GROSS FIBRATIONS, SYZ MIRROR SYMMETRY, AND OPEN GROMOV-WITTEN INVARIANTS FOR TORIC CALABI-YAU ORBIFOLDS

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Abstract. Given a toric Calabi-Yau orbifold $\mathcal{X}$ whose underlying toric variety is semi-projective, we construct and study a non-toric Lagrangian torus fibration on $\mathcal{X}$, which we call the Gross fibration. We apply the Strominger-Yau-Zaslow recipe to the Gross fibration of (a toric modification of) $\mathcal{X}$ to construct its instanton-corrected mirror, where the instanton corrections come from genus 0 open orbifold Gromov-Witten invariants, which are virtual counts of holomorphic orbi-disks in $\mathcal{X}$ bounded by fibers of the Gross fibration.

We explicitly evaluate all these invariants by first proving an open/closed equality and then employing the toric mirror theorem for suitable toric (parital) compactifications of $\mathcal{X}$. Our calculations are then applied to

(1) prove a conjecture of Gross-Siebert on a relation between genus 0 open orbifold Gromov-Witten invariants and mirror maps of $\mathcal{X}$ – this is called the open mirror theorem, which leads to an enumerative meaning of mirror maps, and
(2) demonstrate how open (orbifold) Gromov-Witten invariants for toric Calabi-Yau orbifolds change under toric crepant resolutions – this is an open analogue of Ruan's crepant resolution conjecture.

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1. Introduction

In this paper, we study mirror symmetry for toric Calabi-Yau orbifolds from the SYZ perspective [100]. SYZ mirror symmetry for toric Calabi-Yau manifolds was studied in [20], and it was conjectured that the *SYZ map*, which is defined in terms of *genus 0 open Gromov-Witten invariants, or disk invariants*, is equal to the inverse of a *mirror map* [20, Conjecture 1.1] (see also [23, Conjecture 2]). Such a connection between disk invariants and mirror maps was first envisioned by Gross and Siebert [67, Conjecture 0.2] where they expressed it in terms of tropical, instead of holomorphic, disks. This conjecture leads to explicit formulas for computing disk invariants, and also provides an enumerative meaning to mirror maps, which was originally anticipated in the SYZ proposal.

The conjecture was proved in [23] for the total space of the canonical line bundle over a compact toric Fano manifold in any dimension. In this paper, we generalize the SYZ construction and prove this conjecture for *all semi-projective toric Calabi-Yau orbifolds* (Theorems 7.2 and 7.3), and in particular *all semi-projective toric Calabi-Yau manifolds* (Corollary 7.4). We call this the *open mirror theorem*. The main new ingredients in this generalization include the introduction of *orbi-disk invariants* defined in [28] (cf. also [18]) which are the orbifold analogue of disk invariants, computation of these invariants by constructing various toric (partial) compactifications of $\mathcal{X}$, and a comparison of the mirror maps of these toric compactifications and that of $\mathcal{X}$. Roughly speaking, the use of orbifold techniques allow us to extend the scheme of proof in [23] to all semi-projective toric Calabi-Yau manifolds.

On the other hand, it is natural to work in the orbifold setting since all the techniques involved in [20, 23] adapt naturally to orbifolds. More importantly, the open mirror theorems in this more general orbifold setting can be used to deduce an *open crepant resolution theorem* (Theorem 8.1), which gives a precise relation between the orbi-disk invariants of $\mathcal{X}$ and the (orbi-)disk invariants of its (partial) crepant resolutions. This gives an affirmative answer to Ruan's crepant resolution conjecture [96, 13, 34, 36] in the open sector.

A more detailed introduction and description of our main results are now in order.

1.1. Mirror symmetry for orbifolds.

Mirror symmetry for orbifolds. Mirror symmetry, which was discovered by string-theoretic considerations, may be roughly understood as an equivalence between the symplectic geometry (A-model) of a manifold X and the complex geometry (B-model) of another manifold $\check{X}$ called the mirror of X, and vice versa. Originally formulated for Calabi-Yau manifolds, mirror symmetry for non–Calabi-Yau geometries, such as Fano manifolds and manifolds of general types, has also been investigated extensively, see e.g. [6, 54, 80, 70, 69, 76, 75, 98, 41, 63, 1].

