QUANTUM COHOMOLOGY, SHIFT OPERATORS, AND COULOMB BRANCHES

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Abstract. Given a complex reductive group G and a G-representation $\mathbf{N}$, there is an associated quantized Coulomb branch algebra $\mathcal{A}^{\hbar}_{G,\mathbf{N}}$ defined by [Nak16; BFN18]. In this paper, we give a new interpretation of $\mathcal{A}^{\hbar}_{G,\mathbf{N}}$ as the largest subalgebra of the equivariant Borel–Moore homology of the affine Grassmannian on which shift operators can naturally be defined. As a main application, we show that if X is a smooth semiprojective variety equipped with a G-action, and $X \to \mathbf{N}$ is a G-equivariant proper holomorphic map, then the equivariant big quantum cohomology $QH^{\bullet}_G(X)$ defines a quasi-coherent sheaf of algebras on the Coulomb branch with coisotropic support. Upon specializing the Novikov and bulk parameters, this sheaf becomes coherent with Lagrangian support. We also apply our construction to recover Teleman's gluing construction for Coulomb branches [Tel21] and derive different generalizations of the Peterson isomorphism [Pet97].

Contents

Main results and background. This paper concerns the interaction between *quantum cohomology* [Wit91; KM94; CK99], *shift operators* [Sei97; OP10; Iri17], and *Coulomb branches* [Nak16; BFN18; BDGH16; Tel21] arising from 3d $\mathcal{N}=4$ supersymmetric gauge theories.

Let G be a complex reductive group and $\mathbf{N}$ a representation of G. Let $\mathcal{A}_{G,\mathbf{N}}$ denote the corresponding Braverman-Finkelberg-Nakajima (BFN) Coulomb branch algebra, and $\mathcal{A}^{\hbar}_{G,\mathbf{N}}$ be the quantized Coulomb branch. Let X be a smooth semiprojective variety (see Section 3.1) equipped with a G-action, and let $f : X \to \mathbf{N}$ be a proper G-equivariant holomorphic map. Below are the main results of this paper.

Theorem 1. *There exists a module action*

$$\mathbb{S}_{G,\mathbf{N},X} : \mathcal{A}^{\hbar}_{G,\mathbf{N}} \otimes_{\mathbb{C}[\hbar]} QH^{\bullet}_{G\times\mathbb{C}^{\times}_{\hbar}}(X) \longrightarrow QH^{\bullet}_{G\times\mathbb{C}^{\times}_{\hbar}}(X) \tag{1}$$

of the quantized Coulomb branch algebra $\mathcal{A}^{\hbar}_{G,\mathbf{N}}$ on the $(G\times\mathbb{C}^{\times}_{\hbar})$-equivariant big quantum cohomology of X, such that $\mathbb{S}_{G,\mathbf{N},X}(\Gamma,-)$ commutes with the quantum connections for any $\Gamma\in\mathcal{A}^{\hbar}_{G,\mathbf{N}}$.

Theorem 2. *There exists a ring homomorphism*

$$\Psi_{G,\mathbf{N},X} : \mathcal{A}_{G,\mathbf{N}} \longrightarrow QH^{\bullet}_{G}(X) \tag{2}$$

from the Coulomb branch algebra $\mathcal{A}_{G,\mathbf{N}}$ to the G-equivariant big quantum cohomology of X. Moreover, the support of $QH^{\bullet}_{G}(X)$ in the Coulomb branch $\operatorname{Spec}\mathcal{A}_{G,\mathbf{N}}$ *is coisotropic. Upon specialization of the Novikov and bulk parameters to complex values (whenever such a specialization is possible), the support becomes Lagrangian.*

Theorems 1 and 2 highlight a deep relationship between equivariant quantum cohomology and the geometry of the Coulomb branch. As we will see, the action $\mathbb{S}_{G,\mathbf{N},X}$ is defined using a generalized version of non-abelian shift operators, which encode enumerative data from X into the algebraic structure of $\mathcal{A}^{\hbar}_{G,\mathbf{N}}$. In Theorem 2, the coisotropic property comes from the fact that the module action (1) is a quantization of the ring homomorphism (2); see Proposition 5.20.

By specializing the equivariant parameters $H^{\bullet}_{G}(\mathrm{pt})$ (and $\hbar$) to zero, we obtain the following *non-equivariant limits* of shift operators (Corollary 5.26):

$$\begin{aligned} S_{G,\mathbf{N},X} &: \mathcal{A}^{\hbar}_{G,\mathbf{N}} \otimes QH^{\bullet}(X) \longrightarrow QH^{\bullet}(X)[\hbar], \\ S^{\hbar=0}_{G,\mathbf{N},X} &: \mathcal{A}_{G,\mathbf{N}} \otimes QH^{\bullet}(X) \longrightarrow QH^{\bullet}(X). \end{aligned}$$

New non-equivariant invariants in the quantum cohomology of X can thus be obtained via $S_{G,\mathbf{N},X}$ and $S^{\hbar=0}_{G,\mathbf{N},X}$; see Example 6.2 for an explicit example. One can interpret Theorem 1 as identifying $\mathcal{A}^{\hbar}_{G,\mathbf{N}}$ as the subalgebra in $\mathcal{A}^{\hbar}_{G}$ that captures precisely those shift operators for which a non-equivariant limit exists.

Let us also emphasize that these results depend crucially on our use of *equivariant Novikov variables*, which will be explained in more detail later in this introduction.

To motivate the above theorems, let us review the relevant geometric structures.

Quantum cohomology and shift operators. The equivariant (big) quantum cohomology ring $QH^{\bullet}_{G}(X)$ is a deformation of the classical equivariant cohomology ring $H^{\bullet}_{G}(X)$ over the Novikov and bulk parameters, defined via genus-zero Gromov–Witten invariants.

Shift operators are endomorphisms of equivariant quantum cohomology that play an important role in symplectic topology, mirror symmetry, and representation theory [Sei97; OP10; MO19; Iri17].

When $G=T$ is a complex torus and X is a smooth projective variety equipped with a T-action. Let $\mathbb{C}^{\times}_{\hbar}$ denote an additional one-dimensional torus acting trivially on X. Then for each cocharacter $\lambda : \mathbb{C}^{\times} \to T$, there exists a shift operator

$$\mathbb{S}_{\lambda} : QH^{\bullet}_{T\times\mathbb{C}^{\times}_{\hbar}}(X) \longrightarrow QH^{\bullet}_{T\times\mathbb{C}^{\times}_{\hbar}}(X).$$

These operators satisfy the relations

$$\mathbb{S}_{\lambda_1}(S_{\lambda_2}(\alpha)) = \mathbb{S}_{\lambda_1+\lambda_2}(\alpha), \tag{3}$$

$$\mathbb{S}_{\lambda}(P(a,\hbar)\alpha) = P(a+\lambda(\hbar),\hbar)\,\mathbb{S}_{\lambda}(\alpha), \tag{4}$$

for any cocharacters $\lambda,\lambda_1,\lambda_2 : \mathbb{C}^{\times} \to T$, any $\alpha\in QH^{\bullet}_{T\times\mathbb{C}^{\times}_{\hbar}}(X)$ and any $P(a,\hbar)\in H^{\bullet}_{T\times\mathbb{C}^{\times}_{\hbar}}(\mathrm{pt})$, regarded as a polynomial function on $\mathrm{Lie}(T\times\mathbb{C}^{\times}_{\hbar})$.

The Teleman/González–Mak–Pomerleano generalization. A non-abelian generalization of shift operators was developed in [GMP23b], building on the ideas introduced in Teleman's ICM address [Tel14]. Suppose G is a reductive group acting on a smooth projective variety X (which corresponds to the special case $\mathbf{N} = 0$ in our setting). In their proposal, there should exist a module action

$$\mathbb{S}_{G,X} : H_\bullet^{G_\mathcal{O} \rtimes \mathbb{C}^\times_\hbar}(\mathrm{Gr}_G) \otimes_{\mathbb{C}[\hbar]} QH^\bullet_{G\times\mathbb{C}^\times_\hbar}(X) \longrightarrow QH^\bullet_{G\times\mathbb{C}^\times_\hbar}(X), \tag{5}$$

where $\mathrm{Gr}_G = G_\mathcal{K}/G_\mathcal{O}$[1] is the affine Grassmannian of G, $\mathbb{C}^\times_\hbar$ acts on Gr_G by loop rotation, and $H_\bullet^{G_\mathcal{O} \rtimes \mathbb{C}^\times_\hbar}(\mathrm{Gr}_G)$, also denoted as $\mathcal{A}^\hbar_G$, is the equivariant Borel–Moore homology of Gr_G, equipped with the convolution product; see Section 1 for more details.

When $G = T$ is a torus, the ring $H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}^\times_\hbar}(\mathrm{Gr}_T)$ is a free $H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathrm{pt})$-module with basis $\{t^\lambda\}$ indexed by cocharacters $\lambda : \mathbb{C}^\times \to T$. The convolution product satisfies

$$t^{\lambda_1} * t^{\lambda_2} = t^{\lambda_1+\lambda_2},$$
$$t^\lambda * P(a,\hbar) = P(a + \lambda(\hbar), \hbar) * t^\lambda,$$

for any cocharacters $\lambda, \lambda_1, \lambda_2 : \mathbb{C}^\times \to T$ and any $P(a,\hbar) \in H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathrm{pt})$. Setting $\mathbb{S}_\lambda := \mathbb{S}_{T,X}(t^\lambda, -)$, the module property implies (3) and (4).

Coulomb branches. The motivation behind the works of [Tel14; GMP23b] is that the affine scheme $\operatorname{Spec} H_\bullet^{G_\mathcal{O}}(\mathrm{Gr}_G)$ arises as the Coulomb branch of the 3d $\mathcal{N} = 4$ supersymmetric pure gauge theory associated to the gauge group G. The Coulomb branch is a moduli space of vacua of the theory and dictates the 2d mirror symmetry of G-actions.

More generally, given a representation $\mathbf{N}$ of G, there is a corresponding 3d $\mathcal{N} = 4$ supersymmetric gauge theory with gauge group G and matter $\mathbf{N}$. The Coulomb branch of such a theory was first described in the physics literature [SW], and a rigorous mathematical definition was given much later by Braverman, Finkelberg, and Nakajima in [BFN18]. They defined the quantized Coulomb branch algebra $\mathcal{A}^\hbar_{G,\mathbf{N}}$ as a subalgebra of $\mathcal{A}^\hbar_G$, and the Coulomb branch is the spectrum of its classical limit

$$\mathcal{A}_{G,\mathbf{N}} := \mathcal{A}^\hbar_{G,\mathbf{N}}/\hbar\mathcal{A}^\hbar_{G,\mathbf{N}} \subset \mathcal{A}_G.$$

In [BFN18], it was proved that $\mathcal{A}_{G,\mathbf{N}}$ is a finitely generated commutative algebra, and that the quantization $\mathcal{A}^\hbar_{G,\mathbf{N}}$ induces a Poisson structure on $\mathcal{A}_{G,\mathbf{N}}$, which defines a symplectic structure on the smooth locus of the Coulomb branch. The Coulomb branch $\operatorname{Spec} \mathcal{A}_{G,\mathbf{N}}$ is expected to capture the 2d mirror-symmetric information of G-equivariant fibrations $X \to \mathbf{N}$.

Theorem 1 is an extension of Teleman's proposal to full generality, where we have general gauge group G and matter $\mathbf{N}$, and also general X which may not be projective. A major obstacle is the possible non-compactness of the T-fixed locus X^T (where $T \subset G$ is a maximal torus), which prevents us from applying any of the previous constructions. As we will see shortly in the description of the construction of $\mathbb{S}_{G,\mathbf{N},X}$ below, a new understanding of $\mathcal{A}^\hbar_{G,\mathbf{N}}$ precisely provides the key for overcoming this difficulty.

Relations with earlier constructions.

The case $G = T$ and $\mathbf{N} = \mathbf{0}$. Shift operators were originally introduced by Braverman, Maulik, Okounkov, and Pandharipande in [OP10; BMO11; MO19] for small quantum cohomology. The version on equivariant symplectic cohomology was constructed in [Lie21]. The classical limit as $\hbar \to 0$ corresponds to the so-called Seidel elements or Seidel representations, which had appeared earlier in [Sei97]. A generalization of shift operators to big quantum cohomology was developed by Iritani [Iri17] to study toric mirror symmetry. This corresponds to the "$G = T$ and $\mathbf{N} = \mathbf{0}$" case of Theorem 1.

The case of general G and $\mathbf{N} = \mathbf{0}$. Non-abelian shift operators in the case $\mathbf{N} = \mathbf{0}$ were suggested in [Tel14], and constructed in [GMP23b] using symplectic geometry in the setting of compact monotone symplectic manifolds. This is (the symplecto-geometric version of) the "general G and $\mathbf{N} = \mathbf{0}$" case of Theorem 1 restricted to small quantum cohomology. We also note that the non-abelian generalization of Seidel elements were studied in [Sav08] via the homology of the loop group of the Hamiltonian symplectomorphism group; the ideas therein might have influenced later development. In [GMP23a], a non-abelian version of Seidel representation was also constructed for equivariant symplectic cohomology.

[1] Here, $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$ denote the formal Laurent and power series rings, respectively.

The map $\Psi_{G,X}$ in the special case where G is simple and simply connected, and $X = G/P$ is a partial flag variety was studied in [Cho23], whose setting is closer to ours. However, their treatment of Novikov variables was specific to that particular case. In contrast, our approach uses equivariant Novikov variables, allowing the construction to extend to more general X and arbitrary G.

The main focus of [Cho23] is to show that the non-abelian shift operators recover the Peterson isomorphism. In Section 6.3, we show that our generalization likewise induces an analogue of the Peterson isomorphism, valid for more general G, including groups that are neither simply connected nor semisimple. In this broader setting, equivariant Novikov variables play a crucial role, as certain Lagrangians in $\operatorname{Spec} \mathcal{A}_{G,\mathbf{N}}$ appear only when we incorporate these variables.

The case of general G and $\mathbf{N}$. Our paper is the first to define the shift operators $\mathbb{S}_{G,\mathbf{N},X}$ and $\Psi_{G,\mathbf{N},X}$ for general choices of G, $\mathbf{N}$, and X. In particular, our construction recovers the following special cases:

1. In the case $G = T$ with X^T compact, shift operators were defined in [Iri17] using localization. Under the assumptions of *op. cit.*, there always exists a G-representation $\mathbf{N}$ and a proper holomorphic map $f : X \to \mathbf{N}$. Then, the shift operators of *op. cit.* can be recovered by localizing our construction of $\mathbb{S}_{G,\mathbf{N},X}$ (Remark 5.25).
2. We also recover the case $X = \mathbf{N}$ studied in [GMP23a], which builds on Teleman's description of the Coulomb branch $\operatorname{Spec} \mathcal{A}_{G,\mathbf{N}}$ as a gluing of two copies of the pure gauge Coulomb branch $\operatorname{Spec} \mathcal{A}_G$ (see [Tel21]). Their construction relies on the observation that the gluing maps coincide with certain Seidel elements in the case $X = \mathbf{N}$. In contrast, our result yields a new, self-contained proof of Teleman's gluing construction (Theorem 6.5).

We remark that even in the abelian case, the construction of shift operators when the T-fixed locus X^T is noncompact has not previously appeared in the literature.

Shift operators for quasimaps. In [Tam24], the author provided a construction of non-abelian shift operators using quasimaps for $G = \mathrm{GL}_n$. Their computation offers a new proof that $\mathcal{A}^{\hbar}_{\mathrm{GL}_n}$ is a quotient of the shifted Yangian (see also [BFN19]).

Open-string analogue. The first-named author and Leung have attempted to construct Lagrangians in the Coulomb branch $\mathcal{A}_{G,\mathbf{N}}$ using equivariant 2d mirror symmetry, motivated again by ideas of Teleman [Tel14]. The abelian case was discussed in [CL24a]. The non-abelian case involves studying non-displaceable (real) Lagrangians under a Hamiltonian action. The case of general G with $\mathbf{N} = 0$ was studied in [CL24b]; see Section 1.8 of [CL24a] for a discussion of the case with general G and arbitrary $\mathbf{N}$.

The construction of $\mathbb{S}_{G,\mathbf{N},X}$.

Equivariant Novikov variables. Given G and X, we let $\tau \in H^\bullet_{G\times\mathbb{C}^\times_\hbar}(X)$ be a general point, treated as a formal variable. We define $\mathbb{C}[[q_G,\tau]]$ to a formal completion of $\mathbb{C}[[\tau]][H_2^{\mathrm{ord},G}(X;\mathbb{Z})]$[2] along the cone of effective curve classes (see Section 3.1). In this paper, the equivariant quantum cohomology ring has underlying vector space given by a completed tensor product

$$H^\bullet_G(X)[[q_G,\tau]] \coloneqq H^\bullet_G(X)\widehat{\otimes}\,\mathbb{C}[[q_G,\tau]].$$

The quantum product is defined using equivariant genus-zero Gromov–Witten invariants.

The case $\mathbf{N} = \mathbf{0}$. We begin with the construction in the pure gauge case. Note that when $\mathbf{N} = 0$, the variety X is projective. The first step involves a convolution-type operation.

We define a map (see Definition 2.1)

$$\mathrm{tw} \coloneqq (\pi_R)_* \circ (\pi_L)^* : H_\bullet^{G_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}(\mathrm{Gr}_G) \otimes_{\mathbb{C}[\hbar]} H_\bullet^{G\times\mathbb{C}^\times_\hbar}(X) \longrightarrow H_\bullet^{G\times\mathbb{C}^\times_\hbar}(G_\mathcal{K}\times_{G_\mathcal{O}} X)$$

via the correspondence

$$\begin{tikzcd} & G_\mathcal{K}\times X \arrow[dl, "\pi_L"'] \arrow[dr, "\pi_R"] & \\ \mathrm{Gr}_G\times X & & G_\mathcal{K}\times_{G_\mathcal{O}} X \end{tikzcd}. \tag{6}$$

[2]We use $H_\bullet$ to denote Borel–Moore homology and $H^{\mathrm{ord}}_\bullet$ to denote ordinary (i.e., singular) homology.

The map tw is twisted-linear in the following sense. The projection

$$u : \mathrm{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}} \longrightarrow [\mathrm{pt}/G_{\mathcal{O}}] \tag{7}$$

induces a pullback homomorphism

$$u^* : H^\bullet_{G\times\mathbb{C}^\times_\hbar}(\mathrm{pt}) \longrightarrow H^\bullet_{G_{\mathcal{O}}\rtimes\mathbb{C}^\times_\hbar}(\mathrm{Gr}_G).$$

For any $P \in H^\bullet_{G\times\mathbb{C}^\times_\hbar}(\mathrm{pt})$ and $\Gamma\otimes\alpha \in H_\bullet^{G\times\mathbb{C}^\times_\hbar}(\mathrm{Gr}_G)\otimes H_\bullet^{G\times\mathbb{C}^\times_\hbar}(X)$, we have

$$\mathrm{tw}(\Gamma\otimes P\alpha) = \mathrm{tw}((u^*P\cap\Gamma)\otimes\alpha) = u^*P\cap\mathrm{tw}(\Gamma\otimes\alpha),$$

which in the abelian case reflects the twisted-linearity relation (4) (Proposition 2.9).

The second step involves counting curves in a certain associated X-bundle. By a theorem of Beauville and Laszlo [BL95], there is a canonical principal G-bundle $\mathcal{E}$ over $\mathrm{Gr}_G\times\mathbb{P}^1$ together with a trivialization

$$\varphi : \mathcal{E}|_{\mathrm{Gr}_G\times\mathrm{Spec}(\mathbb{C}[t^{-1}])} \xrightarrow{\sim} G\times\mathrm{Gr}_G\times\mathrm{Spec}(\mathbb{C}[t^{-1}])$$

over $\mathrm{Gr}_G\times\mathrm{Spec}(\mathbb{C}[t^{-1}])$; here we use t to denote the coordinate on $\mathbb{P}^1$. The trivialization φ will play a crucial role in the general $\mathbf{N}$ case below.

Let $\mathcal{E}(X) \coloneqq \mathcal{E}\times_G X$ denote the associated X-bundle. The restrictions of $\mathcal{E}(X)$ to $\mathrm{Gr}_G\times\{0\}$ and $\mathrm{Gr}_G\times\{\infty\}$ are isomorphic to $G_{\mathcal{K}}\times_{G_{\mathcal{O}}} X$ and $\mathrm{Gr}_G\times X$, respectively.

Let $\mathrm{Eff}(\mathcal{E}(X))^{\mathrm{sec}}\subset H_2^{\mathrm{ord}}(\mathcal{E}(X);\mathbb{Z})$ denote the subset of effective section classes, i.e., those effective classes whose image in $H_2^{\mathrm{ord}}(\mathrm{Gr}_G\times\mathbb{P}^1;\mathbb{Z})$ is equal to $[\mathrm{pt}\times\mathbb{P}^1]$.

For $\beta\in\mathrm{Eff}(\mathcal{E}(X))^{\mathrm{sec}}$, let $\mathcal{M}(X,\beta)_n$ be the moduli space of genus-zero stable maps to $\mathcal{E}(X)$ with curve class β, and $n+2$ points $y_0, y_\infty, y_1, \ldots, y_n$, such that y_0 and y_∞ lie over the 0 and ∞ fibres, respectively. Let $\mathrm{ev}_0, \mathrm{ev}_\infty, \mathrm{ev}_1, \ldots$ be the evaluation maps. We define

$$\widetilde{\mathbb{S}}_{G,X} : H_\bullet^{G_{\mathcal{O}}\rtimes\mathbb{C}^\times_\hbar}(G_{\mathcal{K}}\times_{G_{\mathcal{O}}} X) \longrightarrow H_\bullet^{G\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]]$$

by the formula

$$\widetilde{\mathbb{S}}_{G,X}(\gamma) = \sum_{\beta\in\mathrm{Eff}(\mathcal{E}(X))^{\mathrm{sec}}}\sum_{n=0}^{\infty}\frac{q^{\overline{\beta}}}{n!}\,\mathrm{pr}_{X*}\,\mathrm{ev}_{\infty*}\left(\mathrm{ev}_0^*(\gamma)\prod_{\ell=1}^{n}\mathrm{ev}_\ell^*(\hat{\tau})\cap[\mathcal{M}(X,\beta)_n]^{\mathrm{vir}}\right),$$

where $\mathrm{pr}_X : \mathrm{Gr}_G\times X\to X$ is the projection map, $[\mathcal{M}(X,\beta)_n]^{\mathrm{vir}}$ is the virtual fundamental class of the moduli space, and $\overline{\beta}$ is the image of β under the natural map

$$H_2^{\mathrm{ord}}(\mathcal{E}(X);\mathbb{Z}) \longrightarrow H_2^{\mathrm{ord},G}(X;\mathbb{Z})$$

induced by $\mathcal{E}(X) = \mathcal{E}\times_G X\to[X/G]$; and the definition of $\hat{\tau}\in H^\bullet_{G\times\mathbb{C}^\times_\hbar}(\mathcal{E}(X))$ is given in Definition 3.13.

The module map $\mathbb{S}_{G,X}$ is then defined to be the composition

$$\begin{aligned} H_\bullet^{G_{\mathcal{O}}\rtimes\mathbb{C}^\times_\hbar}(\mathrm{Gr}_G)\otimes_{\mathbb{C}[\hbar]} H_\bullet^{G\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]] &\xrightarrow{\mathrm{tw}} H_\bullet^{G\times\mathbb{C}^\times_\hbar}(G_{\mathcal{K}}\times_{G_{\mathcal{O}}} X)[[q_G,\tau]] \\ &\xrightarrow{\widetilde{\mathbb{S}}_{G,X}} H_\bullet^{G\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]], \end{aligned}$$

up to intertwining with Poincaré duality.

We conclude with two remarks about the construction above.

(1) First, to obtain a well-defined notion of virtual fundamental classes, we must approximate the Borel–Moore homology of Gr_G by the homology of resolutions of its affine Schubert varieties.
(2) Second, the use of equivariant Novikov variables is essential in this construction. When $G = T$ is a torus, one can define a map $H_2^{\mathrm{ord}}(\mathcal{E}(X);\mathbb{Z})\to H_2^{\mathrm{ord}}(X;\mathbb{Z})$ depending on the choices of suitable section classes (see [Iri17]). In the cases studied in [Cho23], the natural map $H_2^{\mathrm{ord}}(X;\mathbb{Z})\to H_2^{\mathrm{ord},G}(X;\mathbb{Z})$ is an isomorphism. In [GMP23b; GMP23a], equivariant Novikov variables appeared implicitly by the consideration of vertical Chern classes (cf. Lemma 3.11).

The general case. Since $\mathcal{A}_{G,\mathbf{N}}$ is a subalgebra of $\mathcal{A}_G$, one might be tempted to define $\mathbb{S}_{G,\mathbf{N},X}$ as the restriction of $\mathbb{S}_{G,X}$. This idea works for the map tw, but fails for the map $\widetilde{\mathbb{S}}_{G,X}$.

As mentioned above, the main issue that the T-fixed locus X^T may not be compact. In this case, the evaluation map ev_∞ (or its restriction to the T-fixed locus) may fail to be proper. Hence, the pushforward $\mathrm{ev}_{\infty *}$ is not well-defined in general, even via localization.

Our strategy is to cut down the moduli space $\mathcal{M}(X,\beta)_n$ to a smaller subspace on which the evaluation maps become proper. A key ingredient in our approach is a reformulation of the quantized Coulomb branch algebra $\mathcal{A}^\hbar_{G,\mathbf{N}}$ which we now describe.

Recall that in [BFN18], the authors considered an infinite-rank vector bundle $\mathcal{T}=\mathcal{T}_\mathbf{N}$ over the affine Grassmannian and a fibrewise linear subvariety $\mathcal{R}\subset\mathcal{T}$. The quantized Coulomb branch algebra $\mathcal{A}^\hbar_{G,\mathbf{N}}$ is defined as the equivariant Borel–Moore homology of $\mathcal{R}$. Let $\mathcal{S}:=\mathcal{T}/\mathcal{R}$ be the fibrewise quotient. We prove the following theorem in Section 1.

Theorem 3. *The quantized Coulomb branch algebra $\mathcal{A}^\hbar_{G,\mathbf{N}}$ is isomorphic to the following subalgebra of $\mathcal{A}^\hbar_G$:*

$$e(\mathcal{S})\cap H^{G\times\mathbb{C}^\times_\hbar}_\bullet(\mathrm{Gr}_G)\subset\mathcal{A}^\hbar_G.$$

The heuristic behind Theorem 3 is as follows: if one "resolves" the affine Grassmannian by the vector bundle $\mathcal{T}$, then the pullback of $\mathcal{S}$ admits a canonical section whose zero locus is precisely $\mathcal{R}$. The term "stratified" in Theorem 3 means that $\mathcal{S}$ restricts to a vector bundle over each affine Schubert cell in Gr_G. See Section 1 on how we can make sense of the symbol $e(\mathcal{S})\cap$.

Here we highlight a novel relation between Coulomb branches and shift operators. One can understand Theorem 3 as stating that $\mathcal{A}^\hbar_{G,\mathbf{N}}$ is cut out from $\mathcal{A}^\hbar_G$ by the stratified bundle $\mathcal{S}$. We further show that the pullback of $\mathcal{S}$ to $\mathcal{M}(X,\beta)_n$ admits a canonical section whose zero locus is proper with respect to the evaluation map ev_∞ (Proposition 4.4).

Now we can continue our construction of the shift operators for general $\mathbf{N}$. For simplicity, we will denote the pullback of $\mathcal{S}$ to the various spaces by the same symbol. The twisting map tw now restricts to give a homomorphism

$$\mathrm{tw}:\mathcal{A}_{G,\mathbf{N}}\otimes_{\mathbb{C}[\hbar]}H^{G\times\mathbb{C}^\times_\hbar}_\bullet(X)\longrightarrow e(\mathcal{S})\cap H^{G\times\mathbb{C}^\times_\hbar}_\bullet(G_\mathcal{K}\times_{G_\mathcal{O}}X).$$

As just mentioned, there is a canonical section of $\mathcal{S}$ over the moduli space $\mathcal{M}(X,\beta)_n$, and the restriction of ev_∞ to its zero loci $\mathcal{Z}(X,\beta)_n$ is proper. As an example, the subspace $\mathcal{Z}(X,\mathbf{N})_0\subset\mathcal{M}(X,\mathbf{N})_0=\Gamma(\mathrm{Gr}_G,\mathcal{E}(\mathbf{N}))$ consists of those sections that are constant over $\mathrm{Spec}(\mathbb{C}[t^{-1}])$, with respect to the trivialization of φ.

We may therefore define

$$\widetilde{\mathbb{S}}_{G,\mathbf{N},X}:e(\mathcal{S})\cap H^{G\times\mathbb{C}^\times_\hbar}_\bullet(G_\mathcal{K}\times_{G_\mathcal{O}}X)\longrightarrow H^{G\times\mathbb{C}^\times_\hbar}_\bullet(X)[[q_G,\tau]]$$

by the formula

$$\widetilde{\mathbb{S}}_{G,\mathbf{N},X}(e(\mathcal{S})\cap\gamma)=\sum_{\beta\in\mathrm{Eff}(\mathcal{E}(X))^{\mathrm{sec}}}\sum_{n=0}^\infty\frac{q^{\overline{\beta}}}{n!}\,\mathrm{pr}_{X*}\,\mathrm{ev}_{\infty*}\left(\mathrm{ev}_0^*(\gamma)\prod_{\ell=1}^n\mathrm{ev}_\ell^*(\hat\tau)\cap[\mathcal{Z}(X,\beta)_n]^{\mathrm{vir}}\right),$$

and set $\mathbb{S}_{G,\mathbf{N},X}=\widetilde{\mathbb{S}}_{G,\mathbf{N},X}\circ\mathrm{tw}$, up to intertwining with Poincaré duality.

Remark 0.1 (Independence of the choice of representation). A priori, the construction of $\mathbb{S}_{G,\mathbf{N},X}$ depends on the choice of the representation $\mathbf{N}$ and the map $f\colon X\to\mathbf{N}$. However, we will show that this dependence is in fact superfluous. More precisely, suppose $\mathbf{N}'$ is another representation of G, and $g\colon X\to\mathbf{N}'$ is a proper G-equivariant morphism. Then we show in Corollary 5.2 that $\mathbb{S}_{G,\mathbf{N},X}$ and $\mathbb{S}_{G,\mathbf{N}',X}$ agree on the common domain of definition:

$$(\mathcal{A}^\hbar_{G,\mathbf{N}}\cap\mathcal{A}^\hbar_{G,\mathbf{N}'})\otimes_{\mathbb{C}[\hbar]}QH^\bullet_{G\times\mathbb{C}^\times_\hbar}(X).$$

Therefore, they extend to the same map on the localized algebras

$$\mathcal{A}^\hbar_{G,\mathbf{N},\mathrm{loc}}\otimes_{\mathbb{C}[\hbar]}QH^\bullet_{G\times\mathbb{C}^\times_\hbar}(X)=\mathcal{A}^\hbar_{G,\mathbf{N}',\mathrm{loc}}\otimes_{\mathbb{C}[\hbar]}QH^\bullet_{G\times\mathbb{C}^\times_\hbar}(X).$$

Remark 0.2 (Relation with 2d and 3d mirror symmetry). In the context of 3d mirror symmetry, an equivariant fibration $X\to\mathbf{N}$ corresponds to a 3d brane on the Higgs branch of the associated gauge theory. It is expected that there exists a mirror 3d brane supported on the Coulomb branch, reflecting the 2d mirror of the fibration $X\to\mathbf{N}$ (see [CL24a] for further discussion). This mirror brane is expected to be a Lagrangian in $\mathrm{Spec}\,\mathcal{A}_{G,\mathbf{N}}$ produced from Theorem 2.

In the special case $\mathbf{N}=0$, this perspective is useful in the study of quantum cohomology of GIT quotients [PT24; Iri25]. The categorical generalization of this correspondence, in terms of wrapped Fukaya categories, was also conjectured in [LS25]. If one interprets the quantum cohomology $QH^\bullet_G(X)$ as a closed-string incarnation of the (equivariant) 2d mirror of X, then one may expect a corresponding open-string construction of the Lagrangian

$$\operatorname{Supp} QH^\bullet_G(X) \subset \operatorname{Spec} \mathcal{A}_{G,\mathbf{N}}.$$

As mentioned above, such a construction was achieved by the first-named author and Leung in the abelian case in [CL24a] and in the non-abelian case in [CL24b] (see also [CL24a, Section 1.8]). We conjecture that these Lagrangians coincide with those given by Theorem 2.

Applications.

Rationality of shift operators. Shift operators for non-compact spaces are often defined via localization [BMO11; Iri17], as operators on

$$QH_{G\times\mathbb{C}^\times_\hbar}(X)_{\mathrm{loc}} \coloneqq \operatorname{Frac}(H^\bullet_{G\times\mathbb{C}^\times_\hbar}(\mathrm{pt})) \otimes_{H^\bullet_{G\times\mathbb{C}^\times_\hbar}(\mathrm{pt})} QH^\bullet_{G\times\mathbb{C}^\times_\hbar}(X).$$

In contrast, the subalgebra $\mathcal{A}^\hbar_{G,\mathbf{N}} \subset \mathcal{A}^\hbar_G$ captures those shift operators that can be defined without localizing the equivariant parameters. In the abelian case, this recovers the observation of Iritani [Iri17, Remark 3.10] that the shift operator associated to a semi-negative cocharacter does not require localization, so they admit non-equivariant limits; see Remark 4.10.

A new characterization of the Coulomb branch. In many cases, one can go further and show that $\mathcal{A}_{G,\mathbf{N}}$ is the largest subalgebra of $\mathcal{A}_G$ capturing those shift operators that do not require localization. This is formulated precisely in Theorem 4 below (see Section 6.2). Recall that $T \subset G$ denotes a maximal torus.

Definition 0.3. The G-representation $\mathbf{N}$ is called *gluable* if, for all nonzero T-weights ξ_1, ξ_2 of $\mathbf{N}$, ξ_1 is not a negative multiple of ξ_2.

Theorem 4. *Under the assumptions of Theorem 1, there is a commutative diagram*

$$\begin{tikzcd} \mathcal{A}_{G,\mathbf{N}} \arrow[r, "\Delta_*"] \arrow[d] & \mathcal{A}_G \otimes_{H^\bullet_G(\mathrm{pt})} \mathcal{A}_{G,\mathbf{N}} \arrow[r, "\mathrm{id}\otimes\Psi_{G,\mathbf{N}}"] & \mathcal{A}_G \otimes_{H^\bullet_G(\mathrm{pt})} QH^\bullet_G(X) \arrow[d] \\ \mathcal{A}_G \arrow[r, "\Delta_*"] & \mathcal{A}_G \otimes_{H^\bullet_G(\mathrm{pt})} \mathcal{A}_G \arrow[r, "\mathrm{id}\otimes\Psi_{G,\mathbf{N},\mathrm{loc}}"] & \mathcal{A}_G \otimes_{H^\bullet_G(\mathrm{pt})} QH^\bullet_G(X)_{\mathrm{loc}} \end{tikzcd}. \tag{8}$$

Moreover, if $\mathbf{N}$ is gluable and $X = \mathbf{N}$, then the above diagram is Cartesian.

The gluable assumption is satisfied if one replaces G with $G \times \mathbb{C}^\times_{\mathrm{dil}}$, where the additional $\mathbb{C}^\times_{\mathrm{dil}}$-factor acts on $\mathbf{N}$ by scaling. This recovers Teleman's gluing formula for Coulomb branches [Tel21].

Peterson isomorphisms. The Peterson isomorphism theorem (cf. [Pet97; LS10; LL12; Cho23]) asserts that there is an isomorphism

$$H^{G_\mathcal{O}}_\bullet(\mathrm{Gr}_G)_{\mathrm{loc}} \cong QH^\bullet_G(G/B)|_{\tau=0}$$

when G is a simply connected semisimple group.

In Section 6.3, we compute the map $\Psi_{G,X}$ (after setting $\tau = 0$) for the case where X is a partial flag variety. In particular, this shows that $\Psi_{G,G/B}$ is birational, thereby extending the Peterson isomorphism to general reductive groups. The computations closely follow those in [Cho23], except that we incorporate equivariant Novikov parameters.

In Section 6.4, we prove a version of the Peterson isomorphism under the assumption that all weights of $\mathbf{N}$ are positive (resp. negative) with respect to a central $\mathbb{C}^\times \subset G$; see Corollary 6.13. We also obtain a generalization of Teleman's result on the stratification on pure gauge Coulomb branches to $\operatorname{Spec} \mathcal{A}_{G,\mathbf{N}}$; see Corollary 6.12.

Structure of the paper. In Section 1, we review the definition of the Coulomb branch and prove Theorem 3. The twisting map tw is discussed in Section 2. Sections 3 to 5 are devoted to the definition and properties of shift operators. The proofs of Theorem 1 and Theorem 2 are given in Section 5. Several applications and computational examples of shift operators are presented in Section 6, including Theorem 4 and the different generalizations of the Peterson isomorphism.

Conventions. In this paper, $H_\bullet$ will always denote Borel–Moore homology and $H^{\mathrm{ord}}_\bullet$ denotes the ordinary homology. All varieties, schemes and stacks considered in this paper are over $\mathbb{C}$.

