  (kTW_{\alpha,0}^{-1,0}(k \mathcal{G}_\alpha |_W), k \delta_\alpha) & \xrightarrow{k \mathcal{g}_{\alpha \beta}} & (kTW_{\beta,0}^{-1,0}(k \mathcal{G}_\beta |_W), k \delta_\beta)
\end{array}
\]

here we emphasis that \( k \mathcal{g}_{\alpha \beta} \) identifies the differentials \( k \delta_\alpha \) and \( k \delta_\beta \).

Note that there is an identification \( k \mathcal{G}_\alpha^0(W) = H^0(kTW_{\alpha,0}^p, (k \mathcal{G}_\alpha |_W) [p], k \delta_\alpha) \), enabling us to treat \( k \mathcal{g}_{\alpha \beta} : k \mathcal{G}_\alpha^0(W) \to k \mathcal{G}_\beta^0(W) \) as an isomorphism of Gerstenhaber algebras. These isomorphisms can then be put together to give an isomorphism of sheaves of Gerstenhaber algebras \( k \mathcal{g}_{\alpha \beta} : k \mathcal{G}_\alpha^0 |_{V_{\alpha \beta}} \to k \mathcal{G}_\beta^0 |_{V_{\alpha \beta}} \). Furthermore, the cocycle condition for the gluing morphisms \( k \mathcal{g}_{\alpha \beta} \) (see Definition 3.17) implies the cocycle condition \( k \mathcal{g}_{\alpha \gamma} \circ k \mathcal{g}_{\beta \gamma} \circ k \mathcal{g}_{\alpha \beta} = \text{id} \).

**Definition 5.12.** A set of \( k \)-th order geometric gluing morphisms \( k \mathcal{g} \) consists of, for any pair \( V_\alpha, V_\beta \in \mathcal{V} \), an isomorphism of sheaves of Gerstenhaber algebras \( k \mathcal{g}_{\alpha \beta} : k \mathcal{G}_\alpha^* |_{V_{\alpha \beta}} \to k \mathcal{G}_\beta^* |_{V_{\alpha \beta}} \) satisfying \( k \mathcal{g}_{\alpha \beta} = \text{id} \) (mod \( m \)), and the cocycle condition \( k \mathcal{g}_{\alpha \gamma} \circ k \mathcal{g}_{\beta \gamma} \circ k \mathcal{g}_{\alpha \beta} = \text{id} \). Two such sets of \( k \)-th order geometric gluing morphisms \( k \mathcal{g} \) and \( k \mathcal{h} \) are said to be equivalent if there exists a set of isomorphisms of sheaves of Gerstenhaber algebras \( k a_\alpha : k \mathcal{G}_\alpha^* \to k \mathcal{G}_\beta^* \) with \( k a_\alpha = \text{id} \) (mod \( m \)) fitting into the following

\footnote{We thank Simon Felten for pointing out that this should be an isomorphism of Gerstenhaber algebras, instead of just an isomorphism of graded Lie algebras.}
commutative diagram

If we have two classical Maurer-Cartan elements $^k\psi$ and $^k\tilde{\psi}$ which are gauge equivalent via $^k\theta = (^k\theta_\alpha)_\alpha$, then we can construct as isomorphism $\exp(-[^k\bar{\theta}_\alpha, \cdot]) \circ \exp([^k\theta_\alpha, \cdot]) \circ \exp([^k\partial_\alpha, \cdot]) : (^kTW^s_\alpha, ^k\bar{\theta}_\alpha) \to (^kTW^s_\alpha, ^k\tilde{\theta}_\alpha)$ inducing an isomorphism $^k\alpha \colon ^kG^*_\alpha(V_\alpha) \to ^kG^*_\alpha(V_\alpha)$ by taking $H^0(^kTW^s_\alpha, ^k\tilde{\theta}_\alpha)$, so that the two sets of $k$-th order geometric gluing morphisms $^k\varphi$ and $^k\tilde{\varphi}$ associated to $^k\psi$ and $^k\tilde{\psi}$ respectively are equivalent via $^k\alpha = (^k\alpha_\alpha)_\alpha$. This gives the following:

**Proposition 5.13.** Given classical Maurer-Cartan elements $^k\psi \in ^kPV^{-1,1}$ such that $^k+1\psi = ^k\psi \ (mod \ m^{k+1})$ and $^k\psi = 0 \ (mod \ m)$, there exists an associated set of geometric gluing morphisms $^k\varphi$ for each $k$ satisfying $^k\varphi = ^k\varphi \ (mod \ m^{k+1})$. For two classical Maurer-Cartan elements $^k\psi$ and $^k\tilde{\psi}$ which are gauge equivalent via $^k\theta$ such that $^k+1\theta = ^k\tilde{\theta} \ (mod \ m^{k+1})$, there exists an equivalence $^k\alpha$, satisfying $^k+1\alpha = ^k\alpha \ (mod \ m^{k+1})$ between the associated geometric gluing data $^k\varphi$ and $^k\tilde{\varphi}$.

Lemma 5.11 together with Proposition 5.13 produces a geometric Čech gluing of the sheaves $^kG^*_\alpha$’s, unique up to equivalence, from a gauge equivalence class of the MC elements obtained in Theorem 5.5.

6. Abstract semi-infinite variation of Hodge structures

In this section, we apply techniques developed in [2, 11, 32, 40] to our abstract framework. Under the Assumptions 6.12 (existence of opposite filtration) and 6.17 (nondegeneracy of pairing), this constructs the structure of a logarithmic Frobenius manifold (introduced in [40]) on the formal neighborhood of $X$ in the extended moduli space.

6.1. Brief review of relevant structures.

**Notation 6.1.** Following Notation 5.1, let $\hat{R}_T := \varprojlim_k ^kR_T$ be the completion of $R \otimes T$. We will abuse notations and use $m, I$ and $\mathcal{I}$ to denote the corresponding ideals in $\hat{R}_T$. As in Notation 5.5, we have a monoid homomorphism $Q \to ^kR_T$ sending $m \mapsto q^m$ which equips $^kR_T$ with the structure of a (graded) log ring. We also denote the corresponding formal germ of log space by $^kS^1_T$.

We define the module of log differentials

$$^k\Omega^1_{S^1_T} := ^kR_T \otimes_{\mathcal{K}_C} \text{Sym}^l \left( (\mathcal{K}_C \oplus \mathcal{V}^\vee)[-1] \right),$$

where $\text{Sym}^l$ refers to the graded symmetric product. For $^k\Omega^1_{S^1_T} = ^kR_T \otimes_{\mathcal{K}_C} (\mathcal{K}_C \oplus \mathcal{V}^\vee)[-1]$, we write $d\log q^m$ for the element corresponding to $m \in \mathcal{K}_C$ and $dz$ for that corresponding to $z \in \mathcal{V}^\vee$. Then we have a de Rham differential $d : ^kR_T \to ^k\Omega^1_{S^1_T}$ satisfying the graded Leibniz rule, the relation $d(q^m) = q^m(d\log q^m)$ for $m \in Q$, and $d(z) = dz$ for $z \in \mathcal{V}^\vee$.

\[\text{[1]}\text{Here we abuse notations by writing } q^m \text{ in place of } q^m \otimes 1.\]
We also let \( k\Theta_{S^+_T} := kR_T \otimes_C (K^\vee \oplus \nabla)[1] \) be the space of log derivations, which is equipped with a Lie bracket \([\cdot, \cdot]\) and a natural pairing between \( X \in k\Theta_{S^+_T} \) and \( \alpha \in k\Omega^1_{S^+_T} \). Similarly we can talk about \( \hat{\Omega}_{S^+_T}^* \) and \( \hat{\Theta}_{S^+_T}^* \) by taking inverse limits.

**Definition 6.2.** A log semi-infinite variation of Hodge structure (abbrev. \( \infty^\vee \)-LVHS) over \( \hat{R}_T \) consists of triples \((kH, k\nabla, k\langle \cdot, \cdot \rangle)\) for each \( k \geq 0 \) and \( kR_T \)-linear maps \( k\beta : kH \to kH \) for \( k \geq 1 \), where

1. \( kH = kH^* \) is a graded free \( kR_T[[t]] \) module, called the (sections of the) Hodge bundle;
2. \( k\nabla \) is the Gauss-Manin (partial) connection of the form
   \[(6.1) k\nabla : kH \to \frac{1}{t}(k\Omega^1_{S^+_T}) \otimes_{kR_T} kH \]
   and compatible with the maps \( k\beta \)'s;
3. \( k\langle \cdot, \cdot \rangle : kH \times kH \to kR_T[[t]][-2d] \) is a degree preserving pairing which is compatible with the maps \( k\beta \)'s,

satisfying the following conditions (when there is no danger of confusion, we will omit the dependence on \( k \) and simply write \( \nabla \) and \( \langle \cdot, \cdot \rangle \) instead of \( k\nabla \) and \( k\langle \cdot, \cdot \rangle \)):

1. \( \langle s_1, s_2(t) \rangle = (-1)^{|s_1||s_2|} \langle s_1, s_2 \rangle (-t)^{|\tilde{s}_1|} \)\(^{12}\) where \( |s_i| \) is the degree of the homogeneous element \( s_i \);
2. \( \langle f(t)s_1, s_2 \rangle = (-1)^{|s_1||f|} \langle s_1, f(-t)s_2 \rangle = f(t)\langle s_1, s_2 \rangle \) for \( s_1 \in kH \) and \( f(t) \in kR_T[[t]] \);
3. \( \nabla_X \langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + (-1)^{|s_1|||X|+1|} \langle s_1, \nabla_X s_2 \rangle \) for \( X \in k\Theta_{S^+_T} \);
4. the induced pairing \( g(\cdot, \cdot) : (kH/tkH) \times (kH/tkH) \to kR_T[-2d] \) is non-degenerate.

**Definition 6.3.** Given a \( \infty^\vee \)-LVHS \((kH^*, \nabla, \langle \cdot, \cdot \rangle)\), a grading structure is an extension of the Gauss-Manin connection \( \nabla \) along the \( t \)-coordinate

\[ \nabla_{t^\frac{\partial}{\partial t}} : kH \to t^{-1}(kH), \]

which is compatible with the maps \( k\beta \)'s and such that \( [\nabla_X, t^\frac{\partial}{\partial t}] = 0 \), i.e. it is a flat connection on \( kS^+_T \times (\mathbb{C}, 0) \). We further require the pairing \( \langle \cdot, \cdot \rangle \) to be flat with respect to \( \nabla_{t^\frac{\partial}{\partial t}} \) in the sense that \( t\frac{\partial}{\partial t}\langle s_1, s_2 \rangle = \langle \nabla_{t^\frac{\partial}{\partial t}} s_1, s_2 \rangle + \langle s_1, \nabla_{t^\frac{\partial}{\partial t}} s_2 \rangle \).

**Notation 6.4.** Let \( kH^\pm := kH \otimes_{k[[t]]} [[t]][-t^{-1}] \) be a module over \( kR_T[[t]][-t^{-1}] \) equipped with the natural submodule \( kH_+ := kH \subset kH^+ \) which is closed under multiplication by \( kR_T[[t]] \). There is a natural symplectic structure \( k\omega(\cdot, \cdot) : kH^\pm \times kH^\pm \to kR_T \) defined by \( k\omega(\alpha, \beta) = \text{Res}_{t=0} \langle \alpha, \beta \rangle dt \).