The famous homological mirror symmetry (HMS) conjecture, proposed by Kontsevich in his 1994 ICM address [79], formulates the mirror symmetry phenomenon mathematically and intrinsically as an equivalence between the derived Fukaya category of Lagrangian submanifolds in X and the derived category of coherent sheaves over the mirror $\check{X}$. The HMS conjecture has been proven in various Calabi-Yau geometries, see e.g. [95, 97, 99].

On the other hand, an incredible geometric consequence of mirror symmetry is the computation of Gromov-Witten invariants for a generic quintic 3-fold in terms of Hodge-theoretic data of its mirror. This is the famous *mirror formula*, predicted physically by [14], and proven mathematically by independent works of Givental [55] and Lian-Liu-Yau [88]. Nowadays the mirror formula has been generalized to various settings, including [56], [89, 90, 91], and [33].

In all these developments in mirror symmetry, orbifolds have been playing a significant role, starting with the first constructions of mirrors for Calabi-Yau hypersurfaces in weighted projective spaces [15, 61]. In recent years, it has become even clearer that orbifolds are indispensable in the study of mirror symmetry. For instance, many known constructions of mirrors naturally produce orbifolds. In dimension 3, crepant resolutions of these orbifolds are taken as the mirrors. This, however, cannot be done in general in higher dimensions due to the possibly non-existence of crepant resolutions. It is therefore very natural to consider mirror symmetry for orbifolds.

Indeed, much progress in mirror symmetry for orbifolds has been made in recent years. The HMS conjecture for orbifolds has been proved in various cases, e.g. weighted projective planes [5], weighted projective spaces in general [8], toric orbifolds of toric del Pezzo surfaces [102], toric Deligne-Mumford stacks [43], etc. On the other hand, mirror theorems showing that the A-model (i.e. Gromov-Witten theory) of an orbifold is equivalent to the B-model of its mirror have also been proven for various classes of orbifolds, e.g. $\mathbb{P}^1$-orbifolds [94], weighted projective spaces [35], complete intersection orbifolds [32], toric Deligne-Mumford stacks [31], and the mirror quintic orbifold [85].

1.2. **SYZ mirror construction.** In 1996, Strominger, Yau and Zaslow [100] proposed an intrinsic and geometric way to understand mirror symmetry for Calabi-Yau manifolds via T-duality. Roughly speaking, the Strominger-Yau-Zaslow (SYZ) conjecture asserts that for a pair of Calabi-Yau manifolds X and $\check{X}$ which are mirror to each other, there exist special Lagrangian torus fibrations

$$\begin{array}{ccc} X & & \check{X} \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

which are fiberwise dual to each other. Mathematical approaches to SYZ mirror symmetry have since been extensively studied by many researchers including Kontsevich-Soibelman [81, 82], Leung-Yau-Zaslow [87], Leung [86], Gross-Siebert [64, 65, 66, 67], Auroux [3, 4], Chan-Leung [24], Chan-Lau-Leung [20] and Abouzaid-Auroux-Katzarkov [1].

A very important application of the SYZ conjecture is providing a geometric construction of mirrors: It suggests that, given a Calabi-Yau manifold X, a mirror $\check{X}$ can be constructed by finding a (special) Lagrangian torus fibration $X \to B$ and suitably modifying the complex structure of the total space of the fiberwise dual by instanton corrections. For toric Calabi-Yau manifolds, Gross [62] (and independently Goldstein [57]) constructed such a special Lagrangian torus fibration which we call the *Gross fibration*. In [20], the SYZ construction was applied to the Gross fibration to produce an instanton-corrected mirror family of a toric Calabi-Yau manifold, following the Floer-theoretic approach pioneered by Auroux [3, 4].