Acknowledgements

We thank Hiroshi Iritani, Syu Kato, Conan Leung, Yan-Lung Leon Li, Cheuk Yu Mak, Michael McBreen, Daniel Pomerleano, and Ben Webster for valuable discussions at various stages of this project. We are particularly grateful to Hiroshi Iritani for many useful comments and suggestions.

K. F. Chan was substantially supported by grants from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CUHK14301721, CUHK14306322, and CUHK14305923) and a direct grant from CUHK. He would like to thank the support provided by The Institute of Mathematical Sciences at The Chinese University of Hong Kong. K. Chan and C. H. E. Lam were substantially supported by grants of the Hong Kong Research Grants Council (Project No. CUHK14305322, CUHK14305023 & CUHK14302524).

1. Coulomb branches

In this section, we give a short treatment of the Coulomb branch and set up notations that will be important for later sections. We work over the complex numbers $\mathbb{C}$.

Lie-theoretic notations. We let G denote a connected complex reductive group, $T \subset G$ a Cartan subgroup, and $W = N_G(T)/T$ the corresponding Weyl group. We write Φ for the set of roots of G. We fix a subset $\Phi^+ \subset \Phi$ of positive roots, or equivalently, we fix a Borel subgroup $T \subset B \subset G$.

The coweight lattice of G is denoted by Λ, and its submonoid of dominant coweights is denoted by Λ^+.

Affine Grassmannian. For a $\mathbb{C}$-algebra A, let G_A denote the sheaf[3] on the category of affine schemes over $\mathbb{C}$ (that is, the opposite category of the category of $\mathbb{C}$-algebras) defined by sending a $\mathbb{C}$-algebra R to $G(R \otimes A)$.

Denote $\mathcal{O} = \mathbb{C}[[t]]$ (the ring of formal power series) and $\mathcal{K} = \mathbb{C}(\!(t)\!)$ (the field of Laurent series). In particular, we have

$$G_{\mathcal{O}}(R) = G(R[[t]]),$$
$$G_{\mathcal{K}}(R) = G(R(\!(t)\!)),$$

for any $\mathbb{C}$-algebra R. The affine Grassmannian of G is defined as the quotient

$$\mathrm{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}.$$

Let $\mathcal{I}$ be the Iwahori subgroup

$$\mathcal{I} = \{g \in G_{\mathcal{O}} : g(0) \in B\},$$

that is, $\mathcal{I}$ is the preimage of B under the evaluation map $\mathrm{ev}_{t=0} : G_{\mathcal{O}} \to G$.

For $\lambda \in \Lambda$, let t^λ denote the corresponding point in Gr_G. Define the $\mathcal{I}$-orbit

$$C_\lambda = \mathcal{I} \cdot t^\lambda \subset \mathrm{Gr}_G,$$

and let $C_{\le\lambda}$ denote its closure. Each C_λ is isomorphic to an affine space, and $C_{\le\lambda}$ admits the structure of a projective variety. We define a partial order on Λ by declaring that $\mu \le \lambda$ if and only if $C_\mu \subset C_{\le\lambda}$. In particular,

$$C_{\le\lambda} = \bigsqcup_{\mu\le\lambda} C_\mu.$$

We remark that C_λ (and hence $C_{\le\lambda}$) is $G_{\mathcal{O}}$-invariant if and only if $\lambda \in \Lambda^+$.

Equivariant Borel–Moore homology. In this paper, if X is a complex quasi-projective variety, we use $H_\bullet(X) = H^{-\bullet}(X, \omega_X)$ to denote the Borel–Moore homology groups of X with complex coefficients, using the classical topology. Here, ω_X is the dualizing complex of X. Similarly, if K is an algebraic group acting algebraically on X, we define the equivariant Borel–Moore homology $H^K_\bullet(X) := H^{-\bullet}_K(X, \omega_X)$, where both K and X are considered with the classical topology. We refer to [BL94] for the basics of cohomology of equivariant sheaves.

---

[3] Sheaves are defined with respect to the fppf topology.

The pure gauge Coulomb branch. Let $G_i = G_{\mathcal{O}/t^i\mathcal{O}}$ for any positive integer i. For each $\lambda \in \Lambda^+$, the action of $G_{\mathcal{O}}$ on $C_{\leq\lambda}$ factors through G_i for all sufficiently large i. We set

$$H_\bullet^{G_{\mathcal{O}}}(C_{\leq\lambda}) \coloneqq H_\bullet^{G_i}(C_{\leq\lambda})$$

for any such choice of i. It is easy to see that this definition is independent of i. We define

$$\mathcal{A}_G = H_\bullet^{G_{\mathcal{O}}}(\mathrm{Gr}_G) \coloneqq \varinjlim_{\lambda\in\Lambda^+} H_\bullet^{G_{\mathcal{O}}}(C_{\leq\lambda}).$$

We can similarly define $H_\bullet^{T_{\mathcal{O}}}(\mathrm{Gr}_G)$, $H_\bullet^{G}(\mathrm{Gr}_G)$, and so on.

There is a convolution product $*$ on $\mathcal{A}_G$ turning it into a finitely generated commutative $\mathbb{C}$-algebra [BFM05; BFN18]. The resulting algebra $(\mathcal{A}_G, *)$ is called the *(pure gauge) Coulomb branch algebra*. The affine scheme $\operatorname{Spec}\mathcal{A}_G$ is smooth and is known as the *(pure gauge) Coulomb branch*. We now briefly review the construction of the convolution product.

We follow [MV07; BFN18]. The convolution product on $H_\bullet^{G_{\mathcal{O}}}(\mathrm{Gr}_G)$ is defined using the diagram

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} G_{\mathcal{K}} \times \mathrm{Gr}_G \xrightarrow{q} G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G, \tag{9}$$

where p and q are the natural projections, and m is given by $m([g, g']) = [gg']$. Let $G_{\mathcal{O}}$ act on Gr_G and $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathrm{Gr}_G$ from the left, and let $G_{\mathcal{O}} \times G_{\mathcal{O}}$ act on $G_{\mathcal{K}} \times \mathrm{Gr}_G$ by

$$(g_1, g_2) \cdot (g, [g']) = (g_1 g g_2^{-1}, [g_2 g']).$$

Then p is $G_{\mathcal{O}} \times G_{\mathcal{O}}$-equivariant, m is $G_{\mathcal{O}}$-equivariant, and q is equivariant with respect to the first $G_{\mathcal{O}}$-action on $G_{\mathcal{K}} \times \mathrm{Gr}_G$. The convolution product is given by

$$m_* \circ (q^*)^{-1} \circ p^* : \mathcal{A}_G \otimes \mathcal{A}_G \to \mathcal{A}_G. \tag{10}$$

Since we are dealing with infinite-dimensional spaces and groups, the Borel–Moore homology groups and the homomorphisms among them must be treated carefully. We briefly explain this, and refer to [BFN18] for more details.

For $\lambda \in \Lambda^+$, we write $G_{\mathcal{K}}^{\leq\lambda}$ for the preimage of $C_{\leq\lambda} \subset \mathrm{Gr}_G$ in $G_{\mathcal{K}}$, and let K_i denote the kernel of the natural homomorphism $G_{\mathcal{O}} \to G_i$.

Let $\lambda_1, \lambda_2 \in \Lambda^+$ and set $\lambda_3 = \lambda_1 + \lambda_2$. Choose positive integers $i \gg j \gg 0$ such that the actions of K_j on C_{λ_2}, and the actions of K_i on C_{λ_1}, C_{λ_3}, and $G_{\mathcal{K}}^{\leq\lambda_1}/K_j$ are trivial. The diagram (9) induces the diagram below (we use the same symbols for the induced maps):

$$C_{\leq\lambda_1} \times C_{\leq\lambda_2} \xleftarrow{p} (G_{\mathcal{K}}^{\leq\lambda_1}/K_j) \times C_{\leq\lambda_2} \xrightarrow{q} G_{\mathcal{K}}^{\leq\lambda_1} \times_{G_{\mathcal{O}}} C_{\leq\lambda_2} \xrightarrow{m} C_{\leq\lambda_3}. \tag{11}$$

The product (10) should be understood as the direct limit over $\lambda_1, \lambda_2 \in \Lambda^+$ of

$$m_* \circ (q^*)^{-1} \circ p^* : H_\bullet^{G_i}(C_{\leq\lambda_1}) \otimes H_\bullet^{G_j}(C_{\leq\lambda_2}) \longrightarrow H_\bullet^{G_i}(C_{\leq\lambda_3}),$$

where p^*, q^*, and m^* are homomorphisms induced by the diagram (11). Explicitly:

- The map
$$p^* : H_\bullet^{G_i}(C_{\leq\lambda_1}) \otimes H_\bullet^{G_j}(C_{\leq\lambda_2}) \longrightarrow H_\bullet^{G_i\times G_j}\left((G_{\mathcal{K}}^{\leq\lambda_1}/K_j) \times C_{\leq\lambda_2}\right)$$
is the pullback along p.
- The map
$$q^* : H_\bullet^{G_i}\left(G_{\mathcal{K}}^{\leq\lambda_1} \times_{G_{\mathcal{O}}} C_{\leq\lambda_2}\right) \longrightarrow H_\bullet^{G_i\times G_j}\left((G_{\mathcal{K}}^{\leq\lambda_1}/K_j) \times C_{\leq\lambda_2}\right)$$
is the pullback along q, with respect to the inclusion $G_i \cong G_i \times \{e\} \subset G_i \times G_j$.
- The map
$$m_* : H_\bullet^{G_i}\left(G_{\mathcal{K}}^{\leq\lambda_1} \times_{G_{\mathcal{O}}} C_{\leq\lambda_2}\right) \to H_\bullet^{G_i}(C_{\leq\lambda_3})$$
is the pushforward along m.

Note that q^* is an isomorphism because the G_j-action makes $(G_{\mathcal{K}}^{\leq\lambda_1}/K_j) \times C_{\leq\lambda_2}$ into a principal G_j-bundle over $G_{\mathcal{K}}^{\leq\lambda_1} \times_{G_{\mathcal{O}}} C_{\leq\lambda_2}$.

Quantizations. Let $\mathbb{C}^\times_\hbar$ be a one-dimensional complex torus, which scales the parameter t. It is called the group of loop rotations, and there are induced actions on $G_\mathcal{K}$, $G_\mathcal{O}$, and Gr_G. For $z \in \mathbb{C}^\times_\hbar$ and $g(t) \in G_\mathcal{K}$, the action is given by

$$z \cdot g(t) = g(zt),$$

and the other actions are defined similarly. See Section 3.2 and Appendix B for more discussion on the group of loop rotations.

Note that the diagrams (9) and (11) are equivariant with respect to the loop rotations. Let $\hbar \in H^2_{\mathbb{C}^\times_\hbar}(\mathrm{pt})$ be a generator, so that $H^\bullet_{\mathbb{C}^\times_\hbar}(\mathrm{pt}) \cong \mathbb{C}[\hbar]$. Then we define the *(pure gauge) quantized Coulomb branch algebra* as

$$\mathcal{A}^\hbar_G = H_\bullet^{G_\mathcal{O} \rtimes \mathbb{C}^\times_\hbar}(\mathrm{Gr}_G).$$

The same reasoning as above gives the product map

$$\mathcal{A}^\hbar_G \otimes_{\mathbb{C}[\hbar]} \mathcal{A}^\hbar_G \cong H_\bullet^{(G_\mathcal{O} \times G_\mathcal{O}) \rtimes \mathbb{C}^\times_\hbar}(\mathrm{Gr}_G \times \mathrm{Gr}_G) \longrightarrow H_\bullet^{G_\mathcal{O} \rtimes \mathbb{C}^\times_\hbar}(\mathrm{Gr}_G) \cong \mathcal{A}^\hbar_G.$$

As shown in [BFM05], this defines a noncommutative product structure on $\mathcal{A}^\hbar_G$, with $\hbar$ a central element. The commutator on $\mathcal{A}^\hbar_G$ induces a Poisson bracket on $\mathcal{A}_G = \mathcal{A}^\hbar_G / \hbar \mathcal{A}^\hbar_G$. It was shown in [BFM05] that this Poisson bracket defines a symplectic structure on the pure gauge Coulomb branch $\operatorname{Spec} \mathcal{A}_G$.

The BFN Coulomb branch. Let $\mathbf{N}$ be a complex representation of G, and we fix a decomposition

$$\mathbf{N} = \bigoplus_{i=1}^{N} \mathbb{C}_{\xi_i}$$

into one-dimensional T-representations, where each ξ_i is a character of T, and $\mathbb{C}_{\xi_i}$ denotes the corresponding one-dimensional T-representation. We define the following spaces:

$$\mathcal{T} = \mathcal{T}_{G,\mathbf{N}} = G_\mathcal{K} \times_{G_\mathcal{O}} \mathbf{N}_\mathcal{O},$$
$$\mathcal{R} = \mathcal{R}_{G,\mathbf{N}} = \{(g,s) \in \mathcal{T} : gs \in \mathbf{N}_\mathcal{O}\},$$
$$\mathcal{S} = \mathcal{S}_{G,\mathbf{N}} = \mathcal{T}_{G,\mathbf{N}} / \mathcal{R}_{G,\mathbf{N}}.$$

Suppose d is a positive integer, we write $\mathcal{T}^d$ for the vector bundle

$$\mathcal{T}^d = G_\mathcal{K} \times_{G_\mathcal{O}} (\mathbf{N}_\mathcal{O} / t^d \mathbf{N}_\mathcal{O}).$$

If $\lambda \in \Lambda$, we write $\mathcal{T}^d_{\leq\lambda}$ for the restriction of $\mathcal{T}^d$ to $C_{\leq\lambda}$. If, furthermore, $gt^d\mathbf{N}_\mathcal{O} \subset \mathbf{N}_\mathcal{O}$ for all $[g] \in C_{\leq\lambda}$, then we write $\mathcal{R}^d_{\leq\lambda}$ as the image of $\mathcal{R}_{\leq\lambda} = \mathcal{R}|_{C_{\leq\lambda}}$ in $\mathcal{T}^d_{\leq\lambda}$. Note that the fibrewise quotient

$$\mathcal{S}_{\leq\lambda} := \mathcal{T}^d_{\leq\lambda} / \mathcal{R}^d_{\leq\lambda}$$

is independent of the choice of such d. Moreover, $\mathcal{S}_\lambda := \mathcal{S}_{\leq\lambda}|_{C_\lambda}$ is a vector bundle, whose rank is denoted by d_λ. For $p \in \mathrm{Gr}_G$, we write $\mathcal{S}_p$ for the fibre of $\mathcal{S}_{\leq\lambda}$ at p, for $\lambda \in \Lambda$ satisfying $p \in C_{\leq\lambda}$. This definition is independent of the choice of λ.

For later use, we record the decomposition

$$\mathcal{S}_{t^\lambda} \cong \bigoplus_{\langle \xi_i, \lambda\rangle < 0} \mathbb{C}_{\xi_i}^{-\xi_i(\lambda)}, \tag{12}$$

as T-representations. In particular,

$$d_\lambda = -\sum_{\langle \xi_i, \lambda\rangle < 0} \langle \xi_i, \lambda \rangle. \tag{13}$$

For each $\lambda \in \Lambda^+$, we choose an integer d so that $\mathcal{R}^d_{\leq\lambda}$ is defined, and we use the same symbol z^*_λ to denote the corresponding Gysin pullbacks

$$H^K_\bullet(\mathcal{T}^d_{\leq\lambda}) \longrightarrow H^K_\bullet(C_{\leq\lambda}),$$

where K stands for one of $G_\mathcal{O}$, $G_\mathcal{O} \rtimes \mathbb{C}^\times_\hbar$, $T_\mathcal{O}$, or $T_\mathcal{O} \rtimes \mathbb{C}^\times_\hbar$. The following is a reformulation of the definition of the Coulomb branch in [BFN18], which follows from Lemma 5.11 in *loc. cit.*

Definition 1.1 ([BFN18])**.** The *Coulomb branch algebra* $\mathcal{A}_{G,\mathbf{N}}$ is the sum over $\lambda \in \Lambda^+$ of the images of $H_\bullet^{G_\mathcal{O}}(\mathcal{R}^d_{\leq\lambda})$ under the composition

$$H_\bullet^{G_\mathcal{O}}(\mathcal{R}^d_{\leq\lambda}) \longrightarrow H_\bullet^{G_\mathcal{O}}(\mathcal{T}^d_{\leq\lambda}) \xrightarrow{z_\lambda^*} H_\bullet^{G_\mathcal{O}}(C_{\leq\lambda}) \longrightarrow H_\bullet^{G_\mathcal{O}}(\mathrm{Gr}_G),$$

where the first and last maps are pushforwards along inclusions.

Similarly, the *quantized Coulomb branch algebra* $\mathcal{A}^\hbar_{G,\mathbf{N}}$ is defined as the sum over $\lambda \in \Lambda^+$ of the images of $H_\bullet^{G_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}(\mathcal{R}^d_{\leq\lambda})$ under the composition

$$H_\bullet^{G_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}(\mathcal{R}^d_{\leq\lambda}) \longrightarrow H_\bullet^{G_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}(\mathcal{T}^d_{\leq\lambda}) \xrightarrow{z_\lambda^*} H_\bullet^{G_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}(C_{\leq\lambda}) \longrightarrow H_\bullet^{G_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}(\mathrm{Gr}_G).$$

An alternative description. We now give a new and alternative description of the Coulomb branch algebra $\mathcal{A}_{G,\mathbf{N}}$ and quantized Coulomb branch algebra $\mathcal{A}^\hbar_{G,\mathbf{N}}$.

Lemma 1.2. *There exists an $\mathcal{I} \rtimes \mathbb{C}^\times_\hbar$-equivariant resolution of singularities*[4]

$$\rho_\lambda : \widetilde{C}_{\leq\lambda} \longrightarrow C_{\leq\lambda}$$

such that $\rho_\lambda^{-1}(\mathcal{S}|_{C_\lambda})$ extends to a (necessarily unique) $\mathcal{I} \rtimes \mathbb{C}^\times_\hbar$-equivariant quotient vector bundle $\widetilde{\mathcal{S}}_{\leq\lambda}$ of $\rho_\lambda^{-1}(\mathcal{T}^d_{\leq\lambda})$. Moreover, we may assume that ρ_λ and $\widetilde{\mathcal{S}}_{\leq\lambda}$ are $G_\mathcal{O} \rtimes \mathbb{C}^\times_\hbar$-equivariant if $\lambda \in \Lambda^+$.

Proof. Let $d > 0$ be a sufficiently large integer such that $\mathcal{S}_{\leq\lambda}$ is well-defined. Consider the Grassmannian bundle $\mathrm{Gr}(d_\lambda, \mathcal{T}^d_{\leq\lambda})$, which parameterizes rank d_λ quotients of $\mathcal{T}^d_{\leq\lambda}$. The vector bundle $\mathcal{S}_\lambda$ defines a section of $\mathrm{Gr}(d_\lambda, \mathcal{T}^d_{\leq\lambda})$ over C_λ. Let $\overline{C}_\lambda$ denote the closure of the image of this section. By construction, the section $C_\lambda \to \mathrm{Gr}(d_\lambda, \mathcal{T}^d_{\leq\lambda})$ extends uniquely to a $\mathcal{I} \rtimes \mathbb{C}^\times_\hbar$-equivairant (or $G_\mathcal{O} \rtimes \mathbb{C}^\times_\hbar$-equivairant when $\lambda \in \Lambda^+$) map

$$\overline{C}_\lambda \longrightarrow \mathrm{Gr}(d_\lambda, \mathcal{T}^d_{\leq\lambda}).$$

As a result, $\mathcal{S}_\lambda$ extends to a quotient bundle of the pullback of $\mathcal{T}^d_{\leq\lambda}$ to $\overline{C}_\lambda$. We then choose $\widetilde{C}_{\leq\lambda}$ to be an $\mathcal{I} \rtimes \mathbb{C}^\times_\hbar$-equivariant (or $G_\mathcal{O} \rtimes \mathbb{C}^\times_\hbar$-equivariant) resolution of $\overline{C}_\lambda$, which always exists (see Theorem 3.27 of [Kol07]). ☐

We now fix the $\mathcal{I} \rtimes \mathbb{C}^\times_\hbar$-equivariant resolution $\rho_\lambda : \widetilde{C}_{\leq\lambda} \to C_{\leq\lambda}$ and $\widetilde{\mathcal{S}}_{\leq\lambda}$ for each $\lambda \in \Lambda$, and ρ_λ is assumed to be $G_\mathcal{O} \rtimes \mathbb{C}^\times_\hbar$-equivariant when $\lambda \in \Lambda^+$.

Proposition 1.3. *Let $\lambda \in \Lambda$, and let d be a sufficiently large positive integer such that $\mathcal{R}^d_{\leq\lambda}$ is well-defined. Then the following subspaces of $H_\bullet^{T_\mathcal{O}}(\mathrm{Gr}_G)$ are equal:*

(1) *The sum of the images $e(\widetilde{\mathcal{S}}_{\leq\mu}) \cap H_\bullet^{T_\mathcal{O}}(\widetilde{C}_{\leq\mu})$ in $H_\bullet^{T_\mathcal{O}}(\mathrm{Gr}_G)$ under the pushforward*

$$H_\bullet^{T_\mathcal{O}}(\widetilde{C}_{\leq\mu}) \xrightarrow{\rho_{\mu *}} H_\bullet^{T_\mathcal{O}}(C_{\leq\mu}) \subset H_\bullet^{T_\mathcal{O}}(\mathrm{Gr}_G),$$

taken over all $\mu \leq \lambda$;

(2) *The image of $H_\bullet^{T_\mathcal{O}}(\mathcal{R}^d_{\leq\lambda})$ under the composition*

$$H_\bullet^{T_\mathcal{O}}(\mathcal{R}^d_{\leq\lambda}) \longrightarrow H_\bullet^{T_\mathcal{O}}(\mathcal{T}^d_{\leq\lambda}) \xrightarrow{z_\lambda^*} H_\bullet^{T_\mathcal{O}}(C_{\leq\lambda}) \subset H_\bullet^{T_\mathcal{O}}(\mathrm{Gr}_G);$$

(3) *The direct sum*

$$\bigoplus_{\mu\leq\lambda} H^\bullet_T(\mathrm{pt}) \cdot p_{\mu *}\left(e(\widetilde{\mathcal{S}}_{\leq\mu}) \cap [\widetilde{C}_{\leq\mu}]\right),$$

i.e., the free $H^\bullet_T(\mathrm{pt})$-submodule of $H_\bullet^{T_\mathcal{O}}(\mathrm{Gr}_G)$ with basis $\{p_{\mu *}\left(e(\widetilde{\mathcal{S}}_{\leq\mu}) \cap [\widetilde{C}_{\leq\mu}]\right)\}_{\mu\leq\lambda}$.

Moreover, each element $e(\widetilde{\mathcal{S}}_{\leq\mu}) \cap [\widetilde{C}_{\leq\mu}]$ is independent of the choice of resolution. The same equivalences hold when $T_\mathcal{O}$ is replaced by $T_\mathcal{O} \rtimes \mathbb{C}^\times_\hbar$. Furthermore, (1) *and* (2) *remain equivalent when $T_\mathcal{O}$ is replaced by $G_\mathcal{O}$ or $G_\mathcal{O} \rtimes \mathbb{C}^\times_\hbar$, and only dominant coweights are considered.*

[4] That is, $\widetilde{C}_{\leq\lambda}$ is non-singular and ρ_λ is a proper birational map.

Proof. We only prove the proposition for $T_{\mathcal{O}}$; the case for $T_{\mathcal{O}} \rtimes \mathbb{C}_{\hbar}^{\times}$ proceeds similarly. The cases for $G_{\mathcal{O}}$ and $G_{\mathcal{O}} \rtimes \mathbb{C}_{\hbar}^{\times}$ are obtained by taking Weyl invariants.

We first show that (1) $\subset$ (2). Consider the fibre diagram

$$\begin{tikzcd} \rho_\mu^{-1}\mathcal{T}^d_{\leq\mu} \arrow[r, "\rho'_\mu"] \arrow[d, "\widetilde{\mathrm{pr}}"'] & \mathcal{T}^d_{\leq\mu} \arrow[d, "\mathrm{pr}"] \\ \widetilde{C}_{\leq\mu} \arrow[r, "\rho_\mu"] & C_{\leq\mu} \end{tikzcd}.$$

Let $\widetilde{z_\lambda}^*$ be the Gysin map for the vector bundle $\rho_\mu^{-1}\mathcal{T}^d_{\leq\mu}$ over $\widetilde{C}_{\leq\mu}$, then for $\gamma \in H_\bullet^{T_\mathcal{O}}(\widetilde{C}_{\leq\mu})$, we have

$$\begin{aligned} \rho_{\mu*}\left(e(\widetilde{\mathcal{S}}_{\leq\mu}) \cap \gamma\right) &= \rho_{\mu*}\left(e(\widetilde{\mathcal{S}}_{\leq\mu}) \cap \widetilde{z_\lambda}^*\widetilde{\mathrm{pr}}^*\gamma\right) \\ &= \rho_{\mu*}\,\widetilde{z_\lambda}^*\left(\widetilde{\mathrm{pr}}^*e(\widetilde{\mathcal{S}}_{\leq\mu}) \cap \widetilde{\mathrm{pr}}^*\gamma\right) \\ &= z_\lambda^*\,\rho'_{\mu*}\left(\widetilde{\mathrm{pr}}^*e(\widetilde{\mathcal{S}}_{\leq\mu}) \cap \widetilde{\mathrm{pr}}^*\gamma\right). \end{aligned} \tag{14}$$

Let $\widetilde{\mathcal{R}}^d_{\leq\mu}$ denote the kernel of the projection $\rho_\mu^{-1}\mathcal{T}^d_{\leq\mu} \to \widetilde{\mathcal{S}}_{\leq\mu}$, and $\iota : \widetilde{\mathcal{R}}^d_{\leq\mu} \to \rho_\mu^{-1}\mathcal{T}^d_{\leq\mu}$ be the inclusion. Since $\widetilde{\mathcal{S}}_{\leq\mu} = \rho_\mu^{-1}\mathcal{T}^d_{\leq\mu}/\widetilde{\mathcal{R}}^d_{\leq\mu}$, we have:

$$\widetilde{\mathrm{pr}}^*e(\widetilde{\mathcal{S}}_{\leq\mu}) \cap H_\bullet^{T_\mathcal{O}}(\rho_\mu^{-1}\mathcal{T}^d_{\leq\mu}) \subset \iota_* H_\bullet^{T_\mathcal{O}}(\widetilde{\mathcal{R}}^d_{\leq\mu}).$$

As a result:

$$\rho_{\mu*}\big(e(\widetilde{\mathcal{S}}_{\leq\mu}) \cap \gamma\big) \in z_\lambda^*\,\rho'_{\mu*}\iota_* H_\bullet^{T_\mathcal{O}}(\widetilde{\mathcal{R}}^d_{\leq\mu}) \subset z_\lambda^*\,\iota_* H_\bullet^{T_\mathcal{O}}(\mathcal{R}^d_{\leq\mu}),$$

where the last inclusion follows from $\rho'_\mu(\widetilde{\mathcal{R}}^d_{\leq\mu}) \subset \mathcal{R}^d_{\leq\mu}$. This proves that (1) $\subset$ (2).

To prove (2) $\subset$ (3), note that there is an affine stratification

$$\mathcal{R}^d_{\leq\lambda} = \bigsqcup_{\mu\leq\lambda} \mathcal{R}^d_\mu, \tag{15}$$

where $\mathcal{R}^d_\mu = \mathcal{R}^d_{\leq\lambda}|_{C_\mu}$ and hence

$$H_\bullet^{T_\mathcal{O}}(\mathcal{R}^d_{\leq\lambda}) = \bigoplus_{\mu\leq\lambda} H_T^\bullet(\mathrm{pt}) \cdot [\mathcal{R}^d_{\leq\mu}].$$

Putting $\gamma = [\widetilde{C}_{\leq\mu}]$ in (14), and using $\widetilde{\mathcal{S}}_{\leq\mu} = \rho_\mu^{-1}\mathcal{T}^d_{\leq\mu}/\widetilde{\mathcal{R}}^d_{\leq\mu}$ again, we obtain :

$$\rho_{\mu*}\big(e(\widetilde{\mathcal{S}}_{\leq\mu}) \cap [\widetilde{C}_{\leq\mu}]\big) = z_\lambda^*\,\rho'_{\mu*}[\widetilde{\mathcal{R}}^d_{\leq\mu}] = z_\lambda^*[\mathcal{R}^d_{\leq\mu}],$$

where the last equality follows from the fact that $\widetilde{\mathcal{R}}^d_{\leq\mu}$ is mapped birationally to $\mathcal{R}^d_{\leq\mu}$ under ρ'_μ. This proves (2) $\subset$ (3) and also establishes that the class $\rho_{\mu*}\big(e(\widetilde{\mathcal{S}}_{\leq\mu}) \cap [\widetilde{C}_{\leq\mu}]\big)$ is independent of the choice of resolutions.

To see that the elements $\{\rho_{\mu*}\big(e(\widetilde{\mathcal{S}}_{\leq\mu}) \cap [\widetilde{C}_{\leq\mu}]\}_{\mu\leq\lambda}$ are linearly independent over $H_T^\bullet(\mathrm{pt})$, it suffices to show that both the Gysin map $H_\bullet^{T_\mathcal{O}}(\mathcal{T}^d_{\leq\lambda}) \to H_\bullet^{T_\mathcal{O}}(C_{\leq\lambda})$ and the pushforward $H_\bullet^{T_\mathcal{O}}(\mathcal{R}^d_{\leq\lambda}) \to H_\bullet^{T_\mathcal{O}}(\mathcal{T}^d_{\leq\lambda})$ are injective.

The former is clear because $\mathcal{T}^d_{\leq\lambda}$ is a vector bundle over $C_{\leq\lambda}$. For the latter, note that there is an affine stratification:

$$\mathcal{T}^d_{\leq\lambda} = \bigsqcup_{\mu\leq\lambda} \mathcal{T}^d_\mu,$$

which is compatible with (15). Therefore, it suffices to show that each pushforward $H_\bullet^{T_\mathcal{O}}(\mathcal{R}^d_\mu) \to H_\bullet^{T_\mathcal{O}}(\mathcal{T}^d_\mu)$ is injective. This is equivalent to verifying that the element $e(\mathcal{S}_\mu) = e(\mathcal{T}^d_\mu/\mathcal{R}^d_\mu) \in H_T^\bullet(C_\mu) \cong H_T^\bullet(\mathrm{pt})$ is nonzero. However, by Equation (12), the fibre of $\mathcal{S}_\mu$ at t^μ is a T-representation with no zero weights, hence $e(\mathcal{S}_\mu) \neq 0$.

Finally, it is clear that the subspace defined in (3) is contained in the subspace defined in (1). $\square$

We introduce the notation

$$e(\mathcal{S}) \cap H_\bullet^{G_\mathcal{O}}(\mathrm{Gr}_G) \coloneqq \sum_{\lambda\in\Lambda^+} \mathrm{Im}\left(e(\widetilde{\mathcal{S}}_{\leq\lambda}) \cap H_\bullet^{G_\mathcal{O}}(\widetilde{C}_{\leq\lambda}) \to H_\bullet^{G_\mathcal{O}}(\mathrm{Gr}_G)\right),$$

and similarly for the versions of $G_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times$, $T_\mathcal{O}$, or $T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times$-equivariant Borel–Moore homology.

Now, the following theorem follows immediately.

Theorem 1.4 (=Theorem 3)**.** *We have*

$$\mathcal{A}_{G,\mathbf{N}} = e(\mathcal{S}_{G,\mathbf{N}}) \cap H^{G_\mathcal{O}}_\bullet(\mathrm{Gr}_G) = \left(e(\mathcal{S}_{G,\mathbf{N}}) \cap H^{T_\mathcal{O}}_\bullet(\mathrm{Gr}_G)\right)^W,$$

and similarly

$$\mathcal{A}^\hbar_{G,\mathbf{N}} = e(\mathcal{S}_{G,\mathbf{N}}) \cap H^{G_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}_\bullet(\mathrm{Gr}_G) = \left(e(\mathcal{S}_{G,\mathbf{N}}) \cap H^{T_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}_\bullet(\mathrm{Gr}_G)\right)^W.$$

The following proposition follows from Proposition 1.3 and [BFN18]. An independent proof is given in Appendix A.

Proposition 1.5. *The subspaces $e(\mathcal{S}) \cap H^{G_\mathcal{O}}_\bullet(\mathrm{Gr}_G)$ and $e(\mathcal{S}) \cap H^{G_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}_\bullet(\mathrm{Gr}_G)$ are stable under the convolution product.*

We remark that localization

$$\left(e(\mathcal{S}_\mathbf{N}) \cap H^{T_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}_\bullet(\mathrm{Gr}_G)\right)_{\mathrm{loc}} := \mathrm{Frac}H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathrm{pt}) \otimes_{H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathrm{pt})} \left(e(\mathcal{S}_\mathbf{N}) \cap H^{T_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}_\bullet(\mathrm{Gr}_G)\right) \tag{16}$$

is independent of $\mathbf{N}$ ([BFN18]).

2. Twisting maps and twisted linearity

2.1. **Twisting maps.** Let X be a smooth quasiprojective variety with a G-action. Consider the space $G_\mathcal{K} \times_{G_\mathcal{O}} X$, where $G_\mathcal{O}$ acts on X with via the homomorphism $\mathrm{ev}_{t=0} : G_\mathcal{O} \to G$. It is equipped with a left $G_\mathcal{O}$-action via $h\cdot[g,x] = [hg,x]$. There is a $G_\mathcal{O}$-equivariant morphism $G_\mathcal{K} \times_{G_\mathcal{O}} X \to \mathrm{Gr}_G$ by sending $[g,x] \mapsto [g] \in \mathrm{Gr}_G$. We will consider the $G_\mathcal{O}$-equivariant Borel–Moore homology of $G_\mathcal{K} \times_{G_\mathcal{O}} X$, defined using finite-dimensional approximation. For each dominant coweight $\lambda \in \Lambda^+$, consider the $G_\mathcal{O}$-invariant closed subset

$$G^{\le\lambda}_\mathcal{K} \times_{G_\mathcal{O}} X \subset G_\mathcal{K} \times_{G_\mathcal{O}} X. \tag{17}$$

We define

$$H^{G_\mathcal{O}}_\bullet(G_\mathcal{K} \times_{G_\mathcal{O}} X) := \varinjlim_{\lambda\in\Lambda^+} H^{G_\mathcal{O}}_\bullet(G^{\le\lambda}_\mathcal{K} \times_{G_\mathcal{O}} X).$$

Convolution. We have a correspondence similar to the convolution diagram (9) (we will use the notations $p_{G,X}$ and $q_{G,X}$ if we need to specify G),

$$\mathrm{Gr}_G \times X \xleftarrow{p_X} G_\mathcal{K} \times X \xrightarrow{q_X} G_\mathcal{K} \times_{G_\mathcal{O}} X$$

which induces the following map,

$$(q_X^*)^{-1} \circ p_X^* : H^{G_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}_\bullet(\mathrm{Gr}_G) \otimes_{\mathbb{C}[\hbar]} H^{G_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}_\bullet(X) \longrightarrow H^{G_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}_\bullet(G_\mathcal{K} \times_{G_\mathcal{O}} X). \tag{18}$$

As in the discussion following (10), the above should be understood using finite-dimensional approximations. For $\lambda \in \Lambda^+$, let $i \gg j > 0$ be positive integers such that the actions of K_i on $C_{\le\lambda}$ and $G^{\le\lambda}_\mathcal{K}/K_j$ are trivial. Then there is a corresponding diagram (we use the same notations for the maps in the finite-dimensional approximation):

$$C_{\le\lambda} \times X \xleftarrow{p_X} (G^{\le\lambda}_\mathcal{K}/K_j) \times X \xrightarrow{q_X} G^{\le\lambda}_\mathcal{K} \times_{G_\mathcal{O}} X \tag{19}$$

The map (18) should be understood as the direct limit over $\lambda \in \Lambda^+$ of

$$(q_X^*)^{-1} \circ p_X^* : H^{G_i\rtimes\mathbb{C}^\times_\hbar}_\bullet(C_{\le\lambda}) \otimes_{\mathbb{C}[\hbar]} H^{G_j\rtimes\mathbb{C}^\times_\hbar}_\bullet(X) \to H^{G_i\rtimes\mathbb{C}^\times_\hbar}_\bullet(G^{\le\lambda}_\mathcal{K} \times_{G_\mathcal{O}} X),$$

where p_X^* and q_X^* are homomorphisms induced by the diagram (19). Explicitly:

- The map p_X^* is the pullback along p_X:
$$p_X^* : H^{G_i\rtimes\mathbb{C}^\times_\hbar}_\bullet(C_{\le\lambda}) \otimes_{\mathbb{C}[\hbar]} H^{G_j\rtimes\mathbb{C}^\times_\hbar}_\bullet(X) \longrightarrow H^{(G_i\times G_j)\rtimes\mathbb{C}^\times_\hbar}_\bullet\left((G^{\le\lambda}_\mathcal{K}/K_j) \times X\right).$$
- The map q_X^* is the pullback along q_X, with respect to the inclusion $G_i \cong G_i \times \{e\} \subset G_i \times G_j$:
$$q_X^* : H^{G_i\rtimes\mathbb{C}^\times_\hbar}_\bullet\left(G^{\le\lambda}_\mathcal{K} \times_{G_\mathcal{O}} X\right) \longrightarrow H^{(G_i\times G_j)\rtimes\mathbb{C}^\times_\hbar}_\bullet\left((G^{\le\lambda}_\mathcal{K}/K_j) \times X\right).$$

Note q_X^* is an isomorphism, and that this construction is independent of i and j. Similarly, there is also $T_\mathcal{O} \rtimes \mathbb{C}^\times_\hbar$-equivariant version of (18), as in the following definition.