Also let \( H^\pm := \lim\sup_k kH^\pm \), which is a module over \( \hat{R}_T[[t]][-t^{-1}] := \lim\inf kR_T[[t]][-t^{-1}] \), equipped with a natural \( R_T[[t]] \) submodule \( H_+ := \lim\sup_k kH_+ \) and the symplectic structure \( \omega := \lim\inf k\omega \). We also write \( \hat{R}_T[-t^{-1}] := \lim\inf kR_T[-t^{-1}] \).

**Definition 6.5.** An opposite filtration is a choice of \( kR_T[-t^{-1}] \) submodule \( kH_- \subset kH_\pm \) for each \( k \in \mathbb{Z}_{\geq 0} \), compatible with the maps \( k\beta \)'s and satisfying the following conditions for each \( k \):

1. \( kH_+ \oplus kH_- = kH_\pm \);
2. \( kH_- \) is preserved by \( \nabla_X \) for any \( X \in k\Theta_{S^+_T} \);

\(^{12}\)Here \( f(-t) \in kR_T[[t]] \) is the element obtained from \( f(t) \) by substituting \( t \) with \(-t\).
(3) \( k^\mathcal{H}_- \) is isotropic with respect to the symplectic structure \( k^w(\cdot, \cdot) \);
(4) \( k^\mathcal{H}_- \) is preserved by the \( \nabla_{t^{\frac{k}{m}}} \).

We also write \( \mathcal{H}_- := \lim_{\to k} k^\mathcal{H}_- \).

Given an opposite filtration \( \mathcal{H}_- \), we have a natural isomorphism (cf. [20], [39])
\[
k^{\mathcal{H}_+} / t(k^{\mathcal{H}_+}) \cong k^{\mathcal{H}_+} \cap t(k^{\mathcal{H}_-}) \cong t(k^{\mathcal{H}_-}) / k^{\mathcal{H}_-},
\]
for each \( k \), giving identifications
\[
\tau_+: k^{\mathcal{H}_+} \cap t(k^{\mathcal{H}_-}) \otimes_{\mathbb{C}} \mathbb{C}[i/t] \to k^{\mathcal{H}_+},
\]
\[
\tau_-: k^{\mathcal{H}_+} \cap t(k^{\mathcal{H}_-}) \otimes_{\mathbb{C}} \mathbb{C}[t^{-1}] \to t(k^{\mathcal{H}_-}).
\]

Using arguments from [39], we see that
\[
\langle k^{\mathcal{H}_+} \cap t(k^{\mathcal{H}_-}), k^{\mathcal{H}_+} \cap t(k^{\mathcal{H}_-}) \rangle \in kR_T \text{ and } \langle k^{\mathcal{H}_-}, k^{\mathcal{H}_-} \rangle \in kR_T[t^{-1}]t^{-2}.
\]

Morally speaking, a choice of an opposite filtration \( \mathcal{H}_- \) gives rise to a (trivial) bundle \( \mathcal{H} \) over \( \hat{S}^\dagger_T \times \mathbb{P}^1 \), where \( t \) is a coordinate on \( \mathbb{P}^1 \), as follows. We let the sections of germ of \( k\mathcal{H} \) near \( k\hat{S}^\dagger_T \times (\mathbb{C}, 0) \) be given by \( k^{\mathcal{H}_+} \), and that of \( k\mathcal{H} \) over \( k\hat{S}^\dagger_T \times (\mathbb{P}^1 \setminus \{0\}) \) be given by \( t(k^{\mathcal{H}_-}) \). Then (6.3) and (6.4) give a trivialization of the bundle \( k\mathcal{H} \) over \( k\hat{S}^\dagger_T \times \mathbb{P}^1 \) whose global sections are \( k^{\mathcal{H}_+} \cap t(k^{\mathcal{H}_-}) \).

The pairing \( \langle \cdot, \cdot \rangle \) can be extended to \( \hat{S}^\dagger_T \times \mathbb{P}^1 \) using the trivialization in (6.4), and \( \nabla \) extends to give a flat connection on \( \hat{S}^\dagger_T \times \mathbb{P}^1 \) which preserves the pairing \( \langle \cdot, \cdot \rangle \), has an order 2 irregular singularity at \( t = 0 \) and an order 1 regular pole at \( t = \infty \) (besides those coming from the log structure on \( \hat{S}^\dagger_T \)). The extended pairing and the extended connection give a so-called \( \langle \log D-trTEP(w) \rangle \)-structure [46]. Finally, let us recall the notion of a miniversal element from [46].

**Definition 6.6.** A miniversal section \( \xi = (k\xi)_{k \in \mathbb{Z}_{\geq 0}} \) is an element \( k\xi \in k^{\mathcal{H}_+} \cap t(k^{\mathcal{H}_-}) \) such that

1. \( k^{+1}\xi = k\xi (\mod T^{k+1}) \);
2. \( \nabla_X k\xi = 0 \) for each \( k \) on \( t(k^{\mathcal{H}_-}) / k^{\mathcal{H}_-} \);
3. \( \nabla_{t^{\frac{k}{m}}} k\xi = r(k\xi) \) for each \( k \) on \( t(k^{\mathcal{H}_-}) / k^{\mathcal{H}_-} \), with the same \( r \in \mathbb{C} \);
4. the Kodaira-Spencer map \( K\Sigma: k\Theta_{\hat{S}^\dagger_T} \to k^{\mathcal{H}_+} / t(k^{\mathcal{H}_-}) \) given by \( K\Sigma(X) := t\nabla_X \xi (\mod t(k^{\mathcal{H}_+})) \) is a bundle isomorphism for each \( k \).

By [46] Proposition 1.11, an opposite filtration \( \mathcal{H}_- \) together with a miniversal element \( \xi \) give the structure of a (germ of a) logarithmic Frobenius manifold.

6.2. **Construction of a \( \hat{N} \)-LVHS.** Following [2], [11], [22], [40], we will construct a \( \hat{N} \)-LVHS from the dgBV algebra \( kPV_T^\dagger[t] \) in Notation 5.1 and its unobstructed deformations.

**Condition 6.7.** For the 0\textsuperscript{th}-order Kodaira-Spencer map \( 0\nabla([0]_{\omega}) \): \( K^\dagger_{\hat{S}^\dagger_T} \to \mathcal{F}^{\geq d-1}H_0^0 \) defined after Proposition 2.12, we assume that the induced map \( 0\nabla([0]_{\omega}) : K^\dagger_{\hat{S}^\dagger_T} \to \mathcal{F}^{\geq d-1}H_0^0 / \mathcal{F}^{\geq d-\frac{1}{2}}H_0^0 \) is injective. Furthermore, we fix the choice of the graded vector space \( V^* := Gr_F(\mathbb{H}^\dagger)^*/Im(0\nabla([0]_{\omega})) \).

**Notation 6.8.** From Lemma 4.17 and Remark 5.2, we define the relative de Rham complex with coefficient in \( T \) as \( k\mathcal{A}_T^\dagger := (k\mathcal{A} \otimes_C T) \otimes_{R_T} (R_T/T^{k+1}) \) and, for each \( k \), consider \( H^*(k\mathcal{A}_T^\dagger)[[t^{\frac{1}{2}}]]/[t^{\frac{1}{2}}] \) which is free over \( kR_T[[t^{\frac{1}{2}}]]/[t^{\frac{1}{2}}] \).

Since the ring \( T \) is itself graded, for an element \( \varphi \in (kPV_{\alpha} \otimes T)/T^{k+1} \subset kPV_T \) (resp. \( \alpha \in (k^\alpha\mathcal{A}_{\alpha} \otimes T)/T^{k+1} \subset k\mathcal{A}_T \)), we define the index of \( \varphi (\alpha \text{ resp.}) \) as \( p + q \) and denoted by \( \hat{\varphi} \) (resp. \( \hat{\alpha} \).
6.2.1. Construction of \( \mathcal{H}_+ \).

**Definition 6.9.** We consider the scaling morphism

\[
l_t : H^*(kPV_T[[t]][t^{-1}], k\mathfrak{d}) \to H^*(k\mathfrak{d}T, k\mathfrak{d})[[t^\frac{1}{2}][t^{-\frac{1}{2}}]],
\]

and define (the sections of) the Hodge bundle over \( 0S^1 \) to be \( 0\mathcal{H}_+ := l_t(H^*(0PV[[t]], 0\mathfrak{d})) \), as a submodule of \( 0R_T[[t]] = \mathbb{C}[[t]] \). We further take

\[
k\mathcal{H}_+ := \text{Im}(l_t)[t^{-1}] = \bigoplus \left( H^{d+ev}(kA, k\mathfrak{d})[[t]][t^{-1}] \oplus H^{d+odd}(kA, k\mathfrak{d})[[t]][t^{-1}]t^{\frac{1}{2}} \right)
\]

(resp. \( k\mathcal{H}_- := \lim_{\to k} k\mathcal{H}_+ \), as a module over \( kR_T[[t]] = k\mathbb{C}[[t]] \)).

Given a Maurer-Cartan element \( \varphi = (k\varphi)_k \) as in Theorem 5.5, note that the cohomology \( H^*(kPV_T[[t]], k\mathcal{H}_+ \oplus t(k\Delta) + [k\varphi, -]) \) is again a free module over \( kR_T[[t]] \). We define

\[
k\mathcal{H}_+ := \left\{ (l_t(\alpha) \wedge e^{t(k\varphi)}_\omega)_\omega | \alpha \in H^*(kPV_T[[t]], k\mathcal{H}_+ \oplus t(k\Delta) + [k\varphi, -]) \right\}
\]

as the \( kR_T[[t]] \) submodule of \( k\mathcal{H}_+ \). Similarly, we let \( \mathcal{H}_+ := \lim_{\to k} \mathcal{H}_+ \) be the \( \tilde{R}_T[[t]] \) module.

**Remark 6.10.** Notice that \( H^{d+ev} \) and \( H^{d+odd} \) are modules for \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) in relation to the Hodge filtration given in Definition 2.10. We define

\[
\mathcal{H}_+ := \{ (l_t(\alpha) \wedge e^{t(k\varphi)}_\omega)_\omega | \alpha \in H^*(kPV_T[[t]], k\mathcal{H}_+ \oplus t(k\Delta) + [k\varphi, -]) \}
\]

as the \( kR_T[[t]] \) submodule of \( k\mathcal{H}_+ \). Therefore, we obtain \( \nabla : k\mathcal{H}_+ \to \frac{1}{t}(k\Omega^1_{S^1_T}) \otimes kR_T \cdot k\mathcal{H}_+ \).

**Proof.** It suffices to prove the first statement of lemma. We begin by considering the case \( \alpha = 1 \) and restricting the Maurer-Cartan element \( k\varphi \) to the coefficient ring \( kR \) (because the extra coefficient \( T \) is not involved in the differential \( k\mathfrak{d} \) for defining the Gauss-Manin connection in Definition 4.12). Note that \( l_t(1) \wedge e^{t(k\varphi)}_\omega \) is the element \( k\omega \) for computing the connection \( k\nabla \) via the sequence (4.3). Direct computation shows that

\[
k\mathfrak{d} \left( l_t(e^{t(k\varphi)}_\omega) \right) = \frac{1}{t^2} l_t \left( \sum_i d\log q^{m_i} \otimes e^{t(k\varphi)}_\omega \right) = \sum_i d\log q^{m_i} \otimes t^{-1} (l_t(\psi_i) \wedge e^{t(k\varphi)}_\omega) \wedge k\omega \text{ for some } m_i \in k \text{ and } \psi_i \in kPV[[t]].
\]

Since \( (l_t(\psi_i) \wedge e^{t(k\varphi)}_\omega)_\omega \in k\mathcal{H}_+ \), we have \( k\nabla \left( e^{t(k\varphi)}_\omega \right) = \frac{1}{t}(k\Omega^1_{S^1_T}) \otimes (kR_T)(k\mathcal{H}_+). \)

For the case of \( (l_t(\alpha) \wedge e^{t(k\varphi)}_\omega)_\omega \), we may simply introduce a formal parameter \( \epsilon \) of degree \( -|\alpha| \) such that \( \epsilon^2 = 0 \), and repeat the above argument for the Maurer-Cartan element \( k\varphi + \epsilon \alpha \) over the ring \( kR[\epsilon]/(\epsilon^2) \).