In this paper we consider the SYZ mirror construction for toric Calabi-Yau *orbifolds*. A toric Calabi-Yau orbifold is a (necessarily non-compact) Gorenstein toric orbifold $\mathcal{X}$ whose canonical line bundle $K_{\mathcal{X}}$ is trivial. We also assume that the coarse moduli space of $\mathcal{X}$ is a *semi-projective* toric variety, or equivalently, that $\mathcal{X}$ is as in Setting 4.3. Following [62], we define in Definition 4.7 a special Lagrangian torus fibration

$$\mu : \mathcal{X} \to B$$

which we again call the Gross fibration of $\mathcal{X}$. A special Lagrangian torus fibration

$$\mu' : \mathcal{X}' \to B'$$

on a suitable toric modification $\mathcal{X}'$ of $\mathcal{X}$ is also defined; see Definitions 4.17 and 4.20 for details.

As in the manifold case, the discriminant locus $\Gamma' \subset B'$ (resp. $\Gamma \subset B$) can be described explicitly. In particular, it is a real codimension 2 subset contained in a hyperplane which we call the *wall* in the base B' (resp. B). The wall divides the smooth locus $B'_0 = B' \setminus \Gamma'$ (resp. $B_0 = B \setminus \Gamma$) into two chambers B'_+ and B'_- (resp. B_+ and B_-). Over B'_0, the fibration μ' restricts to a torus bundle $\mu' : \mathcal{X}'_0 \to B'_0$, and the dual torus bundle

$$\check{\mu}' : \check{\mathcal{X}}'_0 \to B'_0$$

admits a natural complex structure, producing the so-called *semi-flat mirror* of $\mathcal{X}$.

This does not give the genuine mirror for $\mathcal{X}$ because the semi-flat complex structure *cannot* be extended further to any partial compactification of $\check{\mathcal{X}}'_0$. This is due to nontrivial monodromy of the affine structure around the discriminant locus Γ'. As instructed by the SYZ proposal, we need to deform the semi-flat complex structure so that it becomes extendable to a suitable partial compactification. More concretely, what we do is (following Auroux [3, 4]) to modify the gluing between the complex charts over the chambers B'_+ and B'_- by *instanton corrections*, which in our case come from genus 0 open orbifold Gromov-Witten invariants, or orbi-disk invariants, of $\mathcal{X}$ (cf. the manifold case [3, 4, 20, 1]). The latter are virtual counts of holomorphic orbi-disks in the toric Calabi-Yau orbifold $\mathcal{X}$ with boundary lying on special Lagrangian torus fibers of μ over the wall in B. A suitable partial compactification then yields the following instanton-corrected mirror, or *SYZ mirror*, of $\mathcal{X}$:

Theorem 1.1 (See Section 5.3)**.** *Let $\mathcal{X}$ be a toric Calabi-Yau orbifold as in Setting 4.3 and equipped with the Gross fibration in Definition 4.7. Then the SYZ mirror of $\mathcal{X}$ (with a hypersurface removed) is the family of non-compact Calabi-Yau manifolds*

$$\check{\mathcal{X}} := \{(u, v, z_1, \ldots, z_{n-1}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} \mid uv = g(z_1, \ldots, z_{n-1})\},$$

where the defining equation $uv = g$ is given by

$$uv = (1 + \delta_0) + \sum_{j=1}^{n-1} (1 + \delta_j) z_j + \sum_{j=n}^{m-1} (1 + \delta_j) q_j z^{\boldsymbol{b}_j} + \sum_{\nu \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} (\tau_\nu + \delta_\nu) q^{-D_\nu^\vee} z^\nu.$$

Here $1 + \delta_j$ and $\tau_\nu + \delta_\nu$ are generating functions of orbi-disk invariants of $(\mathcal{X}, F_r)$ (see Section 5.2 for the reasons why the generating functions are of these forms).

Remark 1.2.

(1) *The SYZ mirror of the toric Calabi-Yau orbifold $\mathcal{X}$ itself (i.e. without removing a hypersurface) is given by the* Landau-Ginzburg model $(\check{\mathcal{X}}, W)$ *where $W : \check{\mathcal{X}} \to \mathbb{C}$ is the holomorphic function $W := u$; see* [20, Section 4.6] *and* [1, Section 7] *for related discussions in the manifold case.*

(2) *Several explicit examples will be discussed in Section 6.5. For instance, when* $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_m]$, *the mirror is given by the equation*

$$uv = \prod_{j=0}^{m-1} (z - \kappa_j),$$

where κ_j *is explicitly defined in* (6.22).