Definition 2.1. The *twisting map* is defined to be the composition $(q_X^*)^{-1} \circ p_X^*$, namely,

$$\mathrm{tw}_G : H_\bullet^{G_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G) \otimes_{\mathbb{C}[\hbar]} H_\bullet^{G_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(X) \longrightarrow H_\bullet^{G_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(G_\mathcal{K} \times_{G_\mathcal{O}} X),$$

as well as its $T_\mathcal{O}$-equivariant counterpart:

$$\mathrm{tw}_G : H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G) \otimes_{\mathbb{C}[\hbar]} H_\bullet^{G_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(X) \longrightarrow H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(G_\mathcal{K} \times_{G_\mathcal{O}} X).$$

Note that we use the same notation tw_G to denote both homomorphisms. This abuse of notation should cause no ambiguity, as the first homomorphism agrees with the restriction of the second to the subspace of W-invariants.

The following proposition is obvious if one notice that both p_X and q_X are quotient maps for a free $G_\mathcal{O}$-action.

Proposition 2.2. *There is a commutative diagram*

$$\begin{tikzcd}
H_\bullet^{G_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G) \otimes_{\mathbb{C}[\hbar]} H_\bullet^{G_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(X) \arrow[r, "\mathrm{tw}_G"] \arrow[d, "\cong"'] & H_\bullet^{G_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(G_\mathcal{K} \times_{G_\mathcal{O}} X) \arrow[d, "\cong"] \\
H_\bullet^{(G_\mathcal{O} \times G_\mathcal{O} \times G_\mathcal{O}) \rtimes \mathbb{C}_\hbar^\times}(G_\mathcal{K} \times X) \arrow[r] & H_\bullet^{(G_\mathcal{O} \times G_\mathcal{O}) \rtimes \mathbb{C}_\hbar^\times}(G_\mathcal{K} \times X)
\end{tikzcd},$$

where the action of $G_\mathcal{O} \times G_\mathcal{O} \times G_\mathcal{O}$ on $G_\mathcal{K} \times X$ is defined as follows: the first two factors act on $G_\mathcal{K}$ from the left and right respectively, while the third factor acts on X. The bottom horizontal map is induced by restricting the equivariance to the subgroup

$$G_\mathcal{O} \times \Delta_{G_\mathcal{O}} \subset G_\mathcal{O} \times G_\mathcal{O} \times G_\mathcal{O}.$$

In other words, one may regard tw_G as identifying the equivariance on X with the equivariance on Gr_G coming from $\mathrm{Gr}_G = G_\mathcal{K}/G_\mathcal{O}$. One can also understand this as saying there is the following fibre product identity:

$$[G_\mathcal{O} \backslash \mathrm{Gr}_G] \times_{[\mathrm{pt}/G_\mathcal{O}]} [G_\mathcal{O} \backslash X] \cong [G_\mathcal{O} \backslash (G_\mathcal{K} \times_{G_\mathcal{O}} X)].$$

Proposition 2.3. *There is a commutative diagram.*

$$\begin{tikzcd}
H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G) \otimes_{\mathbb{C}[\hbar]} H_\bullet^{G_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(X) \arrow[r, "\mathrm{tw}_G"] & H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(G_\mathcal{K} \times_{G_\mathcal{O}} X) \\
H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_T) \otimes_{\mathbb{C}[\hbar]} H_\bullet^{G_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(X) \arrow[u, phantom, "\cup"] \arrow[r] \arrow[d] & H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(T_\mathcal{K} \times_{T_\mathcal{O}} X) \arrow[u] \arrow[d, equal] \\
H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_T) \otimes_{\mathbb{C}[\hbar]} H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(X) \arrow[r, "\mathrm{tw}_T"] & H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(T_\mathcal{K} \times_{T_\mathcal{O}} X)
\end{tikzcd} \tag{20}$$

such that all the maps in the diagram are W-equivariant.

Proof. In the proof below, we omit the finite-dimensional approximations for clarity, as the necessary constructions should now be clear.

Consider the diagram

$$\begin{tikzcd}
\mathrm{Gr}_G \times X & G_\mathcal{K} \times X \arrow[l, "p_{G,X}"'] \arrow[r, "q_{G,X}"] & G_\mathcal{K} \times_{G_\mathcal{O}} X \\
\mathrm{Gr}_T \times X \arrow[u, "\iota''"] \arrow[d, equal] & T_\mathcal{K} G_\mathcal{O} \times X \arrow[l, "p'_X"'] \arrow[r, "q'_X"] \arrow[u, "\iota'"] & T_\mathcal{K} G_\mathcal{O} \times_{G_\mathcal{O}} X \arrow[u, "\iota"] \\
\mathrm{Gr}_T \times X & T_\mathcal{K} \times X \arrow[l, "p_{T,X}"'] \arrow[r, "q_{T,X}"] \arrow[u] & T_\mathcal{K} \times_{T_\mathcal{O}} X \arrow[u, "\cong"]
\end{tikzcd}.$$

Here, the second row is the base change of the first row via the inclusion $\mathrm{Gr}_T \to \mathrm{Gr}_G$. Since pushforward commutes with smooth pullback in a fibre diagram, the restriction of tw_G to $H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_T) \otimes_{\mathbb{C}[\hbar]} H_\bullet^{G_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(X)$ is equal to

$$(q_{G,X}^*)^{-1} \circ p_{G,X}^* \circ \iota''_* = (q_{G,X}^*)^{-1} \circ \iota'_* \circ p'^*_X = \iota_* \circ (q'^*_X)^{-1} \circ p'^*_X.$$

This proves the commutativity of the upper square in (20). For the lower square, note that we have the following commutative diagram:

$$\begin{array}{ccc} H_\bullet^{(T_\mathcal{O}\times G_\mathcal{O}\times G_\mathcal{O})\rtimes\mathbb{C}_\hbar^\times}(T_\mathcal{K}G_\mathcal{O}\times X) & \longrightarrow & H_\bullet^{(T_\mathcal{O}\times\Delta_{G_\mathcal{O}})\rtimes\mathbb{C}_\hbar^\times}(T_\mathcal{K}G_\mathcal{O}\times X) \\ \downarrow & & \downarrow \\ H_\bullet^{(T_\mathcal{O}\times T_\mathcal{O}\times T_\mathcal{O})\rtimes\mathbb{C}_\hbar^\times}(T_\mathcal{K}\times X) & \longrightarrow & H_\bullet^{(T_\mathcal{O}\times\Delta_{T_\mathcal{O}})\rtimes\mathbb{C}_\hbar^\times}(T_\mathcal{K}\times X) \end{array}.$$

Here, the horizontal maps correspond to restrictions of equivariant parameters. The left vertical map is the composition

$$\begin{aligned} H_\bullet^{(T_\mathcal{O}\times G_\mathcal{O}\times G_\mathcal{O})\rtimes\mathbb{C}_\hbar^\times}(T_\mathcal{K}G_\mathcal{O}\times X) \to H_\bullet^{(T_\mathcal{O}\times G_\mathcal{O}\times T_\mathcal{O})\rtimes\mathbb{C}_\hbar^\times}(T_\mathcal{K}G_\mathcal{O}\times X) \\ \cong H_\bullet^{(T_\mathcal{O}\times T_\mathcal{O}\times T_\mathcal{O})\rtimes\mathbb{C}_\hbar^\times}(T_\mathcal{K}\times X), \end{aligned}$$

and similarly for the right vertical map. In view of Proposition 2.2, this proves the commutativity of the lower square in (20). $\square$

For the purpose of defining the shift operators (see Definition 4.8), we require the following variant of the twisting map. Let λ be a coweight, and define the fibre product

$$\widetilde{G}_\mathcal{K}^{\leq\lambda} := G_\mathcal{K}\times_{\mathrm{Gr}_G}\widetilde{C}_{\leq\lambda}. \tag{21}$$

Using the same construction as above, we obtain a map

$$\mathrm{tw}_{\leq\lambda} : H_\bullet^{G_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(\widetilde{C}_{\leq\lambda})\otimes_{\mathbb{C}[\hbar]} H_\bullet^{G_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(X) \longrightarrow H_\bullet^{G_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}\big(\widetilde{G}_\mathcal{K}^{\leq\lambda}\times_{G_\mathcal{O}} X\big).$$

By abuse of notation, we also denote by the same symbol the following map, obtained from $\mathrm{tw}_{\leq\lambda}$ by applying Poincaré duality twice.

$$\mathrm{tw}_{\leq\lambda} : H_\bullet^{G_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(\widetilde{C}_{\leq\lambda})\otimes_{\mathbb{C}[\hbar]} H^\bullet_{G_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(X) \longrightarrow H^\bullet_{G_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}\big(\widetilde{G}_\mathcal{K}^{\leq\lambda}\times_{G_\mathcal{O}} X\big). \tag{22}$$

Similarly, one can define the $T_\mathcal{O}$-equivariant version (and its Poincaré dual version):

$$\mathrm{tw}_{\leq\lambda} : H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(\widetilde{C}_{\leq\lambda})\otimes_{\mathbb{C}[\hbar]} H_\bullet^{G_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(X) \longrightarrow H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}\big(\widetilde{G}_\mathcal{K}^{\leq\lambda}\times_{G_\mathcal{O}} X\big).$$

For later use, we record the following lemmas, whose proofs follow immediately from the compatibility of smooth pullbacks with pushforwards, and cap products.

Lemma 2.4. *The map tw_G is compatible with $\mathrm{tw}_{\leq\lambda}$ in the sense that the following diagram commutes:*

$$\begin{array}{ccc} H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(\widetilde{C}_{\leq\lambda})\otimes_{\mathbb{C}[\hbar]} H_\bullet^{G_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(X) & \xrightarrow{\mathrm{tw}_{\leq\lambda}} & H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}\big(\widetilde{G}_\mathcal{K}^{\leq\lambda}\times_{G_\mathcal{O}} X\big) \\ \downarrow & & \downarrow \\ H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(\mathrm{Gr}_G)\otimes_{\mathbb{C}[\hbar]} H_\bullet^{G_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(X) & \xrightarrow{\mathrm{tw}_G} & H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(G_\mathcal{K}\times_{G_\mathcal{O}} X) \end{array}$$

Lemma 2.5. *Suppose we have another resolution $\widetilde{C}'_{\leq\lambda}\to C_{\leq\lambda}$ which factors through $\widetilde{C}_{\leq\lambda}\to C_{\leq\lambda}$. Then the following diagram commutes:*

$$\begin{array}{ccc} H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(\widetilde{C}'_{\leq\lambda})\otimes_{\mathbb{C}[\hbar]} H^\bullet_{G_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(X) & \xrightarrow{\mathrm{tw}'_{\leq\lambda}} & H^\bullet_{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}\big(\widetilde{G}'^{\leq\lambda}_\mathcal{K}\times_{G_\mathcal{O}} X\big) \\ \downarrow & & \downarrow \\ H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(\widetilde{C}_{\leq\lambda})\otimes_{\mathbb{C}[\hbar]} H^\bullet_{G_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(X) & \xrightarrow{\mathrm{tw}_{\leq\lambda}} & H^\bullet_{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}\big(\widetilde{G}_\mathcal{K}^{\leq\lambda}\times_{G_\mathcal{O}} X\big) \end{array}$$

Here, $\widetilde{G}'^{\leq\lambda}_\mathcal{K}$ and $\mathrm{tw}'_{\leq\lambda}$ are defined analogously. The vertical maps are pushforwards.

Lemma 2.6. *Suppose $\Gamma\in H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(\widetilde{C}_{\leq\lambda})$ and $\alpha\in H_\bullet^{G_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(X)$, we have*

$$\mathrm{tw}_{\leq\lambda}\Big(e(\widetilde{S}_{\leq\lambda})\cap\Gamma\otimes\alpha\Big) = e(\widetilde{S}_{\leq\lambda})\cup\mathrm{tw}_{\leq\lambda}(\Gamma\otimes\alpha),$$

where the bundle $\widetilde{S}_{\leq\lambda}$ on the right-hand side is understood as the pullback along the projection

$$\widetilde{G}_{\mathcal{K}}^{\leq\lambda} \times_{G_{\mathcal{O}}} X \longrightarrow \widetilde{C}_{\leq\lambda}.$$

2.2. **Twisted linearity.** It is clear that $\mathrm{tw}_G(-\otimes -)$ and $\mathrm{tw}_T(-\otimes -)$ are $H^\bullet_{T_{\mathcal{O}}\rtimes\mathbb{C}^\times_\hbar}(\mathrm{pt})$-linear in the first argument. However, linearity in the second argument is more subtle, which we will explain now.

There is a morphism $u : [G_{\mathcal{O}}\backslash\mathrm{Gr}_G] = [G_{\mathcal{O}}\backslash G_{\mathcal{K}}/G_{\mathcal{O}}] \to [\mathrm{pt}/G_{\mathcal{O}}] \to [\mathrm{pt}/G]$, where the second map is induced by $\mathrm{ev}_{t=0} : G_{\mathcal{O}} \to G$. The pullback map on cohomology

$$u^* : H^\bullet_G(\mathrm{pt}) \longrightarrow H^\bullet_{G_{\mathcal{O}}\rtimes\mathbb{C}^\times_\hbar}(\mathrm{Gr}_G)$$

defines a second $H^\bullet_G(\mathrm{pt})$-module structure on $H_\bullet^{G_{\mathcal{O}}\rtimes\mathbb{C}^\times_\hbar}(\mathrm{Gr}_G)$. More specifically, the map u^* sends a characteristic class $c_k(V) \in H^\bullet_G(\mathrm{pt})$ to the cohomology class

$$u^*(c_k(V)) = c_k(G_{\mathcal{K}} \times_{G_{\mathcal{O}}} V) \in H^\bullet_{G_{\mathcal{O}}\times\mathbb{C}^\times_\hbar}(\mathrm{Gr}_G).$$

Here, V is a G-representation, and $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} V$ is the associated vector bundle on Gr_G. Then, the second module structure is given by

$$\begin{aligned} H^\bullet_G(\mathrm{pt}) \otimes H_\bullet^{G_{\mathcal{O}}\rtimes\mathbb{C}^\times_\hbar}(\mathrm{Gr}_G) &\longrightarrow H_\bullet^{G_{\mathcal{O}}\rtimes\mathbb{C}^\times_\hbar}(\mathrm{Gr}_G), \\ P \otimes \Gamma &\longmapsto u^*(P) \cap \Gamma. \end{aligned} \tag{23}$$

By viewing $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} X$ as the quotient of $G_{\mathcal{K}} \times X$ by the action of $G_{\mathcal{O}}$, one similarly obtains the homomorphism

$$u^* : H^\bullet_G(\mathrm{pt}) \longrightarrow H^\bullet_{G_{\mathcal{O}}\rtimes\mathbb{C}^\times_\hbar}(G_{\mathcal{K}} \times_{G_{\mathcal{O}}} X)$$

by considering the map $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} X \to [\mathrm{pt}/G_{\mathcal{O}}]$.

Now we can explain the twisted linearity of tw_G in the second factor.

Proposition 2.7. *Let $P \in H^\bullet_{G\times\mathbb{C}^\times_\hbar}(\mathrm{pt})$. Then*

$$\mathrm{tw}_G(\Gamma \otimes (P \cap \alpha)) = \mathrm{tw}_G((u^*P \cap \Gamma) \otimes \alpha) = u^*P \cap \mathrm{tw}_G(\Gamma \otimes \alpha).$$

Proof. By Proposition 2.2, we may understand tw_G as the map

$$H_\bullet^{(G_{\mathcal{O}}\times G_{\mathcal{O}}\times G_{\mathcal{O}})\rtimes\mathbb{C}^\times_\hbar}(G_{\mathcal{K}} \times X) \longrightarrow H_\bullet^{(G_{\mathcal{O}}\times G_{\mathcal{O}})\rtimes\mathbb{C}^\times_\hbar}(G_{\mathcal{K}} \times X)$$

obtained by identifying the second and third copies of $G_{\mathcal{O}}$. The proposition follows by observing that: 1) the second $G_{\mathcal{O}}$ corresponds to the second $H^\bullet_G(\mathrm{pt})$-module action on $H_\bullet^{G_{\mathcal{O}}\rtimes\mathbb{C}^\times_\hbar}(\mathrm{Gr}_G)$ via u^*; 2) The third $G_{\mathcal{O}}$ corresponds to the standard $H^\bullet_G(\mathrm{pt})$-module structure on $H_\bullet^{G\times\mathbb{C}^\times_\hbar}(X)$. ☐

Remark 2.8. There is a parallel story for $H^\bullet_T(G/T)$ replacing $H_\bullet^{G_{\mathcal{O}}}(\mathrm{Gr}_G)$, as explained in [Pet97]. Specifically,

$$H^\bullet_T(G/T) \cong H^\bullet_T(\mathrm{pt}) \otimes_{H^\bullet_G(\mathrm{pt})} H^\bullet_T(\mathrm{pt})$$

carries two distinct $H^\bullet_T(\mathrm{pt})$-module structures. Moreover, for any T-space X, there is a natural isomorphism

$$H^\bullet_T(G/T) \otimes_{H^\bullet_T(\mathrm{pt})} H^\bullet_T(X) \cong H^\bullet_T(G \times_T X)$$

which mirrors the twisting map construction introduced in this section.

Comparison with the previous construction. Let $G = T$ be a torus. We now compare our version of the twisted map with that of Iritani [Iri17]. For $\lambda \in \Lambda$, let X^λ denote the fibre of $T_{\mathcal{K}} \times_{T_{\mathcal{O}}} X$ over the point $[t^\lambda] \in \mathrm{Gr}_T$. The variety X^λ is isomorphic to X as a T-variety, but carries the loop rotation action:

$$z_\hbar \cdot x = \lambda(z_\hbar) \cdot x, \quad \text{for } z_\hbar \in \mathbb{C}^\times_\hbar,\ x \in X^\lambda.$$

Consider the diagram:

$$\begin{tikzcd} & T \times X \arrow[dl, "a"'] \arrow[dr, "b"] & \\ X & & X^\lambda \end{tikzcd}$$

where the map a is projection to the second factor, and b is the action map of for the T-action on X (not X^λ).

Taking $j = 1$ in (19), one observes that the map

$$\mathrm{tw}_T([t^\lambda] \otimes -)\colon H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X) \longrightarrow H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X^\lambda)$$

is given by the composition

$$H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X) \xrightarrow{a^*} H^\bullet_{T\times T\times\mathbb{C}^\times_\hbar}(T\times X) \xrightarrow{(b^*)^{-1}} H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X^\lambda).$$

Here, $T\times T\times\mathbb{C}^\times_\hbar$ acts on $T\times X$ by the formula

$$(z_1, z_2, z_\hbar)\cdot(z, x) = (\lambda(z_\hbar) z_1 z z_2^{-1}, z_2 x).$$

To compute a^* and b^* explicitly, let $E = ET$ and $E' = E\mathbb{C}^\times_\hbar$ be contractible spaces equipped with free right actions by T and $\mathbb{C}^\times_\hbar$, respectively. Also let $E^\lambda = E$ as a T-space, but $\mathbb{C}^\times_\hbar$ acts on E^λ through λ. Then a^* and b^* correspond to pullbacks along a' and b' in the diagram:

$$\begin{tikzcd}[column sep=small]
& (E^\lambda\times E\times E')\times_{T\times T\times\mathbb{C}^\times_\hbar}(T\times X) \arrow[dl, "a'"'] \arrow[dr, "b'"] & \\
(E\times E')\times_{T\times\mathbb{C}^\times_\hbar} X & & (E^\lambda\times E')\times_{T\times\mathbb{C}^\times_\hbar} X^\lambda
\end{tikzcd}$$

where

$$a'(e_1, e_2, e', z, x) = (e_2, e', x), \qquad b'(e_1, e_2, e', z, x) = (e_1, e', z\cdot x),$$

with $z\cdot x$ denoting the original T-action on X.

The map a' is a weak homotopy equivalence with inverse

$$c' : (E\times E')\times_{T\times\mathbb{C}^\times_\hbar} X \longrightarrow (E^\lambda\times E\times E')\times_{T\times T\times\mathbb{C}^\times_\hbar}(T\times X)$$
$$c'(e, e', x) = (e, e, e', 1, x).$$

One can check that c' is indeed well-defined, and that $b'\circ c' = \mathrm{id}$.

One can summarize the above as follows. The identity map $(E\times E')\times X \to (E^\lambda\times E')\times X^\lambda$ is equivariant with respect to the group automorphism

$$T\times\mathbb{C}^\times_\hbar \longrightarrow T\times\mathbb{C}^\times_\hbar, \quad (z, z_\hbar)\mapsto \big(z\lambda(z_\hbar)^{-1}, z_\hbar\big), \tag{24}$$

and the induced map on $T\times\mathbb{C}^\times_\hbar$-equivariant cohomology is naturally identified with $\mathrm{tw}_T([t^\lambda]\otimes -)$. We record this in the following proposition.

Proposition 2.9. *Let $\Phi_\lambda\colon H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X) \longrightarrow H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X^\lambda)$ be the (twisted) homomorphism induced by the identity map $\mathrm{id}\colon X\to X^\lambda$, which is equivariant with respect to the group automorphism (24). Then Φ_λ agrees with the twisting map $\mathrm{tw}_T([t^\lambda]\otimes -)$. In particular, for $P(a,\hbar)\in H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathrm{pt})$ and $\alpha\in H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X)$, we have*

$$\mathrm{tw}_T([t^\lambda]\otimes P(a,\hbar)\alpha) = P(a+\lambda(\hbar),\hbar)\,\mathrm{tw}_T([t^\lambda]\otimes\alpha).$$

Remark 2.10. Φ_λ is the same as the twisted homorphism defined in [Iri17, Section 3.1].

3. Shift operators I: preparation

3.1. **Quantum cohomology.**

Assumptions on the space. Let X be a smooth, semiprojective complex variety with an algebraic G-action. By *semiprojective*, we mean that X is quasiprojective and that the affinization map

$$\mathrm{aff} : X \longrightarrow X^{\mathrm{aff}} := \mathrm{Spec}\, H^0(X, \mathcal{O}_X)$$

is projective. We assume that X carries an algebraic G-action commuting with an auxiliary $\mathbb{C}^\times_{\mathrm{dil}}$-action satisfying:

(1) all $\mathbb{C}^\times_{\mathrm{dil}}$-weights on $H^0(X,\mathcal{O}_X)$ are non-positive; and
(2) the $\mathbb{C}^\times_{\mathrm{dil}}$-invariants satisfy $H^0(X,\mathcal{O}_X)^{\mathbb{C}^\times_{\mathrm{dil}}} = \mathbb{C}$.

These conditions imply in particular that $X^{\mathbb{C}^\times_{\mathrm{dil}}}$ is compact. We refer to $\mathbb{C}^\times_{\mathrm{dil}}$ as the group of *dilations* or the *conical action*, not to be confused with the group of *loop rotations*, $\mathbb{C}^\times_\hbar$, which acts trivially on X.

These assumptions ensure that X has nice cohomological properties, including Proposition 3.1 below; see [Iri17]. Note that in [Iri17], it is assumed that the conical group $\mathbb{C}^\times_{\mathrm{dil}}$ is a subgroup of G, which is required for defining shift operators in their setting. However, we do not need this assumption in the Coulomb branch setting, as will be explained in Section 4.

Proposition 3.1. *The G-action on X is equivariantly formal.*

Proof. See [Iri17, Proposition 2.1]. In *loc. cit.*, only abelian G were considered, but the same proof applies to any reductive group G. □

Equivariant quantum cohomology. We fix a graded $\mathbb{C}$-basis $\{a_i\}_{i\in I}$ of $H^\bullet_G(\mathrm{pt})$ and a graded $H^\bullet_G(\mathrm{pt})$-basis $\{\phi_j\}_{j=0}^N$ of $H^\bullet_G(X)$. We denote $\phi_{i,j} := a_i\phi_j$. Let $\{\tau^{i,j}\}$ be coordinates on $H^\bullet_G(X)$ dual to the basis $\{\phi_{i,j}\}$, and set $\tau = \sum_{i,j}\tau^{i,j}\phi_{i,j}$. We declare the degrees of the coordinates $\tau^{i,j}$ to be

$$\deg\tau^{i,j} = 2 - \deg(\phi_{i,j}).$$

We write $\mathbb{C}[[\tau]] = \mathbb{C}[[\tau^{i,j}]]$, where the odd variables (i.e., $\tau^{i,j}$ for $\deg\phi_{i,j}$ odd) anti-commute. In other words, this is the tensor product of the formal power series ring in the even variables with the exterior algebra in the odd variables.

Remark 3.2.

(1) If $K \subset G$ is a reductive subgroup, then the restriction of $\{\phi_j\}_{j=0}^N$ gives a graded $H^\bullet_K(\mathrm{pt})$-basis of $H^\bullet_K(X)$. We always assume that such a compatible choice of basis is made.
(2) We may use the notation τ_G to indicate the dependence on G. In this paper, we always assume $\tau = \tau_G$ unless otherwise specified.

Remark 3.3. The introduction of the variables $\tau^{i,j}$ is in order to define the quantum connection $\nabla_{\hbar\partial_\hbar}$ (26) and shift operators (38). More specifically, this is due to the grading operator μ_X (27) and the assignment $\tau\mapsto\hat{\tau}$ (30) not being $H^\bullet_G(\mathrm{pt})$-linear. For the sole purpose of defining equivariant quantum cohomology, it is possible to take $H^\bullet_G(\mathrm{pt})$-valued coordinates instead of the $\mathbb{C}$-valued coordinates $\tau^{i,j}$.

Let $K\subset G$ be a reductive subgroup, we write

$$\mathbb{C}[q_K] = \mathbb{C}[H_2^{\mathrm{ord},K}(X;\mathbb{Z})]$$

for the group algebra of the abelian group $H_2^{\mathrm{ord},K}(X;\mathbb{Z})$. The ring $\mathbb{C}[q_K]$ is equipped with a grading determined by

$$\deg q^\beta = 2\langle c_1^K(X),\beta\rangle.$$

Let $\mathrm{Eff}(X)\subset H_2^{\mathrm{ord}}(X;\mathbb{Z})$ be the subset of effective curve classes. We denote $\iota_*: H_2^{\mathrm{ord}}(X;\mathbb{Z})\to H_2^{\mathrm{ord},K}(X;\mathbb{Z})$ the natural map. We define $H^\bullet_G(X)[[q_K,\tau]]$ to be the graded completion of $H^\bullet_G(X)[[\tau]]\otimes_{\mathbb{C}}\mathbb{C}[q_K]$ along the direction $\mathrm{Eff}(X)$[5]. Concretely, an element of $H^\bullet_G(X)[[q_K,\tau]]$ is a formal sum

$$\sum_{\beta,m} q^\beta c_{\beta,m}\prod_{\substack{i\in I\\ 0\le j\le N}}(\tau^{i,j})^{m_{i,j}}$$

where β ranges over $H_2^{\mathrm{ord},K}(X;\mathbb{Z})$, $m=\{m_{i,j}\}$ runs over multi-indices with finite support on $I\times\{0,1,\dots,N\}$, and the product is taken in some fixed order on this index set. The sum is required to satisfy the following:

(1) Each $c_{\beta,m}$ lies in $H^\bullet_G(X)$.
(2) There exists a finite subset $S\subset H_2^{\mathrm{ord},K}(X;\mathbb{Z})$ such that $c_{\beta,m}=0$ unless $\beta\in S+\iota_*(\mathrm{Eff}(X))$.
(3) There exists a finite subset $R\subset\mathbb{Z}$ such that $(c_{\beta,m})_d = 0$ for $d+\sum m_{i,j}\deg\tau^{i,j}+\deg q^\beta\notin R$, where $(c_{\beta,m})_d$ denotes the degree d component of $c_{\beta,m}$.

[5]One can see that the product on $\mathbb{C}[[q_G]]$ is well-defined as follows. Choose a closed embedding $X\hookrightarrow\mathbb{P}^m\times\mathbb{C}^n$. It suffices to verify that the moduli stack of stable maps to X of degree less than or equal to a fixed number E has only finitely many components. This follows from the fact that this moduli stack is a closed substack of the corresponding moduli stack of stable maps to $\mathbb{P}^m\times\mathbb{C}^n$, which has quasiprojective coarse moduli space. We thank Hiroshi Iritani for explaining this argument to us.

Definition 3.4. The *G-equivariant quantum cohomology* of X with K-equivariant Novikov variables is the ring

$$QH_G^\bullet(X)[[q_K,\tau]],$$

whose underlying vector space is $H_G^\bullet(X)[[q_K,\tau]]$. Its ring structure is defined by the *big quantum product* $\star_\tau$, where

$$\gamma \star_\tau \gamma' = \sum_{\beta\in \mathrm{Eff}(X)} \sum_{n=0}^{\infty} \frac{q^{\iota_*\beta}}{n!} \,\mathrm{PD}\; \mathrm{ev}_{3*}\Big(\mathrm{ev}_1^*(\gamma)\; \mathrm{ev}_2^*(\gamma') \prod_{\ell=4}^{n+3} \mathrm{ev}_\ell^*(\tau) \cap [\overline{M}_{0,n+3}(X,\beta)]^{\mathrm{vir}}\Big)$$

for $\gamma,\gamma'\in H_G^\bullet(X)$. Here, $\overline{M}_{0,n+3}(X,\beta)$ is the moduli stack of genus-zero stable maps to X with $n+3$ marked points and curve class β, and $[\overline{M}_{0,n+3}(X,\beta)]^{\mathrm{vir}}$ denotes its virtual fundamental class. Note that ev_3 is proper, thanks to X being semiprojective. The product $\star_\tau$ on a general element of $H_G^\bullet(X)[[q_K,\tau]]$ is then defined termwise on its power series expansion.

From now on, we will simply write $q^{\iota_*\beta}$ as q^β whenever no confusion is likely to arise. It is clear that if $K_1\subset K_2\subset G$ are reductive subgroups of G, then there is a natural ring homomorphism

$$QH_G^\bullet(X)[[q_{K_1},\tau]] \longrightarrow QH_G^\bullet(X)[[q_{K_2},\tau]]. \tag{25}$$

Quantum connection. We consider the trivial $\mathbb{C}_\hbar^\times$-action on X. We follow the notations in [Iri25, Section 2.1].

Definition 3.5 (Quantum connection)**.** The *equivariant quantum connections* are the operators

$$\nabla_{\tau^{i,j}}, \nabla_{\hbar\partial_\hbar}, \nabla_D : H^\bullet_{G\times\mathbb{C}_\hbar^\times}(X)[[q_K,\tau]] \to H^\bullet_{G\times\mathbb{C}_\hbar^\times}(X)[[q_K,\tau]][\hbar^{-1}],$$

defined as

$$\begin{aligned} \nabla_{\tau^{i,j}} &= \partial_{\tau^{i,j}} + \hbar^{-1}(\phi_{i,j}\star_\tau), \\ \nabla_{\hbar\partial_\hbar} &= \hbar\partial_\hbar - \hbar^{-1}(E\star_\tau) + \mu_X, \\ \nabla_{Dq\partial_q} &= Dq\partial_q + \hbar^{-1}(D\star), \end{aligned} \tag{26}$$

where $D\in H_G^2(X)$ and $Dq\partial_q$ is the derivation on $\mathbb{C}[[q_K]]$ with $Dq\partial_q q^\beta = \langle D,\beta\rangle q^\beta$, while E_X is the *Euler vector field* and μ_X is the *grading operator*, defined respectively by the formulas

$$E_X = c_1^G(X) + \sum_{i,j} \frac{\deg(\tau^{i,j})}{2}\tau^{i,j}\phi_{i,j}$$

and

$$\mu_X(\phi_{i,j}) = \frac{1}{2}\left(\deg(\phi_{i,j}) - \dim X\right)\phi_{i,j}. \tag{27}$$

Definition 3.6 (Fundamental solution)**.** The *fundamental solution* to the quantum differential equation is the operator

$$\mathbb{M}_X : H^\bullet_{G\times\mathbb{C}_\hbar^\times}(X)[[q_G,\tau]][[\hbar^{-1}]] \to H^\bullet_{G\times\mathbb{C}_\hbar^\times}(X)[[q_G,\tau]][[\hbar^{-1}]]$$

defined by

$$\mathbb{M}_X(\gamma) = \gamma + \sum_{0\neq\beta\in\mathrm{Eff}(X)} \sum_{n=0}^{\infty} \frac{q^\beta}{n!}\,\mathrm{PD}\circ \mathrm{ev}_{2*}\left(\frac{\mathrm{ev}_1^*(\gamma)}{\hbar-\psi_1}\prod_{\ell=3}^{n+2}\mathrm{ev}_\ell^*(\tau)\cap[\overline{M}_{0,n+2}(X,\beta)]^{\mathrm{vir}}\right),$$

where ψ_1 is the equivariant first Chern class of the universal cotangent line bundle at the first marked point.

The following proposition is well-known (see [Giv96; Pan98; CIJ18; Iri25]).

Proposition 3.7. *The operator $\mathbb{M}_X$ is invertible and satisfies the identities*

$$\begin{aligned} \nabla_{\tau^{i,j}} \circ \mathbb{M}_X &= \mathbb{M}_X \circ \partial_{\tau^{i,j}}, \\ \nabla_{\hbar\partial_\hbar} \circ \mathbb{M}_X &= \mathbb{M}_X \circ \left(\hbar\partial_\hbar - \hbar^{-1}(c_1^G(X)\cup) + \mu_X\right), \\ \nabla_{Dq\partial_q} \circ \mathbb{M}_X &= \mathbb{M}_X \circ \left(Dq\partial_q + \hbar^{-1}(D\cup)\right). \end{aligned} \tag{28}$$

3.2. **Universal G-torsor.** We fix the standard affine chart $\mathbb{C} = \operatorname{Spec} \mathbb{C}[t] \subset \mathbb{P}^1$ and take the base point $0 \in \mathbb{C} \subset \mathbb{P}^1$. By the Beauville–Laszlo theorem [BL95] (see also [Zhu17]), the affine Grassmannian Gr_G represents the functor on the category of $\mathbb{C}$-scheme.

$$Z \longmapsto \left\{ (\mathcal{P}, \varphi) \,\middle|\, \begin{array}{l} \mathcal{P} \text{ is a } G\text{-torsor over } \mathbb{P}^1_Z, \\ \varphi_\infty : \mathcal{P}|_{(\mathbb{P}^1 \setminus \{0\})_Z} \xrightarrow{\sim} (\mathbb{P}^1 \setminus \{0\})_Z \times G \text{ is a trivialization} \end{array} \right\} / \cong .$$

In particular, there exists a universal G-torsor $\mathcal{E} \to \mathrm{Gr}_G \times \mathbb{P}^1$. If $p \in \mathrm{Gr}_G$, we denote $\mathcal{E}_p$ the restriction of $\mathcal{E}$ to $p \times \mathbb{P}^1$.

We now make the correspondence between a map $f : Z \to \mathrm{Gr}_G$ and the pair $(\mathcal{P}, \varphi_\infty)$ more explicit. Suppose first that f factors through a map $\widetilde{f} : Z \to G_{\mathcal{K}}$. Then $\mathcal{P}$ is the unique (up to isomorphism) G-torsor over $\mathbb{P}^1_R$ with local trivializations

$$\varphi_\infty : \mathcal{P}|_{Z \times (\mathbb{P}^1 \setminus \{0\})} \xrightarrow{\sim} (Z \times (\mathbb{P}^1 \setminus \{0\})) \times G,$$

and

$$\varphi_0 : \mathcal{P}|_{Z \times \operatorname{Spec} \mathcal{O}} \xrightarrow{\sim} (Z \times \operatorname{Spec} \mathcal{O}) \times G,$$

such that the induced automorphism

$$\varphi_\infty|_{Z \times \operatorname{Spec} \mathcal{K}} \circ (\varphi_0|_{Z \times \operatorname{Spec} \mathcal{K}})^{-1} : (Z \times \operatorname{Spec} \mathcal{K}) \times G \to (Z \times \operatorname{Spec} \mathcal{K}) \times G$$

is given by left multiplication by the map

$$(Z \times \operatorname{Spec} \mathcal{K}) \times G \to Z \times \operatorname{Spec} \mathcal{K} \xrightarrow{\widetilde{f}} G.$$

The existence of such a torsor $\mathcal{P}$ is the content of the Beauville–Laszlo theorem. In general, there exists an fppf cover $\bigsqcup Z_i \to Z$ such that the pullback of f to each Z_i factors through $G_{\mathcal{K}}$. One can then construct $\mathcal{P}$ and φ_∞ by gluing the corresponding G-torsors and local trivializations over the cover, using fppf descent.

Under the above correspondence, the subgroup $G_{\mathbb{C}[t^{-1}]} \subset G_{\mathcal{K}}$ acts on Gr_G by modifying the section φ over $\mathbb{P}^1 \setminus \{0\}$, while loop rotation acts by scaling the coordinate on $\mathbb{P}^1$. More explicitly, for $z \in \mathbb{C}^\times_\hbar$, let $m_z : \mathbb{P}^1 \to \mathbb{P}^1$ be the morphism given by $t \mapsto z^{-1}t$. There is an action of $G_{\mathbb{C}[t^{-1}]} \rtimes \mathbb{C}^\times_\hbar$ on Gr_G given by

$$g = (g_1, z) \cdot (\mathcal{P}, \varphi) = (m_z^* \mathcal{P},\ g \cdot m_z^* \varphi)\,,$$

where g_1 is regarded as a morphism $\operatorname{Spec} \mathbb{C}[t^{-1}] \to G$.