**6.2.2. Construction of \( \mathcal{H}_- \).** The 0th-order Gauss-Manin connection defined in Definition 2.11 induces an endomorphism \( \nu : H^*(0A, 0\mathfrak{d}) \to H^*(0A, 0\mathfrak{d}) \) for every element \( \nu \in K^V \). The flatness of the Gauss-Manin connection (Proposition 4.14) then implies that these are commuting operators: \( N_{\nu_1}N_{\nu_2} = N_{\nu_2}N_{\nu_1} \).

**Assumption 6.12.** We assume there is an increasing filtration \( \mathcal{W}_\leq \cdot H^*(0A, 0\mathfrak{d}) \)

\[
\{ 0 \} \subset \mathcal{W}_{\leq 0} \subset \cdots \subset \mathcal{W}_{\leq r} \subset \cdots \subset \mathcal{W}_d = H^*(0A, 0\mathfrak{d})
\]

indexed by \( r \in \frac{1}{2}Z_{\geq 0} \) which is

\footnote{We follow Barannikov \cite{1} in using half integers \( r \in \frac{1}{2}Z \) as the weights for the filtration \( \mathcal{W}_\leq \); multiplying by 2 gives the usual indices \( \mathcal{W}_{\leq 0} \subset \cdots \subset \mathcal{W}_{\leq r} \subset \cdots \subset \mathcal{W}_{\leq 2d} \) with \( r \in Z \).}
• preserved by the 0th-order Gauss-Manin connection in the sense that \( N_{\nu} W_r \subset W_{r-1} \) for any \( \nu \in K^\vee \), and
• an opposite filtration to the Hodge filtration \( \mathcal{F}^{2r} \) in the sense that \( \mathcal{F}^{2r} \oplus W_{r-1} = H^*(\| A, 0 d) \).

Under this assumption, the commuting operators \( N_{\nu} \)'s are nilpotent.

**Lemma 6.13.** Under Assumption 6.12, there exists an index (introduced in Notation 6.8) and degree preserving trivialization

\[ \varkappa : H^*(\| A, 0 d) \otimes C R_T \to H^*(\| A_T, d) \]

which identifies the connection form of the Gauss-Manin connection \( \nabla \) with the nilpotent operator \( N \), i.e. for any \( \nu \in K^\vee \), we have \( \nabla_{\nu} (s \otimes 1) = N_{\nu} (s) \otimes 1 \) for \( s \in H^*(\| A, 0 d) \).

**Proof.** Since the extra coefficient ring \( T \) does not couple with the differential \( d \), we only need to construct inductively a trivialization \( k \varkappa : H^*(\| A, 0 d) \otimes C k R \rightarrow H^*(\| A, k d) \) for every \( k \) such that \( k+1 \varkappa = k \varkappa \) (mod \( m^{k+1} \)) and which identifies \( k \nabla \) with the nilpotent operator \( N \).

To prove the induction step, we assume that \( k-1 \varkappa \) has been constructed and the aim is to construct its lifting \( k \varkappa \). We first choose an arbitrary lifting \( k \varkappa \) and a filtered basis \( e_1, \ldots, e_m \) of the finite dimensional vector space \( H^*(\| A, 0 d) \), meaning that it is a lifting of a basis in the associated quotient \( \text{Gr}_{\nu}(H^*(\| A, 0 d)) \). We also write \( \tilde{e}_i \) for \( k \varkappa(e_i \otimes 1) \). With respect to the frame \( \tilde{e}_i \)'s of \( H^*(\| A, k d) \), we define a connection \( \tilde{\nabla} \) with \( \tilde{\nabla}_{\nu}(\tilde{e}_j) = \sum_j (N_{\nu})_i^j (\tilde{e}_j) \) for \( \nu \in K^\vee \), where \( (N_{\nu})_i^j \)'s are the matrix coefficients of the operator \( N_{\nu} \) with respect to the basis \( \{e_i\} \). We may also treat \( N = (N_1^j) \) as \( K^\vee \)-valued endomorphisms on \( H^*(\| A, 0 d) \).

From the induction hypothesis, we have \( k \nabla - \tilde{\nabla} = 0 \) (mod \( m^k \)) and hence \( (k \nabla - \tilde{\nabla})(\tilde{e}_i) = \sum_m \sum_j \alpha^j_m e_j q^m \in 0 \Omega_{1,1} \otimes C H^*(\| A) \otimes C (m^k / m^{k+1}) \). From the flatness of both \( k \nabla \) and \( \tilde{\nabla} \), we notice that \( (d \log q^m) \wedge \alpha^j_m = 0 \) and hence \( \alpha^j_m = c^j_m d \log q^m \) for some constant \( c^j_m \in C \) for every \( m \) and \( j \). We will use \( c_m \) to denote the endomorphism on \( H^*(\| A, 0 d) \) whose matrix coefficients are given by \( c_m = (c^j_m) \) with respect to the basis \( e_i \)'s.

As a result, if we define a new frame \( \tilde{e}_i^{(0)} := \tilde{e}_i - \sum_{m,j} c^j_m e_j q^m \) and a new connection \( \tilde{\nabla}^{(0)} \) by \( \tilde{\nabla}^{(0)}(\tilde{e}_i^{(0)}) = \sum_j (N_1^j)(\tilde{e}_j^{(0)}) \), then

\[ (k \nabla - \tilde{\nabla}^{(0)}) \tilde{e}_i^{(0)} = \sum_{j,m} [c_m, N_1^j] \tilde{e}_j^{(0)} q^m, \]

where \([c_m, N]\) is the usual Lie bracket with its \( K^\vee \)-valued matrix coefficient given by \( ([c_m, N_1^j]) = c^j_m N_1^i - c^i_m N_1^j \). Once again using flatness of both \( k \nabla \) and \( \tilde{\nabla}^{(0)} \), we get some constant \( c^{(1)}_{m} \) such that \( [c_m, N_1^j] = c^{(1)}_{m} d \log q^m \). Taking an element \( \nu_m \in K^\vee \) with \( \{m, \nu_m\} \neq 0 \), we obtain \( c^{(1)}_{m} = \frac{1}{(m, \nu_m)}[c_m, N_{\nu_m}] \). Now if we define a new frame \( \tilde{e}_i^{(1)} := \tilde{e}_i^{(0)} - \sum_{j,m} c^{(1)}_{m} e_j q^m \) and a new connection \( \nabla^{(1)}(\tilde{e}_i^{(1)}) := \sum_j N_1^j \tilde{e}_j^{(1)} \), then we have \( c^{(2)}_{m} = \frac{1}{(m, \nu_m)}[c_m^{(1)}, N_{\nu_m}] = \frac{1}{(m, \nu_m)^2}[(c_m, N_{\nu_m}), N_{\nu_m}] \) such that \( c^{(2)}_{m} d \log q^m = [c_m^{(1)}, N_{\nu_m}] \).

Repeating this argument produces a frame \( \{e_i^{(2d)}\} \) such that if we let \( \nabla^{(2d)}(e_i^{(2d)}) = \sum_j N_1^j e_j^{(2d)} \), we have \( c^{(2d+1)}_{m} = \frac{1}{(m, \nu_m)^{2d+1}}(-[N_{\nu_m}, \cdot])^{2d+1}(c_m) = 0 \). Therefore letting \( k \varkappa(e_i \otimes 1) = e_i^{(2d)} \) gives the desired trivialization for the Hodge bundle. \( \square \)
With Assumption \textit{6.12} we can take a filtered basis \{e_{r;i}\}_{0 \leq 2r \leq 2d} of the vector space \(H^*(\| A, 0 \mathfrak{d})\) such that \(e_{r;i} \in \mathcal{W}_{\leq r} \cap (H^{d+ev}(\| A))\) if \(r \in \mathbb{Z}\) and \(e_{r;i} \in \mathcal{W}_{< r} \cap (H^{d+od}(\| A))\) if \(r \in \mathbb{Z} + \frac{1}{2}\), and \{e_{r;i}\}_{0 \leq i \leq m_r}\) forms a basis of \(\mathcal{W}_{\leq r}/\mathcal{W}_{< r - \frac{1}{2}}\).

\textbf{Definition 6.14.} Using the trivialization \(\pi\) in Lemma \textit{6.13} we let \(\epsilon_{r;i} := \pi(e_{r;i} \otimes 1)\), as a section of the Hodge bundle \(H^*(\| A_T\)). The collection \(\{\epsilon_{r;i}\}\), which forms a frame of the Hodge bundle, is called the set of elementary sections (cf. Deligne’s canonical extension \$8\$).

Note that the index of \(\epsilon_{r;i}\), introduced in Notation \textit{6.8}, is the same as that of \(e_{r;i}\).

\textbf{Lemma 6.15.} If we let

\[
0\mathcal{H}_{d+ev} := \bigoplus_{r=0}^{d} \left( \mathcal{W}_{\leq r} \cap H^{d+ev}(\| A) \right) \mathbb{C}[t^{-1}]t^{-r+d+2} \subset 0\mathcal{H}_{d+ev},
\]

\[
0\mathcal{H}_{d+od} := \bigoplus_{r=0}^{d-1} \left( \mathcal{W}_{\leq r+\frac{1}{2}} \cap H^{d+od}(\| A) \right) \mathbb{C}[t^{-1}]t^{-(r+\frac{1}{2})+d+2} \subset 0\mathcal{H}_{d+od},
\]

and use \(0\mathcal{H}_{-} := 0\mathcal{H}_{d+ev} \oplus 0\mathcal{H}_{d+od}\) as the \(\mathbb{C}[t^{-1}]\) submodule of \(0\mathcal{H}_{\pm}\), then there exists a unique free \(\mathcal{R}_T[t^{-1}]\) submodule \(\mathcal{H}_{-} = \lim_{\rightarrow k}^k \mathcal{H}_{-}\) of \(\mathcal{H}_{\pm}\), which is preserved by \(\nabla_X\) for any \(X \in \tilde{\Theta}_{S_T}\) and satisfies \(k+1\mathcal{H}_{-} = k\mathcal{H}_{-}\) \((\text{mod } k+1)\).

\textbf{Proof.} For existence, we take a set of elementary sections \(\{\epsilon_{r;i}\}_{0 \leq 2r \leq 2d}\) as in Definition \textit{6.14}. We can take the free submodules \(\mathcal{W}_{d+ev} := \bigoplus_{r \leq d} \mathcal{W}_{\leq r} \oplus \bigoplus_{r \leq d} \mathcal{W}_{< r} \cap kR_T \cdot \epsilon_{l,i}\), \(\mathcal{W}_{d+od} := \bigoplus_{r \leq d} \mathcal{W}_{\leq r+\frac{1}{2}} \cap kR_T \cdot \epsilon_{l,i}\), and let

\[
\mathcal{H}_{-} := \bigoplus_{0 \leq r \leq d} k\mathcal{W}_{\leq r} \mathbb{C}[t^{-1}]t^{-r+d+2} \oplus \bigoplus_{0 \leq r \leq d-1} k\mathcal{W}_{\leq r+\frac{1}{2}} \mathbb{C}[t^{-1}]t^{-(r+\frac{1}{2})+d+2}
\]

be the desired \(\nabla_X\) invariant subspace.