The action of $G_{\mathbb{C}[t^{-1}]} \rtimes \mathbb{C}^\times_\hbar$ lifts to the universal torsor $\mathcal{E}$, in the sense of the next lemma, which follows from the functorial description of Gr_G.

Lemma 3.8. *Suppose $f : Z \to \mathrm{Gr}_G$ is a morphism from a scheme Z corresponding to the pair $(\mathcal{P}, \varphi_\infty)$, and let $g = (g_1, z) \in G_{\mathbb{C}[t^{-1}]} \rtimes \mathbb{C}^\times_\hbar$. If $(\mathcal{P}', \varphi'_\infty)$ corresponds to the map $g \cdot f : Z \to \mathrm{Gr}_G$, then there exists a G-equivariant isomorphism*

$$g_f : \mathcal{P} \xrightarrow{\sim} m_z^* \mathcal{P}'$$

such that $m_z^* \varphi'_\infty \circ g_f = g_1 \cdot \varphi_\infty$.

In particular, if $H \subset G_{\mathbb{C}[t^{-1}]} \rtimes \mathbb{C}^\times_\hbar$ and $f : Z \to \mathrm{Gr}_G$ is an H-equivariant morphism corresponding to the pair $(\mathcal{P}, \varphi_\infty)$, then there is an induced H-action on $\mathcal{P}$ such that the projection $\mathcal{P} \to Z \times \mathbb{P}^1$ is H-equivariant (where $\mathbb{C}^\times_\hbar$ acts on $\mathbb{P}^1$ by scalar multiplication).

Strictly speaking, we have defined the action on $\mathcal{E}$ only at the level of $\mathbb{C}$-points of $G_{\mathbb{C}[t^{-1}]} \rtimes \mathbb{C}^\times_\hbar$. In Appendix B, we will describe the action of $G_{\mathbb{C}[t^{-1}]} \rtimes \mathbb{C}^\times_\hbar$ as a group scheme by considering its R-points, and in particular show that the actions defined above are algebraic.

Example 3.9. Suppose $G = T = \mathbb{C}^\times$ and $\lambda = 1$. Then

$$\mathcal{E}_t := \mathcal{E}|_t \cong \mathcal{O}_{\mathbb{P}^1}(-1)^\times \cong \mathbb{C}^2 \setminus \{0\}.$$

Let u, v be coordinates on $\mathbb{C}^2$, so that the projection $\mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ is given by $t = v/u$. Indeed, there are trivializations

$$\varphi_0 \colon \mathcal{O}_{\mathbb{P}^1}(-1)^\times|_{\operatorname{Spec} \mathbb{C}[t]} \xrightarrow{\sim} \operatorname{Spec} \mathbb{C}[t] \times \mathbb{C}^\times,$$

$$\varphi_\infty \colon \mathcal{O}_{\mathbb{P}^1}(-1)^\times|_{\operatorname{Spec} \mathbb{C}[t^{-1}]} \xrightarrow{\sim} \operatorname{Spec} \mathbb{C}[t^{-1}] \times \mathbb{C}^\times,$$

given by

$$\varphi_0(u, v) = \left(\frac{v}{u}, u \right), \qquad \varphi_\infty(u, v) = \left(\frac{u}{v}, v \right).$$

It is clear that the transition function $\varphi_\infty \circ \varphi_0^{-1}$ corresponds to left multiplication by $t = v/u$. The subgroup[6] $\mathbb{C}^\times \times \mathbb{C}^\times_\hbar \subset \mathbb{C}^\times_{\mathbb{C}[t^{-1}]} \rtimes \mathbb{C}^\times_\hbar$ acts on $\mathcal{O}_{\mathbb{P}^1}(-1)^\times$ by $(g, z_\hbar) \cdot (u, v) = (z_\hbar g u, g v)$.

More generally, if $G = T$ is abelian and $\lambda \in \Lambda$ is arbitrary, then

$$\mathcal{E}_{t^\lambda} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^\times \times_{\mathbb{C}^\times} T,$$

where $\mathbb{C}^\times$ acts on T via λ, and $T \times \mathbb{C}^\times_\hbar$ acts on $\mathcal{E}_{t^\lambda}$ via $(g, z_\hbar) \cdot (u, v, g') = (z_\hbar u, v, g g')$. □

3.3. **Seidel spaces.** For each $\lambda \in \Lambda$, we fix an $\mathcal{I}$-equivariant resolution of singularities $\rho_\lambda \colon \widetilde{C}_{\leq\lambda} \to C_{\leq\lambda}$ satisfying the conditions in Lemma 1.2. Let $\mathcal{E}_{\leq\lambda}$ be the principal G-bundle on $\widetilde{C}_{\leq\lambda} \times \mathbb{P}^1$ induced by the morphism

$$\widetilde{C}_{\leq\lambda} \xrightarrow{\rho_\lambda} C_{\leq\lambda} \subset \mathrm{Gr}_G.$$

If $p \in \widetilde{C}_{\leq\lambda}$, we write $\mathcal{E}_p$ for the restriction of $\mathcal{E}_{\leq\lambda}$ to $p \times \mathbb{P}^1$. In particular, there is a canonical isomorphism $\mathcal{E}_p \cong \mathcal{E}_{\rho_\lambda(p)}$.

If X is a G-variety, we call

$$\mathcal{E}_{\leq\lambda}(X) := \mathcal{E}_{\leq\lambda} \times_G X$$

the *Seidel space associated to* X *and* $\widetilde{C}_{\leq\lambda}$.

By construction, there is a $T \times \mathbb{C}^\times_\hbar$-action on $\mathcal{E}_{\leq\lambda}(X)$ such that the projection

$$\pi : \mathcal{E}_{\leq\lambda}(X) \to \widetilde{C}_{\leq\lambda} \times \mathbb{P}^1$$

is equivariant, where $\mathbb{C}^\times_\hbar$ acts on $\mathbb{P}^1$ by scalar multiplication. Moreover, if $\lambda \in \Lambda^+$, this action can be upgraded to a $G \times \mathbb{C}^\times_\hbar$-action.

Example 3.10. The bundle $\mathcal{E}_{t^\lambda}(X)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^\times \times_{\mathbb{C}^\times} X$, where $\mathbb{C}^\times$ acts on X via the cocharacter $\lambda \colon \mathbb{C}^\times \to T$. The action of $T \times \mathbb{C}^\times_\hbar$ on $\mathcal{E}_{t^\lambda}(X)$ is given by (cf. Example 3.9)

$$(g, z_\hbar) \cdot (u, v, x) = (z_\hbar u, v, g x).$$

This space is $T \times \mathbb{C}^\times_\hbar$-equivariantly isomorphic to the Seidel space E_λ defined in Section 3.2 of [Iri17], but with the roles of 0 and ∞ swapped. □

Section classes. An effective curve class $\beta \in H_2^{\mathrm{ord}}(\mathcal{E}_{\leq\lambda}(X); \mathbb{Z})$ is called a *section class* if $\pi_*\beta = [\mathrm{pt} \times \mathbb{P}^1]$. Denote by $\mathrm{Eff}(\mathcal{E}_{\leq\lambda}(X))^{\mathrm{sec}}$ the subset of section classes.

The projection $\mathcal{E}_{\leq\lambda} \times X \to X$ is G-equivariant, and hence induces a pushforward map in equivariant homology:

$$H_2^{\mathrm{ord}}(\mathcal{E}_{\leq\lambda}(X); \mathbb{Z}) \longrightarrow H_2^{\mathrm{ord},G}(X; \mathbb{Z}).$$

For any $\beta \in H_2^{\mathrm{ord}}(\mathcal{E}_{\leq\lambda}(X); \mathbb{Z})$, we denote by $\overline{\beta} \in H_2^{\mathrm{ord},G}(X, \mathbb{Z})$ its image under this map.

Let $T^{\mathrm{vert}}\mathcal{E}_{\leq\lambda}(X) := \ker(d\pi) \cong \mathcal{E}_{\leq\lambda}(TX)$ be the relative tangent bundle of π. The following lemma is immediate.

Lemma 3.11. $\deg(q^{\overline{\beta}}) = 2\, c_1(T^{\mathrm{vert}}\mathcal{E}_{\leq\lambda}(X)) \cdot \beta$.

Fibres at 0 *and* ∞. There are $G_{\mathbb{C}[t^{-1}]} \rtimes \mathbb{C}^\times_\hbar$-equivariant isomorphisms

$$\begin{aligned} \mathcal{E}(X)|_{\mathrm{Gr}_G \times \{0\}} &\cong G_{\mathcal{K}} \times_{G_{\mathcal{O}}} X, \\ \mathcal{E}(X)|_{\mathrm{Gr}_G \times \{\infty\}} &\cong \mathrm{Gr}_G \times X. \end{aligned} \tag{29}$$

These are straightforward consequences of the construction of $\mathcal{E}$, see (65) in Appendix B. We denote

$$\iota_0 : \mathcal{X}_{\leq\lambda,0} := \pi^{-1}(\widetilde{C}_{\leq\lambda} \times \{0\}) \hookrightarrow \mathcal{E}_{\leq\lambda}(X),$$
$$\iota_\infty : \mathcal{X}_{\leq\lambda,\infty} := \pi^{-1}(\widetilde{C}_{\leq\lambda} \times \{\infty\}) \hookrightarrow \mathcal{E}_{\leq\lambda}(X).$$

Recall that $\widetilde{G}^{\leq\lambda}_{\mathcal{K}}$ denotes $G_{\mathcal{K}} \times_{\mathrm{Gr}_G} \widetilde{C}_{\leq\lambda}$ (see (21)). The following lemma follows immediately from Equation (29).

Lemma 3.12. *There are $T \times \mathbb{C}^\times_\hbar$-equivariant isomorphisms*

$$\mathcal{X}_{\leq\lambda,0} \cong \widetilde{G}^{\leq\lambda}_{\mathcal{K}} \times_{G_{\mathcal{O}}} X,$$
$$\mathcal{X}_{\leq\lambda,\infty} \cong \widetilde{C}_{\leq\lambda} \times X.$$

Moreover, these isomorphisms are $G \times \mathbb{C}^\times_\hbar$-equivariant if $\lambda \in \Lambda^+$.

---

[6] $\mathbb{C}^\times$ and $\mathbb{C}^\times_{\mathbb{C}[t^{-1}]}$ have the same underlying $\mathbb{C}$-points

In particular, $\rho_{X,\lambda}\colon \mathcal{X}_{\leq\lambda,0} \to G^{\leq\lambda}_{\mathcal{K}} \times_{G_{\mathcal{O}}} X$ is a resolution of singularities.

Definition 3.13. We define a map

$$H^\bullet_{G\times\mathbb{C}^\times_\hbar}(X) \to H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathcal{E}_{\leq\lambda}(X)), \qquad \tau \mapsto \hat{\tau}, \tag{30}$$

as the composition of the following maps:

$$H^\bullet_{G_{\mathcal{O}}\rtimes\mathbb{C}^\times_\hbar}(X) \xrightarrow{\mathrm{pr}_X^*} H^\bullet_{(T_{\mathcal{O}}\times G_{\mathcal{O}}\times G_{\mathcal{O}})\rtimes\mathbb{C}^\times_\hbar}(\mathcal{E}_{\leq\lambda}\times X) \to H^\bullet_{(T_{\mathcal{O}}\times\Delta_{G_{\mathcal{O}}})\rtimes\mathbb{C}^\times_\hbar}(\mathcal{E}_{\leq\lambda}\times X) \xrightarrow{\sim} H^\bullet_{T_{\mathcal{O}}\rtimes\mathbb{C}^\times_\hbar}(\mathcal{E}_{\leq\lambda}(X)),$$

where we identify $H^\bullet_{G\times\mathbb{C}^\times_\hbar}(X)$ with $H^\bullet_{G_{\mathcal{O}}\rtimes\mathbb{C}^\times_\hbar}(X)$, and similarly for the other terms.

The following lemma is an immediate consequence of Definition 3.13 and the considerations in Proposition 2.2.

Lemma 3.14. *Let*

$$\widetilde{\tau} = 1\otimes\tau \in H^\bullet_{(T\times G_{\mathcal{O}}\times G)\rtimes\mathbb{C}^\times_\hbar}(\widetilde{G}^{\leq\lambda}_{\mathcal{K}}\times X) \cong H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\widetilde{C}_{\leq\lambda}) \otimes_{\mathbb{C}[\hbar]} H^\bullet_{G\times\mathbb{C}^\times_\hbar}(X).$$

Then the restriction of $\hat{\tau}$ to $\mathcal{X}_{\leq\lambda,0}$ is equal to $\mathrm{tw}_{\leq\lambda}(\widetilde{\tau})$, and the restriction of $\hat{\tau}$ to $\mathcal{X}_{\leq\lambda,0}$ is equal to the image of $\widetilde{\tau}$ under the restriction of the equivariant parameter

$$H^\bullet_{(T\times G_{\mathcal{O}}\times G)\rtimes\mathbb{C}^\times_\hbar}(\widetilde{G}^{\leq\lambda}_{\mathcal{K}}\times X) \longrightarrow H^\bullet_{(T\times G_{\mathcal{O}}\times 1)\rtimes\mathbb{C}^\times_\hbar}(\widetilde{G}^{\leq\lambda}_{\mathcal{K}}\times X) \cong H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\widetilde{C}_{\leq\lambda}\times X).$$

Remark 3.15. It follows from Lemma 3.14 that when $G = T$ is abelian, the class $\hat{\tau}$ agrees with [Iri17, Notation 3.8], except that the roles of the zero and infinity fibres are reversed (cf. Example 3.10).

Moduli spaces and virtual fundamental classes. Let $\beta \in \mathrm{Eff}(\mathcal{E}_{\leq\lambda}(X))^{\mathrm{sec}}$. Recall that $\overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X),\beta)$ denotes the moduli stack of genus-zero stable maps to $\mathcal{E}_{\leq\lambda}(X)$ with $n+2$ marked points and curve class β. A typical object in this moduli stack is a stable map

$$\sigma\colon (\Sigma, y_0, y_\infty, y_1, \dots, y_n) \to \mathcal{E}_{\leq\lambda}(X),$$

where Σ is a genus-zero nodal curve and $y_0, y_\infty, y_1 \dots, y_n \in \Sigma$ are the marked points.

Since $\mathcal{E}_{\leq\lambda}(X)$ is smooth, the moduli stack admits a virtual fundamental class ([BF97]),

$$[\overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X),\beta)]^{\mathrm{vir}} \in H^{T\times\mathbb{C}^\times_\hbar}_\bullet(\overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X),\beta)).$$

Definition 3.16. Let $\mathcal{M}_{\leq\lambda}(X,\beta)_n \subset \overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X),\beta)$ denote the substack consisting of those stable maps σ such that $\sigma(y_0)$ lies over $0\in\mathbb{P}^1$ and $\sigma(y_\infty)$ lies over $\infty\in\mathbb{P}^1$. In other words, we have the fibre diagram:

$$\begin{tikzcd} \mathcal{M}_{\leq\lambda}(X,\beta)_n \arrow[r, "j_{\mathcal{M}}"] \arrow[d] & \overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X),\beta) \arrow[d, "(\mathrm{pr}_{\mathbb{P}^1}\circ\mathrm{Ev}_0)\times(\mathrm{pr}_{\mathbb{P}^1}\circ\mathrm{Ev}_\infty)"] \\ \{(0,\infty)\} \arrow[r] & \mathbb{P}^1\times\mathbb{P}^1 \end{tikzcd} \tag{31}$$

Here, Ev_0 and Ev_∞ denote the *evaluation maps at the marked points y_0 and y_∞*, respectively. We also denote

$$\mathcal{M}_{\leq\lambda}(X)_n := \bigsqcup_{\beta\in\mathrm{Eff}(\mathcal{E}_{\leq\lambda}(X))^{\mathrm{sec}}} \mathcal{M}_{\leq\lambda}(X,\beta)_n.$$

In particular, the evaluation maps restrict to

$$\mathrm{ev}_0 : \mathcal{M}_{\leq\lambda}(X)_n \longrightarrow \mathcal{X}_{\leq\lambda,0}, \qquad \mathrm{ev}_\infty : \mathcal{M}_{\leq\lambda}(X)_n \longrightarrow \mathcal{X}_{\leq\lambda,\infty}.$$

In order to distinguish between the evaluation maps on $\mathcal{M}$ and $\overline{M}$, we use the different symbols ev and Ev, respectively.

We define the virtual fundamental class of $\mathcal{M}_{\leq\lambda}(X,\beta)_n$ to be

$$[\mathcal{M}_{\leq\lambda}(X,\beta)_n]^{\mathrm{vir}} := j^!_{\mathcal{M}}\left[\overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X),\beta)\right]^{\mathrm{vir}}, \tag{32}$$

where $j^!_{\mathcal{M}}$ is the refined Gysin pullback

$$j^!_{\mathcal{M}}\colon H^{T\times\mathbb{C}^\times_\hbar}_\bullet\big(\overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X),\beta)\big) \to H^{T\times\mathbb{C}^\times_\hbar}_{\bullet-4}(\mathcal{M}_{\leq\lambda}(X,\beta)_n)$$

defined via the fibre diagram (31) (see [8.3.21] in [CG97] or part 3(ii)(b) of [BFN18]). The degree of the virtual fundamental class equals twice the virtual dimension of $\mathcal{M}_{\leq\lambda}(X,\beta)_n$, which is computed as follows (cf. [FP97; CK99]):

$$\begin{aligned}\operatorname{vdim}\bigl(\mathcal{M}_{\leq\lambda}(X,\beta)_n\bigr) &= \dim\mathcal{E}_{\leq\lambda}(X) + c_1\bigl(T\mathcal{E}_{\leq\lambda}(X)\bigr)\cdot\beta + n - 3\\ &= \dim X + \dim C_\lambda + c_1\bigl(T^{\mathrm{vert}}\mathcal{E}_{\leq\lambda}(X)\bigr)\cdot\beta + n\\ &= \dim X + \dim C_\lambda + c_1^G(X)\cdot\overline{\beta} + n.\end{aligned}\tag{33}$$

If p is a point in Gr_G or in $\widetilde{C}_{\leq\lambda}$, the substack $\mathcal{M}_p(X,\beta)_n \subset \overline{M}_{0,n+2}(\mathcal{E}_p(X),\beta)$ is defined similarly. We write $\mathrm{pr}_{\widetilde{C}_{\leq\lambda}} : \mathcal{M}_{\leq\lambda}(X,\beta)_n \to \widetilde{C}_{\leq\lambda}$ for the projection.

4. Shift operators II: definition

We assume that X satisfies the conditions in Section 3.1, that $\mathbf{N}$ is a representation of G, and that there exists an equivariant proper morphism

$$f\colon X \to \mathbf{N}.$$

Our goal is to construct a homomorphism

$$\mathbb{S}_{G,\mathbf{N},X}\colon \mathcal{A}^{\hbar}_{G,\mathbf{N}} \otimes_{\mathbb{C}[\hbar]} H^\bullet_{G\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]] \longrightarrow H^\bullet_{G\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]]$$

that endows $H^\bullet_{G\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]]$ with the structure of an $\mathcal{A}^{\hbar}_{G,\mathbf{N}}$-module.

4.1. The tautological section and its zero locus.

Throughout this section, β will always denote a section class.

A lemma on $\mathcal{E}_p(\mathbf{N})$. Recall in Section 3.2 that we have a trivialization of $\mathcal{E}$ over $\mathrm{Gr}_G\times\{\infty\}$. Let $p\in\mathrm{Gr}_G$, there is thus an induced local trivialization of the vector bundle $\mathcal{E}_p(\mathbf{N})$ over $\mathbb{P}^1$

$$\varphi_\infty(\mathbf{N}) : \mathcal{E}_p(\mathbf{N})|_{\operatorname{Spec}\mathbb{C}[t^{-1}]} \xrightarrow{\sim} \operatorname{Spec}\mathbb{C}[t^{-1}]\times\mathbf{N}$$

over $\operatorname{Spec}\mathbb{C}[t^{-1}]$. We write $\varphi_\infty(\mathbf{N})$ simply as φ_∞ when no confusion arises. Suppose s is a section of the vector bundle $\mathcal{E}_p(\mathbf{N})$, which corresponds to a morphism $s_\infty : \operatorname{Spec}\mathbb{C}[t^{-1}]\to\mathbf{N}$ under φ_∞. We regard s_∞ as an element of $\mathbf{N}_{\mathbb{C}[t^{-1}]}$. Recall the bundle $\mathcal{T} = G_{\mathcal{K}}\times_{G_{\mathcal{O}}}\mathbf{N}_{\mathcal{O}}$.

Lemma 4.1. *$s_\infty\in\mathcal{T}_p$.*

Proof. Let $\widetilde{p}$ be a lift of p in $G_{\mathcal{K}}$, and recall that $\mathcal{T}_p = \widetilde{p}\,\mathbf{N}_{\mathcal{O}}\subset\mathbf{N}_{\mathcal{K}}$. There is a trivialization of $\mathcal{E}_p(\mathbf{N})$ over $\operatorname{Spec}\mathcal{O}$:

$$\varphi_0 : \mathcal{E}_p(\mathbf{N})|_{\operatorname{Spec}\mathcal{O}} \xrightarrow{\sim} \operatorname{Spec}\mathcal{O}\times\mathbf{N}$$

such that the transition function $\varphi_\infty\circ\varphi_0^{-1} : \mathbf{N}_{\mathcal{K}}\to\mathbf{N}_{\mathcal{K}}$ is given by the action of $\widetilde{p}\in G_{\mathcal{K}}$. Let $s_0 : \operatorname{Spec}\mathcal{O}\to\mathbf{N}$ be the morphism induced by s under φ_0, and regard it as an element of $\mathbf{N}_{\mathcal{O}}$. Then we have $s_\infty = \widetilde{p}s_0\in\mathcal{T}_p$, as claimed. ☐

The tautological section. Let $\lambda\in\Lambda$, and let $\rho_\lambda\colon\widetilde{C}_{\leq\lambda}\to C_{\leq\lambda}$ be a resolution as in Lemma 1.2. We will construct a section of $\widetilde{\mathcal{S}}_{\leq\lambda}$ over $\mathcal{M}_{\leq\lambda}(X)_n$[7].

Let $\sigma : (\Sigma, y_0, y_\infty, y_1,\dots,y_n)\to\mathcal{E}_{\leq\lambda}(X)$ be a stable map representing a point in $\mathcal{M}_{\leq\lambda}(X)_n$. By the definition of the section class, there exists a point $p_\sigma\in\widetilde{C}_{\leq\lambda}$ and a unique irreducible component Σ_0 of Σ such that the composition

$$\Sigma\xrightarrow{\sigma}\mathcal{E}_p(X)\to\widetilde{C}_{\leq\lambda}\times\mathbb{P}^1$$

restricts to an isomorphism $r:\Sigma_0\xrightarrow{\sim} p_\sigma\times\mathbb{P}^1$. Composing $\sigma\circ r^{-1}$ with the projection $\mathcal{M}_{\leq\lambda}(X)_n\to\mathcal{M}_{\leq\lambda}(\mathbf{N})_n$ gives a section $\overline{\sigma}$ of the vector bundle $\mathcal{E}_p(\mathbf{N}) \coloneqq \mathcal{E}_{\rho_\lambda(p)}(\mathbf{N})$.

By Lemma 4.1, the image $\overline{\sigma}_\infty$ lies in $\mathcal{T}_{p_\sigma}\coloneqq\mathcal{T}_{\rho_\lambda(p_\sigma)}$.

Definition 4.2. Let $\mathrm{can}_{\leq\lambda}(X)$ be the section of $\widetilde{\mathcal{S}}_{\leq\lambda}$ over $\mathcal{M}_{\leq\lambda}(X)_n$ given by the assignment

$$\sigma\longmapsto[\overline{\sigma}_\infty],$$

where $\sigma\in\mathcal{M}_{\leq\lambda}(X)_n$, and $[\overline{\sigma}_\infty]$ denotes the image of $\overline{\sigma}_\infty$ under the projection $\mathcal{T}_{p_\sigma}\to(\widetilde{\mathcal{S}}_{\leq\lambda})_{p_\sigma}$.

Moreover, if β is a section class of $\mathcal{E}_{\leq\lambda}(X)$, we denote by $\mathrm{can}_{\leq\lambda}(X,\beta)$ the restriction of $\mathrm{can}_{\leq\lambda}(X)$ to $\mathcal{M}_{\leq\lambda}(X,\beta)_n$.

If $p\in\widetilde{C}_{\leq\lambda}$, then the maps $\mathrm{can}_p(X)\colon\mathcal{M}_p(X)_n\to\widetilde{\mathcal{S}}_{\leq\lambda}|_p$ are defined analogously.

[7] That is, a morphism $\mathcal{M}_{\leq\lambda}(X)_n\to\widetilde{\mathcal{S}}_{\leq\lambda}$ of spaces over $\widetilde{C}_{\leq\lambda}$.

The zero locus.

Definition 4.3. We denote by $\mathcal{Z}_{\leq\lambda}(X,\beta)_n$ and $\mathcal{Z}_{\leq\lambda}(X)_n$ the zero loci of $\mathrm{can}_{\leq\lambda}(X,\beta)$ and $\mathrm{can}_{\leq\lambda}(X)$ respectively. We also write $\mathcal{Z}^{\mathbf{N}}_{\leq\lambda}(X,\beta)_n$ and $\mathcal{Z}^{\mathbf{N}}_{\leq\lambda}(X)_n$ if we want to emphasize the representation $\mathbf{N}$.

In contexts involving multiple representations, we include a superscript to indicate the representation, denoting, for example, $\mathcal{Z}^{\mathbf{N}}_{\leq\lambda}(X)_n$ and $\mathrm{can}^{\mathbf{N}}_{\leq\lambda}(X)_n$.

If p is a point in Gr_G or in $\widetilde{C}_{\leq\lambda}$, the substack $\mathcal{Z}_p(X,\beta)_n \subset \mathcal{M}_p(X,\beta)_n$ is defined similarly. We regard ev_∞ as a morphism $\mathcal{M}_{\leq\lambda}(X)_n \to \widetilde{C}_{\leq\lambda} \times X$.

Proposition 4.4. *The restriction of the evaluation map* $\mathrm{ev}_\infty : \mathcal{M}_{\leq\lambda}(X)_n \to \widetilde{C}_{\leq\lambda} \times X$ *to* $\mathcal{Z}_{\leq\lambda}(X)_n$ *is proper.*

Proof. Since the morphism $\mathcal{M}_{\leq\lambda}(X)_{n+1} \to \mathcal{M}_{\leq\lambda}(X)_n$ that forgets the last marked point (and stabilizes) is proper, it suffices to consider the case $n = 0$. For simplicity, we write $\mathcal{M}_{\leq\lambda}(X)_0 = \mathcal{M}_{\leq\lambda}(X)$.

We first consider the case $X = \mathbf{N}$. It suffices to prove that for any $p \in \widetilde{C}_{\leq\lambda}$, the restriction of the evaluation map $\mathrm{ev}_\infty : \mathcal{M}_p(\mathbf{N}) \to \mathcal{E}_p(\mathbf{N})$ to the zero locus of $\mathrm{can}_p(\mathbf{N})$ is proper. We regard $\mathrm{can}_p(\mathbf{N})$ as a map $\mathcal{M}_p(\mathbf{N}) \to (\widetilde{\mathcal{S}}_{\leq\lambda})_p$.

Note that we may identify the moduli space $\mathcal{M}_p(\mathbf{N})$ with the vector space of global sections of the vector bundle $\mathcal{E}_p(\mathbf{N})$ over $p \times \mathbb{P}^1$. Under this identification, the map $\mathrm{can}_p(\mathbf{N})$ sends a section s to $[s_\infty]$. It is clear from this description that $\mathrm{can}_p(\mathbf{N})$ is a linear map. Note that the composition

$$\mathcal{R}_p \to \mathcal{T}_p \to (\widetilde{\mathcal{S}}_{\leq\lambda})_p$$

is zero. If $\mathrm{can}_p(\mathbf{N})(s) = 0$, then we must have $s_\infty \in \mathcal{R}_p$, which means $s_\infty \in \mathbf{N}_{\mathcal{O}} \cap \mathbf{N}_{\mathbb{C}[t^{-1}]} = \mathbf{N}$. We can summarize this by saying that the assignment $s \mapsto s_\infty$ gives an injective linear map of vector spaces $\ker \mathrm{can}_p(\mathbf{N}) \hookrightarrow \mathbf{N}$. However, the evaluation map ev_∞ also sends $s \mapsto s_\infty$, so this proves the lemma in the case $X = \mathbf{N}$.

For the general case, we consider the commutative diagram:

$$\begin{tikzcd}
\mathcal{Z}_{\leq\lambda}(X,\beta) \arrow[r, hook, "j_X"] \arrow[d, "\mathcal{Z}(f)"] & \mathcal{M}_{\leq\lambda}(X,\beta) \arrow[r, "\mathrm{ev}_\infty"] \arrow[d, "\mathcal{M}(f)"] & \mathcal{E}_{\leq\lambda}(X) \arrow[d, "\mathcal{E}(f)"] \\
\mathcal{Z}_{\leq\lambda}(\mathbf{N}) \arrow[r, hook, "j_{\mathbf{N}}"] & \mathcal{M}_{\leq\lambda}(\mathbf{N}) \arrow[r, "\mathrm{ev}_\infty"] & \mathcal{E}_{\leq\lambda}(\mathbf{N})
\end{tikzcd}$$

It suffices to show that $\mathcal{M}(f)$ is proper, since this implies the properness of $\mathcal{Z}(f)$, and hence of the composition $\mathcal{E}(f) \circ \mathrm{ev}_\infty \circ j_X = \mathrm{ev}_\infty \circ j_{\mathbf{N}} \circ \mathcal{Z}(f)$. In particular, this shows that $\mathrm{ev}_\infty \circ j_X$ is proper.

Properness of $\mathcal{M}(f)$ follows from the semiprojectivity of X. Indeed, $f : X \to \mathbf{N}$ factors equivariantly as

$$X \xhookrightarrow{h} \mathbb{P}^n \times \mathbf{N} \longrightarrow \mathbf{N},$$

where h is a closed embedding. Then there is a corresponding factorization of $\mathcal{M}(f)$:

$$\mathcal{M}_{\leq\lambda}(X,\beta) \xrightarrow{\mathcal{M}(h)} \mathcal{M}_{\leq\lambda}(\mathbb{P}^n \times \mathbf{N}, h_*\beta) \longrightarrow \mathcal{M}_{\leq\lambda}(\mathbf{N}).$$

The natural map induces an isomorphism

$$\mathcal{M}_{\leq\lambda}(\mathbb{P}^n \times \mathbf{N}, h_*\beta) \cong \mathcal{M}_{\leq\lambda}(\mathbb{P}^n, (\mathrm{pr}_{\mathbb{P}^n})_* h_*\beta) \times \mathcal{M}_{\leq\lambda}(\mathbf{N}).$$

Since $\overline{h}$ is a closed embedding (cf. [FP97, Section 5.1]) and $\mathcal{M}_{\leq\lambda}(\mathbb{P}^n, (\mathrm{pr}_{\mathbb{P}^n})_* h_*\beta)$ is proper, the map $\mathcal{M}(f)$ is also proper, as claimed. ☐

4.2. **Section-counting map.**

Virtual fundamental classes. Consider the fibre diagram

$$\begin{tikzcd}
\mathcal{Z}_{\leq\lambda}(X,\beta)_n \arrow[r, "j_X"] \arrow[d] & \mathcal{M}_{\leq\lambda}(X,\beta)_n \arrow[d, "\mathrm{can}_{\leq\lambda}(X)"] \\
\widetilde{C}_{\leq\lambda} \arrow[r] & \widetilde{\mathcal{S}}_{\leq\lambda}
\end{tikzcd}, \tag{34}$$

where the bottom row is the inclusion of the zero section. We define the virtual fundamental class

$$[\mathcal{Z}_{\leq\lambda}(X,\beta)_n]^{\mathrm{vir}} := j_X^! \left[\mathcal{M}_{\leq\lambda}(X,\beta)_n\right]^{\mathrm{vir}}, \tag{35}$$

where $j_X^!$ denotes the Gysin pullback

$$j_X^!: H_\bullet^{T\times\mathbb{C}_\hbar^\times}\left(\mathcal{M}_{\leq\lambda}(X,\beta)_n\right) \longrightarrow H_{\bullet-2d_\lambda}^{T\times\mathbb{C}_\hbar^\times}\left(\mathcal{Z}_{\leq\lambda}(X,\beta)_n\right),$$

taken relative to the diagram (34). Here, d_λ denotes the rank of the vector bundle $\widetilde{\mathcal{S}}_{\leq\lambda}$ (see (13)). We record the following.

$$\operatorname{vdim}\left(\mathcal{Z}_{\leq\lambda}(X,\beta)_n\right) = \dim X + \dim C_\lambda + c_1^G(X)\cdot\overline{\beta} - d_\lambda + n. \tag{36}$$

We write ev_∞ in place of $\mathrm{ev}_\infty \circ j_X$ for simplicity.

Definition 4.5. We define the *section-counting map*

$$\widetilde{\mathbb{S}}_{\mathbf{N},\leq\lambda}: e(\widetilde{\mathcal{S}}_{\leq\lambda}) \cup H^\bullet_{T\times\mathbb{C}_\hbar^\times}(\mathcal{X}_{\leq\lambda,0})[[q_G,\tau]] \longrightarrow H^\bullet_{T\times\mathbb{C}_\hbar^\times}(X)[[q_G,\tau]]$$

by setting

$$\widetilde{\mathbb{S}}_{\mathbf{N},\leq\lambda}(e(\widetilde{\mathcal{S}}_{\leq\lambda})\cup\gamma) := \sum_{\beta\in\mathrm{Eff}(\mathcal{E}_{\leq\lambda}(X))^{\mathrm{sec}}} \sum_{n=0}^{\infty} \frac{q^{\overline{\beta}}}{n!} \mathrm{PD}\circ \mathrm{pr}_{X*}\,\mathrm{ev}_{\infty*}\left(\mathrm{ev}_0^*(\gamma)\prod_{\ell=1}^n \mathrm{ev}_\ell^*(\hat{\tau})\cap[\mathcal{Z}_{\leq\lambda}(X,\beta)_n]^{\mathrm{vir}}\right) \tag{37}$$

for $\gamma\in H^\bullet_{T\times\mathbb{C}_\hbar^\times}(\mathcal{X}_{\leq\lambda,0})$. Here, $\mathrm{pr}_X: \mathcal{X}_{\leq\lambda,\infty}\cong\widetilde{C}_{\leq\lambda}\times X\to X$ is the projection map, and $\mathrm{PD}: H_\bullet^{T\times\mathbb{C}_\hbar^\times}(X)\to H^{2\dim X-\bullet}_{T\times\mathbb{C}_\hbar^\times}(X)$ is the Poincaré duality map. The formula for an arbitrary $\gamma\in H^\bullet_{T\times\mathbb{C}_\hbar^\times}(\mathcal{X}_{\leq\lambda,0})[[q_G,\tau]]$ is defined termwise on its power series expansion.

The above definition makes sense due to the following two lemmas.

Lemma 4.6. *There exists a finite subset $S\subset H_2^{\mathrm{ord},G}(X;\mathbb{Z})$ such that*

$$\left\{\overline{\beta}\in H_2^{\mathrm{ord},G}(X;\mathbb{Z})\,\middle|\,\beta\in\mathrm{Eff}(\mathcal{E}_{\leq\lambda}(X))^{\mathrm{sec}}\right\}\subset S+i_*\left(\mathrm{Eff}(X)\right),$$

where $i: X\hookrightarrow\mathcal{E}_{\leq\lambda}(X)$ is the inclusion of a fibre.

Lemma 4.7. *Let $\gamma\in H^\bullet_{T\times\mathbb{C}_\hbar^\times}(\mathcal{X}_{\leq\lambda,0})$ be a homogeneous element. Then, each summand in the power series expansion of the right-hand side of (37) is of degree $\deg e((\widetilde{\mathcal{S}}_{\leq\lambda})\cup\gamma)-2\dim C_\lambda = 2d_\lambda-2\dim C_\lambda+\deg\gamma$, which is independent of β and n.*

Definition 4.8. We define *the shift operator with matter* $\mathbf{N}$ to be the map

$$\mathbb{S}_{G,\mathbf{N},X}: e(\mathcal{S}_\mathbf{N})\cap H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(\mathrm{Gr}_G)\otimes_{\mathbb{C}[\hbar]} H^\bullet_{G\times\mathbb{C}_\hbar^\times}(X)[[q_G,\tau]]\longrightarrow H^\bullet_{T\times\mathbb{C}_\hbar^\times}(X)[[q_G,\tau]]$$

which sends

$$e(\mathcal{S})\cap[C_{\leq\lambda}]\otimes\alpha\longmapsto\widetilde{\mathbb{S}}_{\mathbf{N},\leq\lambda}\left(e(\widetilde{\mathcal{S}}_{\leq\lambda})\cup\mathrm{tw}_{\leq\lambda}\left([\widetilde{C}_{\leq\lambda}]\otimes\alpha\right)\right). \tag{38}$$

The map $\mathbb{S}_{G,\mathbf{N},X}$ is then extended $H^\bullet_{T\times\mathbb{C}_\hbar^\times}(\mathrm{pt})$-linearly in the first argument. And for a general element of $H^\bullet_{G\times\mathbb{C}_\hbar^\times}(X)[[q_G,\tau]]$, the assignment $\mathbb{S}_{G,\mathbf{N},X}$ is then defined termwise on its power series expansion. Here $\mathrm{tw}_{\leq\lambda}$ is the twisting map defined in (22).