For uniqueness, we will prove the uniqueness of \(kR_T[t^{-1}]\) submodule \(k\mathcal{H}_{-}\) of \(k\mathcal{H}_{\pm}\) for each \(k\) by induction. Again since the coefficient ring \(T\) does not couple with the differential \(d\), we only need to consider the corresponding statement for Hodge bundle over \(kR\). For the induction step we assume that there is another increasing filtration \(\mathcal{W}_{\leq r}\) of \(H^*(\| A, k\mathfrak{d})\) satisfying the desired properties such that they agree when passing to \(H^*(\| A, k-1\mathfrak{d})\). We should prove that \(\mathcal{W}_{\leq r} \subset k\mathcal{W}_{\leq r}\) for each \(r\).

We take \(r \in \frac{1}{2}\mathbb{Z}\) with \(k\mathcal{W}_{\leq r} \neq 0\). The proof of Lemma \textit{6.13} gives a trivialization \(0\mathcal{W}_{\leq r} \otimes \mathbb{C} \to k\mathcal{W}_{\leq r}\) using the frame \(\{\tilde{\epsilon}_{r;i}\}\) which identifies \(k\nabla\) with \(N\); in particular we must have \(r \geq 0\). Let \(l_r \geq r\) be the minimum half integer such that \(k\mathcal{W}_{\leq r} \subset k\mathcal{W}_{\leq l_r}\), and take the frame \(\{\epsilon_{l,i}\}_{0 \leq 2l \leq 2l_r}\) for the submodule \(k\mathcal{W}_{\leq l_r}\). Then we can write

\[
\tilde{\epsilon}_{r;i} = \sum_{0 \leq 2l < 2r} f_{l,i} \epsilon_{l,i} + \sum_{2r+1 < 2l \leq 2l_r} f_{l,i} \epsilon_{l,i}
\]

for some \(f_{l,i} \in kR\) with \(f_{l,i} = 0\) \((\text{mod } m^k)\) for \(r + \frac{1}{2} \leq l\).
We start with \( r = 0 \) and assume on the contrary that \( l_0 > 0 \). As \( \tilde{e}_{0;i} = \sum_{0 \leq i \leq m_0} f_{0;0} \epsilon_{0;i} + \sum_{1 \leq i \leq m_1} f_{i;i} \epsilon_{i;i} \), applying the connection \( k \nabla \) gives

\[
0 = \sum_{0 \leq i \leq m_0} \partial(f_{0;i}) \epsilon_{0;i} + \sum_{1 \leq i \leq m_1} \partial(f_{i;i}) \epsilon_{i;i} + \sum_{1 \leq i \leq m_2} f_{i;i} N(\epsilon_{i;i}).
\]

Passing to the quotient \( k \mathcal{W}_{l \leq 0} / k \mathcal{W}_{l \leq -\frac{1}{2}} \) yields \( 0 = \sum_{0 \leq i \leq m_0} \partial(f_{i;i}) \epsilon_{i;i} \) which implies that \( f_{i;i} = 0 \) for \( l \geq 1 \) and hence \( l_0 = 0 \). By induction on \( r \), we have \( \mathcal{W}_{l \leq r - \frac{1}{2}} \subset k \mathcal{W}_{l \leq r} \) by the induction hypothesis. We assume on the contrary that \( l_r > r \) and consider

\[
N(\tilde{e}_{r;i}) = \sum_{0 \leq 2 \leq 2r} k \nabla(f_{i;i} \epsilon_{i;i}) + \sum_{2r+1 \leq i \leq m_2} \partial(f_{i;i}) \epsilon_{i;i} + \sum_{2r+1 \leq i \leq m_2} f_{i;i} N(\epsilon_{i;i}).
\]

This gives \( 0 = \sum_{2r+1 \leq i \leq m_2} \partial(f_{i;i}) \epsilon_{i;i} \) by passing to the quotient \( k \mathcal{W}_{l \leq r} / k \mathcal{W}_{l \leq r - \frac{1}{2}} \) and thus \( f_{i;i} = 0 \) for \( l \geq r + \frac{1}{2} \). This gives \( l_r = r \), which is a contradiction and hence completes the induction step and the proof of the lemma.

**Remark 6.16.** Lemma 6.15 says that the opposite filtration \( \mathcal{H}_- \) is determined uniquely by \( 0 \mathcal{H}_- \) which is given by the opposite filtration in Assumption 6.12. Applying this to the case of maximally degenerate log Calabi-Yau varieties studied by Gross-Siebert gives the weight filtration which is canonically determined by the nilpotent operators \( N_\nu \) for any \( \nu \in \text{int}(Q^d_\mathbb{R}) \cap K^\vee \).

### 6.2.3. Construction of the pairing \( \langle \cdot, \cdot \rangle \)

The next assumption concerns the existence of the pairing \( \langle \cdot, \cdot \rangle \) in Definition 6.2.

**Assumption 6.17.** We assume that

- \( H^*(\mathbb{A}, \mathbb{A}, d) \) is nontrivial only when \( 0 \leq * \leq 2d \);
- \( H^{p,d}(\mathbb{A}, \mathbb{A}, \bar{\partial}) = 0 \) for all \( 0 \leq p \leq d \);
- there is a trace map \( \text{tr} : H^{d,d}(\mathbb{A}, \mathbb{A}, \bar{\partial}) = H^{2d}(\mathbb{A}, \mathbb{A}, 0) \rightarrow \mathbb{C} \), so that we can define a pairing \( \langle \cdot, \cdot \rangle \) on \( H^*(\mathbb{A}, \mathbb{A}, 0) \) by
  \[
  \langle \alpha, \beta \rangle := \text{tr} (\alpha \wedge \beta)
  \]
  for \( \alpha, \beta \in H^*(\mathbb{A}, \mathbb{A}, 0) \);
- the trace map \( \text{tr} \) and the corresponding pairing \( \langle \cdot, \cdot \rangle \), when descended to \( \text{Gr}_{\mathcal{F}}(H^*(\mathbb{A}, \mathbb{A})) \) are non-degenerate.

We will denote by \( \text{tr} : H^*(\mathbb{A}, \mathbb{A}, 0) \rightarrow \mathbb{C} \) the map which extends \( \text{tr} : H^{2d}(\mathbb{A}, \mathbb{A}, 0) \rightarrow \mathbb{C} \) and is trivial on \( H^{<2d}(\mathbb{A}, \mathbb{A}, 0) \). Note that by definition \( \mathcal{F}^{>\bullet} \) is isotropic with respect to the pairing \( \langle \cdot, \cdot \rangle \), i.e. \( \langle \mathcal{F}^{>\bullet}, \mathcal{F}^{>\bullet} \rangle = 0 \) when \( r + s > d \). We have \( \text{tr}(N_\nu(\alpha)) = 0 \) for any \( \alpha \in H^*(\mathbb{A}) \) and \( \nu \in K^\vee_\mathbb{C} \), and since \( \mathcal{W}_< \) is opposite to \( \mathcal{F}^{>\bullet} \), the filtration \( \mathcal{W}_< \) is isotropic with respect to \( \langle \cdot, \cdot \rangle \), i.e. \( \langle \mathcal{W}_<, \mathcal{W}_< \rangle = 0 \) when \( r + s < d \).

**Lemma 6.18.** Take the truncation \( k \tau(A)^* := k \tau(A)^{*,<d} \) of the complex \( (k \tau(A)^*, k \mathcal{D}) \) as a quotient complex. Then under Assumption 6.17, the natural map between the cohomology group \( H^*(k \tau(A), k \mathcal{D}) \rightarrow H^*(k \tau(A), k \mathcal{D}) \) is an isomorphism for all \( k \).
Proof. Consider the exact sequence of complexes $0 \to \mathbb{K}_{\mathbb{C}} \to \mathbb{K}_{\mathbb{C}}^* \to \mathbb{K}_{\mathbb{C}}^* \to 0$. We need to show that $\mathbb{K}_{\mathbb{C}}^*$ is acyclic. Exactly the same argument as in [4,3,2] tells us that the cohomology $H^*(\mathbb{K}_{\mathbb{C}}^*,\mathbb{K})$ is a free $\mathbb{K}$ module. For $k = 0$, using the second item in Assumption 6.17 and a standard zig-zag argument, we see that $H^*(\mathbb{K}_{\mathbb{C}}^*,\mathbb{K}) = 0$. Thus we have $H^*(\mathbb{K}_{\mathbb{C}}^*,\mathbb{K}) = 0$. □

Lemma 6.18 allows us to work with $H^*(\mathbb{K}_{\mathbb{C}}^*,\mathbb{K})$ in defining the pairing $(\cdot,\cdot)$.

Definition 6.19. Using the elementary sections $\{e_{r,i}\}$ in Definition 6.14, we extend the trace map $\text{tr}$ to the Hodge bundle $H^*(\mathbb{K}_{\mathbb{C}}^* \langle t^\frac{1}{2} \rangle)$ by the formula

$$\text{tr}(f e_{r,i}) := f \text{tr}(e_{r,i})$$

for $f \in kR_T[[t^\frac{1}{2}]]$, and extending linearly.

We also extend the pairing $p$ to $H^*(\mathbb{K}_{\mathbb{C}}^* \langle t^\frac{1}{2} \rangle)$ by the formula

$$p(f(t)e_{r,i},g(t)e_{l,j}) := (-1)^{|f||e_{r,i}|} f(t) g(t) p(e_{r,i},e_{l,j})$$

for $f(t), g(t) \in kR_T[[t^\frac{1}{2}]]$, and extending linearly.

Finally we define a pairing $(\cdot,\cdot)$ on $k\mathcal{H}_\pm$ by the formula

$$(l_t(\alpha(t)), l_t(\beta(t))) := (-1)^{-|d||t|} p(l_t(\alpha(t)), l_t(\beta(t))))$$

for $l_t(\alpha(t)), l_t(\beta(t)) \in k\mathcal{H}_\pm$, where $\tilde{t}$ is the index of $\beta \in kPV[[t]]$ (see Notation 6.8).

Lemma 6.20. We have the identification $p(\alpha,\beta) = \text{tr}(\alpha \wedge \beta)$ between the pairing and the trace map in Definition 6.19. Furthermore, the pairing $p$ is flat, i.e.

$$X(p(\alpha,\beta)) = p(\nabla_X \alpha,\beta) + p(\alpha,\nabla_X \beta)$$

for any $\alpha, \beta \in H^*(\mathbb{K}_{\mathbb{C}})$ and $X \in k\Theta S^1_t$.