Note that by Lemma 2.6, the expression (38) can also be written as the composition $\mathbb{S}_{\mathbf{N},\leq\lambda}\circ\mathrm{tw}_{\leq\lambda}(e(\widetilde{\mathcal{S}}_{\leq\lambda})\cap[\widetilde{C}_{\leq\lambda}]\otimes\alpha)$.

We will simply write $\mathbb{S}_{G,\mathbf{N}}$ when the space X is clear from the context, and we write $\mathbb{S}_G$ if the representation $\mathbf{N}=\mathbf{0}$. We will show in Corollary 5.9 that $\mathbb{S}_{G,\mathbf{N},X}$ is W-equivariant and in particular its restriction to the subspace of W-invariants gives the map (1).

Proof of Lemma 4.6. We first reduce to the case where X is compact. Indeed, if C is a curve on $\mathcal{E}_{\leq\lambda}(X)$ representing a section class, we may use the $\mathbb{C}^\times_{\mathrm{dil}}$-action to move σ so that it lies in $\mathcal{E}_{\leq\lambda}(X^{\mathbb{C}^\times})$. Hence, we may replace X by $X^{\mathbb{C}^\times}$, which is compact.

Next, we use the T-action to move the curve C to a T-invariant curve. In particular, its projection $p=\mathrm{pr}_{\widetilde{C}_{\leq\lambda}}(C)$ must be a T-fixed point in $\widetilde{C}_{\leq\lambda}$. We have

$$\mathcal{E}_p(X)=\mathcal{E}^T_{\rho_\lambda(p)}(X),$$

where $\mathcal{E}^T$ denotes the universal T-torsor over $\mathrm{Gr}_T\times\mathbb{P}^1$. Therefore, we may reduce to the case where $G=T$ is abelian.

Now let $C \subset \mathcal{E}_p^T(X)$ be a T-invariant curve representing a section class with irreducible components $C_0, \ldots, C_m$. Then one of them, say C_0, must be the constant section

$$\mathrm{Const}_x = \mathcal{E}_p^T \times_T x$$

for some $x \in X^T$, and all other components lie in the fibres of $\mathcal{E}_p^T(X) \to p \times \mathbb{P}^1$. Let $F_1, \ldots, F_k$ be the connected components of X^T, and let $\beta_i \in \mathrm{Eff}(\mathcal{E}_{\leq\lambda}(X))^{\mathrm{sec}}$ be the class represented by Const_x for some (any) $y \in F_i$. Then the above reasoning gives

$$[C] \in \mathbb{Z}_{\geq 0}\beta_1 + \cdots + \mathbb{Z}_{\geq 0}\beta_k + \mathrm{Eff}(X).$$

Therefore, we may conclude the lemma by taking $S = \{\overline{\beta_1}, \ldots, \overline{\beta_k}\}$. □

Proof of Lemma 4.7. Using (36) and Lemma 3.11, we see that for any $i_\ell \in I$ and $j_\ell \in \{0, 1, \ldots, N\}$, the degree of

$$\frac{q^{\overline{\beta}}}{n!} \mathrm{PD} \circ \mathrm{pr}_{X*} \mathrm{ev}_{\infty *} \left(\mathrm{ev}_0^*(\gamma) \prod_{\ell=1}^{n} \mathrm{ev}_\ell^*(\tau^{i_\ell, j_\ell} \hat{\phi}_{i_\ell, j_\ell}) \cap [\mathcal{Z}_{\leq\lambda}(X, \beta)_n]^{\mathrm{vir}} \right)$$

is equal to

$$\begin{aligned}&2c_1^G(X) \cdot \overline{\beta} + 2\dim X - 2\mathrm{vdim}(\mathcal{Z}_{\leq\lambda}(X,\beta)_n) + \deg\gamma + \sum_{\ell=1}^{n} \left(\deg \tau^{i_\ell, j_\ell} + \deg \phi_{i_\ell, j_\ell}\right) \\ =\ & 2d_\lambda - 2\dim C_\lambda + \deg\gamma.\end{aligned}$$

□

We adopt the cohomological convention for degrees in Borel–Moore homology. In particular, an element of $H_k^{T\times\mathbb{C}_\hbar^\times}(\mathrm{Gr}_G)$ has degree $-k$.

Since $\mathrm{tw}_{\leq\lambda}$ increases the degree by $2\dim C_\lambda$ (due to Poincaré duality), we obtain the following corollary of Lemma 4.7.

Corollary 4.9. *$\mathbb{S}_{G,\mathbf{N}}$ preserves cohomological degrees.*

Remark 4.10. Suppose $G = T$ is abelian, and let $\lambda \in \Lambda$ be semi-negative with respect to X, in the sense that $\xi(\lambda) \leq 0$ for every weight ξ of $H^0(X, \mathcal{O}_X)$ (cf. [Iri17, Definition 3.3]).

It is easy to show that there exists a representation $\mathbf{N}$ with λ-non-positive weights and a proper T-equivariant morphism $f\colon X \to \mathbf{N}$. By Equation (12), we have $\mathcal{S}_{t^\lambda} = 0$, and hence $\mathcal{Z}_{\leq\lambda}(X)_n = \mathcal{M}_{\leq\lambda}(X)_n$. In this case, the properness statement in Proposition 4.4 follows from [Iri17, Lemma 3.5].

Remark 3.10 of [Iri17] also notes that the semi-negativity condition implies that the shift operator is defined without localization. One may regard Proposition 4.4 and Definition 4.8 as generalizations of this observation. In Section 5 below, we will explain in detail how our construction of shift operators compares with that in [Iri17].

By the $H^\bullet_{T\times\mathbb{C}_\hbar^\times}(\mathrm{pt})$-linearities of $\mathrm{tw}_{\leq\lambda}$ and $\widetilde{\mathbb{S}}_{\mathbf{N},\leq\lambda}$, we can extend the shift operators to the localized Coulomb branch algebras (see (16)).

Definition 4.11. Define

$$\mathbb{S}_{G,\mathbf{N},\mathrm{loc}} : \left(e(\mathcal{S}_\mathbf{N}) \cap H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(\mathrm{Gr}_G)\right)_{\mathrm{loc}} \otimes_{\mathbb{C}[\hbar]} H^\bullet_{G\times\mathbb{C}_\hbar^\times}(X)[[q_G,\tau]] \longrightarrow H^\bullet_{T\times\mathbb{C}_\hbar^\times}(X)_{\mathrm{loc}}[[q_G,\tau]]$$

by extending $\mathrm{Frac}\, H^\bullet_{T\times\mathbb{C}_\hbar^\times}(\mathrm{pt})$-linearly in the first argument. Here $H^\bullet_{T\times\mathbb{C}_\hbar^\times}(X)_{\mathrm{loc}} := \mathrm{Frac} H^\bullet_{T\times\mathbb{C}_\hbar^\times}(\mathrm{pt}) \otimes_{H^\bullet_{T\times\mathbb{C}_\hbar^\times}(\mathrm{pt})} H^\bullet_{T\times\mathbb{C}_\hbar^\times}(X)$ is the localized cohomology.

The Seidel representation.

Definition 4.12. Setting $\hbar = 0$ defines the *equivariant Seidel representation*,

$$\mathbb{S}_{G,\mathbf{N}}^{\hbar=0} \colon e(\mathcal{S}_\mathbf{N}) \cap H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(\mathrm{Gr}_G) \otimes_\mathbb{C} QH_G^\bullet(X)[[q_G,\tau]] \longrightarrow QH_G^\bullet(X)[[q_G,\tau]].$$

The *equivariant Seidel map* is then defined as the map

$$\begin{aligned}\Psi_{G,\mathbf{N},X} \colon e(\mathcal{S}_\mathbf{N}) \cap H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}_\hbar^\times}(\mathrm{Gr}_G) &\longrightarrow QH_G^\bullet(X)[[q_G,\tau]], \\ \Gamma &\longmapsto \mathbb{S}_{G,\mathbf{N}}^{\hbar=0}(\Gamma, 1).\end{aligned}$$

Remark 4.13. Similar to Definition 4.11, we may define the localized Seidel representation and the localized Seidel map by extending $\mathrm{Frac}\, H^\bullet_{T\times\mathbb{C}_\hbar^\times}(\mathrm{pt})$-linearly. We will denote them by $\mathbb{S}_{G,\mathbf{N},\mathrm{loc}}^{\hbar=0}$ and $\Psi_{G,\mathbf{N},\mathrm{loc}}$ respectively.

4.3. **Independence of resolutions.** In this subsection, we prove basic properties of $\mathbb{S}_{G,\mathbf{N}}$, including the independence of choice of resolutions $\widetilde{C}_{\leq\lambda}$.

Proposition 4.14. *Let $P \in H^\bullet_{G\times\mathbb{C}^\times_\hbar}(\mathrm{pt})$. Then*
$$\mathbb{S}_{G,\mathbf{N}}(\Gamma\otimes(P\cup\alpha)) = \mathbb{S}_{G,\mathbf{N}}((u^*P\cap\Gamma)\otimes\alpha),$$
where $u^* : H^\bullet_G(\mathrm{pt}) \to H^\bullet_{T_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}(\mathrm{Gr}_G)$ is the second module structure defined in Equation (23).

Proof. It follows from Proposition 2.7 and the projection formula. $\square$

The following proposition explains the origin of the name "shift operator".

Proposition 4.15. *The map $\mathbb{S}_{T,\mathbf{N}}$ is twisted-linear, i.e., for $P(a,\hbar) \in H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathrm{pt})$ and any cocharacter λ,*
$$\mathbb{S}_{T,\mathbf{N}}(e(\mathcal{S}_{t^\lambda})\cap[t^\lambda]\otimes P(a,\hbar)\alpha) = P(a+\lambda(\hbar),\hbar)\,\mathbb{S}_{T,\mathbf{N}}(e(\mathcal{S}_{t^\lambda})\cap[t^\lambda]\otimes\alpha).$$

Proof. It follows from Proposition 2.9 and the fact that $\widetilde{\mathbb{S}}_{\mathbf{N}\leq\lambda}$ is $H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathrm{pt})$-linear. $\square$

Let 1 denote the class $[C_{\leq 0}] \in H^{T_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}_\bullet(\mathrm{Gr}_G)$. For any G-representation $\mathbf{N}$, we have $\mathcal{S}_{\leq 0} = 0$, so that $1 \in e(\mathcal{S}_\mathbf{N})\cap H^{T_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}_\bullet(\mathrm{Gr}_G)$.

Proposition 4.16. *The operator $\mathbb{S}_{G,\mathbf{N}}(1,-)$ is equal to the inclusion*
$$H^\bullet_{G\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]] \subset H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]].$$

Proof. Note that $\mathcal{S}_{t^0} = 0$, and by definition, $\mathrm{tw}_{\leq 0}$ is the inclusion map. Therefore, it suffices to show that the section-counting map
$$\widetilde{\mathbb{S}}_{\leq 0} : H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]] \longrightarrow H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]]$$
is the identity. Let $\gamma,\gamma' \in H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X)$. Since
$$\overline{M}_{0,2}(X\times\mathbb{P}^1,(0,1)) = X\times\overline{M}_{0,2}(\mathbb{P}^1,1),$$
it is immediate that that $\langle\iota_{0*}\gamma,\iota_{\infty*}\gamma'\rangle^{X\times\mathbb{P}^1}_{0,(0,1)} = (\gamma,\gamma')$. Note that $\hat{\tau} = \tau\otimes 1$. Hence, it suffices to prove that the Gromov–Witten invariant
$$\left\langle \gamma\otimes\mathrm{PD}([0]),\ \underbrace{\tau\otimes 1,\ldots,\tau\otimes 1}_{n\text{ times}},\ \gamma'\otimes\mathrm{PD}([\infty])\right\rangle^{X\times\mathbb{P}^1}_{0,(\beta,1)} \tag{39}$$
vanishes whenever $\beta\neq 0$ or $n\neq 0$. Indeed, the forgetful morphism
$$\mathrm{pr} : \overline{M}_{0,n+2}(X\times\mathbb{P}^1,(\beta,1)) \to \overline{M}_{0,n+2}(X,\beta)$$
has relative dimension 3 if $\beta\neq 0$ or $n\neq 0$. Therefore,
$$\mathrm{pr}_*\left(\mathrm{Ev}_0^*(1\otimes\mathrm{PD}([0]))\cup\mathrm{Ev}_\infty^*(1\otimes\mathrm{PD}([\infty]))\right) = 0,$$
and by the projection formula, the invariant in (39) vanishes when $\beta\neq 0$ or $n\neq 0$. $\square$

Proposition 4.17. *The operator $\mathbb{S}_{G,\mathbf{N}}$ is independent of the choices of resolutions $\widetilde{C}_{\leq\lambda}$.*

Proof. Suppose $\widetilde{C}'_{\leq\lambda} \to C_{\leq\lambda}$ is another resolution of $C_{\leq\lambda}$ on which $\mathcal{S}_\lambda$ extends as in Lemma 1.2. Replacing $\widetilde{C}'_{\leq\lambda}$ by a resolution of the fibre product $\widetilde{C}'_{\leq\lambda}\times_{C_{\leq\lambda}}\widetilde{C}_{\leq\lambda}$ if necessary, we may assume $\widetilde{C}'_{\leq\lambda}\to C_{\leq\lambda}$ factors as
$$\widetilde{C}'_{\leq\lambda} \xrightarrow{r} \widetilde{C}_{\leq\lambda} \longrightarrow C_{\leq\lambda}.$$
Let us denote
$$\widetilde{\alpha}_{\leq\lambda} = \mathrm{tw}_{\leq\lambda}\left([\widetilde{C}_{\leq\lambda}]\otimes\alpha\right),\quad \widetilde{\alpha}'_{\leq\lambda} = \mathrm{tw}'_{\leq\lambda}\left([\widetilde{C}'_{\leq\lambda}]\otimes\alpha\right).$$

Let $\mathcal{E}_{\leq\lambda}(X)$ and $\mathcal{E}'_{\leq\lambda}(X)$ denote the Seidel spaces corresponding to $\widetilde{C}_{\leq\lambda}$ and $\widetilde{C}'_{\leq\lambda}$, respectively. We write $\hat{\tau} \in H^{\bullet}_{T\times\mathbb{C}^{\times}_{\hbar}}(\mathcal{E}_{\leq\lambda}(X))$ and $\hat{\tau}' \in H^{\bullet}_{T\times\mathbb{C}^{\times}_{\hbar}}(\mathcal{E}'_{\leq\lambda}(X))$ for the assignments (30) associated to $\mathcal{E}_{\leq\lambda}(X)$ and $\mathcal{E}'_{\leq\lambda}(X)$, respectively. By Lemma 2.5 and the projection formula, these classes satisfy

$$\widetilde{\alpha}_{\leq\lambda} = r_*\widetilde{\alpha}'_{\leq\lambda}, \quad \hat{\tau} = r_*\hat{\tau}'. \tag{40}$$

Here, we use the same symbol r_* to denote the pushforward maps $H^{\bullet}_{T_{\mathcal{O}}\rtimes\mathbb{C}^{\times}_{\hbar}}(\widetilde{G}'^{\leq\lambda}_{\mathcal{K}} \times_{G_{\mathcal{O}}} X) \to H^{\bullet}_{T_{\mathcal{O}}\rtimes\mathbb{C}^{\times}_{\hbar}}(\widetilde{G}^{\leq\lambda}_{\mathcal{K}} \times_{G_{\mathcal{O}}} X)$ and $H^{\bullet}_{T\times\mathbb{C}^{\times}_{\hbar}}(\mathcal{E}'_{\leq\lambda}(X)) \to H^{\bullet}_{T\times\mathbb{C}^{\times}_{\hbar}}(\mathcal{E}_{\leq\lambda}(X))$, whenever there is no confusion.

Write $\widetilde{\mathbb{S}}$ and $\widetilde{\mathbb{S}}'$ for the respective section-counting maps (4.5). It remains to verify that

$$\widetilde{\mathbb{S}}(e(\widetilde{\mathcal{S}}_{\leq\lambda}) \cup \widetilde{\alpha}'_{\leq\lambda}) = \widetilde{\mathbb{S}}'(e(\widetilde{\mathcal{S}}_{\leq\lambda}) \cup \widetilde{\alpha}_{\leq\lambda}).$$

Note that the moduli spaces are related by the fibre diagram

$$\begin{tikzcd}
\overline{M}_{0,n+2}(\mathcal{E}'_{\leq\lambda}(X),\beta) \arrow[r] \arrow[d, "\mathrm{pr}_{\widetilde{C}'_{\leq\lambda}}"'] & \overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X, r_*\beta) \arrow[d, "\mathrm{pr}_{\widetilde{C}_{\leq\lambda}}"] \\
\widetilde{C}'_{\leq\lambda} \arrow[r, "r"] & \widetilde{C}_{\leq\lambda}.
\end{tikzcd}$$

Therefore, by functoriality (see [BF97, Proposition 5.10]), the virtual fundamental classes are related by Gysin pullback

$$[\overline{M}_{0,n+2}(\mathcal{E}'_{\leq\lambda}(X),\beta)]^{\mathrm{vir}} = r^![\overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X), r_*\beta)]^{\mathrm{vir}},$$

and hence the same is true for the zero loci (see [Ful98, Section 6.4])

$$[\mathcal{Z}'_{\leq\lambda}(X,\beta)_n]^{\mathrm{vir}} = r^![\mathcal{Z}_{\leq\lambda}(X, r_*\beta)_n]^{\mathrm{vir}}.$$

Denote by $\mathrm{ev}'_0, \mathrm{ev}'_\infty, \mathrm{pr}'_X$ the evaluation and projection maps for the spaces associated with $\widetilde{C}'_{\leq\lambda}$. Using the above, we then compute

$$\begin{aligned}
\widetilde{\mathbb{S}}'(e(\widetilde{\mathcal{S}}_{\leq\lambda}) \cup \widetilde{\alpha}'_{\leq\lambda}) &= \sum_{\beta}\sum_{n=1}^{\infty} \frac{q^{\overline{\beta}}}{n!} \mathrm{PD} \circ \mathrm{pr}'_{X*}\, \mathrm{ev}'_{\infty*} \left(\mathrm{ev}'^*_0(\widetilde{\alpha}'_{\leq\lambda}) \prod_{\ell=1}^{n} \mathrm{ev}'^*_\ell(\hat{\tau}') \cap [\mathcal{Z}'_{\leq\lambda}(X,\beta)_n]^{\mathrm{vir}} \right) \\
&= \sum_{\beta}\sum_{n=1}^{\infty} \frac{q^{\overline{\beta}}}{n!} \mathrm{PD} \circ \mathrm{pr}_{X*}\, \mathrm{ev}_{\infty*}\, r_* \left(\mathrm{ev}'^*_0(\widetilde{\alpha}'_{\leq\lambda}) \prod_{\ell=1}^{n} \mathrm{ev}'^*_\ell(\hat{\tau}') \cap r^![\mathcal{Z}_{\leq\lambda}(X, r_*\beta)_n]^{\mathrm{vir}} \right) \\
&= \sum_{\beta}\sum_{n=1}^{\infty} \frac{q^{\overline{\beta}}}{n!} \mathrm{PD} \circ \mathrm{pr}_{X*}\, \mathrm{ev}_{\infty*} \left(\mathrm{ev}^*_0(\widetilde{\alpha}_{\leq\lambda}) \prod_{\ell=1}^{n} \mathrm{ev}^*_\ell(\hat{\tau}) \cap [\mathcal{Z}_{\leq\lambda}(X,\beta)_n]^{\mathrm{vir}} \right) \\
&= \widetilde{\mathbb{S}}(e(\widetilde{\mathcal{S}}_{\leq\lambda}) \cup \widetilde{\alpha}_{\leq\lambda}).
\end{aligned}$$

Here, we have used the identification $\overline{r_*\beta} = \overline{\beta}$ and the last equality is by projection formula and (40). □

5. Shift operators III: properties

In this section, we establish several key properties of the shift operators. In particular, we prove Theorem 1 and Theorem 2 as stated in the introduction. Throughout, we assume that the pair $(X, f\colon X \to \mathbf{N})$ satisfies the assumptions outlined in Section 4.

5.1. Change of representations and groups.

In this subsection, we study the compatibility of shift operators under changes of the representation or the group. All the properties proved in this subsection are stated for $\mathbb{S}_{G,\mathbf{N}}$, but the analogous statements all hold for $\mathbb{S}^{\hbar=0}_{G,\mathbf{N}}$ and $\Psi_{G,\mathbf{N}}$ with similar proofs (e.g., by specializing $\hbar = 0$ in all the constructions and maps).

Proposition 5.1 (Change of representations)**.** *Suppose $\mathbf{V}$ is another G-representation, and either of the following holds:*

- *(a) $\mathbf{V}$ contains $\mathbf{N}$ as a subrepresentation.*
- *(b) $\mathbf{N}$ is a quotient representation of $\mathbf{V}$, and the morphism $f : X \to \mathbf{N}$ factors through $\mathbf{V} \to \mathbf{N}$.*

Then $\mathbb{S}_{G,\mathbf{V}}$ *is equal to the restriction of* $\mathbb{S}_{G,\mathbf{N}}$ *along*

$$e(\mathcal{S}_{\mathbf{V}}) \cap H_\bullet^{T_{\mathcal{O}} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G) \subset e(\mathcal{S}_{\mathbf{N}}) \cap H_\bullet^{T_{\mathcal{O}} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G).$$

Proof. We prove case (a); the other case proceeds similarly.

Write $\mathbf{V} = \mathbf{N} \oplus \mathbf{N}'$, and choose a resolution $\widetilde{C}_{\leq\lambda}$ such that both $\mathcal{S}_{\mathbf{N}}$ and $\mathcal{S}_{\mathbf{N}'}$ extend over it. Since the composition $X \to \mathbf{V} \to \mathbf{N}'$ is zero, we have $\mathrm{can}^{\mathbf{N}'}_{\leq\lambda}(X) = 0$. Moreover, since $\mathrm{can}^{\mathbf{V}}_{\leq\lambda}(X) = (\mathrm{can}^{\mathbf{N}}_{\leq\lambda}(X), \mathrm{can}^{\mathbf{N}'}_{\leq\lambda}(X))$, it follows that

$$\mathcal{Z}^{\mathbf{V}}_{\leq\lambda}(X,\beta)_n = \mathcal{Z}^{\mathbf{N}}_{\leq\lambda}(X,\beta)_n$$

for any $\beta \in \mathrm{Eff}(\mathcal{E}_{\leq\lambda}(X))^{\mathrm{sec}}$.

We also have the identity

$$[\mathcal{Z}^{\mathbf{V}}_{\leq\lambda}(X,\beta)_n]^{\mathrm{vir}} = e(\widetilde{\mathcal{S}}_{\mathbf{N}',\leq\lambda}) \cap [\mathcal{Z}^{\mathbf{N}}_{\leq\lambda}(X,\beta)_n]^{\mathrm{vir}},$$

which follows from (35) and excess intersection formula associated to the diagram:

$$\begin{tikzcd}
\mathcal{Z}_{\leq\lambda}(X,\beta)_n \arrow[r, "j_X"] \arrow[d] & \mathcal{M}_{\leq\lambda}(X,\beta)_n \arrow[d, "\mathrm{can}^{\mathbf{N}}_{\leq\lambda}(X)"] \\
\widetilde{C}_{\leq\lambda} \arrow[r] \arrow[d, equal] & \widetilde{\mathcal{S}}_{\mathbf{N},\leq\lambda} \arrow[d] \\
\widetilde{C}_{\leq\lambda} \arrow[r] & \widetilde{\mathcal{S}}_{\mathbf{V},\leq\lambda} = \widetilde{\mathcal{S}}_{\mathbf{N},\leq\lambda} \oplus \widetilde{\mathcal{S}}_{\mathbf{N}',\leq\lambda}
\end{tikzcd}$$

Let ev'_0 and ev'_∞ denote the evaluation maps on $\mathcal{Z}^{\mathbf{V}}_{\leq\lambda}(X,\beta)_n$, and writing $\widetilde{\alpha}_{\leq\lambda} = \mathrm{tw}_{\leq\lambda}([\widetilde{C}_{\leq\lambda}] \otimes \alpha)$. Then we verify

$$\begin{aligned}
&\mathbb{S}_{G,\mathbf{N}}(e(\mathcal{S}_{\mathbf{V}}) \cap [C_{\leq\lambda}] \otimes \alpha) \\
&= \widetilde{\mathbb{S}}_{\mathbf{N},\leq\lambda}\left(e(\widetilde{\mathcal{S}}_{\mathbf{N},\leq\lambda}) \cup e(\widetilde{\mathcal{S}}_{\mathbf{N}',\leq\lambda}) \cup \widetilde{\alpha}_{\leq\lambda}\right) \\
&= \sum_{n=0}^{\infty}\sum_{n=0}^{\infty} \frac{q^{\overline{\beta}}}{n!} \mathrm{PD} \circ \mathrm{pr}_{X*}\,\mathrm{ev}_{\infty*}\left(\mathrm{ev}_0^*\left(\widetilde{\alpha}_{\leq\lambda} \cup e(\widetilde{\mathcal{S}}_{\mathbf{N}',\leq\lambda})\right)\prod_{\ell=1}^{n} \mathrm{ev}_\ell^*(\hat{\tau}) \cap [\mathcal{Z}^{\mathbf{N}}_{\leq\lambda}(X,\beta)_n]^{\mathrm{vir}}\right) \\
&= \sum_{\beta}\sum_{n=0}^{\infty} \frac{q^{\overline{\beta}}}{n!} \mathrm{PD} \circ \mathrm{pr}_{X*}\,\mathrm{ev}'_{\infty*}\left(\mathrm{ev}_0'^*(\widetilde{\alpha}_{\leq\lambda})\prod_{\ell=1}^{n} \mathrm{ev}_\ell'^*(\hat{\tau}) \cap [\mathcal{Z}^{\mathbf{V}}_{\leq\lambda}(X,\beta)_n]^{\mathrm{vir}}\right) \\
&= \mathbb{S}_{G,\mathbf{V}}(e(\mathcal{S}_{\mathbf{V}}) \cap [C_{\leq\lambda}] \otimes \alpha).
\end{aligned}$$

□

Corollary 5.2. *Suppose* $\mathbf{N}'$ *is another representation of* G*, and* $g\colon X \to \mathbf{N}'$ *is a* G*-equivariant proper morphism. Then*

$$\mathbb{S}_{G,\mathbf{N}}(\Gamma \otimes \alpha) = \mathbb{S}_{G,\mathbf{N}'}(\Gamma \otimes \alpha)$$

for any Γ *in the intersection of* $(e(\mathcal{S}_{\mathbf{N}}) \cap H_\bullet^{T_{\mathcal{O}} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G))$ *and* $(e(\mathcal{S}_{\mathbf{N}'}) \cap H_\bullet^{T_{\mathcal{O}} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G))$*, and* $\alpha \in H^\bullet_{G\times\mathbb{C}_\hbar^\times}(X)$.

Proof. Let $\mathbf{V} = \mathbf{N} \oplus \mathbf{N}'$, and let $P \in H^\bullet_{T\times\mathbb{C}_\hbar^\times}(\mathrm{pt})$ be such that $P\Gamma \in e(\mathcal{S}_{\mathbf{V}}) \cap H_\bullet^{T_{\mathcal{O}} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G)$. By Proposition 5.1, we have

$$\mathbb{S}_{G,\mathbf{N}}(P\Gamma \otimes \alpha) = \mathbb{S}_{G,\mathbf{V}}(P\Gamma \otimes \alpha) = \mathbb{S}_{G,\mathbf{N}'}(P\Gamma \otimes \alpha).$$

The result then follows from the lemma below by taking $R = H^\bullet_{T\times\mathbb{C}_\hbar^\times}(\mathrm{pt})$. □

Lemma 5.3. *Let* R *be an integral domain, and let* $\zeta, \zeta'\colon A \to B$ *be homomorphisms of torsion-free* R*-modules. Suppose that* ζ *and* ζ' *agree on a submodule* $A' \subset A$ *such that*

$$\mathrm{Frac}(R) \otimes_R A = \mathrm{Frac}(R) \otimes_R A'.$$

Then $\zeta = \zeta'$.

Proof. Let $a \in A$. Since $\mathrm{Frac}(R) \otimes_R A = \mathrm{Frac}(R) \otimes_R A'$, there exists $r \in R \setminus \{0\}$ such that $ra \in A'$. Then

$$r\zeta(a) = \zeta(ra) = \zeta'(ra) = r\zeta'(a),$$

and since B is torsion-free over R, it follows that $\zeta(a) = \zeta'(a)$. $\square$

Recall that the localization $\left(e(\mathcal{S}_{\mathbf{N}}) \cap H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G)\right)_{\mathrm{loc}}$ is independent of the representation $\mathbf{N}$ (see (16)). The following is now immediate.

Corollary 5.4 (Independence of $\mathbf{N}$ and f)**.** *The localized shift operators $\mathbb{S}_{G,\mathbf{N},\mathrm{loc}}$ is independent of $f : X \to \mathbf{N}$.*

Change of groups. Next, we study the functoriality with respect to changing groups.

Following [BFN18, 3(vii)(c)], let $G' \to G$ be a finite covering, and suppose that $\mathbf{N}$ is a G-representation, so it also has an action of G'. Then the Pontrjagin dual F of $\pi_1(G)/\pi_1(G')$ acts on $e(\mathcal{S}_{\mathbf{N}}) \cap H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G)$ such that

$$e(\mathcal{S}_{G',\mathbf{N}}) \cap H_\bullet^{T'_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_{G'}) = \left(e(\mathcal{S}_{G,\mathbf{N}}) \cap H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G)\right)^F.$$

Proposition 5.5. *Let $G' \to G$ be a finite covering, and let $\mathbf{N}$ be a G-representation, then $\mathbb{S}_{G',\mathbf{N}'}$ is the restriction of $\mathbb{S}_{G,\mathbf{N}}$ to $e(\mathcal{S}_{\mathbf{N}'}) \cap H_\bullet^{T'_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_{G'})$.*

Proof. Note that the equivariant cohomology with respect to G and G' coincides since the coefficient is $\mathbb{C}$. The map $G' \to G$ induces a closed embedding $\mathrm{Gr}_{G'} \hookrightarrow \mathrm{Gr}_G$. For a coweight $\lambda' \in \Lambda_{G'}$ mapping to $\lambda \in \Lambda_G$, there is a natural identification $C_{\leq\lambda'} \cong C_{\leq\lambda}$, which induces an isomorphism $\widetilde{\mathcal{S}}_{G',\leq\lambda'} \cong \widetilde{\mathcal{S}}_{G,\leq\lambda}$. Therefore, $\mathbb{S}_{G',\mathbf{N}}(e(\widetilde{\mathcal{S}}_{\leq\lambda'}) \cap [\widetilde{C}_{\leq\lambda'}] \otimes \alpha)$ coincides with $\mathbb{S}_{G,\mathbf{N}}(e(\widetilde{\mathcal{S}}_{\leq\lambda}) \cap [\widetilde{C}_{\leq\lambda}] \otimes \alpha)$. $\square$

Next, consider a short exact sequence $1 \to G \to \widehat{G} \to T_F \to 1$ of connected reductive groups, where T_F is a torus. Let $\mathbf{N}$ be a $\widehat{G}$-representation. Recall that the quantized Coulomb branch algebras for $(\widehat{G}, \mathbf{N})$ and $(G, \mathbf{N})$ are related via *quantum Hamiltonian reduction* with respect to $T_F^\vee$, the Pontrjagin dual of $\pi_1(T_F)$ (see [BFN18, Proposition 3.18]). Specifically, we have

$$\left(e(\mathcal{S}_{\widehat{G},\mathbf{N}}) \cap H_\bullet^{\widehat{T}_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_{\widehat{G}})\right)^{T_F^\vee} \;\cong\; e(\mathcal{S}_{\widehat{G},\mathbf{N}}) \cap H_\bullet^{\widehat{G}_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G), \tag{41}$$

and

$$e(\mathcal{S}_{G,\mathbf{N}}) \cap H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G) \cong \left(e(\mathcal{S}_{\widehat{G},\mathbf{N}}) \cap H_\bullet^{\widehat{G}_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G)\right) \otimes_{H^\bullet_{\widehat{G} \times \mathbb{C}_\hbar^\times}(\mathrm{pt})} H^\bullet_{G \times \mathbb{C}_\hbar^\times}(\mathrm{pt}). \tag{42}$$

Now suppose that X has a $\widehat{G}$-action such that $f : X \to \mathbf{N}$ is $\widehat{G}$-equivariant and proper. Consider the shift operator $\mathbb{S}_{\widehat{G},\mathbf{N}}$. We will construct a reduced operator $\mathbb{S}^{\mathrm{red}}_{\widehat{G},\mathbf{N}}$ as follows.

First, we restrict $\mathbb{S}_{\widehat{G},\mathbf{N}}$ to the subalgebra (41). This map descends to $H^\bullet_{\widehat{G} \times \mathbb{C}^\times}(X)[[q_G, \tau]]$ (rather than $q_{\widehat{G}}$), since for any section class β of the Seidel space corresponding to a coweight $\lambda \in \Lambda_G \subset \Lambda_{\widehat{G}}$, the assignment $\beta \mapsto \overline{\beta}$ factors through $H_2^G(X, \mathbb{Z}) \to H_2^{\widehat{G}}(X, \mathbb{Z})$. Next, we quotient out the T_F-equivariant parameters by composing with the natural projection map $H^\bullet_{\widehat{T} \times \mathbb{C}_\hbar^\times}(X)[[q_G, \tau]] \to H^\bullet_{T \times \mathbb{C}_\hbar^\times}(X)[[q_G, \tau]]$. Finally, by Proposition 4.15, the map descends to

$$\mathbb{S}^{\mathrm{red}}_{\widehat{G},\mathbf{N}} : e(\mathcal{S}_{G,\mathbf{N}}) \cap H_\bullet^{T_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G) \otimes_{\mathbb{C}[\hbar]} H^\bullet_{G \times \mathbb{C}_\hbar^\times}(X)[[q_G, \tau]] \longrightarrow H^\bullet_{T \times \mathbb{C}_\hbar^\times}(X)[[q_G, \tau]].$$

Proposition 5.6 (Quantum Hamiltonian reduction)**.** *The shift operator $\mathbb{S}_{G,\mathbf{N}}$ coincides with the reduced operator $\mathbb{S}^{\mathrm{red}}_{\widehat{G},\mathbf{N}}$.*

Proof. This follows directly from the construction, after specializing the twisting map, the evaluation maps, and the projection map from $\widehat{G}$- (or $\widehat{T}$-)equivariance to G- (or T-)equivariance. The only map that requires verification is $\mathrm{tw}_{\leq\lambda}$. Note that for any $f \in H^\bullet_{T_F}(\mathrm{pt})$ and $\Gamma \subset H_\bullet^{\widehat{T}_\mathcal{O} \rtimes \mathbb{C}_\hbar^\times}(\mathrm{Gr}_G)$ we have

$$\mathrm{tw}_{\leq\lambda}(\Gamma \otimes (f \cup \alpha)) = f \cup \mathrm{tw}_{\leq\lambda}(\Gamma \otimes \alpha).$$

In other words, the map $\mathrm{tw}_{\leq\lambda}$ for the group $\widehat{G}$, when restricted to the subalgebra in (41), descends to the quotient in (42). $\square$

5.2. **Properties of shift operators: the T-compact case.** In this subsection, we assume that the T-fixed locus X^T is compact. We will explain in the next subsection how to remove this assumption.

Extension to localizations. Let $\lambda \in \Lambda$ and $\beta \in \mathrm{Eff}(\mathcal{E}_{\leq\lambda}(X))^{\mathrm{sec}}$. By comparing (32) and (35), and applying the excess intersection formula ([Ful98, Theorem 6.3]), we obtain:

$$\begin{aligned}(j_X)_*[\mathcal{Z}_{\leq\lambda}(X,\beta)_n]^{\mathrm{vir}} &= e(\widetilde{\mathcal{S}}_{\leq\lambda}) \cap [\mathcal{M}_{\leq\lambda}(X,\beta)_n]^{\mathrm{vir}} \\ &= e(\widetilde{\mathcal{S}}_{\leq\lambda}) \cap j^!_{\mathcal{M}}\left[\overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X),\beta)\right]^{\mathrm{vir}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\widetilde{\mathbb{S}}_{\mathbf{N},\leq\lambda}(e(\widetilde{\mathcal{S}}_{\leq\lambda}) \cup \gamma) \\ &= \sum_{\beta}\sum_{n=0}^{\infty}\frac{q^{\overline{\beta}}}{n!}\,\mathrm{PD}\circ \mathrm{pr}_{X*}\,\mathrm{ev}_{\infty*}\Big(\mathrm{ev}_0^*(\gamma)\prod_{\ell=1}^{n}\mathrm{ev}_\ell^*(\hat{\tau}) \cap [\mathcal{Z}_{\leq\lambda}(X,\beta)_n]^{\mathrm{vir}}\Big) \\ &= \sum_{\beta}\sum_{n=0}^{\infty}\frac{q^{\overline{\beta}}}{n!}\,\mathrm{PD}\circ \mathrm{pr}_{X*}\,\mathrm{ev}_{\infty*}\Big(\mathrm{ev}_0^*\big(\gamma \cup e(\widetilde{\mathcal{S}}_{\leq\lambda})\big)\prod_{\ell=1}^{n}\mathrm{ev}_\ell^*(\hat{\tau}) \cap j^!_{\mathcal{M}}\big[\overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X),\beta)\big]^{\mathrm{vir}}\Big) \\ &= \sum_{\beta}\sum_{n=0}^{\infty}\frac{q^{\overline{\beta}}}{n!}\,\mathrm{PD}\circ \mathrm{pr}_{X*}\,\iota_\infty^*\,\mathrm{Ev}_{\infty*}\Big(\mathrm{Ev}_0^*\,\iota_{0*}\big(\gamma \cup e(\widetilde{\mathcal{S}}_{\leq\lambda})\big)\prod_{\ell=1}^{n}\mathrm{Ev}_\ell^*(\hat{\tau}) \cap \big[\overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X),\beta)\big]^{\mathrm{vir}}\Big).\end{aligned}$$

Here, all the pushforwards are understood as using localization.