Proof. Using the short exact sequence $0 \to K_{\mathbb{C}} \otimes _{\mathbb{C}} \mathbb{K}_{\mathbb{C}} \to \mathbb{K}_{\mathbb{C}}^* \to \mathbb{K}_{\mathbb{C}}^* \to 0$ which defines the $k$th-order Gauss-Manin connection $k\nabla$, we have the flatness of the product:

$$k\nabla(\alpha \wedge \beta) = (k\nabla \alpha) \wedge \beta + (-1)^{|\alpha|} \alpha \wedge (k\nabla \beta).$$

To prove the identity $p(\alpha,\beta) = \text{tr}(\alpha \wedge \beta)$, we choose a basis $\{e_{r,i}\}$ of $H^*(\mathbb{K}_{\mathbb{C}}^*,\mathbb{K})$ and the corresponding elementary sections $\{e_{r,i}\}$ as in Definition 6.14. We claim that the relation $e_{r,i} \wedge e_{l,j} = \sum_{s,k} c^s_{r,i} e_{s,k}$ holds for some constant $c^s_{r,i} \in \mathbb{C}$. This can be proved by an induction on the order $k$, followed by an induction on the lexicographical order: $(r,l) < (r',l')$ if $r < r'$, or $r = r'$ and $l < l'$ for each fixed $k$. So we fix $r$ and $l$ and assume the product $e_{r,i} \wedge e_{l,j}$, and assume that the above relation holds for any $(r',l') < (r,l)$. Writing $e_{r,i} \wedge e_{l,j} = \sum_{s,k} c^s_{r,i} e_{s,k}$ and applying the Gauss-Manin connection gives $k\nabla p(e_{r,i},e_{l,j}) = (N_{e_{r,i}} e_{r,i} \wedge e_{l,j}) + e_{r,i} \wedge e_{r,i} \wedge e_{l,j}$. The induction hypothesis then forces $\sum_{s,k} c^s_{r,i} e_{s,k} \sum_{m} b_m e_{s,k} q^m = 0$. As a result we have $p(e_{r,i},e_{l,j}) = \text{tr}(e_{r,i} \wedge e_{l,j})$ and the general relation $p(\alpha,\beta) = \text{tr}(\alpha \wedge \beta)$ follows. Flatness of the pairing $p$ now follows from that of $\text{tr}$. □

Lemma 6.21. The pairing in Definition 6.19 satisfies $(s_1, s_2) \in kR_T[[t]]$ for any $s_1, s_2 \in k\mathcal{H}_+$, and $(s_1, s_2) \in kR_T[t^{-1}]$ for any $s_1, s_2 \in k\mathcal{H}_-$. Furthermore, it descends to a non-degenerate pairing $g(\cdot,\cdot) : k\mathcal{H}_+ / t(k\mathcal{H}_+) \times k\mathcal{H}_+ / t(k\mathcal{H}_+) \to kR_T[-2d].$
**Proof.** The statement for \( kH^- \) follows from the second item in Assumption 6.17 and Definition 6.19.

To prove the statement for \( kH^+ \), we work with the truncated complex \( (k\tau(A))_T[[t^{\frac{1}{2}}][t^{-\frac{1}{2}}], k^d) \). For two cohomology classes of \( H^*(k\tau(A))_T[[t^{\frac{1}{2}}][t^{-\frac{1}{2}}] \) represented by \( l_t(\alpha) \land k\omega, l_t(\beta) \land k\omega \in (k\tau(A))_T \) for some elements \( \alpha, \beta \in kPV_T[[t]][t^{-1}] \) with fixed indices \( \bar{\alpha} = l_1, \bar{\beta} = l_2 \), [1] Proposition 5.9.4 gives the following formula

\[
(6.5) \quad (-1)^{-d(\beta) + \frac{1}{2}(\beta)} (l_t(\alpha(t)) \land k\omega) \land (l_t(-\beta(-t)) \land k\omega) = ((l_t(\alpha(t)) \land l_t(\beta(-t))) \land k\omega) \land k\omega.
\]

in \( H^*(k\tau(A))_T[[t^{\frac{1}{2}}][t^{-\frac{1}{2}}] \). (Note that this does not hold for the full complex, which is the reason for introducing the truncated complex.) The result follows from such a formula because the same argument as in [1] Proposition 5.9.4 shows that the RHS lies in \( kR_T[[t]] \).

For non-degeneracy it suffices to consider the pairing for \( 0H^+ \), which follows from the non-degeneracy condition in Assumption 6.17. \( \square \)

The above constructions give a \( \nabla^\tau \)-LVHS \( (H^+ \nabla, \langle \cdot, \cdot \rangle) \) together with an opposite filtration \( H^- \) satisfying (1)-(3) in Definition 6.5. It remains to construct the grading structure.

### 6.2.4. Construction of the grading structure.

**Definition 6.22.** For each \( k \), we define the extended connection \( \nabla^{\alpha}_{t\frac{\partial}{\partial t}} \) acting on \( H^*[(\alpha)]_T[[t^{\frac{1}{2}}][t^{-\frac{1}{2}}]] \) by the rule that \( \nabla^{\alpha}_{t\frac{\partial}{\partial t}}(s) = 2-d \frac{\partial}{\partial t} s \) for \( s \in H^*[(\alpha)]_T[[t^{\frac{1}{2}}][t^{-\frac{1}{2}}]] \) and \( \nabla^{\alpha}_{t\frac{\partial}{\partial t}}(f s) = t\frac{\partial}{\partial t}(f) s + f(\nabla^{\alpha}_{t\frac{\partial}{\partial t}}(s)) \).

**Proposition 6.23.** The extended connection \( \nabla^{\alpha} \) is a flat connection acting on \( kH^+ \), i.e. we have \( [\nabla^{\alpha}_{t\frac{\partial}{\partial t}}, \nabla_X] = 0 \) for any \( X \in k\Theta_{S_T}^{1} \). The submodule \( kH^- \) is preserved by \( \nabla^{\alpha}_{t\frac{\partial}{\partial t}} \) and we have \( \nabla^{\alpha}_{t\frac{\partial}{\partial t}}(H^+) \subseteq t^{-1}(kH^+) \). Furthermore, the pairing \( \langle \cdot, \cdot \rangle \) is flat with respect to \( \nabla^{\alpha}_{t\frac{\partial}{\partial t}} \).

**Proof.** Beside \( \nabla^{\alpha}_{t\frac{\partial}{\partial t}}(H^+) \subseteq t^{-1}(kH^+) \), the other properties simply follow from definitions. Take \( \alpha \in kPV_T[[t]] \) and consider \( (l_t(\alpha) \land e^{l_t(k\varphi)}) \land k\omega \). Then

\[
\nabla^{\alpha}_{t\frac{\partial}{\partial t}}(l_t(\alpha) \land e^{l_t(k\varphi)}) \land k\omega = ((\nabla^{\alpha}_{t\frac{\partial}{\partial t}}l_t(\alpha)) \land l_t(k\varphi) + l_t(\alpha) \land (\nabla^{\alpha}_{t\frac{\partial}{\partial t}}l_t(k\varphi)) \land e^{l_t(k\varphi)}) \land k\omega.
\]

Since we can write both \( \nabla^{\alpha}_{t\frac{\partial}{\partial t}}l_t(\alpha) = l_t(\beta) \) and \( \nabla^{\alpha}_{t\frac{\partial}{\partial t}}l_t(k\varphi) = l_t(\gamma) \) for some \( \beta, \gamma \in kPV_T[[t]] \), we may rewrite

\[
\nabla^{\alpha}_{t\frac{\partial}{\partial t}}(l_t(\alpha) \land e^{l_t(k\varphi)}) \land k\omega = ((l_t(\beta) + t^{-1}l_t(\alpha \land \gamma))e^{l_t(k\varphi)}) \land k\omega,
\]

which gives the desired result. \( \square \)

### 6.3. Construction of a miniversal section.

**Notation 6.24.** Consider the cohomology class \( [0] \omega \in F^{\leq d} \cap W_{\leq d} \). We let \( k\mu \) be the extension of the cohomology classes \( [\omega] \in t^{-1}(0H^+) \cap t^2(0H^-) \) by first expressing it as a linear combination of the filtered basis \( [\omega] = \sum_{r;i} e_{r;i} \epsilon_{r;i}, \) and then extend it by elementary sections in Definition 6.14 to \( t^2(kH^-) \) using the formula \( k\omega = \sum_{r;i} e_{r;i} \epsilon_{r;i} \) for each \( k \).

**Notation 6.25.** By our choice of the graded vector space \( \nabla^\tau = Gr_T(H^*(0A)) / Im(0\nabla([0] \omega])) \) in Condition 6.7, we further make a choice of a degree 0 element \( \psi \in 0PV_T[[t]] \otimes \nabla^\tau \) such that its cohomology class \( [l_t(\psi)] = 0 \in (0H^+/t(0H^+)) \otimes \nabla^\tau = Gr_T(H^*(0A)) \otimes \nabla^\tau \) maps to the identity element \( id \in \nabla \otimes \nabla^\tau \) under the natural quotient \( Gr_T(H^*(0A)) \otimes \nabla^\tau \to \nabla \otimes \nabla^\tau \).
Definition 6.26. For the $\psi$ chosen in Notation 6.25, let $\varphi = (^k\varphi)_k$ be the corresponding Maurer-Cartan element constructed in Theorem 5.5. Then $\left( t^{-1}e^{(^k\varphi)_k} k\omega \right)_k$ is called a primitive section if it further satisfies the condition
\[ t^{-1}e^{(^k\varphi)_k} k\omega - k\mu \in k\mathcal{H}_- \]
for each $k$, where $k\omega$ is the element constructed in Proposition 4.8.

Proposition 6.27. We can modify the Maurer-Cartan element $\varphi = (^k\varphi)_k$ constructed in Theorem 5.5 by $\varphi \mapsto \varphi + t\zeta$ for some $\zeta = (^k\zeta)_k \in \lim_{\leftarrow} k\mathcal{PV}_\mathcal{T}[t]$ to get a primitive section. Furthermore, $\mathcal{H}_+$ is unchanged under this modification.

Proof. The proof is a refinement of that of Theorem 5.5 by the same argument as in [1, Theorem 1]. \qed

The following theorem concludes this section:

Theorem 6.28. The triple $(^k\mathcal{H}_+, \nabla, \langle \cdot, \cdot \rangle)$ is a $\mathbb{R}$-LVHS, and $^k\mathcal{H}_-$ is an opposite filtration. Furthermore, the element $^k\xi := t^{-1}e^{(^k\varphi)_k} (k\omega)$ constructed in Proposition 6.27 is a miniversal section in the sense of Definition 6.6.

Proof. It remains to check that $\xi$ is a miniversal section. We write $\xi = \lim_{\leftarrow} ^k\xi$ and prove the condition for each $k$. First of all, we have $^k\xi \in ^k\mathcal{H}_+ \cap t(^k\mathcal{H}_-)$ from its construction, and that $^k\xi = t^{-1}(^k\mu)$ in $t(^k\mathcal{H}_-)/^k\mathcal{H}_-$. So $\nabla_\nu (^k\xi) = t^{-1}(\nabla_\nu (^k\mu)) = t^{-1}N_\nu (^k\mu) \in ^k\mathcal{H}_-$ for any $\nu \in K^\lambda$. We have computed the action of $\nabla_{t\frac{\partial}{\partial t}}$ in the proof of Proposition 6.23, and the formula gives $\nabla_{t\frac{\partial}{\partial t}} (t^{-1}e^{(^k\varphi)}) \in (1 - d)e^{(^k\varphi)} + (^k\mathcal{H}_-)$. Therefore we have $\nabla_{t\frac{\partial}{\partial t}} (^k\xi) = (1 - d)(^k\xi) \in (^k\mathcal{H}_-)/^k\mathcal{H}_-$. Finally, to check that the Kodaira-Spencer map is an isomorphism, we only need to show this for $^0\mathcal{S}_T^\dagger$, which follows from our choice of the input $\psi$ for solving the Maurer-Cartan equation (5.1) in Theorem 5.5. \qed

Remark 6.29. Following [1, 39], we can define the semi-infinite period map $\Phi : ^0\mathcal{S}_T^\dagger \to t\mathcal{H}_-/^k\mathcal{H}_-$ as $\Phi(s) := [e^{\varphi(s, \cdot)}_k \omega - \mu]$. In the case of maximally degenerate log Calabi-Yau varieties studied in [23], this gives the canonical coordinates on the (extended) moduli space.

7. Abstract deformation data from log smooth Calabi-Yau varieties

In this section, we apply our results to the case of log smooth Calabi-Yau varieties studied by Friedman [16] and Kawamata-Namikawa in [34]. We assume the reader has some familiarity with the papers [30, 34].

Notation 7.1. Following [34, §2], we take a projective $d$-dimensional simple normal crossing variety $(X, \mathcal{O}_X)$. Let $Q = \mathbb{N}^s$ and write $R = \mathbb{C}[[t_1, \ldots, t_s]]$, where $s$ is the number of connected components of $D = \cup_{i=1}^s D_i = \text{Sing}(X)$. There is a log structure on $X$ over the $Q$-log point $^0\mathcal{S}_T^\dagger$, making it a log smooth variety $X^\dagger$ over $^0\mathcal{S}_T^\dagger$; we further require it to be log Calabi-Yau.

7.1. The $0^{th}$ order deformation data. The $0^{th}$-order deformation data in Definition 2.9 is described as follows:

Definition 7.2. • the $0^{th}$-order complex of polyvector fields is given by the analytic sheaf of relative log polyvector fields $^0\mathcal{G}_X := \wedge^* \mathcal{O}_{X^\dagger/0\mathcal{S}_T^\dagger}$ equipped with the natural product structure.\[15\]
• the 0th-order de Rham complex is given by the sheaf of relative log differential forms $\Omega^*_X/\mathcal{C}$ which is a locally free sheaf (in particular coherent) of dga’s, and equipped with a natural dga structure over $\mathcal{O}_{X^1/\mathcal{C}}$ inducing the filtration as in Definition 2.9.

• the volume element $0\omega$ is given via the trivialization $\mathcal{O}_{X^1/\mathcal{C}}^d \cong \mathcal{O}_X$ coming from the Calabi-Yau condition, and then $0\mathcal{G}^*$ is equipped with the BV operator defined by $0\Delta(\varphi) \cdot 0\omega := 0\partial(\varphi) \cdot 0\omega$.

These data satisfies all the conditions in Definition 2.9. For example, the map $0\sigma : 0\mathcal{O}_{S^1}^* \otimes \mathcal{C} (0\mathcal{K}^*/0\mathcal{K}^*[-r]) \to 0\mathcal{K}^*/r + 1\mathcal{K}^*$ given by taking wedge product in $\Omega^*_X/\mathcal{C}$ is an isomorphism of sheaves of BV modules.

7.2. The higher order deformation data. As discussed in [34, §1], every point $\bar{x} \in X^1$ is covered by a log chart $V$ which is biholomorphic to an open neighborhood of $(0, \ldots, 0)$ in $\{z_0 \cdots z_r = 0 \mid (z_0, \ldots, z_d) \in \mathbb{C}^{d+1}\}$. From the theory of log deformations in [34, §2], we obtain a smoothing $V^\dagger$ of $V$ given by a neighborhood of $(0, \ldots, 0)$ in $\{z_0 \cdots z_r = s_i \mid (z_0, \ldots, z_d) \in \mathbb{C}^{d+1}\}$ if $V \cap D_i \neq \emptyset$. We choose a covering $\mathcal{V} = \{V_{\alpha}\}_\alpha$ of $X$ by such log charts together with a local smoothing $V_{\alpha}^\dagger$ of each $V_{\alpha}$. We further assume that each $V_{\alpha}$ is Stein, and let $kV_{\alpha}^\dagger$ be the $k$th order thickening of the local model $V_{\alpha}$.

The higher order deformation data in Definitions 2.13 and 2.17 are described as follows:

**Definition 7.3.** For each $k \in \mathbb{Z}_{\geq 0},$

• the sheaf of $k$th-order polyvector fields is given by $k\mathcal{G}^*_{\alpha} := \bigwedge^{-\bullet} \Theta_k V_{\alpha}/S^\dagger$ equipped with the natural product structure;

• the $k$th-order de Rham complex is given by $k\mathcal{K}^*_{\alpha} := \Omega^*_{kV_{\alpha}/\mathcal{C}}$;

• the local $k$th-order volume element is given by a lifting $\omega_{\alpha}$ of $0\omega$ as an element in $\Omega^*_{V_{\alpha}^\dagger/S^\dagger}$ and taking $k\omega_{\alpha} = \omega_{\alpha} (\text{mod } m^{k+1})$, and then the BV operator $k\Delta_{\alpha}$ on $k\mathcal{G}^*_{\alpha}$ is induced by the volume form $k\omega_{\alpha}$;

• the isomorphism $k\mathcal{p}_{\alpha}$ of sheaves of BV modules is induced by taking wedge product as in Definition 2.12.

Finally, for each $k \in \mathbb{Z}_{\geq 0}$ and triple $(U_i; V_{\alpha}, V_{\beta})$ with $U_i \subset V_{\alpha} V_{\beta} := V_{\alpha} \cap V_{\beta}$, [34, Theorem 2.2] says that the two log deformations $kV_{\alpha}$ and $kV_{\beta}$ are isomorphic over $U_i$ via $k\psi_{\alpha,\beta,i}$, which induces the corresponding patching isomorphisms $k\psi_{\alpha,\beta,i} : k\mathcal{G}^*_{\alpha}|_{U_i} \to k\mathcal{G}^*_{\beta}|_{U_i}$ in Definition 2.15 and $k\psi_{\alpha,\beta,i} : k\mathcal{K}^*_{\alpha}|_{U_i} \to k\mathcal{K}^*_{\beta}|_{U_i}$ in Definition 2.18. The existence of the log vector fields $k\mathcal{p}_{\alpha,\beta,i}$'s, $k\omega_{\alpha,\beta,i}$'s and $k\mathcal{p}_{\alpha,\beta,i}$'s follows from the fact that any automorphism of a log deformation over $U_i$ or $U_{ij}$ comes from exponential action of vector fields. The difference between volume elements is compared by $k\psi_{\alpha,\beta,i}(k\omega_{\beta}) = \exp(k\mathcal{w}_{\alpha,\beta,i})$ for some holomorphic function $k\mathcal{w}_{\alpha,\beta,i}$.

Assumption 4.15, which is local in nature, can be checked by simply taking base change of the family $\pi : V_{\alpha}^\dagger \to S^\dagger$ with $i_\alpha : A_{\text{aff}}^1 \to S^\dagger$, and working on $k\mathcal{K}^*_{\text{aff}}$, using the local computations from [30, §2]. Alternatively, and more conveniently, one can use (analytification of) the local computations in [23, proof of Theorem 4.1] because indeed Assumption 4.15 is motivated from [23, proof of Theorem 4.1], and the log smooth local model $kV_{\alpha}$ is included as a special case.

7.3. The Hodge theoretic data.

---

10 In [34], the sheaf of relative log differential forms was denoted by $\Omega^*_X/\mathcal{C}(\log)$. 
7.3.1. Hodge-to-de Rham degeneracy. In \[21\] proof of Lemma 4.1, p.406, a cohomological mixed Hodge complex of sheaves \((A_\mathbb{Z}, (A^*_\mathbb{Q}, W), (A^*_\mathbb{C}, W, F))\), in the sense of \[15\] Definition 3.13., is constructed, where

\[
A^*_C = \bigoplus_{p+q=k} A^*_{C, q} := \bigoplus_{p+q=k} (\Omega^{p+q+1}_{X^1/C}/W_q \Omega^{p+q+1}_{X^1/C}),
\]

\[
F^{\geq r} A^*_C := \bigoplus_p \bigoplus_{q \geq r} A^*_{C, q}
\]

and \(W_{\leq r} A^*_{C, q} := W_{r+2p+1} \Omega^{p+q+1}_{X^1/C}/W_q \Omega^{p+q+1}_{X^1/C};\)

here \(W_q\) refers to the subsheaf with at most \(q\) log poles. There is a natural quasi-isomorphism \(\mu : (\Omega^*_{X^1/\mathbb{Q}S^1}, F^{\geq r} := \Omega^{\geq r}_{X^1/\mathbb{Q}S^1}) \to (A^*_C, F)\) preserving the Hodge filtration \(F\). Applying \[15\] Theorem 3.18] gives a mixed Hodge structure \((\mathbb{H}^*(A^*_C), (\mathbb{H}^*(A^*_\mathbb{Q}), W), (\mathbb{H}^*(A^*_\mathbb{C}), W, F))\), as well as the Hodge-to-de Rham degeneracy (Assumption 6.12) as in \[15\] proof of Lemma 4.1. Injectivity of the Kodaira-Spencer map \(^0\nabla(0, \omega)\) in Condition 6.7 can also be easily verified using the cohomological mixed Hodge complex \((A^*_C, F, W)\).

7.3.2. Opposite filtrations. On each \(\mathbb{H}^l(A^*_C)\), we have Deligne’s splitting \(\mathbb{H}^l(A^*_C) = \bigoplus_{s,t} I^{s,t}\) such that \(W_{\leq r} = \bigoplus_{s+t \leq r} I^{s,t}\) and \(F^{\geq r} = \bigoplus_{s \geq r} I^{s,t}\). Since the nilpotent operator \(N_\nu\) is defined over \(\mathbb{Q}\) and satisfies \(N_{\nu} W_{\leq r} C \subseteq W_{\leq r-2}\), we deduce that \(N_{\nu} I^{s,t} \subseteq I^{s-1,t-1}\). As the Hodge filtration \(F^{\geq r} \mathbb{H}^l(A^*_C)\) in Definition 2.10 is related to \(F^{\geq r} \mathbb{H}^l(A^*_C)\) by a shift: \(F^{\geq r - \frac{l+q}{2}} = F^{\geq r}\), letting \(W_{\leq r - \frac{l+q}{2}} := \bigoplus_{s \leq r} I^{s,t}\) and \(W_{\leq r} (\mathbb{H}^l(A^*_C)) := \bigoplus_{s} W_{\leq r} (\mathbb{H}^l(A^*_C))\) gives an opposite filtration satisfying Assumption 6.12.

7.3.3. The trace and pairing. For Assumption 6.17, we use the trace map \(\text{tr}\) defined in [17] Definition 7.11, which induces a pairing \(\mathbb{H}^l((\Omega^*_{X^1/\mathbb{Q}S^1}) \otimes \mathbb{H}^l((\Omega^*_{X^1/\mathbb{Q}S^1}) \to \mathbb{C}\) by the product structure on \(\Omega^*_{X^1/\mathbb{Q}S^1}\); this was denoted by \(Q_K\) in [17] Definition 7.13. The pairing is compatible with the weight filtration \(W_{\leq r}\) on \(\mathbb{H}^l((\Omega^*_{X^1/\mathbb{Q}S^1})\) by [17] Lemma 7.18. Furthermore, non-degeneracy of \(\mathbb{H}^l((\Omega^*_{X^1/\mathbb{Q}S^1})\) follows from that of the induced pairing \(\langle \cdot, \cdot \rangle\) on \(L_C = \text{Gr}_W(\mathbb{H}^l((\Omega^*_{X^1/\mathbb{Q}S^1})\) defined in [17] Definition 8.10], which in turn is a consequence of [17] Theorem 8.11] (where projectivity of \(X\) was used).

As a result, Theorem 5.5 Proposition 5.13 and Theorem 6.28 together gives the following corollary.