Originally, $\mathbb{S}_{\mathbf{N},\leq\lambda}$ is defined on the subspace $e(\widetilde{\mathcal{S}}_{\leq\lambda}) \cap H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathcal{X}_{\leq\lambda,0})$. By the above computation, we may extend the definition of $\mathbb{S}_{\mathbf{N},\leq\lambda}$ to $H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathcal{X}_{\leq\lambda,0})_{\mathrm{loc}}$ to obtain a map $\widetilde{\mathbb{S}}_{\leq\lambda}$ that is independent of $\mathbf{N}$. For $\gamma \in H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathcal{X}_{\leq\lambda,0})_{\mathrm{loc}}$, we set

$$\widetilde{\mathbb{S}}_{\leq\lambda}(\gamma) := \sum_{\beta}\sum_{n=0}^{\infty}\frac{q^{\overline{\beta}}}{n!}\,\mathrm{PD}\circ \mathrm{pr}_{X*}\,\iota_\infty^*\,\mathrm{Ev}_{\infty*}\Big(\mathrm{Ev}_0^*\,\iota_{0*}(\gamma)\prod_{\ell=1}^{n}\mathrm{Ev}_\ell^*(\hat{\tau}) \cap \big[\overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X),\beta)\big]^{\mathrm{vir}}\Big).$$

Equivalently, let $\{\eta_i\}$ is a $H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathrm{pt})$-basis of $H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathcal{X}_{\leq\lambda,0})$, and $\{\eta^i\}$ the dual basis under the Poincaré pairing. Then

$$\widetilde{\mathbb{S}}_{\leq\lambda}(\gamma) = \sum_{\beta,i}\sum_{n=0}^{\infty}\frac{q^{\overline{\beta}}}{n!}\,\eta_i\int_{[\overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X),\beta)]^{\mathrm{vir}}}\mathrm{Ev}_0^*\,\iota_{0*}(\gamma)\prod_{\ell=1}^{n}\mathrm{Ev}_\ell^*(\hat{\tau})\,\mathrm{Ev}_\infty^*\,\iota_{\infty*}\,\mathrm{pr}_X^*(\eta^i). \tag{43}$$

Similarly, for any $\mu \in \Lambda$ and $\gamma \in H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathcal{X}_{\leq\lambda,0})_{\mathrm{loc}}$, we define:

$$\widetilde{\mathbb{S}}_{t^\mu}(\gamma) = \sum_{\beta,i}\sum_{n=0}^{\infty}\frac{q^{\overline{\beta}}}{n!}\,\eta_i\int_{[\overline{M}_{0,2}(\mathcal{E}_{t^\mu}(X),\beta)]^{\mathrm{vir}}}\mathrm{Ev}_0^*\,\iota_{0*}(\gamma)\prod_{\ell=1}^{n}\mathrm{Ev}_\ell^*(\hat{\tau})\,\mathrm{Ev}_\infty^*\,\iota_{\infty*}(\eta^i). \tag{44}$$

A localization formula. Fix $\lambda \in \Lambda$. For any $\mu \leq \lambda$, let F_μ denote the $T \times \mathbb{C}^\times_\hbar$-fixed locus of $\rho_\lambda^{-1}(t^\mu)$. Let $e_\mu \in H^\bullet_{T\times\mathbb{C}^\times_\hbar}(F_\mu)$ be the Euler class of the normal bundle of F_μ in $\widetilde{C}_{\leq\lambda}$.

Recall that we denote $X^\mu = t^\mu G_\mathcal{O} \times_{G_\mathcal{O}} X \subset G_\mathcal{K} \times_{G_\mathcal{O}} X$. The restriction of the X-bundle $\mathcal{X}_{\leq\lambda,0} \to \widetilde{C}_{\leq\lambda}$ to F_μ is naturally identified with $F_\mu \times X^\mu$. Let $\rho_{X,\lambda}\colon \mathcal{X}_{\leq\lambda,0} \to G_\mathcal{K}^{\leq\lambda} \times_{G_\mathcal{O}} X$ be the resolution map induced by ρ_λ (see Lemma 3.12), whose restriction to $F_\mu \times X^\mu$ is equal to the projection map

$$\mathrm{pr}_{X^\mu}\colon F_\mu \times X^\mu \to X^\mu.$$

Since each F_μ is proper, we can define the pushforward $(\rho_{X,\lambda})_*$ in cohomology via localization.

Let $\gamma \in H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathcal{X}_{\leq\lambda,0})_{\mathrm{loc}}$, and define

$$\gamma_\mu = (\mathrm{pr}_{X^\mu})_*\left(\frac{1}{e_\mu}\cdot \gamma\big|_{F_\mu\times X^\mu}\right). \tag{45}$$

Then we have the equality

$$(\rho_{X,\lambda})_*\gamma = \sum_{\mu\leq\lambda}\gamma_\mu$$

in the localized cohomology ring

$$H^\bullet_{T\times\mathbb{C}^\times_\hbar}(G_\mathcal{K}^{\leq\lambda}\times_{G_\mathcal{O}} X)_{\mathrm{loc}} \cong \bigoplus_{\mu\leq\lambda} H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X^\mu)_{\mathrm{loc}}.$$

Proposition 5.7. *For any $\gamma\in H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathcal{X}_{\leq\lambda,0})$, let γ_μ be defined by (45). Then we have*

$$\widetilde{\mathbb{S}}_{\leq\lambda}(\gamma)=\sum_\mu \widetilde{\mathbb{S}}_{t^\mu}(\gamma_\mu).$$

Proof. We apply the virtual localization formula [GP99], following closely the proof of [Cho23, Proposition 3.12].

We write $\overline{M}_n=\overline{M}_{0,n+2}(\mathcal{E}_{\leq\lambda}(X),\beta)$ and $\overline{M}_{\mu,n}=\overline{M}_{0,n+2}(\mathcal{E}_{t^\mu}(X),\beta)$ for a section class β in $\mathcal{E}_{\leq\lambda}(X)$. The latter is empty if β is not represented by a curve contained in $\mathcal{E}_{t^\mu}(X)$. We have

$$j_\mu : F_\mu\times\mathcal{E}_{t^\mu}(X)\cong F_\mu\times_{\widetilde{C}_{\leq\lambda}}\mathcal{E}_{\leq\lambda}(X)\hookrightarrow\mathcal{E}_{\leq\lambda}(X),$$

and hence inducing

$$\bar{j}_\mu : F_\mu\times\overline{M}_{\mu,n}\cong F_\mu\times_{\widetilde{C}_{\leq\lambda}}\overline{M}_n\hookrightarrow\overline{M}_n.$$

Taking $T\times\mathbb{C}^\times_\hbar$-fixed loci, we obtain

$$\overline{M}_n^{T\times\mathbb{C}^\times_\hbar}=\bigsqcup_{\mu\leq\lambda}F_\mu\times\overline{M}_{\mu,n}^{T\times\mathbb{C}^\times_\hbar}.$$

Therefore, the virtual localization formula implies that

$$[\overline{M}_n]^{\mathrm{vir}}=\sum_{\mu\leq\lambda}\bar{j}_{\mu*}\left(\frac{1}{e_\mu}[F_\mu]\times[\overline{M}_{\mu,n}]^{\mathrm{vir}}\right). \tag{46}$$

Now let $\phi\in H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X)$. Let us write $\mathrm{Ev}_{\mu,i}:\overline{M}_{\mu,n}\to\mathcal{E}_{t^\mu}(X)$ for the evaluation maps on $\overline{M}_\mu$, and $\mathrm{pr}_{\overline{M}_\mu}$: $F_\mu\times\overline{M}_{\mu,n}\to\overline{M}_{\mu,n}$ be the projection. We compute:

$$\begin{aligned}
&\int_{[\overline{M}_n]^{\mathrm{vir}}}\mathrm{Ev}_0^*\,\iota_{0*}(\gamma)\prod_{\ell=1}^n\mathrm{Ev}_\ell^*(\hat{\tau})\ \mathrm{Ev}_\infty^*\,\iota_{\infty*}\,\mathrm{pr}_X^*(\phi)\\
&=\sum_{\mu\leq\lambda}\int_{[F_\mu]\times[\overline{M}_{\mu,n}]^{\mathrm{vir}}}\frac{1}{e_\mu}\,\bar{j}_\mu^*\,\mathrm{Ev}_0^*\,\iota_{0*}(\gamma)\prod_{\ell=1}^n\mathrm{Ev}_\ell^*(\hat{\tau})\,j_\mu^*\,\mathrm{Ev}_\infty^*\,\iota_{\infty*}\,\mathrm{pr}_X^*(\phi)\\
&=\sum_{\mu\leq\lambda}\int_{[F_\mu]\times[\overline{M}_{\mu,n}]^{\mathrm{vir}}}\frac{1}{e_\mu}\ \mathrm{Ev}_{\mu,0}^*\,j_\mu^*\iota_{0*}(\gamma)\prod_{\ell=1}^n\mathrm{Ev}_\ell^*(\hat{\tau})\ \mathrm{pr}_{\overline{M}_\mu}^*\ \mathrm{Ev}_{\mu,\infty}^*\,\iota_{\infty*}(\phi)\\
&=\sum_{\mu\leq\lambda}\int_{[\overline{M}_{\mu,n}]^{\mathrm{vir}}}\mathrm{Ev}_{\mu,0}^*\,\iota_{0*}(\gamma_\mu)\prod_{\ell=1}^n\mathrm{Ev}_{\mu,\ell}^*(\hat{\tau})\ \mathrm{Ev}_{\mu,\infty}^*\,\iota_{\infty*}(\phi).
\end{aligned}$$

Here, the first equality follows from (46) and the last equality follows from (45). In view of equations (43) and (44), the result follows. ☐

Corollary 5.8. *For any $\Gamma\in H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}(\mathrm{Gr}_T)_{\mathrm{loc}}\cong H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}(\mathrm{Gr}_G)_{\mathrm{loc}}$ and any $\alpha\in H^\bullet_{G\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]]$, we have*

$$\mathbb{S}_{G,\mathbf{N},\mathrm{loc}}(\Gamma\otimes\alpha)=\mathbb{S}_{T,\mathbf{N},\mathrm{loc}}(\Gamma\otimes\alpha).$$

Proof. This follows from Proposition 2.3 and Proposition 5.7. ☐

Corollary 5.9. *The map $\mathbb{S}_{G,\mathbf{N},X}$ is W-equivariant.*

Proof. By Corollary 5.8 and Proposition 4.17, $\mathbb{S}_{G,\mathbf{N}}$ and $\mathbb{S}_{T,\mathbf{N},\mathrm{loc}}$ agrees on their common domain of definition.

By Proposition 2.3, the map tw_T is W-equivariant, so it remains to show that $\bigoplus\widetilde{\mathbb{S}}_{t^\lambda}$ is also W-equivariant. Observe that the $G_{\mathbb{C}[t^{-1}]}\rtimes\mathbb{C}^\times_\hbar$-action on $\mathcal{E}(X)$ induces an $N(T)$-action on

$$\bigsqcup_{\mu\in\Lambda}\mathcal{E}_{t^\mu}(X),$$

which is compatible with the natural $N(T)$-actions on both $T_\mathcal{K}\times_{T_\mathcal{O}}X$ and on Gr_T.

This implies that $\bigoplus \widetilde{\mathbb{S}}_{t^\lambda}$ is W-equivariant, because the integrand in (44) involves Gromov–Witten invariants of $\bigsqcup_{\mu\in\Lambda} \mathcal{E}_{t^\mu}(X)$, which are invariant under the $N(T)$-action. $\square$

In view of Corollary 5.9, it makes sense to take the W-invariant part of $\mathbb{S}_{G,\mathbf{N},X}$. This gives a map, still denoted by the same symbol,

$$\mathbb{S}_{G,\mathbf{N},X} : \mathcal{A}^{\hbar}_{G,\mathbf{N}} \otimes_{\mathbb{C}[\hbar]} H^{\bullet}_{G\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]] \longrightarrow H^{\bullet}_{G\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]].$$

Module property. For each λ, consider

$$\mathbb{S}_{t^\lambda} := \widetilde{\mathbb{S}}_{t^\lambda} \circ \Phi_\lambda : H^{\bullet}_{T\times\mathbb{C}^\times_\hbar}(X)_{\mathrm{loc}}[[q_G,\tau]] \longrightarrow H^{\bullet}_{T\times\mathbb{C}^\times_\hbar}(X)_{\mathrm{loc}}[[q_G,\tau]],$$

where $\Phi_\lambda = \mathrm{tw}_T([t^\lambda]\otimes -)$ is the twisting map (see Proposition 2.9), and $\widetilde{\mathbb{S}}_{t^\lambda}$ is defined in (44).

Proposition 5.10. *For $\lambda,\mu\in\Lambda$, $\alpha\in H^{\bullet}_{T\times\mathbb{C}^\times_\hbar}(X)$ and $P\in H^{\bullet}_{T\times\mathbb{C}^\times_\hbar}(\mathrm{pt})$, we have*

$$\mathbb{S}_{t^{\lambda+\mu}}(\alpha) = \mathbb{S}_{t^\lambda}(\mathbb{S}_{t^\mu}(\alpha)), \quad \mathbb{S}_{t^\lambda}(P\alpha) = \Phi_\lambda(P)\,\mathbb{S}_{t^\lambda}(\alpha).$$

Remark 5.11. Consider the pairing $(-,-)_\lambda$ on $H^{\bullet}_{T\times\mathbb{C}^\times_\hbar}(X)$ given by $(\alpha,\alpha')_\lambda := \Phi_\lambda(\alpha,\alpha')$, where $(-,-)$ is the original Poincaré pairing. Let $\mathbb{S}^{\mathrm{Iri}}_\lambda$ denote the "adjoint" of $\mathbb{S}_{t^\lambda}$ in the sense that $(\mathbb{S}^{\mathrm{Iri}}_\lambda(\alpha),\alpha') = (\alpha,\mathbb{S}_{t^\lambda}(\alpha'))_\lambda$.

The operator $\mathbb{S}^{\mathrm{Iri}}_\lambda$ agrees with the shift operators defined in [Iri17, Definition 3.9], provided that in *loc. cit.* one replaces all $\hat{d}-\sigma_{\min}$ with $\overline{d}$. The appearance of the adjoint is due to a difference in the conventions for the zero and infinity fibres in our definition of Seidel spaces versus that in Iritani's (cf. Example 3.10).

The following is a slight modification of [Iri17, Definition 3.13].

Definition 5.12 (Shift operator on the Givental space)**.** Let $F_1,F_2,\dots$ be the fixed components of X^T. Let $N_i = \bigoplus_\alpha N_{i,\alpha}$ denote the normal bundle to F_i in X, where $N_{i,\alpha}$ is the subbundle on which T acts via the character $\alpha\in\Lambda^\vee$. Let $r_{i,\alpha,1}, r_{i,\alpha,2},\dots$ denote the Chern roots of $N_{i,\alpha}$.

Let $\lambda\in\Lambda$, and $\beta_i=\beta_{\lambda,i}\in\mathrm{Eff}(\mathcal{E}_{t^\lambda}(X))^{\mathrm{sec}}$ be the class of the section $\mathcal{E}_{t^\lambda}(x)$ for a point $x\in F_i$. We define

$$\Delta_i(\lambda) := q^{\overline{\beta_i}} \prod_{\alpha,j} \frac{\prod_{c=-\infty}^{0}(r_{i,\alpha,j}+\alpha+c\hbar)}{\prod_{c=-\infty}^{-\alpha(\lambda)}(r_{i,\alpha,j}+\alpha+c\hbar)} \;\in\; H^{\bullet}_{T\times\mathbb{C}^\times_\hbar}(F_i)_{\mathrm{loc}}[[q_G,\tau]].$$

The *shift operator on the Givental space*

$$S_{t^\lambda} : H^{\bullet}_{T\times\mathbb{C}^\times_\hbar}(X)_{\mathrm{loc}}[[q_G,\tau]] \to H^{\bullet}_{T\times\mathbb{C}^\times_\hbar}(X)_{\mathrm{loc}}[[q_G,\tau]]$$

is defined via the following commutative diagram:

$$\begin{tikzcd}
H^{\bullet}_{T\times\mathbb{C}^\times_\hbar}(X)_{\mathrm{loc}}[[q_G,\tau]] \arrow[r, "S_{t^\lambda}"] \arrow[d] & H^{\bullet}_{T\times\mathbb{C}^\times_\hbar}(X)_{\mathrm{loc}}[[q_G,\tau]] \arrow[d] \\
H^{\bullet}_{T\times\mathbb{C}^\times_\hbar}(X^T)_{\mathrm{loc}}[[q_G,\tau]] \arrow[r, "\bigoplus_i \Phi_\lambda\circ\Delta_i(\lambda)"] & H^{\bullet}_{T\times\mathbb{C}^\times_\hbar}(X^T)_{\mathrm{loc}}[[q_G,\tau]]
\end{tikzcd}$$

Here, the vertical arrows are the natural restriction maps, and the bottom arrow uses the decomposition

$$H^{\bullet}_{T\times\mathbb{C}^\times_\hbar}(X^T)_{\mathrm{loc}} \cong \bigoplus_i \mathrm{Frac}(H^{\bullet}_{T\times\mathbb{C}^\times_\hbar}(\mathrm{pt}))\otimes H^{\bullet}(F_i),$$

where $\Delta_i(\lambda)$ is interpreted as the operator of multiplication by $\Delta_i(\lambda)$ on the i-th summand.

Lemma 5.13. *We have $\mathbb{S}_{t^\lambda} = \mathbb{M}_X S_{t^\lambda} \mathbb{M}_X^{-1}$.*

Proof. This is essentially [Iri17, Theorem 3.14]. After addressing the differences in conventions for 0 and ∞ (see Example 3.10), the argument in *loc. cit.* carries over to our setting upon replacing each occurrence of $\sigma_i-\sigma_{\min}$ with $\overline{\beta_i}$. $\square$

Proof of Proposition 5.10. The second equality follows from Proposition 4.15, so it suffices to prove the first. By Lemma 5.13, it is enough to show that $S_{t^{\lambda+\mu}}(\alpha) = S_{t^\lambda}(S_{t^\mu}(\alpha))$. By localization, we may assume that α is the pushforward of a class in $H^\bullet_{T\times\mathbb{C}^\times_\hbar}(F_i)$. This reduces the problem to checking the identity

$$\Phi_{\lambda+\mu}(\Delta_i(\lambda+\mu)) = \Phi_\lambda(\Delta_i(\lambda)) \cdot \Phi_{\lambda+\mu}(\Delta_i(\mu)),$$

which follows from a direct computation. $\square$

Theorem 5.14. *The operator $\mathbb{S}_{G,\mathbf{N},X}$ defines an $\mathcal{A}^\hbar_{G,\mathbf{N}}$-module structure on the equivariant quantum cohomology ring $QH^\bullet_{G\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]]$.*

Proof. Let $\Gamma = \sum_\lambda a_\lambda[t^\lambda]$ and $\Gamma' = \sum_\mu b_\mu[t^\mu]$, where $a_\mu, b_\mu \in H^\bullet_{T\times\mathbb{C}^\times_\hbar}(\mathrm{pt})$. By Corollary 5.8 and Proposition 5.10, we have

$$\begin{aligned}\mathbb{S}_{G,\mathbf{N}}(\Gamma, \mathbb{S}_{G,\mathbf{N}}(\Gamma',\alpha)) &= \sum_\lambda a_\lambda \mathbb{S}_{t^\lambda}\left(\sum_\mu b_\mu \mathbb{S}_{t^\mu}(\alpha)\right)\\ &= \sum_{\lambda,\mu} a_\lambda \Phi_\lambda(b_\mu)\, \mathbb{S}_{t^{\lambda+\mu}}(\alpha)\\ &= \mathbb{S}_{G,\mathbf{N}}(\Gamma * \Gamma', \alpha).\end{aligned}$$

Here, we used the formula

$$[t^\lambda] * b_\mu = \Phi_\lambda(b_\mu) * [t^\lambda],$$

which is twisted linearity of convolution product (11) (see [BFN18, Section 4(ii)]).

Together with Proposition 4.16, the theorem follows. $\square$

More properties. We now collect other properties of $\mathbb{S}_{G,\mathbf{N},X}$ that follow from the localization result of Corollary 5.8.

Proposition 5.15. *For any $\Gamma \in \mathcal{A}^\hbar_{G,\mathbf{N}}$, the shift operator $\mathbb{S}_{G,\mathbf{N}}(\Gamma,-)$ commutes with the quantum connections (see Definition 3.5), i.e.,*

$$[\nabla_{\tau^{i,j}}, \mathbb{S}_{G,\mathbf{N}}(\Gamma,-)] = 0, \quad [\nabla_{\hbar\partial_\hbar}, \mathbb{S}_{G,\mathbf{N}}(\Gamma,-)] = 0, \quad [\nabla_{Dq\partial_q}, \mathbb{S}_{G,\mathbf{N}}(\Gamma,-)] = 0.$$

Proof. By localization, it suffices to consider the case where $G = T$ is abelian. By (28) and Lemma 5.13, the claim reduces to showing that S_{t^λ} commutes with $\partial_{\tau^{i,j}}$, $\hbar\partial_\hbar - \hbar^{-1}(c_1^T(X)\cup) + \mu_X$, and $Dq\partial_q + \hbar^{-1}(D\cup)$, which follows easily from a direct computation (cf. [Iri23, Corollary 2.11]). $\square$

Proposition 5.16. *There is a commutative diagram:*

$$\begin{array}{ccc} \mathcal{A}^\hbar_{T,\mathbf{N}} \otimes_{\mathbb{C}[\hbar]} H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X)[[q_T,\tau]] & \xrightarrow{\mathbb{S}_{T,\mathbf{N}}} & H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X)[[q_T,\tau]] \\ \downarrow & & \downarrow \\ e(\mathcal{S}_\mathbf{N}) \cap H_\bullet^{T_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}(\mathrm{Gr}_G) \otimes_{\mathbb{C}[\hbar]} H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]] & \xrightarrow{\mathbb{S}_{G,\mathbf{N}}} & H^\bullet_{T\times\mathbb{C}^\times_\hbar}(X)[[q_G,\tau]] \end{array}$$

Proof. This proposition asserts that $\mathbb{S}_{G,\mathbf{N}} = \mathbb{S}_{T,\mathbf{N}} \otimes_{\mathbb{C}[[q_T,\tau]]} \mathbb{C}[[q_G,\tau]]$, when restricted to the common domain of definition. This follows directly from Corollary 5.8, applied both for G and for T. $\square$

By specializing $\hbar = 0$, we also obtain properties for Seidel representation and Seidel homomorphisms.

Corollary 5.17. *The map $\mathbb{S}^{\hbar=0}_{G,\mathbf{N}}$ is compatible with the quantum product, in the sense that*

$$\mathbb{S}^{\hbar=0}_{G,\mathbf{N}}(\Gamma, \alpha_1 \star_\tau \alpha_2) = \mathbb{S}^{\hbar=0}_{G,\mathbf{N}}(\Gamma,\alpha_1) \star_\tau \alpha_2.$$

In particular,

$$\mathbb{S}^{\hbar=0}_{G,\mathbf{N}}(\Gamma,\alpha) = \Psi_{G,\mathbf{N}}(\Gamma) \star_\tau \alpha.$$

Proof. This follows from Proposition 5.15. Setting $\hbar = 0$ in the equality $[\hbar\nabla_{\tau^{i,j}}, \mathbb{S}_{G,\mathbf{N}}(\Gamma,-)] = 0$ yields the desired result. $\square$

Corollary 5.18. *The map $\Psi_{G,\mathbf{N}}$ is a ring homomorphism.*

Proof. This follows directly from Theorem 5.14 and Corollary 5.17:

$$\mathbb{S}^{\hbar=0}_{G,\mathbf{N}}(\Gamma * \Gamma', 1) = \mathbb{S}^{\hbar=0}_{G,\mathbf{N}}(\Gamma, \mathbb{S}^{\hbar=0}_{G,\mathbf{N}}(\Gamma', 1)) = \Psi_{G,\mathbf{N}}(\Gamma) \star_\tau \Psi_{G,\mathbf{N}}(\Gamma'). \qquad \square$$

For $a, b \in \mathcal{A}_{G,\mathbf{N}}$, choose lifts $\widetilde{a}, \widetilde{b} \in \mathcal{A}^\hbar_{G,\mathbf{N}}$. There is a Poisson algebra structure on $\mathcal{A}_{G,\mathbf{N}}$ given by the bracket

$$\{a, b\} = \frac{1}{\hbar}(\widetilde{a}\widetilde{b} - \widetilde{b}\widetilde{a}) \mod \hbar.$$

It is known that this Poisson bracket induces a symplectic structure on the smooth locus of $\operatorname{Spec} \mathcal{A}_{G,\mathbf{N}}$ (see [BFN18, Proposition 6.15]).

Definition 5.19. A closed subscheme $Z \subset \operatorname{Spec} \mathcal{A}_{G,\mathbf{N}}$ is said to be *coisotropic* if the radical of its defining ideal is a Lie subalgebra of $\mathcal{A}_{G,\mathbf{N}}$. A coisotropic subscheme $Z \subset \operatorname{Spec} \mathcal{A}_{G,\mathbf{N}}$ is said to be *Lagrangian* if $\dim Z = \frac{1}{2} \dim \operatorname{Spec} \mathcal{A}_{G,\mathbf{N}} = \dim T$.

Note that these definition agree with the classical notions of coisotropic and Lagrangian subvarieties in the smooth case (see [CG97, Proposition 1.5.1]).

Proposition 5.20. *The subscheme $V(\ker \Psi_{G,\mathbf{N},X}) \subset \operatorname{Spec} \mathcal{A}_{G,\mathbf{N}}$ is coisotropic. Moreover, if all infinite sums involved in the quantum product and shift operators converge upon evaluation at a homomorphism*

$$q_0 \colon \mathbb{C}[q_G, \tau] \to \mathbb{C},$$

and if we denote by

$$\Psi^{q_0}_{G,\mathbf{N},X} \colon \mathcal{A}_{G,\mathbf{N}} \to QH^\bullet_G(X)$$

the resulting specialization, then $V(\ker \Psi^{q_0}_{G,\mathbf{N},X})$ is Lagrangian.

Proof. This follows from [Gab81, Theorem I] (see also [CG97, page 56]). $\square$

5.3. **Properties of shift operators: the general case.** In this subsection, we show that the results in the previous subsection hold without the T-compact assumption.

Set $\widehat{G} = G \times \mathbb{C}^\times_{\mathrm{dil}}$, where $\mathbb{C}^\times_{\mathrm{dil}}$ is the group of conical action (see Section 3).

Lemma 5.21. *There exists a $\widehat{G}$-representation $\mathbf{V}$ and a $\widehat{G}$-equivariant proper morphism $g \colon X \to \mathbf{V}$.*

Proof. Let $\mathbf{V}^\vee$ be a finite-dimensional $\widehat{G}$-invariant subspace of $H^0(X, \mathcal{O}_X)$ such that the corresponding map $\operatorname{Sym}^\bullet \mathbf{V}^\vee \to H^0(X, \mathcal{O}_X)$ is surjective. In other words, the induced morphism $\operatorname{Spec} H^0(X, \mathcal{O}_X) \to \mathbf{V}$ is a closed embedding. The composition

$$X \to \operatorname{Spec} H^0(X, \mathcal{O}_X) \to \mathbf{V}$$

is clearly $\widehat{G}$-equivariant and proper, as required. $\square$

Theorem 5.22. *Corollary 5.8, Corollary 5.9, Theorem 5.14, Proposition 5.15, and Proposition 5.16 all hold without the T-compactness assumption.*

Proof. Let $\mathbf{V}$ and $g \colon X \to \mathbf{V}$ be as in Lemma 5.21. All the stated properties have been established for $\mathbb{S}_{\widehat{G},\mathbf{V},X}$ in the last subsection. By Proposition 5.6, these properties also hold for $\mathbb{S}_{G,\mathbf{V},X}$.

Now, by (5.4), $\mathbb{S}_{G,\mathbf{N},X}$ and $\mathbb{S}_{G,\mathbf{V},X,\mathrm{loc}}$ agree on $\mathcal{A}^\hbar_{G,\mathbf{N}} \subset \mathcal{A}^\hbar_{G,\mathbf{V},\mathrm{loc}}$. Therefore, the desired properties of $\mathbb{S}_{G,\mathbf{N},X}$ follows from the corresponding properties of $\mathbb{S}_{G,\mathbf{V},X,\mathrm{loc}}$. $\square$

Corollary 5.23. *Corollary 5.17, Corollary 5.18 and Proposition 5.20 also hold without the T-compactness assumption.*

Proof. The same arguments as in the proofs of Corollary 5.17, Corollary 5.18 and Proposition 5.20 apply verbatim. $\square$

Corollary 5.24. *Theorem 1 and Theorem 2 are valid.*

Remark 5.25. Suppose X^T-compact, Iritani in [Iri17] defined shift operators for the T-action on X. By the proof of Lemma 5.21, there always exists a T-representaion $\mathbf{N}$ with a equivariant proper map $f : X \to \mathbf{N}$. Therefore, the shift operators $\mathbb{S}_{T,\mathbf{N}}$ defined as in Equation (38) recovers Iritani's definition, up to adjoint and Novikov variables (see Remark 5.11).

The following corollary follows from Theorem 1 immediately.

Corollary 5.26. *There are linear maps*

$$S_{G,\mathbf{N},X} \colon \mathcal{A}^{\hbar}_{G,\mathbf{N}} \otimes QH^{\bullet}(X) \longrightarrow QH^{\bullet}(X)[[q_G,\tau]][\hbar],$$
$$S^{\hbar=0}_{G,\mathbf{N},X} \colon \mathcal{A}_{G,\mathbf{N}} \otimes QH^{\bullet}(X) \longrightarrow QH^{\bullet}(X)[[q_G,\tau]].$$

obtained by specializing the equivariant parameters $H^{\bullet}_G(\mathrm{pt})$ to zero. Moreover, $S^{\hbar=0}_{G,\mathbf{N},X}$ defines a module action.

Note that $\mathbb{C}^{\times}_{\hbar}$ acts on X trivially, so it makes sense to refer to $S_{G,\mathbf{N},X}$ and $S^{\hbar}_{G,\mathbf{N},X}$ as the non-equivariant limits of the shift operator $\mathbb{S}_{G,\mathbf{N},X}$. See Example 6.2 for a sample calculation.

5.4. **Product formula.** Recall that for a product variety, its quantum cohomology satisfies a Künneth formula [Beh99]. The equivariant version also holds: if X_1 and X_2 are smooth semiprojective G-varieties, then

$$QH^{\bullet}_{G\times\mathbb{C}^{\times}_{\hbar}}(X_1\times X_2)[[q_{G\times G},\tau_{G\times G}]] \cong QH^{\bullet}_{G\times\mathbb{C}^{\times}_{\hbar}}(X_1)[[q_G,\tau_G]] \otimes_{H^{\bullet}_{G\times\mathbb{C}^{\times}_{\hbar}}(\mathrm{pt})} QH^{\bullet}_{G\times\mathbb{C}^{\times}_{\hbar}}(X_2)[[q_G,\tau_G]]. \tag{47}$$

Here, $\tau_{G\times G}$ is the bulk parameters associated to the $G\times G$-action on $X_1\times X_2$. The left-hand side is defined to be the $G\times G\times\mathbb{C}^{\times}_{\hbar}$-equivariant quantum cohomology under specialization of equivariant parameters along $\Delta G\subset G\times G$.

This compatibility reflects a similar structure on the level of Coulomb branches. Indeed, the diagonal map

$$\Delta : \mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G$$

induces a pushforward homomorphism

$$\Delta_* : H^{G_{\mathcal{O}}\rtimes\mathbb{C}^{\times}_{\hbar}}_{\bullet}(\mathrm{Gr}_G) \longrightarrow H^{G_{\mathcal{O}}\rtimes\mathbb{C}^{\times}_{\hbar}}_{\bullet}(\mathrm{Gr}_G) \otimes_{H^{\bullet}_{G\times\mathbb{C}^{\times}_{\hbar}}(\mathrm{pt})} H^{G_{\mathcal{O}}\rtimes\mathbb{C}^{\times}_{\hbar}}_{\bullet}(\mathrm{Gr}_G). \tag{48}$$

Setting $\hbar=0$, the spectrum of this map gives the flat $\operatorname{Spec} H^{\bullet}_G(\mathrm{pt})$-group scheme structure on $\operatorname{Spec}\mathcal{A}_{G,0}$, realized as the regular centralizer of the Langlands dual group $\check{G}$ ([BFM05]).

Moreover, if $\mathbf{N}_1$ and $\mathbf{N}_2$ are two representations of G, it is clear that $\Delta^{-1}(\mathcal{S}_{G,\mathbf{N}_1}\boxtimes\mathcal{S}_{G,\mathbf{N}_2})\cong\mathcal{S}_{G,\mathbf{N}_1\oplus\mathbf{N}_2}$. Therefore, (48) restricts to give a homomorphism (denoted by the same symbol)

$$\Delta_* : \mathcal{A}^{\hbar}_{G,\mathbf{N}_1\oplus\mathbf{N}_2} \longrightarrow \mathcal{A}^{\hbar}_{G,\mathbf{N}_1} \otimes_{H^{\bullet}_{G\times\mathbb{C}^{\times}_{\hbar}}(\mathrm{pt})} \mathcal{A}^{\hbar}_{G,\mathbf{N}_2}.$$

In particular, for each representation $\mathbf{N}$, there is a group scheme action of $\operatorname{Spec}\mathcal{A}_G$ on $\operatorname{Spec}\mathcal{A}_{G,\mathbf{N}}$.

Proposition 5.27. *Let $f_i : X_i \to \mathbf{N}_i$ be a G-equivariant proper morphism for $i=1,2$, satisfying the assumptions of Section 4. Then we have*

$$\mathbb{S}_{G,\mathbf{N}_1\oplus\mathbf{N}_2}(\Gamma\otimes\gamma_1\otimes\gamma_2) = (\mathbb{S}_{G,\mathbf{N}_1}\otimes\mathbb{S}_{G,\mathbf{N}_2})(\Delta_*(\Gamma)\otimes\gamma_1\otimes\gamma_2), \tag{49}$$

for any $\Gamma\in\mathcal{A}^{\hbar}_{G,\mathbf{N}_1\oplus\mathbf{N}_2}$, any $\gamma_i\in H^{\bullet}_{G\times\mathbb{C}^{\times}_{\hbar}}(X_i)[[q_G,\tau_G]]$, for $i=1,2$. Here the tensor product between the γ_i's on the left-hand side is taken over $H^{\bullet}_{G\times\mathbb{C}^{\times}_{\hbar}}(\mathrm{pt})$, and on the right-hand side taken over $\mathbb{C}$. We have used the identification given by Künneth formula (47).

Proof. Using the same reasoning as in Theorem 5.22, we only need to prove the $G=T$ case.

To simplify notation, we write $a^{\lambda} := e(\mathcal{S}_{\mathbf{N}_1,t^{\lambda}})\cap[t^{\lambda}]$, $b^{\lambda} := e(\mathcal{S}_{\mathbf{N}_2,t^{\lambda}})\cap[t^{\lambda}]$ for $\lambda\in\Lambda$, and $c^{\lambda} := e(\mathcal{S}_{\mathbf{N}_1\oplus\mathbf{N}_2,t^{\lambda}})\cap[t^{\lambda}]$. Then we have

$$\Delta_*(c^{\lambda}) = a^{\lambda}\otimes b^{\lambda}.$$

For any $\gamma_1\in H^{\bullet}_{T\times\mathbb{C}^{\times}_{\hbar}}(X_1)[[q_T,\tau]]$ and $\gamma_2\in H^{\bullet}_{T\times\mathbb{C}^{\times}_{\hbar}}(X_2)[[q_T,\tau]]$, we have

$$(\mathbb{S}_{T,X_1}\otimes\mathbb{S}_{T,X_2})\circ(\Delta_*(c^{\lambda})\otimes\gamma_1\otimes\gamma_2) = \mathbb{S}_{T,X_1}(a^{\lambda}\otimes\gamma_1)\otimes\mathbb{S}_{T,X_2}(b^{\lambda}\otimes\gamma_2). \tag{50}$$

On the other hand, we have

$$\begin{aligned}\mathbb{S}_{T\times T,X_1\times X_2}((a^{\lambda}\otimes b^{\lambda})\otimes(\gamma_1\otimes\gamma_2)) &= \mathbb{S}_{T\times T,X_1\times X_2}((a^{\lambda}\otimes 1)\otimes\mathbb{S}_{T\times T,X_1\times X_2}((1\otimes b^{\lambda}),\gamma_1\otimes\gamma_2))\\ &= \mathbb{S}_{T\times T,X_1\times X_2}((a^{\lambda}\otimes 1)\otimes\gamma_1\otimes\mathbb{S}_{T,X_2}(b^{\lambda},\gamma_2))\\ &= \mathbb{S}_{T,X_1}(a^{\lambda}\otimes\gamma_1)\otimes\mathbb{S}_{T,X_2}(b^{\lambda}\otimes\gamma_2).\end{aligned} \tag{51}$$

Here, the first equality follows from Theorem 5.14; the second equality follows from the fact that $\mathcal{E}^{G\times G}_{t^{(0,\lambda)}}(X_1\times X_2)\cong X_1\times\mathcal{E}_{t^{\lambda}}(X_2)$, and so only sections that are trivial in the X_1-direction contribute (see Proposition 4.16 for a similar argument). The third equality holds for similar reasons. Now (50) and (51) agree. Moreover, $\mathbb{S}_{T\times T,X_1\times X_2}((a^{\lambda}\otimes b^{\lambda})\otimes$

$(\gamma_1 \otimes \gamma_2))$ is equal to the right-hand side of (49) after specialization of equivariant parameters along $\Delta T \subset T \times T$. This completes the proof. □

6. Applications/calculations

6.1. Examples. To illustrate the rationality of $\Psi_{T,\mathbf{N}}$, we compute from definition (38) the map $\Psi_{T,\mathbf{N}}$ for $X = \mathbf{N}$.

Proposition 6.1. *Let $G = T = (\mathbb{C}^\times)^k$ and $X = \mathbf{N} = \bigoplus_{j=1}^n \mathbb{C}_{\xi_j}$, then*

$$\Psi_{T,\mathbf{N}}(e(\mathcal{S}_{t^\lambda}) \cap [t^\lambda]) = q^\lambda \prod_{j:\xi_j(\lambda)>0} \xi_j^{\xi_j(\lambda)}, \text{ and}$$

$$\Psi_{T,\mathbf{N},\mathrm{loc}}([t^\lambda]) = q^\lambda \frac{\prod_{j:\xi_j(\lambda)>0} \xi_j^{\xi_j(\lambda)}}{\prod_{j:\xi_j(\lambda)<0} \xi_j^{-\xi_j(\lambda)}}. \tag{52}$$

Proof. Let $[t^\lambda] \in \mathrm{Gr}_T$, by Example 3.10, the associated Seidel space is given by

$$\mathcal{E}_{t^\lambda}(\mathbf{N}) \cong \bigoplus_{j=1}^n \mathcal{O}_{\mathbb{P}^1}(-\xi_j(\lambda)).$$

It is straightforward to verify that the only contribution comes from 2-points invariants. The associated moduli space of sections, with marked points on the fibres over 0 and ∞, is

$$\mathcal{M}_{t^\lambda}(\mathbf{N})_0 = H^0\left(\mathbb{P}^1, \mathcal{E}_{t^\lambda}(\mathbf{N})\right) = \bigoplus_{j:\xi_j(\lambda)\leq 0} \mathbb{C}_{\xi_j}^{-\xi_j(\lambda)+1},$$

and its virtual fundamental class is

$$[\mathcal{M}_{t^\lambda}(\mathbf{N})_0]^{\mathrm{vir}} = e(H^1(\mathbb{P}^1, \mathcal{E}_{t^\lambda}(\mathbf{N})) \cap [\mathcal{M}_{t^\lambda}(\mathbf{N})_0] = \prod_{j:\xi_j(\lambda)>0} \xi_j^{\xi_j(\lambda)-1} \cap [\mathcal{M}_{t^\lambda}(\mathbf{N})_0].$$

By the proof of Proposition 4.4 in the special case $X = \mathbf{N}$, we have

$$\mathcal{Z}_{t^\lambda}(\mathbf{N})_0 \cong \prod_{j:\xi_j(\lambda)\leq 0} \mathbb{C}_{\xi_j}.$$

The inclusion $\mathcal{Z}_{t^\lambda}(\mathbf{N})_0 \hookrightarrow \mathcal{M}_{t^\lambda}(\mathbf{N})_0$ is a regular embedding. Consequently,

$$[\mathcal{Z}_{t^\lambda}(\mathbf{N})_0]^{\mathrm{vir}} = \prod_{j:\xi_j(\lambda)>0} \xi_j^{\xi_j(\lambda)-1} \cap [\mathcal{Z}_{t^\lambda}(\mathbf{N})_0].$$

Finally, the evaluation map $\mathrm{ev}_\infty : \mathcal{Z}_{t^\lambda}(\mathbf{N}) \to \mathbf{N}$ is simply the inclusion. Thus, we obtain

$$\Psi_{T,\mathbf{N}}(e(\mathcal{S}_{t^\lambda}) \cap [t^\lambda]) = q^\lambda \prod_{j:\xi_j(\lambda)>0} \xi_j^{\xi_j(\lambda)} \in H_T^\bullet(\mathbf{N})[q_T][[\tau]],$$

as desired. By (12), we obtain (52). □

Next, let us consider a slightly more sophisticated example. This demonstrates the existence of non-equivariant limits as in Corollary 5.26. As we will see, the statement of Corollary 5.26 cannot possibly hold without the introduction of G-equivariant Novikov variables.

Example 6.2. Let $G = \mathrm{GL}_2$ act on $\mathbb{P}^1$ by fractional linear transformations, and consider the induced action on $X = T^*\mathbb{P}^1$. Let $T = (\mathbb{C}^\times)^2 \subset \mathrm{GL}(2, \mathbb{C})$ denote the subgroup of diagonal matrices. The cohomology of $T^*\mathbb{P}^1$ admits the following presentation

$$H^\bullet_{(\mathbb{C}^\times)^2 \times \mathbb{C}^\times_\hbar}(T^*\mathbb{P}^1) = \frac{\mathbb{C}[x, a_1, a_2, \hbar]}{\langle (x + a_1)(x + a_2)\rangle},$$

where x is the pullback of $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ along $T^*\mathbb{P}^1 \to \mathbb{P}^1$, and a_i are the equivariant parameters of T.

Let us denote $q^0_{T,\lambda}$ (resp. $q^\infty_{T,\lambda}$) to be the T-equivariant Novikov variables associated to the constant section at 0 (resp. ∞) of the Seidel space $\mathcal{E}_{t^\lambda}(X)$. One can compute that

$$\mathbb{S}_{t^{(1,0)}}(1) = \frac{a_1 - a_2 + \hbar}{(a_1-a_2)^2}\left(q^0_{T,(1,0)}(x+a_2) - q^\infty_{T,(1,0)}(x+a_1)\right),$$
$$\mathbb{S}_{t^{(0,1)}}(1) = \frac{a_2 - a_1 + \hbar}{(a_1-a_2)^2}\left(q^\infty_{T,(0,1)}(x+a_1) - q^0_{T,(0,1)}(x+a_2)\right).$$

Notice that there exists a G-equivariant proper morphism $T^*\mathbb{P}^1 \to \mathfrak{gl}_2$. For simplicity, let us denote the fixed point classes $z_1 = [t^{(1,0)}]$ and $z_2 = [t^{(0,1)}]$. In this case, a class $p(a_1,a_2)z_1 \in \mathcal{A}^\hbar_{(\mathbb{C}^\times)^2,\mathfrak{gl}_2}$ if and only if p is divisible by $(a_1-a_2)^2$. On the other hand, we find

$$S_{(\mathbb{C}^\times)^2,\mathfrak{gl}_2,T^*\mathbb{P}^1}((a_1-a_2)^2 z_1 \otimes 1) = (q^0_{T,(1,0)} - q^\infty_{T,(1,0)})\hbar x,$$
$$S_{(\mathbb{C}^\times)^2,\mathfrak{gl}_2,T^*\mathbb{P}^1}((a_1-a_2)^2 z_2 \otimes 1) = (q^\infty_{T,(0,1)} - q^0_{T,(0,1)})\hbar x.$$

It is more interesting to consider the non-abelian shift operators. Consider the class

$$e(\mathcal{S}) \cap [C_{\le (1,0)}] = (a_1-a_2)(z_1-z_2) \in \mathcal{A}^\hbar_{\mathrm{GL}_2,\mathfrak{gl}_2}.$$

Let $\eta : \mathbb{C}[q_T] \to \mathbb{C}[q_G]$ be the base change map, then the relevant G-equivariant Novikov variables are identified as follows

$$q_G \coloneqq \eta(q^0_{T,(1,0)}) = \eta(q^\infty_{T,(0,1)}), \quad q'_G \coloneqq \eta(q^\infty_{T,(0,1)}) = \eta(q^0_{T,(0,1)}).$$

Then, we find

$$\mathbb{S}_{\mathrm{GL}_2,\mathfrak{gl}_2,T^*\mathbb{P}^1}(e(\mathcal{S}) \cap [C_{\le(1,0)}] \otimes 1) = (q_G - q'_G)(2x + a_1 + a_2) - (q_G + q'_G)\hbar.$$

Note that $2x + a_1 + a_2$ is the negative of equivariant Euler class of $T^*\mathbb{P}^1$. So,

$$S_{\mathrm{GL}_2,\mathfrak{gl}_2,T^*\mathbb{P}^1}(e(\mathcal{S}) \cap [C_{\le(1,0)}] \otimes 1) = -(q_G - q'_G)e(T^*\mathbb{P}^1) - (q_G + q'_G)\hbar.$$

In particular, we have

$$S^{\hbar=0}_{\mathrm{GL}_2,\mathfrak{gl}_2,T^*\mathbb{P}^1}(e(\mathcal{S}) \cap [C_{\le(1,0)}] \otimes 1) = -(q_G - q'_G)e(T^*\mathbb{P}^1).$$

Note that the expression admits a non-equivariant limit only after identifying the q_T variables via η. Without this identification, the cancellation needed for existence of the limit does not occur. □

Example 6.3. It is known that for adjoint matter $\mathfrak{g}$, the Coulomb branch $\mathcal{A}_{G,\mathfrak{g}}$ is isomorphic to $\mathbb{C}[T^*\check{T}]^W$ (see [BFN18, Section 6(vi)]). Continuing from Example 6.2, we may compute explicitly the Seidel map

$$\Psi_{\mathrm{GL}_2,\mathfrak{gl}_2,T^*\mathbb{P}^1} : \mathbb{C}[T^*(\mathbb{C}^\times)^2]^{\mathbb{Z}_2} \longrightarrow QH^\bullet_{\mathrm{GL}_2}(T^*\mathbb{P}^1)[[q_G]].$$

It sends

$$\begin{array}{rclrcl} z_1 z_2 & \mapsto & q_G q'_G, & z_1^{-1}z_2^{-1} & \mapsto & q_G^{-1}q_G'^{-1}, \\ z_1 + z_2 & \mapsto & -q_G - q'_G, & (a_1-a_2)(z_1-z_2) & \mapsto & -(q_G - q'_G)\, e^{\mathrm{GL}_2}(T^*\mathbb{P}^1). \end{array}$$

□

6.2. **A new characterization of the Coulomb branch.** In this section, we characterize the Coulomb branch algebra as the largest subalgebra of the pure gauge Coulomb branch algebra for which the shift operators are defined before localization. First, let us introduce the following definition, which appeared in [CL24a] and [GW25].

Definition 6.4. The G-representation $\mathbf{N}$ is called *gluable* if for all nonzero T-weights ξ_1, ξ_2, we have ξ_1 is not a negative multiple of ξ_2.

In particular, the $G \times \mathbb{C}^\times_{\mathrm{dil}}$-representation, where $\mathbb{C}^\times_{\mathrm{dil}}$ acts on $\mathbf{N}$ via scaling, is gluable.

Theorem 6.5 (=Theorem 4)**.** *Let $\mathbf{N}$ be a gluable G-representation. Then the following diagram commutes:*

$$\begin{tikzcd} \mathcal{A}_{G,\mathbf{N}} \arrow[r, "\Delta_*"] \arrow[d] & \mathcal{A}_G \otimes_{H^\bullet_G(\mathrm{pt})} \mathcal{A}_{G,\mathbf{N}} \arrow[r, "\mathrm{id}\otimes\Psi_{G,\mathbf{N}}"] & \mathcal{A}_G \otimes_{H^\bullet_G(\mathrm{pt})} QH^\bullet_G(X)[[q_G,\tau]] \arrow[d] \\ \mathcal{A}_G \arrow[r, "\Delta_*"] & \mathcal{A}_G \otimes_{H^\bullet_G(\mathrm{pt})} \mathcal{A}_G \arrow[r, "\mathrm{id}\otimes\Psi_{G,\mathbf{N},\mathrm{loc}}"] & \mathcal{A}_G \otimes_{H^\bullet_G(\mathrm{pt})} QH^\bullet_G(X)_{\mathrm{loc}}[[q_G,\tau]]. \end{tikzcd} \tag{53}$$

Moreover, if $\mathbf{N}$ is gluable and $X = \mathbf{N}$, then the outer square is Cartesian.

We remark that the last statement of Theorem 6.5 recovers Teleman's result that the Coulomb branch $\operatorname{Spec}\mathcal{A}_{G\times\mathbb{C}^\times_\hbar,\mathbf{N}}$ is the affinization of the scheme obtained by gluing two copies of $\operatorname{Spec}\mathcal{A}_{G\times\mathbb{C}^\times_\hbar}$ along a rational map.

Proof. The first assertion follows directly from the construction.

For the second assertion, it suffices to prove the claim for the $T_\mathcal{O}$-equivariant version of the diagram. The $G_\mathcal{O}$-equivariant case then follows by taking W-invariant parts of the homomorphisms.

We begin by recalling that the Coulomb branch algebra admits a natural filtration (see Proposition 1.5):

$$e(\mathcal{S}_\mathbf{N})\cap H^{T_\mathcal{O}}_\bullet(\mathrm{Gr}_G)=\bigcup_{\lambda\in\Lambda}\operatorname{Im}\left(e(\widetilde{\mathcal{S}}_{\le\lambda})\cap H^{T_\mathcal{O}}_\bullet(\widetilde{C}_{\le\lambda})\right).$$

The diagram (53) consists of homomorphisms of filtered $H^\bullet_T(\mathrm{pt})$-algebras. To show that it is Cartesian, it suffices to verify this on each filtered piece.

For clarity, denote the entries in the $T_\mathcal{O}$-eqvuiariant version of (53) as follows: let A, B, C, and D correspond to the top-left, top-right, bottom-left and bottom-right entries, respectively. Define the pullback $P:=B\times_D C=\ker(B\oplus C\to D)$ in the category of filtered $H^\bullet_T(\mathrm{pt})$-modules.

Since both $B\to D$, $C\to D$ and $A\to C$ are all injective, the induced map $A\to P$ is injective. So, it suffices to show that $A\to P$ is surjective. By induction on λ, we may reduce to showing that the map on associated graded, $\mathrm{gr}A\to\mathrm{gr}P$, is surjective.

Recall that the filtration on A is given by

$$F^\lambda A=H^{T_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}_\bullet(\mathcal{R}^d_{\le\lambda}),\quad F^{<\lambda}A=H^{T_\mathcal{O}\rtimes\mathbb{C}^\times_\hbar}_\bullet(\mathcal{R}^d_{<\lambda}).$$

Let $e\in H^\bullet_T(\mathrm{pt})$ be the Euler class of the tangent space of C_λ at $[t^\lambda]$. The natural map $F^{<\lambda}A\to F^\lambda A$ is the pushforward map associated to the closed subset $\mathcal{R}^d_{<\lambda}\to\mathcal{R}^d_{\le\lambda}$. The long exact sequence in Borel–Moore homology yields an isomorphism

$$\mathrm{gr}^\lambda A\xrightarrow{\sim}H^{T_\mathcal{O}}_\bullet(\mathcal{R}^d_\lambda)\cong H^\bullet_T(\mathrm{pt})\cdot e(\mathcal{S}_{t^\lambda})\cdot\frac{1}{e}[t^\lambda].$$

On the associated graded level, the bottom horizontal arrow in the diagram sends

$$[t^\lambda]\longmapsto[t^\lambda]\otimes[t^\lambda]\longmapsto[t^\lambda]\otimes\Psi_T([t^\lambda]).$$

Meanwhile, for the pullback P,

$$\mathrm{gr}^\lambda P\subset\mathrm{gr}^\lambda H^{T_\mathcal{O}}_\bullet(\mathrm{Gr}_G)\cong H^\bullet_T(\mathrm{pt})\cdot\frac{1}{e}[t^\lambda].$$

Suppose that $c[t^\lambda]\in\mathrm{gr}^\lambda P$. By definition of pullback, we must have

$$ce\Psi_{T,\mathbf{N},\mathrm{loc}}([t^\lambda])\in H^\bullet_T(\mathbf{N})[[q_T,\tau]].$$

When $\mathbf{N}$ is gluable, by Proposition 6.1, the denominator of $\Psi_{T,\mathbf{N},\mathrm{loc}}([t^\lambda])$ is exactly $e(\mathcal{S}_{t^\lambda})$, and no cancellation of equivariant parameters occurs. Thus, ce is divisible by $e(\mathcal{S}_{t^\lambda})$.

Hence, $c[t^\lambda]$ lies in the image of $\mathrm{gr}^\lambda A$, and the map $\mathrm{gr}^\lambda A\to\mathrm{gr}^\lambda P$ is surjective. This completes the proof. □

6.3. **Peterson isomorphism for reductive groups.** In this subsection, we set $\tau=0$ and consider only the small quantum cohomology.

We compute the homomorphism $\Psi_{G,X}=\Psi_{G,\mathbf{0},X}$ in the case where X is a partial flag variety. The main result is Theorem 6.9, which generalizes the Peterson isomorphism for simply connected groups [Pet97; LS10; Cho23].

The ideas for the computations presented here are all from [Cho23], and we do not claim originality for it. The only new input here is the use of equivariant Novikov parameters. We hope that Theorem 6.9 offers compelling evidence that working with equivariant Novikov variables is the natural setting for studying quantum cohomology.

We begin by introducing some notations. Let R be the set of roots of G, and let $R^+\subset R$ be the set of positive roots. Let $P\supset B$ be a standard parabolic subgroup of G, and let $R_P\subset R$ be the set of roots of the Levi subgroup of P, so that $R^+_P=R_P\cap R^+$. In particular, we have the decomposition

$$\operatorname{Lie}P=\operatorname{Lie}B\oplus\bigoplus_{\alpha\in R^+_P}\mathfrak{g}_{-\alpha}.$$

We denote the Weyl group of P by W_P. Throughout this section, we regard the quantum cohomology vector space as

$$QH^\bullet_G(G/P)[q_G]=H^\bullet_G(G/P)\otimes\mathbb{C}[H^{\mathrm{ord},G}_2(G/P,\mathbb{Z})],$$

and we will show that the image of $\Psi_{G,G/P}$ is actually contained in this space.

Let $X = G/P$ and $v \in W/W_P$. Let $\ell_P(v)$ denote the length of the minimal representative of v in W, respectively. Define

$$\sigma(v) = \overline{BvP} \subset G/P, \quad \sigma^-(v) = \overline{B^- vP} \subset G/P.$$

Then $\dim(\sigma(v)) = \ell_P(v)$ and $\dim(\sigma^-(v)) = \dim G/P - \ell_P(v)$.

In Appendix C, we will show that $\mathcal{M}_{\leq\lambda}(G/P,\beta)$ is smooth of expected dimension for any $\beta \in \mathrm{Eff}(\mathcal{E}_{\leq\lambda}(G/P))^{\mathrm{sec}}$. Moreover, since $\mathrm{ev}_\infty : \mathcal{M}_{\leq\lambda}(G/P,\beta) \to G/P$ is B-equivariant, it is transverse to all B^--orbits. Therefore,

$$\Psi_{G,G/P}([C_{\leq\lambda}]) = \sum_\beta q^{\overline{\beta}} \sum_{v\in W/W_P} \#\left(\mathcal{M}_{\leq\lambda}(G/P,\beta) \times_{\mathrm{ev}_\infty} \sigma^-(v)\right) \cdot \sigma(v), \tag{54}$$

where $\#(S)$ for a set S is 0 if S is infinite, and is the cardinality of S otherwise.

Let $v \in W$, and consider the point $s_{\lambda,v} \in \mathcal{E}_{\leq\lambda}(G/P)$ representing the constant section

$$\mathcal{E}^T_{t^\lambda} \times_T v \subset \mathcal{E}_\lambda \times_G (G/P),$$

where $\mathcal{E}^T$ is the universal T-torsor over $\mathrm{Gr}_T \times \mathbb{P}^1$. Let $\beta_{\lambda,v} \in H_2^{\mathrm{ord}}(\mathcal{E}_{\leq\lambda}(G/P);\mathbb{Z})$ be the corresponding section class. We write $\lambda = w_\lambda(\lambda^-)$, where λ^- is antidominant, and w_λ is the longest-length element in the coset $w_\lambda \operatorname{Stab}_W(\lambda^-)$.

Definition 6.6. We say that $\lambda \in \Lambda$ is *P-allowed* if for any $\alpha \in R_P^+$,

$$\langle \alpha, \lambda^- \rangle = \begin{cases} 0 & \text{if } w_\lambda(\alpha) < 0, \\ -1 & \text{if } w_\lambda(\alpha) > 0. \end{cases}$$

Lemma 6.7. *Let $\beta \in \mathrm{Eff}(\mathcal{E}_{\leq\lambda}(G/P))^{\mathrm{sec}}$, and let $v \in W$. Suppose*

$$\mathcal{M}_{\leq\lambda}(G/P,\beta) \times_{\mathrm{ev}_\infty} \sigma^-(v)$$

is finite. Then $v \in w_\lambda W_P$, λ is P-allowed, and

$$\mathcal{M}_{\leq\lambda}(G/P,\beta_{\lambda,w_\lambda}) \times_{\mathrm{ev}_\infty} \sigma^-(w_\lambda) = \{s_{\lambda,w_\lambda}\}.$$

Proof. Let $s \in \mathcal{M}_{\leq\lambda}(G/P,\beta) \times_{\mathrm{ev}_\infty} \sigma^-(v)$. Since the space is T-invariant, s must be a T-fixed point, hence lying over some $t^\mu \in C_{\leq\lambda}$.

1) Showing that $\mu = \lambda$: Suppose $\mu < \lambda$. Then s descends to give a point in

$$\mathcal{M}_{\leq\mu}(G/P,[s]) \times_{\mathrm{ev}_\infty} \sigma^-(v),$$

but

$$\dim \mathcal{M}_{\leq\mu}(G/P,[s]) \times_{\mathrm{ev}_\infty} \sigma^-(v) < \dim \mathcal{M}_{\leq\lambda}(G/P,[s]) \times_{\mathrm{ev}_\infty} \sigma^-(v) = 0,$$

a contradiction. Therefore, $\mu = \lambda$, and we must have $s = s_{\lambda,w}$ for some $w \in W$.

2) Showing that $v \in wW_P$: Since $\mathrm{ev}_\infty(s_{\lambda,w}) = wP$, we have $wP \in \sigma^-(v)$. On the other hand, since ev_∞ is B-equivariant and $\sigma^-(v)$ is B^--invariant, we must have $\sigma(w) \cap \sigma^-(v) = wP$, this shows that $v \in wW_P$.

3) Showing that $v \in w_\lambda W_P$: This condition is equivalent to requiring that for all $\alpha \in R^+ \setminus R_P^+$,

$$\langle v(\alpha), \lambda \rangle \leq 0, \quad \text{and equality holds only if } v(\alpha) < 0.$$

Let $\alpha \in R^+ \setminus R_P^+$. Then the normal bundle to $s_{\lambda,v}$ in the direction of $-v(\alpha)$ is

$$\mathcal{O}_{\mathbb{P}^1}(\langle v(\alpha), \lambda \rangle).$$

We must have $\langle v(\alpha), \lambda \rangle \leq 0$, because otherwise we can deform $s_{\lambda,v}$ while fixing the infinite point.

Moreover, if $\langle v(\alpha), \lambda \rangle = 0$, then we can deform $s_{\lambda,v}$ while moving the infinite point in the $-v(\alpha)$ direction. We must have $v(\alpha) < 0$, otherwise this deformation remains contained in $\mathrm{ev}_\infty^{-1}(B^-(vP))$.

4) Verifying that λ is P-allowed: We introduce some notations. For $\alpha \in R$, we write $U_\alpha : \mathbb{C} \to G$ the inclusion of the corresponding root subgroup. If $f \in \mathcal{K}$, we also write $U_\alpha(f)$ for the corresponding element in $G_{\mathcal{K}}$.

Let $\alpha \in R_P^+$. If $w_\lambda(\alpha) < 0$, then $U_{-w_\lambda(\alpha)}(z)t^\lambda \cdot w_\lambda = w_\lambda$, for any $z \in \mathbb{C}$, and $t \in \mathbb{C}^\times$. Therefore, s can deform to the constant section s_z over $U_{-w_\lambda(\alpha)}(z)t^\lambda \in C_\lambda$ with value w_λ. This contradicts to our assumption that $\mathcal{M}_{\leq\lambda}(G/P,\beta) \times_{\mathrm{ev}_\infty} \sigma^-(v)$ is finite unless $U_{-w_\lambda(\alpha)}(z)t^\lambda = t^\lambda$ for any $z \in \mathbb{C}$. This means

$$\langle \lambda^-, \alpha \rangle = \langle \lambda, w_\lambda(\alpha) \rangle \geq 0,$$

and must in fact equal 0 because λ^- is antidominant.

If $w_\lambda(\alpha) > 0$, then $U_{-w_\lambda(\alpha)}(zt)t^\lambda \cdot w_\lambda = w_\lambda$, for any $z \in \mathbb{C}$, and $t \in \mathbb{C}^\times$. Similar to the above case, we must have

$$\langle \lambda^-, \alpha \rangle = \langle \lambda, w_\lambda(\alpha) \rangle \geq -1.$$

Since λ^- is antidominant, $\langle \lambda^-, \alpha \rangle$ can only be 0 or -1. The former case, however, implies $w_\lambda(\alpha) < 0$ by definition of w_λ. Therefore, we must have $\langle \lambda^-, \alpha \rangle = 0$.

This confirms that λ is P-allowed. □

We want to calculate the class $\overline{\beta_{\lambda,v}} \in H_2^{\mathrm{ord},G}(G/P;\mathbb{Z}) = H_2^P(\mathrm{pt};\mathbb{Z})$. Let EP be a contractible space with a free P-action, and $BP = EP/P$. The long exact sequence of homotopy groups associated to the fibration $P \to EP \to BP$ shows that $\pi_1(BP) = 1$ and $\pi_2(P) \cong \pi_1(P)$. In particular, the Hurewicz theorem in algebraic topology implies $H_2^{\mathrm{ord},P}(\mathrm{pt};\mathbb{Z}) = H_2^{\mathrm{ord}}(BP;\mathbb{Z}) \cong \pi_1(P)$.

We identify $\pi_1(T)$ with Λ; and for $\lambda \in \Lambda$, we write λ_P for the image of λ under the natural map $\pi_1(T) \to \pi_1(P)$. We write λ_P^- for $(\lambda^-)_P$

Lemma 6.8. $\overline{\beta_{\lambda,v}} = v^{-1}(\lambda)_P \in \pi_1(P)$.

Theorem 6.9.

$$\Psi_{G,G/P}([C_{\leq\lambda}]) = \begin{cases} q^{\lambda_P^-}\sigma(w_\lambda) & \textit{if } \lambda \textit{ is } P\textit{-allowed}, \\ 0 & \textit{otherwise}. \end{cases}$$

In particular, the homomorphism

$$\mathcal{A}_G \to QH_G^\bullet(G/B)[q_G]$$

becomes an isomorphism if we localize $\mathcal{A}_G$ *by all class* $[C_{w_0(\lambda^-)}]$ *for all* $\lambda^- \in \Lambda^-$.

Proof. The last statement follows from the first. To prove the first statement, we observe that

$$\dim \mathcal{M}_{\leq\lambda}(G/P, \beta_{\lambda,w_\lambda}) \times_{\mathrm{ev}_\infty} \sigma^-(w_\lambda) = \dim C_\lambda + \dim G/P + \sum_{\alpha \in R^+ \setminus R_P^+} \langle \alpha, \lambda^- \rangle - \ell^-(w_\lambda),$$

and

$$\dim C_\lambda = -\sum_{\alpha \in R^+} \langle \alpha, \lambda^- \rangle - |R^+| + \ell(w_\lambda).$$

If λ is P-allowed, then

$$\ell(w_\lambda) = \ell^-(w_\lambda) + |R_P^+| + \sum_{\alpha \in R_P^+} \langle \alpha, \lambda^- \rangle.$$

To see this, notice that $\ell(w_\lambda)$ (resp. $\ell_P(w_\lambda)$) is the number of $\alpha \in R^+$ (resp. $\alpha \in R^+ \setminus R_P^+$) with $w_\lambda(\alpha) < 0$, and $\sum_{\alpha \in R_P^+} \langle \alpha, \lambda \rangle$ is minus the number of $\alpha \in R_P^+$ with $w_\lambda(\alpha) > 0$. Combining the above formula together, we see that $\dim \mathcal{M}_{\leq\lambda}(G/P, \beta_{\lambda,w_\lambda}) \times_{\mathrm{ev}_\infty} \sigma^-(w_\lambda) = 0$ when λ is P-allowed.

Now the theorem follows from Equation (54), Lemma 6.7, and Lemma 6.8. □

Proof of Lemma 6.8. Consider the diagram

$$\begin{tikzcd}
\pi_2(\mathcal{E}_{\leq\lambda} \times (G/P)) \arrow[r] & \pi_2(\mathcal{E}_{\leq\lambda}(G/P)) \arrow[r] \arrow[d] & \pi_1(P) \arrow[r, "1"] \arrow[d, equal] & \pi_1(\mathcal{E}_{\leq\lambda} \times (G/P)) \\
\pi_2(EP) \arrow[r] & \pi_2(BP) \arrow[r, "\cong"] & \pi_1(P) \arrow[r] & \pi_1(EP),
\end{tikzcd}$$

where the rows are the long exact sequence associated to the fibrations $P \to \mathcal{E}_{\leq\lambda} \to \mathcal{E}_{\leq\lambda}(G/P)$ and $P \to EP \to BP$ respectively. The vertical homomorphisms are induced by the classifying map $\mathcal{E}_{\leq\lambda}(G/P) \to BP$ corresponding to the principal P-bundle $\mathcal{E}_{\leq\lambda} \to \mathcal{E}_{\leq\lambda}(G/P)$. By a diagram tracing, it remains to show that the boundary homomorphism $\pi_2(\mathcal{E}_{\leq\lambda}(G/P)) \to \pi_1(P)$ sends $\beta_{\lambda,v}$ to $v^{-1}(\lambda)$.

Let $\mathcal{O}_{\mathbb{P}^1}(-1)^\times$ be the $\mathbb{C}^\times$ bundle associated to the tautological bundle of $\mathbb{P}^1$. By checking the transition functions, it is easy to see that the restriction of $\mathcal{E}_{\leq\lambda}$ to $\beta_{\lambda,v}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^\times \times_{v^{-1}(\lambda)} P$.

This implies that there is a commutative diagram

$$\begin{tikzcd}
\pi_2(\mathbb{P}^1) \arrow[r] \arrow[d, equal] & \pi_1(\mathbb{C}^\times) \arrow[d] \\
\pi_2(\mathbb{P}^1) \arrow[r] \arrow[d] & \pi_1(P) \arrow[d, equal] \\
\pi_2(\mathcal{E}_{\leq\lambda}(G/P)) \arrow[r] & \pi_1(P),
\end{tikzcd}$$

where the horizontal maps are boundary homomorphism associated to the fibrations $\mathbb{C}^\times \to \mathcal{O}_{\mathbb{P}^1}^\times \to \mathbb{P}^1$, $\mathbb{C}^\times \to \mathcal{O}_{\mathbb{P}^1}(-1)^\times \times_{v^{-1}(\lambda)} P \to \mathbb{P}^1$, and $P \to \mathcal{E}_{\leq\lambda} \to \mathcal{E}_{\leq\lambda}(G/P)$ respectively. Note that the left lower map sends $[\mathbb{P}^1]$ to $\beta_{\lambda,v}$ and the right upper map sends the generator $1 \in \mathbb{Z} \cong \pi_1(\mathbb{C}^\times)$ to $v^{-1}(\lambda)_P$. Now the lemma follows from the convention that the upper boundary homomorphism sends $[\mathbb{P}^1]$ to the generator $1 \in \pi_1(\mathbb{C}^\times)$. $\square$

Remark 6.10. There is a ring homomorphism

$$\mathbb{C}[\Lambda]^W \cong \mathbb{C}[\Lambda^-] \longrightarrow \mathcal{A}_G,$$

which sends q^{λ^-} to the class $[C_{\leq -w_0(\lambda^-)}]$. The induced morphism

$$\operatorname{Spec}\mathcal{A}_G \longrightarrow \operatorname{Spec}\mathbb{C}[\Lambda]^W$$

has fibres that are, set-theoretically, equal to the supports of $QH^\bullet_G(G/P)$ (after specializing q_G accordingly). This gives a foliation of the Coulomb branch by the Lagrangians corresponding to partial flag varieties (cf. [Tel14]). This statement can be proved using ideas similar to those in [Cho24] for simply connected G.

6.4. **Peterson isomorphism with matters.** In the case when $\mathbf{N} = 0$ and $X = G/P$ is a partial flag variety, by a theorem of [LS10] and also [Cho23], the Novikov variables q^β lies in the image of the homomorphism $H^{T_{\mathcal{O}}}_\bullet(\mathrm{Gr}_G) \to QH^\bullet_T(G/P)[q_G]$. In particular, this implies that for different G/P and G/P', or for G/P equipped with different symplectic forms, the corresponding Lagrangians $\mathbb{L}_{G,\mathbf{N}}(G/P)$ and $\mathbb{L}_{G,\mathbf{N}}(G/P')$ have empty intersections.

The analogous statement for general $\mathbf{N}$ is not true. The Novikov variables $q^{\overline{\beta}}$ may not lie in the image of $e(\mathcal{S}_\mathbf{N}) \cap H^{T_{\mathcal{O}}}_\bullet(\mathrm{Gr}_G) \to QH^\bullet_T(G/P \times \mathbf{N})[q_G]$. Nevertheless, Corollary 6.13 provides partial results, identifying a subset of quantum Schubert classes which do lie in the image of the Seidel map.

Proposition 6.11. *Let Y be a smooth projective G-variety, let $\mathbf{N}$ be a G-representation and $\mathbf{N}^\vee$ be its dual representation. For any $\lambda \in \Lambda$, we have*

$$\Psi_{G,\mathbf{N},Y\times\mathbf{N}}(e(\mathcal{S}_\mathbf{N}) \cap C_{\leq\lambda}) = \Psi_{G,\mathbf{N}^\vee,Y}(e(\mathcal{S}_{\mathbf{N}^\vee}) \cap C_{\leq\lambda}), \tag{55}$$

under the pullback isomorphism $H^\bullet_G(Y) \cong H^\bullet_G(Y \times \mathbf{N})$.

Proof. By Proposition 5.27, we reduce to the case $Y = \mathrm{pt}$. By Proposition 5.6, we may replace G with $G \times \mathbb{C}^\times$, where $\mathbb{C}^\times$ acts on $\mathbf{N}$ via the scaling action. Now, we may apply Corollary 5.8 to reduce to the abelian case, which follows from the calculation in Proposition 6.1. $\square$

Now we set $\tau = 0$ and only consider small quantum cohomology.

Corollary 6.12. *The equivariant Seidel map $\Psi_{G,\mathbf{N},G/B\times\mathbf{N}}$ induces a birational morphism* $\operatorname{Spec} QH^\bullet_G(G/B \times \mathbf{N})[q_G] \to \operatorname{Spec}\mathcal{A}_{G,\mathbf{N}}$.

Proof. This follows from (55) and the $\mathbf{N} = 0$ case (Theorem 6.9). $\square$

The following proposition is a generalization of Section 6.3. The assumption is satisfied if $G = G' \times \mathbb{C}^\times_{\mathrm{dil}}$, where the $\mathbb{C}^\times_{\mathrm{dil}}$-factor acts on $\mathbf{N}$ by scaling.

Corollary 6.13. *Let $\rho : \mathbb{C}^\times \subset G$ be the inclusion of a central subgroup, and let $\mathbf{N}$ be a representation whose $\mathbb{C}^\times$-weights are all positive (resp. negative). Let $X = G/P \times \mathbf{N}$, and fix $\lambda \in \Lambda$. Then for sufficiently large $n > 0$, we have*

$$\Psi_{G,\mathbf{N},X}\big(e(\widetilde{\mathcal{S}}_{\leq\lambda-n\rho}) \cap [\widetilde{C}_{\lambda-n\rho}]\big) = q^{\overline{(\lambda-n\rho)_P}}\sigma(w_\lambda),$$

$$\textit{resp.}\quad \Psi_{G,\mathbf{N},X}\big(e(\widetilde{\mathcal{S}}_{\leq\lambda+n\rho}) \cap [\widetilde{C}_{\lambda+n\rho}]\big) = q^{\overline{(\lambda+n\rho)_P}}\sigma(w_\lambda),$$

under the pullback isomorphism $H^\bullet_G(G/P) \cong H^\bullet_G(G/P \times \mathbf{N})$.