**Corollary 7.4.** The complex analytic space \((X, \mathcal{O}_X)\) is smoothable, i.e. there exists a \(k\)-th order thickening \((^kX, ^k\mathcal{O})\) over \(^kS^1\) locally modeled on \(^kV^\alpha\) (which is log smooth) for each \(k \in \mathbb{Z}_{\geq 0}\), and these thickenings are compatible. Furthermore, there is a structure of logarithmic Frobenius manifold on the formal extended moduli \(\hat{\mathcal{S}}^\alpha_{T}\) near \((X, \mathcal{O}_X)\).

8. Abstract deformation data from maximally degenerate log Calabi-Yau varieties

In this section, we apply our results to the case of maximally degenerate log Calabi-Yau varieties studied by Kontsevich-Soibelman \[37\] and Gross-Siebert in \[22\] 23 24. We will mainly follow \[22\] and assume the reader is familiar with these papers.

**Notation 8.1.** The characteristic \(0\) algebraically closed field \(\mathbb{C}\) in \[22\] is always chosen to be \(\mathbb{C}\). We work with a \(d\)-dimensional integral affine manifold \(B\) with holonomy in \(\mathbb{Z}^d \rtimes SL_d(\mathbb{Z})\) and codimension 2 singularities \(\Delta\) as in \[22\] Definition 1.15, together with a toric polyhedral decomposition \(\mathcal{P}\) of \(B\) into lattice polytopes as in \[22\] Definition 1.22. Following \[22\], we take \(Q = \mathbb{N}\) for simplicity. We also fix a lifted gluing data \(s\) as in Definition \[22\] Definition 5.1 for the pair \((B, \mathcal{P})\).
Assumption 8.2. We assume that \((B, \mathcal{P})\) satisfies the assumption in \cite{22} Theorem 3.21] (in order to get Hodge-to-de Rham degeneracy using results from \cite{22}). We further assume that there is a multivalued integral strictly convex piecewise affine function \(\varphi : B \to \mathbb{N}\) as defined in \cite{22} §1 for applying Serre’s GAGA \cite{19} to projective varieties.

Definition 8.3. Given \((B, \mathcal{P}, s)\), we let \((X, O_X)\) be the \(d\)-dimensional complex analytic space given by the analytification of the log scheme \(X_0(B, \mathcal{P}, s)\) constructed in \cite{22} Theorem 5.2]. It is equipped with a log structure over the \(Q\)-log point \(0^\dagger\). The existence of \(\varphi\) ensures that \(X\) is projective.

We denote the log-space by \(X^\dagger\) if we want to emphasize the log-structure. Let \(Z \subset X\) be the codimension 2 singular locus of the log-structure (i.e. \(X^\dagger\) is log-smooth away from \(Z\)) and \(j : X \setminus Z \to X\) be the inclusion as in \cite{22}. Note that \(X_0(B, \mathcal{P}, s)\) is projective.

8.1. The 0th-order deformation data. Following the notations from \cite{22, 23}, the 0th-order deformation data in Definition 2.9 is described as follows:

\textbf{Definition 8.4.} • the 0th-order complex of polyvector fields is given by the pushforward of the analytic sheaf of relative polyvector fields \(\mathcal{G}^* = j_*(\wedge^{-*} \Theta_{X^\dagger/0^\dagger})\) equipped with the natural wedge product;
• the 0th-order de Rham complex is given by the pushforward of the analytic sheaf of de Rham differential forms \(\mathcal{K}^* = j_*(\Omega^*_{X^\dagger/\mathbb{C}})\), equipped with the de Rham differential \(\partial\) as in \cite{23} first paragraph of §3.2;
• the volume element \(\omega^0\) is given via the trivialization \(j_*(\Omega^d_{X^\dagger/0^\dagger}) \cong O_X\) by \cite{23} Theorem 3.23, and then the BV operator is defined by \(\Delta(\varphi) \omega^0 := \partial(\varphi \partial \omega^0)\).

The map \(\sigma^{-1} : \Omega^{\dagger}_{X^\dagger/0^\dagger} \otimes_{\mathbb{C}} \Omega^{\dagger}_{\mathcal{K}^*/\mathcal{K}^*[-r]}) \to \Omega^{\dagger}_{\mathcal{K}^*/\mathcal{K}^*[-r]} \Omega^{\dagger}_{\mathcal{K}^*/\mathcal{K}^*}\) given by taking wedge product in \(j_*(\Omega^*_{X^\dagger/\mathbb{C}})\) is a morphism of sheaves of BV modules.

To show that the data in Definition 8.4 satisfies all the conditions in Definition 2.9, we need to verify that \(\mathcal{G}^*\) and \(\mathcal{K}^*\) are coherent, \(\dagger\sigma\) is an isomorphism and we have an identification \(\|\mathcal{K}^* = \\mathcal{K}^*/\mathcal{K}^* \cong j_*(\Omega^*_{X^\dagger/0^\dagger})\). Let us briefly explain how to obtain such statements from \cite{23}.

\textbf{Notation 8.5.} Following \cite{23} Construction 2.1], we consider the monoids \(Q, P\), the corresponding toric varieties \(V = \text{Spec} (\mathbb{C}[P])\) and \(\mathcal{V} = \text{Spec} (\mathbb{C}[Q])\) and the associated analytic spaces \(V = V^{an}\) and \(\mathcal{V} = \mathcal{V}^{an}\) respectively. \(\mathcal{V}\) is equipped with a divisorial log structure induced from the divisor \(\mathcal{V}\), and \(\mathcal{V}\) is equipped with the pull back of the log structure from \(\mathcal{V}\). \cite{23} Theorem 2.6 shows that for every geometric point \(\tilde{x} \in X_0(B, \mathcal{P}, s)\), there is an étale neighborhood \(\mathcal{W}\) of \(\tilde{x}\) which can be identified with an étale neighborhood of \(\mathcal{V}\) as a log scheme in the sense that there are étale maps

\[
\begin{tikzcd}
\mathcal{W} \ar[dr] & \ar[dl] \mathcal{V} \\
& X_0(B, \mathcal{P}, s).
\end{tikzcd}
\]

Taking analytification of these maps, then we can find an open subset (or a sheet) \(W \subset W^{an}\) mapping homeomorphically to both an open subset in \(V^{an}\) and an open subset in \(V \subset X\).

The desired statements is local on \(X\), and we work on the local model \(\mathcal{V}\). As in \cite{23} proof of Proposition 1.12], we let \(\mathcal{V}\) be the log scheme equipped with the smooth divisorial log structure induced by the boundary toric divisor \(\text{Spec} (\mathbb{C}[\partial P])\). Then we have \(j_*(\Omega^*_{\mathcal{V}^{an}}) \cong j_*(\Omega^*_{\mathcal{V}^{an}}) \mathcal{V} \cap Z = (\Omega^*_{\mathcal{V}^{an}}) \mathcal{V}\) (where the notation \(\Omega^*_{\mathcal{V}^{an}}\) refers to algebraic sheaves). From these we obtain the identification
For each $k \in \mathbb{Z}_{\geq 0}$,

- the sheaf of $k$th-order polyvector fields is given by $kG^*_\alpha := j_*(\Omega_{kV^\dagger}^{\dagger}/\mathcal{O}_{S^\dagger})$ (i.e. polyvector fields on $kV^\dagger$);
- the $k$th-order de Rham complex is given by $kK^*_\alpha := j_*(\Omega_{kV^\dagger}^{\dagger}/\mathcal{O}_{S^\dagger})$ (i.e. the space of log de Rham differentials) equipped with the de Rham differential $k\partial_\alpha = \partial$ which is naturally a dg module over $k\mathcal{O}_{S^\dagger}$;
- the local $k$th-order volume element is given by a lifting $\omega_\alpha$ of $0_\omega$ as an element in $j_*(\Omega^{d}_{kV^\dagger})$ and taking $k\omega_\alpha = \omega_\alpha \pmod{m^{k+1}}$, and then the BV operator is defined by $k\Delta_\alpha(\varphi) := k\partial_\alpha(\varphi, k\omega_\alpha)$;
- the morphism of sheaves of BV modules $k_\sigma^{-1} : k\mathcal{O}_{S^\dagger}^{\dagger} \otimes_k k(0_k\mathcal{K}_\alpha/k\mathcal{K}^-_{\alpha}[-r]) \rightarrow k\mathcal{K}^*_\alpha/r_{k+1}\mathcal{K}_\alpha$ is given by taking wedge product.

For both $k\mathcal{K}^*_\alpha$’s and $kG^*_\alpha$’s, the natural restriction map $k^{+1,k}_\alpha$ is given by the isomorphism $kV^\dagger_\alpha \cong k^{-1,V^\dagger_{k+1,S^\dagger}}_\alpha$.  

Similar to the 0th-order case, we need to check that $kG^*_\alpha$ and $kK^*_\alpha$ are coherent sheaves which are free over $kR$ for each $k$, and that $k\sigma$ is an isomorphism which induces an identification $k\mathcal{K}^*_\alpha \cong j_*(\Omega^{d}_{kV^\dagger/kS^\dagger})$. Such verification can be done using [23] Proposition 1.12 and Corollary 1.13, with similar argument as in [8.1]
8.2.1. *Higher order patching data.* To obtain the patching data we need to take suitable analytification of statements from [23]. Given \( \bar{x} \in V_{\alpha\beta} \), we consider the following diagram of étale neighborhoods

\[
\begin{array}{ccc}
W_\alpha \times X_0 & \xrightarrow{kV_\alpha \supset V_\alpha} & W_\beta \\
\downarrow kV_\alpha \supset V_\alpha & & \downarrow kV_\beta \supset V_\beta \\
X_0 & \xrightarrow{\bar{x}} & W_\beta \\
\end{array}
\]

where \( X_0 = X_0(B, \mathcal{P}, s) \), and \( kV_\alpha \) (resp. \( kV_\beta \)) is the \( k \)-th order neighborhood of \( V_\alpha \) (resp. \( V_\beta \)). Using [23, Lemma 2.15] on local uniqueness of thickening (see also [17] for a more detailed study on local uniqueness), and further passing to an étale cover \( W_{\alpha\beta} \) of \( W_\alpha \times X_0 \times W_\beta \), we get an isomorphism

\[
k^{\Xi}_{\alpha\beta,i} : W_{\alpha\beta} \times V_{\alpha} \xrightarrow{k\psi_{\alpha\beta,i}} W_{\alpha\beta} \times V_{\beta} \xrightarrow{kV_{\alpha}} W_{\beta}.
\]

Taking analytification, we can find a (small enough) open subset in \( (W_{\alpha\beta})^{an} \) mapping homeomorphically onto a Stein open neighborhood \( U_i \subset V_{\alpha\beta} \) of \( \bar{x} \).

**Definition 8.7.** Restriction of the analytification of \( k^{\Xi}_{\alpha\beta,i} \) on \( U_i \) gives the gluing map \( k^{\Psi}_{\alpha\beta,i} \):

\[
k^{\Psi}_{\alpha\beta,i} = k^{\psi}_{\alpha\beta,i} \circ k^{\Xi}_{\alpha\beta,i} : k^{\psi}_{\alpha\beta,i} : j_*(\Lambda^{-*} \Theta_{kV_\alpha})|_{U_i} \to j_*(\Lambda^{-*} \Theta_{kV_\beta})|_{U_i} \text{ and } k^{\psi}_{\alpha\beta,i}
\]

The patching isomorphisms \( k^{\psi}_{\alpha\beta,i} : j_*(\Lambda^{-*} \Theta_{kV_\alpha})|_{U_i} \to j_*(\Lambda^{-*} \Theta_{kV_\beta})|_{U_i} \) are then induced by \( k^{\Psi}_{\alpha\beta,i} \).