Proof. This follows from Theorem 6.9 and Proposition 6.11 applied to $Y = G/P$, along with the observation that $\mathcal{S}_{\mathbf{N}^\vee, \le \lambda - n\rho} = 0$ (resp. $\mathcal{S}_{\mathbf{N}^\vee, \le \lambda + n\rho} = 0$) for n sufficiently large. $\square$

Remark 6.14. For each specialization $\varsigma \colon \mathbb{C}[q_G] \to \mathbb{C}$, we obtain a Lagrangian subvariety

$$\mathbb{L}_\varsigma = \operatorname{Supp}\left(QH^\bullet_G(G/B \times \mathbf{N}) \otimes_{\mathbb{C}[q_G]} \mathbb{C}\right).$$

For any two distinct specializations ς_1 and ς_2, it is easy to see that the corresponding Lagrangians $\mathbb{L}_{\varsigma_1}$ and $\mathbb{L}_{\varsigma_2}$ are disjoint. Therefore, the previous corollary may be viewed as a generalization of Teleman's result in the case $\mathbf{N} = 0$ [Tel14, Theorem 6.8].

Appendix A. Stability under the convolution product

Proof of Proposition 1.5. Let $\lambda_1, \lambda_2 \in \Lambda^+$ be dominant coweights, and let $d > 0$ be a sufficiently large positive integer such that $\mathcal{R}^d_\mu$ is defined for all μ satisfying $\mu \le \lambda_1$, $\mu \le \lambda_2$, or $\mu \le \lambda_1 + \lambda_2$. It suffices to show that under the product map (11), the image of

$$\left(z^* H^{G_\mathcal{O}}_\bullet(\mathcal{R}^d_{\le \lambda_1})\right) \otimes \left(z^* H^{G_\mathcal{O}}_\bullet(\mathcal{R}^d_{\le \lambda_2})\right)$$

is contained in $z^* H^{G_\mathcal{O}}_\bullet(\mathcal{R}^d_{\le \lambda_1 + \lambda_2})$, where $z^* : H^{G_\mathcal{O}}_\bullet(\mathcal{T}^d) \to H^{G_\mathcal{O}}_\bullet(\mathrm{Gr}_G)$ is the Gysin map.

Consider the following diagram

$$\begin{tikzcd}
\mathcal{R}^d_{\le\lambda_1} \times \mathcal{R}^d_{\le\lambda_2} \arrow[d, phantom, "\cap"] & p^{-1}(\mathcal{R}^d_{\le\lambda_1} \times \mathcal{R}^d_{\le\lambda_2}) \arrow[l, "p''"'] \arrow[d, phantom, "\cap"] & Z \arrow[l, "j"] \arrow[d, phantom, "\cap"] \\
\mathcal{T}^d_{\le\lambda_1} \times \mathcal{T}^d_{\le\lambda_2} & p^{-1}(\mathcal{T}^d_{\le\lambda_1} \times \mathcal{T}^d_{\le\lambda_2}) \arrow[l, "p'"'] & (G_\mathcal{K})_{\le\lambda_1} \times \mathcal{T}^d_{\le\lambda_2} \\
C_{\le\lambda_1} \times C_{\le\lambda_2} \arrow[u, "z_1"] & (G_\mathcal{K})_{\le\lambda_1} \times C_{\le\lambda_2} \arrow[l, "p"'] \arrow[u, "z_2"] \arrow[r, equal] & (G_\mathcal{K})_{\le\lambda_1} \times C_{\le\lambda_2}. \arrow[u, "z_3"]
\end{tikzcd}$$

Here,

$$Z := \{(g_1, [g_2, s]) \in (G_\mathcal{K})_{\le\lambda_1} \times \mathcal{R}^d_{\le\lambda_2} : g_1 g_2 s \in \mathbf{N}_\mathcal{O}\},$$

and j is defined by

$$j(g_1, [g_2, s]) = (g_1, g_2 s, [g_2, s]).$$

The $G_\mathcal{O} \times G_\mathcal{O}$-action on $p^{-1}(\mathcal{T}^d_{\le\lambda_1} \times \mathcal{T}^d_{\le\lambda_2})$ is given by:

$$(g, g') \cdot (g_1, s_1, [g_2, s_2]) = (g g_1 (g')^{-1}, g' s_1, [g' g_2, s_2]),$$

so j is $G_\mathcal{O} \times G_\mathcal{O}$-equivariant. There is a section ϕ of (the pullback of) $\mathcal{T}^d_{\le\lambda_1}$ over $p^{-1}(\mathcal{R}^d_{\le\lambda_1} \times \mathcal{R}^d_{\le\lambda_2})$ defined by

$$\phi(g_1, s_1, [g_2, s_2]) = [g_1, s_1 - g_2 s_2],$$

whose vanishing locus is Z. Therefore,

$$\begin{aligned}
p^* z_1^* H^{G_\mathcal{O} \times G_\mathcal{O}}_\bullet(\mathcal{R}^d_{\le\lambda_1} \times \mathcal{R}^d_{\le\lambda_2}) &= z_2^* (p')^* H^{G_\mathcal{O} \times G_\mathcal{O}}_\bullet(\mathcal{R}^d_{\le\lambda_1} \times \mathcal{R}^d_{\le\lambda_2}) \\
&\subset z_2^* H^{G_\mathcal{O} \times G_\mathcal{O}}_\bullet(p^{-1}(\mathcal{R}^d_{\le\lambda_1} \times \mathcal{R}^d_{\le\lambda_2})) \\
&\subset z_3^* H^{G_\mathcal{O} \times G_\mathcal{O}}_\bullet(Z).
\end{aligned} \tag{56}$$

Next, consider the diagram

$$\begin{tikzcd}
Z' \arrow[r, "m''"] \arrow[d, phantom, "\cap"] & \mathcal{R}^d_{\le\lambda_1+\lambda_2} \arrow[d, phantom, "\cap"] \\
(\mathrm{Gr}_\mathcal{K})_{\le\lambda_1} \times_{G_\mathcal{O}} \mathcal{T}^d_{\le\lambda_2} \arrow[r, "m'"] & \mathcal{T}^d_{\le\lambda_1+\lambda_2} \\
(G_\mathcal{K})_{\le\lambda_1} \times_{G_\mathcal{O}} C_{\le\lambda_2} \arrow[u, "z_4"] \arrow[r, "m"] & \overline{C}_{\lambda_1+\lambda_2}. \arrow[u, "z_5"]
\end{tikzcd}$$

Here, $Z' = Z/G_{\mathcal{O}}$ is the image of Z under the canonical map

$$(\mathrm{Gr}_{\mathcal{K}})_{\leq\lambda_1} \times \mathcal{T}^d_{\leq\lambda_2} \to (\mathrm{Gr}_{\mathcal{K}})_{\leq\lambda_1} \times_{G_{\mathcal{O}}} \mathcal{T}^d_{\leq\lambda_2}.$$

Hence,

$$\begin{aligned} m_*(q^*)^{-1} z_3^* H_\bullet^{G_{\mathcal{O}}\times G_{\mathcal{O}}}(Z) &= m_* z_4^* H_\bullet^{G_{\mathcal{O}}}(Z') \\ &= z_5^*(m')_* H_\bullet^{G_{\mathcal{O}}}(Z') \\ &\subset z_5^* H_\bullet^{G_{\mathcal{O}}}(\mathcal{R}^d_{\leq\lambda_1+\lambda_2}). \end{aligned} \tag{57}$$

Combining (56) and (57), we obtain

$$m_*(q^*)^{-1} p^* z_1^* H_\bullet^{G_{\mathcal{O}}\times G_{\mathcal{O}}}(\mathcal{R}^d_{\leq\lambda_1} \times \mathcal{R}^d_{\leq\lambda_2}) \subset z_5^* H_\bullet^{G_{\mathcal{O}}}(\mathcal{R}^d_{\leq\lambda_1+\lambda_2}),$$

as desired. □

Appendix B. Universal G-torsor

In this subsection, we will define in terms of functor of points an ind-scheme $\mathcal{E}$ equipped with an action of $G_{\mathbb{C}[t^{-1}]} \rtimes \mathbb{C}^\times_\hbar$ which is a G-torsor over $\mathrm{Gr}_G \times \mathbb{P}^1$. It is best to work in the language of functors of points, since the algebraic loop group $G_{\mathcal{K}}$ is highly non-reduced (see [Zhu17, Remark 1.3.10]).

Universal G-torsors on the loop group. Here $G_{\mathbb{C}[t^{-1}]}$ is the fppf sheaf which sends any $\mathbb{C}$-algebra R to $G(R[t^{-1}])$. The space $\mathcal{E}$ is understood as the universal G-torsor.

Let R be a $\mathbb{C}$-algebra, and $\gamma \in G(R((t)))$. By the theorem of Beauville and Laszlo ([BL95]), there exists a unique G-torsor $\hat{\mathcal{E}}_{R,\gamma}$ on $\mathbb{P}^1_R$, equipped with trivializations

$$\varphi_0^\gamma \colon \hat{\mathcal{E}}_{R,\gamma}|_{\mathrm{Spec}\, R[[t]]} \xrightarrow{\sim} \mathrm{Spec}\, R[[t]] \times G,$$
$$\varphi_\infty^\gamma \colon \hat{\mathcal{E}}_{R,\gamma}|_{\mathrm{Spec}\, R[t^{-1}]} \xrightarrow{\sim} \mathrm{Spec}\, R[t^{-1}] \times G.$$

$\varphi_\infty^\gamma = \gamma \cdot \varphi_0^\gamma$ on $\hat{\mathcal{E}}_{R,\gamma}|_{\mathrm{Spec}\, R((t))}$. In simple terms, $\hat{\mathcal{E}}_{R,\gamma}$ is obtained by gluing trivial G-torsors over $\mathrm{Spec}\, R[[t]]$ and $\mathrm{Spec}\, R[t^{-1}]$ using γ as the transition function.

Moreover, if $f \colon R \to S$ is a $\mathbb{C}$-algebra homomorphism, and $\gamma' \in G(S((t)))$ is the image of γ under the map $G_{\mathcal{K}}(f) \colon G(R((t))) \to G(S((t)))$, then there is an isomorphism

$$\hat{\mathcal{E}}_{S,\gamma'} \cong \hat{\mathcal{E}}_{R,\gamma} \times_R S \tag{58}$$

such that $\varphi_0^{\gamma'} = \varphi_0^\gamma \times_R S$ and $\varphi_\infty^{\gamma'} = \varphi_\infty^\gamma \times_R S$.

Conversely, if $\mathcal{P}$ is a G-torsor over $\mathbb{P}^1_R$, with trivializations φ_0 and φ_∞ over $\mathrm{Spec}\, R[[t]]$ and $\mathrm{Spec}\, R[t^{-1}]$ respectively, then there exists a unique $\gamma \in G_{\mathcal{K}}(R) = G(R((t)))$ such that $\varphi_\infty = \varphi_0$ when restricted to $\mathrm{Spec}\, R((t))$. The following lemma summarizes the above discussion.

Lemma B.1. *The loop group $G_{\mathcal{K}}$ represents the functor*

$$R \longmapsto \left\{ \begin{array}{l} \textit{Isomorphism classes of the pair } (\mathcal{P}, \varphi_0, \varphi_\infty) : \\ \mathcal{P} \textit{ is a } G\textit{-torsor over } \mathbb{P}^1_R, \\ \varphi_0 \textit{ is a trivialization of } \mathcal{P} \textit{ over } \mathrm{Spec}\, R[[t]], \\ \varphi_\infty \textit{ is a trivialization of } \mathcal{P} \textit{ over } \mathrm{Spec}\, R[t^{-1}] \end{array} \right\}.$$

By abstract nonsense, there exists a universal bundle $\hat{\mathcal{E}} \to G_{\mathcal{K}} \times \mathbb{P}^1$ with trivializations φ_0 and φ_∞ of $\hat{\mathcal{E}}$ over $G_{\mathcal{K}} \times \mathrm{Spec}\, \mathbb{C}[[t]]$ and $G_{\mathcal{K}} \times \mathrm{Spec}\, \mathbb{C}[t^{-1}]$ respectively. We will now describe them explcitly.

We understand schemes as functors from the category $\mathbb{C}-\mathrm{Alg}$ of $\mathbb{C}$-algebras to the category Set of sets, via the functor of points construction. If $f \colon R \to S$ is a $\mathbb{C}$-algebra homomorphism, we let $\hat{\mathcal{E}}_{R,\gamma}(S)_f$ denote the preimage of $f \in \mathrm{Spec}(R)(S)$ under the natural projection $\hat{\mathcal{E}}_{R,\gamma} \to \mathrm{Spec}\, R$. In particular, the isomorphism (58) implies that

$$\hat{\mathcal{E}}_{S,G_{\mathcal{K}}(f)(\gamma)}(S)_{\mathrm{id}_S} = \hat{\mathcal{E}}_{R,\gamma}(S)_f \tag{59}$$

for any $\gamma \in G_{\mathcal{K}}(R)$.

Now $\hat{\mathcal{E}} \colon \mathbb{C}-\mathrm{Alg} \to \mathrm{Sets}$ can be defined as follows. We set

$$\hat{\mathcal{E}}(R) = \{(\gamma, x) \mid \gamma \in G(R((t))),\ x \in \hat{\mathcal{E}}_{R,\gamma}(R)_{\mathrm{id}_R}\}$$

for each $\mathbb{C}$-algebra R; and

$$\hat{\mathcal{E}}(f)(\gamma, x) = \big(G_{\mathcal{K}}(f)(\gamma),\ \hat{\mathcal{E}}_{R,\gamma}(f)(x)\big)$$

for each $\mathbb{C}$-algebra homomorphism $f\colon R \to S$. Note that $\hat{\mathcal{E}}_{R,\gamma}(f)(x) \in \hat{\mathcal{E}}_{S,G_{\mathcal{K}}(f)(\gamma)}(S)_{\mathrm{id}_S}$ in view of (59).

The $(G_{\mathbb{C}[t^{-1}]} \times G_{\mathcal{O}}) \rtimes \mathbb{C}^{\times}_{\hbar}$*-actions on* $\hat{\mathcal{E}}$. We let $\mathbb{C}^{\times}_{\hbar}$ act on $G_{\mathcal{O}}$, $G_{\mathcal{K}}$, and $G_{\mathbb{C}[t^{-1}]}$ by loop rotations defined as follows. Let R be a $\mathbb{C}$-algebra. Each element $z \in \mathbb{C}^{\times}_{\hbar}(R) = R^{\times}$ induces an R-algebra automorphism m_z^* of $R[[t]]$ by sending t to $z^{-1}t$. By abuse of notation, we denote the composition

$$G_{\mathcal{O}}(R) = G(R[[t]]) \xrightarrow{G(m_z^*)} G(R[[t]]) = G_{\mathcal{O}}(R),$$

also by m_z^*.

The action of $\mathbb{C}^{\times}_{\hbar}$ on $G_{\mathcal{O}}$ is then given by

$$\begin{aligned} \mathbb{C}^{\times}_{\hbar}(R) \times G_{\mathcal{O}}(R) &\to G_{\mathcal{O}}(R) \\ (z, g) &\mapsto m_z^{*-1}(g). \end{aligned}$$

It is notationally more instructive to write $g = g(t)$ and $m_z^{*-1}(g) = g(zt)$. The loop rotation actions on $G_{\mathcal{K}}$ and $G_{\mathbb{C}[t^{-1}]}$ are defined similarly.

We write $(G_{\mathbb{C}[t^{-1}]} \times G_{\mathcal{O}}) \rtimes \mathbb{C}^{\times}_{\hbar}$ for the semidirect product in which $\mathbb{C}^{\times}_{\hbar}$ acts on $G_{\mathbb{C}[t^{-1}]} \times G_{\mathcal{O}}$ via loop rotation. It is clear that $(G_{\mathbb{C}[t^{-1}]} \times G_{\mathcal{O}}) \rtimes \mathbb{C}^{\times}_{\hbar}$ acts on $G_{\mathcal{K}}$ by

$$(g(t), h(t), z) \cdot \gamma(t) = g(t)\,\gamma(zt)\,h(t)^{-1}$$

for any $\mathbb{C}$-algebra R, any $(g(t), h(t), z) \in \big(G(R[t^{-1}]) \times G(R[[t]])\big) \rtimes R^{\times}$, and any $\gamma(t) \in G(R(\!(t)\!))$. In view of Lemma B.1, one can understand the action of $G(R[t^{-1}])$ (resp. $G(R[[t]])$) as changing the trivialization φ_∞ (resp. φ_0), and $z \in R^{\times}$ acts via the pullback along $m_z^{-1}\colon \mathbb{P}^1_R \to \mathbb{P}^1_R$.

We are going to show that this action lifts to an action of $(G_{\mathbb{C}[t^{-1}]} \times G_{\mathcal{O}}) \rtimes \mathbb{C}^{\times}_{\hbar}$ on $\hat{\mathcal{E}}$. Let R be a $\mathbb{C}$-algebra, $(g(t), h(t), z) \in (G(R[t^{-1}]) \times G(R[[t]])) \rtimes R^{\times}$, and let $\gamma(t) \in G_{\mathcal{K}}(R)$, $\gamma' = (g, h, z) \cdot \gamma$. Namely

$$\gamma'(t) = g(t)\gamma(zt)h(t)^{-1}.$$

Note that $\varphi_0' = m_z^* h \cdot \varphi_0^{\gamma}$ and $\varphi_\infty' = m_z^* g \cdot \varphi_\infty^{\gamma}$ are local trivializations of $\hat{\mathcal{E}}_{R,\gamma}$ over $\operatorname{Spec} R[[t]]$ and $\operatorname{Spec} R[t^{-1}]$, respectively. One checks immediately that $\varphi_\infty' = m_z^*\gamma' \cdot \varphi_0'$ over $\operatorname{Spec} R(\!(t)\!)$. By the theorem of Beauville and Laszlo, there exists a unique isomorphism

$$\vartheta_{g,h,z}\colon \hat{\mathcal{E}}_{R,\gamma} \xrightarrow{\sim} m_z^*\hat{\mathcal{E}}_{R,\gamma'},$$

such that $m_z^*\varphi_0^{\gamma'} \circ \vartheta_{g,h,z} = \varphi_0'$ and $m_z^*\varphi_\infty^{\gamma'} \circ \vartheta_{g,h,z} = \varphi_\infty'$.

Now let $(g'(t), h'(t), z') \in (G(R[t^{-1}]) \times G(R[[t]])) \rtimes R^{\times}$, then we have

$$(g'(t), h'(t), z') \cdot (g(t), h(t), z) = (g'(t)g(z't),\ h'(t)h(z't),\ z'z). \tag{60}$$

We denote the right-hand side of (60) by (g'', h'', z''), and write $\gamma'' = (g'', h'', z'') \cdot \gamma \in G(R(\!(t)\!))$. We claim that

$$m_z^*\vartheta_{g',h',z'} \circ \vartheta_{g,h,z} = \vartheta_{g'',h'',z''}. \tag{61}$$

To see this, it suffices to check that both sides of (61) agree with the unique isomorphism $\vartheta\colon \hat{\mathcal{E}}_{R,\gamma} \xrightarrow{\sim} m_{z'z}^*\hat{\mathcal{E}}_{R,\gamma''}$ satisfying

$$\begin{aligned} m_{z'z}^*\varphi_0^{\gamma''} \circ \vartheta &= m_{z'z}^* h'' \cdot \varphi_0^{\gamma}, \\ m_{z'z}^*\varphi_\infty^{\gamma''} \circ \vartheta &= m_{z'z}^* g'' \cdot \varphi_\infty^{\gamma}. \end{aligned}$$

This is immediate for the right-hand side of (61). For the left-hand side, we compute

$$\begin{aligned} m_{z'z}^*\varphi_0^{\gamma''} \circ m_z^*\vartheta_{g',h',z'} \circ \vartheta_{g,h,z} &= m_z^*(m_{z'}^*\varphi_0^{\gamma''} \circ \vartheta_{g',h',z'}) \circ \vartheta_{g,h,z} \\ &= m_z^*(m_{z'}^* h' \cdot \varphi_0^{\gamma'}) \circ \vartheta_{g,h,z} \\ &= m_{z'z}^* h' \cdot m_z^*\varphi_0^{\gamma'} \circ \vartheta_{g,h,z} \\ &= m_{z'z}^* h' \cdot m_z^* h \cdot \varphi_0^{\gamma} \\ &= m_{z'z}^* h'' \cdot \varphi_0^{\gamma}. \end{aligned}$$

The other equality $m_{z'z}^*\varphi_\infty^{\gamma''} \circ m_z^*\vartheta_{g',h',z'} \circ \vartheta_{g,h,z} = m_{z'z}^* h'' \cdot \varphi_\infty^{\gamma}$ can be checked in the same way.

Moreover, let $f\colon R \to S$ be a $\mathbb{C}$-algebra homomorphism. We write $\gamma_S = G_{\mathcal{K}}(f)(\gamma)$, $g_S = G_{\mathbb{C}[t^{-1}]}(f)(g)$, etc. Then we have

$$\vartheta_{g_S,h_S,z_S} = \vartheta_{g,h,z} \times_R S, \tag{62}$$

because both sides agree with the unique isomorphism $\vartheta : \hat{\mathcal{E}}_{S,\gamma_S} \xrightarrow{\sim} m_{z_S}^*\hat{\mathcal{E}}_{S,\gamma'_S}$ with $m_{z_S}^*(\varphi_0^{\gamma'})_S \circ \vartheta = m_{z_S}^* h_S \cdot (\varphi_0^{\gamma})_S$ and $m_{z_S}^*(\varphi_\infty^{\gamma'})_S \circ \vartheta = m_{z_S}^* h_S \cdot (\varphi_\infty^{\gamma})_S$

Now we can define the action

$$\Phi_R : (G_{\mathbb{C}}[t^{-1}](R)) \times G_{\mathcal{O}}(R)) \rtimes \mathbb{C}_\hbar^\times(R) \times \hat{\mathcal{E}}(R) \longrightarrow \hat{\mathcal{E}}(R)$$

by

$$\Phi_R(g(t), h(t), z, \gamma(t), x) = (g(t)\gamma(zt)h^{-1}(t), \vartheta_{(g,h,z)}(x)). \tag{63}$$

By (61), this defines an action of $(G_{\mathbb{C}}[t^{-1}](R)) \times G_{\mathcal{O}}(R)) \rtimes \mathbb{C}_\hbar^\times(R)$ on $\hat{\mathcal{E}}(R)$. In view of (62), we obtain an action of $(G_{\mathbb{C}}[t^{-1}] \times G_{\mathcal{O}}) \rtimes \mathbb{C}_\hbar^\times$ on $\hat{\mathcal{E}}$.

Since the projection $\hat{\mathcal{E}} \to G_{\mathcal{K}}$ is clearly $(G_{\mathbb{C}[t^{-1}]} \times G_{\mathcal{O}}) \rtimes \mathbb{C}_\hbar^\times$-equivariant, and $\mathrm{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}$,

$$\mathcal{E} = \hat{\mathcal{E}}/G_{\mathcal{O}} \tag{64}$$

is a G-torsor over Gr_G. Moreover, there is a canonical trivialization φ_∞ of $\mathcal{E}$ over $\mathrm{Gr}_G \times \operatorname{Spec}\mathbb{C}[t^{-1}]$. Moreover, there is a remaining action of $G_{\mathbb{C}[t^{-1}]} \rtimes \mathbb{C}_\hbar^\times$ on $\mathcal{E}$.

The trivializations φ_0 and φ_∞ induce the isomorphisms

$$\hat{\mathcal{E}}|_{G_{\mathcal{K}} \times \{0\}} \cong G_{\mathcal{K}} \times G, \quad \hat{\mathcal{E}}|_{G_{\mathcal{K}} \times \{\infty\}} \cong G_{\mathcal{K}} \times G.$$

Since $G_{\mathcal{O}}$ acts by changing the trivialization φ_0, these identifications descend to isomorphisms

$$\mathcal{E}|_{\mathrm{Gr}_G \times \{0\}} \cong G_{\mathcal{K}} \times_{G_{\mathcal{O}}} G, \quad \mathcal{E}|_{\mathrm{Gr}_G \times \{\infty\}} \cong \mathrm{Gr}_G \times G. \tag{65}$$

Appendix C. Regularity of $\mathcal{M}_{\le\lambda}(G/P)$

In this appendix, we show that the restriction of $T\mathcal{E}_{\le\lambda}(G/P)$ to any stable map $\sigma : \Sigma \to \mathcal{E}_{\le\lambda}(G/P)$ has vanishing higher cohomology. By a standard argument (e.g. [Cho23]), this implies that the moduli space $\mathcal{M}_{\le\lambda}(G/P)$ is smooth of the expected dimension.

Using semicontinuity of the dimension of cohomology, one may assume that σ is T-invariant. Moreover, since G/P is convex, it suffices to verify the case where $\Sigma \cong \mathbb{P}^1$, and $\sigma(\Sigma)$ is a genuine section over some T-fixed point $p \in \widetilde{C}_{\le\lambda}$ lying over a point $t^\mu \in C_{\le\lambda}$.

Under this assumption, there exists $v \in W/W_P$ such that σ maps Σ isomorphically onto the constant section curve $\mathrm{Sec}_{p,v}$ over the point $p \in \widetilde{C}_{\le\lambda}$, with value $[v] \in G/P$.

Let $v(R_P^+) = \{\alpha_1, \ldots, \alpha_k\}$. Suppose that $\alpha_i(\mu) > 0$ if and only if $i \le r$.

Consider the map

$$\begin{aligned} V = \mathbb{C}^{\sum_{i=1}^r(\alpha_i(\mu)-1)} &\to \widetilde{C}_{\le\lambda} \\ s = (s_{i,j})_{\substack{1\le i\le r \\ 1\le j<\alpha_i(\mu)}} &\mapsto \exp\left(\sum_{i=1}^r \sum_{j=1}^{\alpha_i(\mu)-1} s_{i,j} t^j X_{\alpha_i}\right) \cdot p. \end{aligned}$$

Let $\mathcal{E}_V(G/P)$ be the pullback of $\mathcal{E}_{\le\lambda}(G/P)$ along $V \to \widetilde{C}_{\le\lambda}$.

Since $H^1(\mathrm{Sec}_{p,v}, T\mathrm{Sec}_{p,v}) = 0$, it suffices to show that the normal bundle to the curve $\mathrm{Sec}_{p,v}$ in $\mathcal{E}_V(G/P)$ has vanishing higher cohomology.

In fact, $\mathcal{E}_V(G/P)$ is obtained by gluing $V \times \operatorname{Spec}\mathbb{C}[t] \times G/P$ with $V \times \operatorname{Spec}\mathbb{C}[t^{-1}] \times G/P$ via the transition function

$$(s, t, x) \mapsto \left(s, t, \exp\left(\sum_{i=1}^{r} \sum_{j=1}^{\alpha_i(\mu)-1} s_{i,j} t^j X_{\alpha_i}\right) t^{\mu} \cdot x\right).$$

The transition function for the normal bundle is therefore equivalent to a block-diagonal matrix with blocks $B_1, \dots, B_k$, where:

- For $i \le r$, the block B_i is the matrix $D_{\alpha_i(\mu)}$ defined below;
- For $i > r$, the block B_i is the 1×1 matrix with entry $t^{\alpha_i(\mu)}$.

We define

$$D_n := \left[\begin{array}{c|ccc} t^n & t^{n-1} & \cdots & t \\ \hline 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{array}\right]$$

We claim that the double coset $G_{\mathbb{C}[t^{-1}]} D_n G_{\mathcal{O}}$ contains the scalar matrix $t \cdot I_n$. To see this, observe that

$$\begin{pmatrix} t^{1-n} & -1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} D_n \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -t & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} t & 0 & 0 & \cdots & 0 \\ 0 & t^{n-1} & t^{n-2} & \cdots & t \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

The claim now follows by induction on n.

To conclude, the normal bundle is a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(\ell)$ with $\ell \ge -1$, and hence has vanishing higher cohomology.

References

[BL95] A. Beauville and Y. Laszlo. *Un lemme de descente*. C. R. Acad. Sci. Paris Sér. I Math. 320.3 (1995), pp. 335–340.

[BF97] K. Behrend and B. Fantechi. *The intrinsic normal cone*. Inventiones Mathematicae 128.1 (Mar. 1997), pp. 45–88. ISSN: 1432-1297. DOI: 10.1007/s002220050136.

[Beh99] K. Behrend. *The product formula for Gromov-Witten invariants*. Journal of Algebraic Geometry 8.3 (1999), pp. 529–541.

[BL94] J. Bernstein and V. Lunts. *Equivariant Sheaves and Functors*. Vol. 1578. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 1994. ISBN: 978-3-540-58071-3. DOI: 10.1007/BFb0073549.

[BFM05] R. Bezrukavnikov, M. Finkelberg, and I. Mirković. *Equivariant homology and K-theory of affine Grassmannians and Toda lattices*. Compos. Math. 141.3 (2005), pp. 746–768.

[BFN18] A. Braverman, M. Finkelberg, and H. Nakajima. *Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II*. Adv. Theor. Math. Phys. 22.5 (2018), pp. 1071–1147.

[BFN19] A. Braverman, M. Finkelberg, and H. Nakajima. *Coulomb branches of 3d $\mathcal{N} = 4$ quiver gauge theories and slices in the affine Grassmannian*. Advances in Theoretical and Mathematical Physics 23.1 (2019), pp. 75–166.

[BMO11] A. Braverman, D. Maulik, and A. Okounkov. *Quantum cohomology of the Springer resolution*. Advances in Mathematics 227.1 (May 2011), pp. 421–458. ISSN: 0001-8708. DOI: 10.1016/j.aim.2011.01.021.

[BDGH16] M. Bullimore, T. Dimofte, D. Gaiotto, and J. Hilburn. *Boundaries, mirror symmetry, and symplectic duality in 3d $\mathcal{N} = 4$ gauge theory*. Journal of High Energy Physics 2016.10 (Oct. 2016). ISSN: 1029-8479. DOI: 10.1007/jhep10(2016)108.

[CL24a] K. F. Chan and N. C. Leung. *3d Mirror Symmetry is Mirror Symmetry*. 2024. arXiv: 2410.03611.

[CL24b] K. F. Chan and N. C. Leung. *SYZ Mirrors in non-Abelian 3d Mirror Symmetry*. 2024. arXiv: 2408.09479.

[Cho23] C. H. Chow. *Peterson-Lam-Shimozono's theorem is an affine analogue of quantum Chevalley formula*. arXiv preprint arXiv:2110.09985 (2023). URL: https://arxiv.org/abs/2110.09985.

[Cho24] C. H. Chow. *On D. Peterson's presentation of quantum cohomology of G/P*. 2024. arXiv: 2210.17382.

[CG97] N. Chriss and V. Ginzburg. *Representation Theory and Complex Geometry*. Modern Birkhäuser Classics. Birkhäuser, 1997.

[CIJ18] T. Coates, H. Iritani, and Y. Jiang. *The crepant transformation conjecture for toric complete intersections*. Adv. Math. 329 (2018), pp. 1002–1087.

[CK99] D. Cox and S. Katz. *Mirror Symmetry and Algebraic Geometry*. American Mathematical Society, Mar. 1999. ISBN: 9781470412951. DOI: 10.1090/surv/068.

[FP97] W. Fulton and R. Pandharipande. *Notes on stable maps and quantum cohomology*. Proc. Sympos. Pure Math. 62 (1997), pp. 45–96.

[Ful98] W. Fulton. *Intersection theory*. Second. Vol. 2. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1998, pp. xiv+470.

[Gab81] O. Gabber. *The integrability of the characteristic variety*. American Journal of Mathematics 103.3 (1981), pp. 445–468.

[GW25] T. Gannon and B. Webster. *Functoriality of Coulomb branches*. arXiv preprint arXiv:2501.09962 (2025). URL: https://arxiv.org/abs/2501.09962.

[Giv96] A. Givental. *Equivariant Gromov-Witten invariants*. International Mathematics Research Notices 1996.13 (1996), p. 613. ISSN: 1073-7928. DOI: 10.1155/s1073792896000414.

[GMP23a] E. Gonzalez, C. Y. Mak, and D. Pomerleano. *Coulomb branch algebras via symplectic cohomology*. 2023. arXiv: 2305.04387. URL: https://arxiv.org/abs/2305.04387.

[GMP23b] E. González, C. Y. Mak, and D. Pomerleano. *Affine nil-Hecke algebras and quantum cohomology*. Adv. Math. 415 (2023), Paper No. 108867, 47.

[GP99] T. Graber and R. Pandharipande. *Localization of virtual classes*. Invent. Math. 135.2 (1999), pp. 487–518.

[Iri17] H. Iritani. *Shift operators and toric mirror theorem*. Geometry & Topology 21.1 (2017), pp. 315–343.

[Iri23] H. Iritani. *Quantum cohomology of blowups*. 2023. DOI: 10.48550/ARXIV.2307.13555.

[Iri25] H. Iritani. *Fourier analysis of equivariant quantum cohomology*. 2025. arXiv: 2501.18849 [math.AG]. URL: https://arxiv.org/abs/2501.18849.

[Kol07] J. Kollár. *Lectures on Resolution of Singularities*. Vol. 166. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2007, pp. vi+208.

[KM94] M. Kontsevich and Y. Manin. *Gromov-Witten classes, quantum cohomology, and enumerative geometry*. Communications in Mathematical Physics 164.3 (Aug. 1994), pp. 525–562. ISSN: 1432-0916. DOI: 10.1007/bf02101490.

[LS10] T. Lam and M. Shimozono. *Quantum cohomology of G/P and homology of affine Grassmannian*. Acta Math. 204.1 (2010), pp. 49–90.

[LS25] Y. Lekili and E. Segal. *Equivariant Fukaya categories at singular values*. Glasgow Mathematical Journal (2025). to appear.

[LL12] N. C. Leung and C. Li. *Gromov-Witten invariants for G/B and Pontryagin product for ΩK*. Transactions of the American Mathematical Society 364.5 (May 2012), pp. 2567–2599. ISSN: 1088-6850. DOI: 10.1090/s0002-9947-2012-05438-9.

[Lie21] T. Liebenschutz-Jones. *Shift operators and connections on equivariant symplectic cohomology*. arXiv preprint arXiv:2104.01891 (2021).

[MO19] D. Maulik and A. Okounkov. *Quantum groups and quantum cohomology*. Vol. 408. Astérisque. Paris: Société Mathématique de France (SMF), 2019. ISBN: 978-2-85629-900-5.

[MV07] I. Mirković and K. Vilonen. *Geometric Langlands duality and representations of algebraic groups over commutative rings*. Ann. of Math. (2) 166.1 (2007), pp. 95–143.

[Nak16] H. Nakajima. *Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, I*. Adv. Theor. Math. Phys. 20.3 (2016), pp. 595–669.

[OP10] A. Okounkov and R. Pandharipande. *The quantum differential equation of the Hilbert scheme of points in the plane*. Transform. Groups 15.4 (2010), pp. 965–982.

[Pan98] R. Pandharipande. *Rational curves on hypersurfaces (after A. Givental)*. Astérisque 252 (1998). Séminaire Bourbaki, Vol. 1997/98, Exp. No. 848, 5, 307–340.

[Pet97] D. Peterson. *Quantum cohomology of G/P*. Lecture notes, Spring Term 1997, Massachusetts Institute of Technology. Lecture course; notes compiled by J. Lu and K. Rietsch. 1997.

[PT24] D. Pomerleano and C. Teleman. *Quantization commutes with reduction again: the quantum GIT conjecture I*. 2024. arXiv: 2405.20301 [math.SG]. URL: https://arxiv.org/abs/2405.20301.

[Sav08] Y. Savelyev. *Quantum characteristic classes and the Hofer metric*. Geometry & Topology 12.4 (Sept. 2008), pp. 2277–2326. ISSN: 1465-3060. DOI: 10.2140/gt.2008.12.2277.

[SW] N. Seiberg and E. Witten. "Gauge dynamics and compactification to three dimensions". *The mathematical beauty of physics (Saclay, 1996)*. Vol. 24. Adv. Ser. Math. Phys. Pp. 333–366.

[Sei97] P. Seidel. π_1 *of symplectic automorphism groups and invertibles in quantum homology rings*. Geom. Funct. Anal. 7.6 (1997), pp. 1046–1095.

[Tam24] S. Tamagni. *Nonabelian shift operators and shifted Yangians*. 2024. eprint: arXiv:2412.17906.

[Tel14] C. Teleman. "Gauge theory and mirror symmetry". *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*. Kyung Moon Sa, Seoul, 2014, pp. 1309–1332.

[Tel21] C. Teleman. *The rôle of Coulomb branches in 2D gauge theory*. Journal of the European Mathematical Society 23.11 (2021), pp. 3497–3520. DOI: 10.4171/JEMS/1071.

[Wit91] E. Witten. "Two-dimensional gravity and intersection theory on moduli space". *Surveys in differential geometry (Cambridge, MA, 1990)*. Lehigh Univ., Bethlehem, PA, 1991, pp. 243–310.

[Zhu17] X. Zhu. "An introduction to affine Grassmannians and the geometric Satake equivalence". *Geometry of moduli spaces and representation theory*. Vol. 24. IAS/Park City Math. Ser. American Mathematical Society, Providence, RI, 2017, pp. 59–154.

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