The existence of the vector fields \( k^{\beta}_{\alpha\beta,i} \), \( k^{\beta}_{\alpha\beta,ij} \) and \( k^{\beta}_{\alpha\gamma,ij} \) in Definition 2.15 follows from the analytic version of [23, Theorem 2.11] which says that any log automorphism of the space \( kV_\alpha|_{U_i} \) (resp. \( kV_\beta|_{U_i} \)) fixing \( X|_{U_i} \) (or \( X|_{U_i} \)) is obtained by exponentiating the action of a vector field in \( \Theta_{kV_\alpha|_{U_i}} \) (resp. \( \Theta_{kV_\beta|_{U_i}} \)). The element \( k^{\beta}_{\alpha\beta,i} \) in Definition 2.18 indeed measures the difference between the volume elements, namely, \( k^{\beta}_{\alpha\beta,i} \) given by \( k^{\beta}_{\alpha\beta,i} = \exp(k^{\beta}_{\alpha\beta,i}) \).

8.2.2. *Criterion for freeness of the Hodge bundle.* To verify Assumption 4.15 which is needed for proving the freeness of the Hodge bundle in [4.3.2], notice that by taking \( Q = \mathbb{N} \), we are already in the situation of a 1-parameter family. The holomorphic Poincaré Lemma in Assumption 4.15 follows by taking the analytification of the results from [23, proof of Theorem 4.1].

8.3. *The Hodge theoretic data.* Since we have the relation \( j_*((\Omega^{\text{alg},*}_{X_0(B, \mathcal{P}, s)}|_{V_{\alpha\beta}})_{an}) = j_*((\Omega^*_{X_{an}})_{\text{alg},*}) \), the Hodge-to-de Rham degeneracy (Assumption 5.4) follows by applying Serre’s GAGA [19] to [23, Theorem 3.26] using the same argument as in the proof of Grothendieck’s algebraic de Rham theorem. Applying Theorem 5.5 and Proposition 5.13 we obtain an alternative proof of the following unobstructedness result due to Gross-Siebert [24]:

**Corollary 8.8.** Under Assumption 8.3, the complex analytic space \( (X, \mathcal{O}_X) \) is smoothable, i.e. there exists a \( k \)-th order thickening \( (kX, k\mathcal{O}_X) \) over \( k^2 \) locally modeled on \( kV_{\alpha} \) for each \( k \in \mathbb{Z}_{\geq 0} \), and these thickenings are compatible.
8.4. F-manifold structure near a LCSL. Finally we demonstrate how to apply Theorem 6.28 to the Gross-Siebert setting.

8.4.1. The universal monoid \( Q \). First of all, we consider \( (B, \mathcal{P}) \) as in Notation 8.1 and work with the cone picture as in [24]. We also need the notion of an multivalued integral piecewise affine function on \( B \) as described before [24, Remark 1.15]. Let \( \text{MPA}(B, \mathbb{N}) \) be the monoid of multivalued convex integral piecewise affine function on \( B \), take \( Q = \text{Hom}(\text{MPA}(B, \mathbb{N}), \mathbb{N}) \) to be the universal monoid and consider the universal multivalued strictly convex piecewise affine function \( \varphi : B \rightarrow Q \) as in [21, equation A.2] (it was denoted as \( \tilde{\varphi} \) there). Since we work in the cone picture, we fix an open gluing data \( s \) as in [24, Definition 1.18] and replace the monodromy polytopes in Assumption 8.2 by the dual monodromy polytopes associated to each \( \tau \in \mathcal{P} \).

8.4.2. Construction of \( X^\dagger = X_0(B, \mathcal{P}, s, \varphi)^\dagger \). We now take an element \( n \in \text{int}_{re}(Q_{\mathbb{R}}^\dagger) \cap \mathbb{K}^\dagger \) and define a multivalued strictly convex piecewise affine function \( \varphi_n : B \rightarrow \mathbb{R} \). The cone picture construction described in [24, Construction 1.17] gives a log scheme \( X_0^\dagger = X_0(B, \mathcal{P}, s, \varphi_n)^\dagger \) over \( \mathbb{C}^\dagger \) (here \( \mathbb{C}^\dagger \) is the standard \( \mathbb{Z}_+ \)-log point) which is log smooth away from a codimension 2 locus \( i : \mathbb{Z} \rightarrow X \). [21, Construction A.6] then gives a log scheme \( X^\dagger \) (with the same underlying scheme as \( X_0^\dagger \)) over \( 0^\dagger \mathcal{S}^\dagger \). Definition 8.4 can be carried through.

8.4.3. Local model on thickening of \( X^\dagger \). For each \( \tau \in \mathcal{P} \), let \( \mathcal{Q}_\tau \) be the normal lattice as defined in [22, Definition 1.33]. We denote by \( \Sigma_\tau \) the normal fan of \( \tau \) defined in [22, Definition 1.35] on \( \mathcal{Q}_{\tau, \mathbb{R}} \), equipped with the strictly convex piecewise linear function \( \varphi : |\Sigma_\tau| = \mathcal{Q}_{\tau, \mathbb{R}} \rightarrow \mathcal{Q}_{\mathbb{R}}^\dagger \) induced by \( \varphi \). We let \( \Delta_1, \ldots, \Delta_r \) be the dual monodromy polytopes associated to \( \tau \) as defined in [22, Definitions 1.58 & 1.60], and \( \psi_i(m) := -\inf\{\langle m, n \rangle \mid m \in \mathcal{Q}_{\tau, \mathbb{R}}, n \in \Delta_i \} \) be the integral piecewise linear function on \( \mathcal{Q}_{\tau, \mathbb{R}} \).

We define monoids \( P_\tau \) and \( \mathcal{Q}_\tau \) by

\[
P_\tau := \{(m, a_0, a_1, \ldots, a_r) \mid m \in \mathcal{Q}_\tau, a_0 \in \mathcal{Q}_{\mathbb{R}}^\dagger, a_i \in \mathbb{Z}, a_0 - \varphi(m) \in Q, a_i - \psi_i(m) \geq 0 \text{ for } 0 \leq i \leq r\},
\]

\[
\mathcal{Q}_\tau := \{(m, a_0, a_1, \ldots, a_r) \mid m \in \mathcal{Q}_\tau, a_0 \in \mathcal{Q}_{\mathbb{R}}^\dagger, a_i \in \mathbb{Z}, a_0 - \varphi(m) \in Q \} \cup \{\infty\},
\]

where the monoid structure on \( \mathcal{Q}_\tau \) is given as in [23, p. 22 in Construction 2.1]. Also let \( \mathbb{V}_\tau := \text{Spec}(\mathbb{C}[P_\tau]) \) which comes with a natural family \( \pi : \mathbb{V}_\tau \rightarrow \text{Spec}(\mathbb{C}[Q]) = \mathbb{S}^\dagger \), \( \mathbb{V}_\tau = \pi^{-1}(0) = \text{Spec}(\mathbb{C}[\mathcal{Q}_\tau]) \) and \( k\mathbb{V}_\tau = \pi^{-1}(k\mathbb{S}^\dagger) \) be the \( k \)-th order thickening of \( \mathbb{V}_\tau \) in \( \mathbb{V}_\tau \).

For \( i = 1, \ldots, r \) and a vertex \( v \in \Delta_i \), we define a submonoid \( D_{i,v} := w_{i,v} \cap P_\tau \), where \( w_{i,v} = v + e_i^\gamma \), and let \( D_{i,v} \) be the corresponding toric divisor of \( \mathbb{V}_\tau \). To simplify notations, we often omit the dependence on \( v \) and write \( w_i, D_i, D_i \) instead of \( w_{i,v}, D_{i,v}, D_{i,v} \). Let \( v_1, \ldots, v_l \) be the generators of 1-dimensional cones in the dual cone \( P_\gamma^\vee \) other than the \( w_j \)'s, with corresponding toric divisors \( D_1, \ldots, D_l \). Writing \( \mathbb{D} = \bigcup_j D_j \), we equip \( \mathbb{V}_\tau \) with the divisorial log structure induced by the divisor \( \mathbb{D} \leftrightarrow \mathbb{V}_\tau \), which is denoted as \( \mathbb{V}_\tau \). Pull back the log structure from \( \mathbb{V}_\tau \) give the log schemes \( \mathbb{V}_\tau \) and \( k\mathbb{V}_\tau \). [23, Theorem 2.6] holds for this setting as described in Notation 8.5 by taking \( P = P_\tau \) and \( Q = \mathcal{Q}_\tau \) for some \( \tau \in \mathcal{P} \).

As in [8.2] analytification of the log schemes \( \mathbb{V}_\tau \) and \( k\mathbb{V}_\tau \) give the log analytic schemes \( \mathbb{V}_\tau \) and \( k\mathbb{V}_\tau \) respectively, and Definition 8.6 can be carried through. We can deduce that \( k\mathcal{G}_\alpha \) and \( k\mathcal{K}_\alpha \) are coherent sheaves which are also sheaves of free modules over \( k\mathbb{R} \), and that \( k\sigma^{-1} \) is an isomorphism, by using the following variant of [23 Proposition 1.12 & Corollary 1.13].
Proposition 8.9. Let $\mathcal{Z} := \mathcal{V}_\tau \cap D_{\text{sing}} \hookrightarrow |k\mathcal{V}_\tau| = |\mathcal{V}_\tau|$ be the inclusion. Then we have the following decomposition into $P_\tau$-homogeneous pieces as

$$
\Gamma(\mathcal{V}_\tau \setminus \mathcal{Z}, \Omega^{r}_{k\mathcal{V}_\tau}) = \bigoplus_{p \in P_\tau \setminus kQ^+} z^p \cdot \bigotimes_{\{j \mid p \in D_j\}} \mathcal{D}_{\mathcal{V}_\tau}^{gp} \otimes \mathbb{C},
$$

$$
\Gamma(\mathcal{V}_\tau \setminus \mathcal{Z}, \Omega^{r}_{k\mathcal{V}_\tau \setminus kS^1}) = \bigoplus_{p \in P_\tau \setminus kQ^+} z^p \cdot \bigotimes_{\{j \mid p \in D_j\}} (\mathcal{D}_{\mathcal{V}_\tau}^{gp}/Q^{gp}) \otimes \mathbb{C}).
$$

The construction of the higher order patching data described in §8.2.1 can be carried through because divisorial deformations over $0S^1$ can be defined as in [23] Definition 2.7, and [23] Theorem 2.11 & Lemma 2.15 hold accordingly with the local models $\mathcal{V}_\tau$’s.

8.4.4. Opposite filtration and pairing. The weight filtration in Assumption 6.12 is taken to be the filtration described in [23] Remark 5.7, which is opposite to the Hodge filtration and preserved by the nilpotent operators $N_\nu$’s. The trace map $\text{tr}$ in Assumption 6.17 can be defined via the isomorphism $\text{tr} : H^d(X, j_*(\Omega^d_X/\mathcal{O}_X)) \cong H^d(X, j_*(\Omega^d_X/\mathcal{O}_{S^1})) \cong \mathbb{C}$. We conjecture that the induced pairing $^0\mathcal{P}$ is non-degenerate.

Corollary 8.10. There is a structure of log $F$-manifold on the formal extended moduli $\mathcal{S}_P^\dagger$ of complex structures near $(X, \mathcal{O}_X)$. If the pairing $^0\mathcal{P}$ is non-degenerate, it can be enhanced to a logarithmic Frobenius manifold structure.


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