To the best of our knowledge, this is the first time the SYZ mirror construction is applied systematically to construct mirrors for *orbifolds*.

1.3. **Orbi-disk invariants.** To demonstrate that $\check{\mathcal{X}}$ is indeed mirror to the toric Calabi-Yau orbifold $\mathcal{X}$, we would like to show that the family $\check{\mathcal{X}}$ is written in *canonical coordinates*. This can be rephrased as the conjecture that the SYZ map, defined in terms of orbi-disk invariants, is inverse to the toric mirror map of $\mathcal{X}$ (cf. [67, Conjecture 0.2], [20, Conjecture 1.1] and [23, Conjecture 2]). To prove this conjecture, knowledge about the orbi-disk invariants is absolutely crucial.

One major advance of this paper is the complete calculation of these orbi-disk invariants, or genus 0 open orbifold Gromov-Witten invariants, for moment-map Lagrangian torus fibers in toric Calabi-Yau orbifolds. Our calculation is based on the following *open/closed equality*:

Theorem 1.3 (See Theorem 6.12 and equation (6.1))**.** *Let* $\mathcal{X}$ *be a toric Calabi-Yau orbifold as in Setting 4.3 and equipped with a toric Kähler structure. Let* $L \subset \mathcal{X}$ *be a Lagrangian torus fiber of the moment map of* $\mathcal{X}$, *and let* $\beta \in \pi_2(\mathcal{X}, L)$ *be a holomorphic (orbi-)disk class of Chern-Weil Maslov index 2. Let* $\bar{\mathcal{X}}$ *be the toric partial compactification of* $\mathcal{X}$ *constructed in Construction 6.1 which depends on* β. *Then we have the following equality between genus 0 open orbifold Gromov-Witten invariants of* $(\mathcal{X}, L)$ *and closed orbifold Gromov-Witten invariants of* $\bar{\mathcal{X}}$:

$$n^{\mathcal{X}}_{1,l,\beta}([\mathrm{pt}]_L; \mathbf{1}_{\nu_1}, \ldots, \mathbf{1}_{\nu_l}) = \langle [\mathrm{pt}], \mathbf{1}_{\bar{\nu}_1}, \ldots, \mathbf{1}_{\bar{\nu}_l} \rangle^{\bar{\mathcal{X}}}_{0,1+l,\bar{\beta}}. \tag{1.1}$$

This theorem is proved by showing that the relevant moduli space of stable (orbi-)disks in $\mathcal{X}$ is isomorphic to the relevant moduli space of stable maps to $\bar{\mathcal{X}}$ *as Kuranishi spaces*. The key geometric ingredient underlying the proof is that the toric compactification $\bar{\mathcal{X}}$ is constructed in such a way that (orbi-)disks in $\mathcal{X}$ can be "capped off" in $\bar{\mathcal{X}}$ to obtain (orbi-)spheres, and that the deformation and obstruction theories of the two moduli problems can naturally be identified.

The closed orbifold Gromov-Witten invariants of $\bar{\mathcal{X}}$ appearing in (1.1) are encoded in the J-function of $\bar{\mathcal{X}}$. We will evaluate these invariants via the toric mirror theorem, but we remark that this requires extra care since $\bar{\mathcal{X}}$ can be non-compact. Fortunately, $\bar{\mathcal{X}}$ is *semi-Fano* (see Definition 2.8) and semi-projective, so the *equivariant* toric mirror theorem of [31] still applies to give an explicit formula for the equivariant J-function of $\bar{\mathcal{X}}$. Extracting the relevant equivariant closed orbifold Gromov-Witten invariants from this formula and taking non-equivariant limits then yield explicit formulas for the genus 0 open orbifold Gromov-Witten invariants of $\mathcal{X}$ and hence the generating functions which appear in the defining equation of the SYZ mirror $\check{\mathcal{X}}$:

Theorem 1.4 (See Theorems 6.19 and 6.20)**.** *Let $\mathcal{X}$ be a toric Calabi-Yau orbifold as in Setting 4.3. Let F_r be a Lagrangian torus fiber of the Gross fibration of $\mathcal{X}$ lying above a point r in the chamber $B_+ \subset B_0$.*

(1) Let $1+\delta_i$ be the generating function of genus 0 open orbifold Gromov-Witten invariants of $\mathcal{X}$ in classes $\beta_i(r) + \alpha$, with $\alpha \in H_2^{\text{eff}}(\mathcal{X})$ satisfying $c_1(\mathcal{X}) \cdot \alpha = 0$ and $\beta_i(r) \in \pi_2(\mathcal{X}, F_r)$ the basic smooth disk class corresponding to the primitive generator $\boldsymbol{b}_i$ of a ray in Σ. Then

$$1 + \delta_i = \exp\left(-A_i^{\mathcal{X}}(y)\right),$$

after inverting the toric mirror map (6.15)*.*

(2) Let $\tau_\nu + \delta_\nu$ be the generating function of genus 0 open orbifold Gromov-Witten invariants of $\mathcal{X}$ in classes $\beta_\nu(r) + \alpha$, with $\alpha \in H_2^{\text{eff}}(\mathcal{X})$ satisfying $c_1(\mathcal{X}) \cdot \alpha = 0$ and $\beta_\nu(r) \in \pi_2(\mathcal{X}, F_r)$ the basic orbi-disk class corresponding to a Box element ν of age one. Then

$$\tau_\nu + \delta_\nu = y^{D_\nu^\vee} \exp\left(-\sum_{i \notin I_\nu} c_{\nu i} A_i^{\mathcal{X}}(y)\right),$$

after inverting the toric mirror map (6.15)*.*

Here the functions $A_i^{\mathcal{X}}(y)$'s are given explicitly in (6.13)*.*

These generalize the corresponding results in [23] to all semi-projective toric Calabi-Yau orbifolds. In particular, this applies to the toric Calabi-Yau 3-fold $\mathcal{X} = K_{\mathbb{F}_2}$ which cannot be handled by [23] (see Example (4) in Section 6.5).

1.4. **Applications.** We discuss two major applications of the explicit calculations of orbi-disk invariants in this paper.

1.4.1. *Open mirror theorems.* The first application, as we mentioned above, is to show that the mirror family $\check{\mathcal{X}}$ is written in canonical coordinates. This concerns the comparison of several mirror maps for a toric Calabi-Yau orbifold $\mathcal{X}$. More precisely, the SYZ construction yields what we call the *SYZ map* $\mathcal{F}^{\text{SYZ}}$, defined in terms of genus 0 open orbifold Gromov-Witten invariants (see the precise definition in (7.2)). In closed Gromov-Witten theory, the toric mirror theorem of [31] involves a combinatorially defined *toric mirror map* $\mathcal{F}^{\text{mirror}}$ (see Section 2.4 and the formula (6.15)). We prove the following *open mirror theorem*:

Theorem 1.5 (See Theorem 7.2)**.** *For a toric Calabi-Yau orbifold $\mathcal{X}$ as in Setting 4.3, the SYZ map is inverse to the toric mirror map, i.e. we have*

$$\mathcal{F}^{\text{SYZ}} = \left(\mathcal{F}^{\text{mirror}}\right)^{-1}$$

near the large volume limit of $\mathcal{X}$.

We remark that an open mirror theorem was proved for compact semi-Fano toric manifolds in [21, 22] and some examples of compact semi-Fano toric orbifolds in [18]. On the other hand, open mirror theorems for 3-dimensional toric Calabi-Yau geometries relative to Aganagic-Vafa type Lagrangian branes were proved in various degrees of generality in [60, 11, 42, 44].

By combining the above open mirror theorem together with the analysis of the relations between *period integrals* and the *GKZ hypergeometric system* associated to $\mathcal{X}$ done in [23], we obtain another version of the open mirror theorem, linking the SYZ map to period integrals:

Theorem 1.6 (See Theorem 7.3)**.** *For a toric Calabi-Yau orbifold $\mathcal{X}$ as in Setting 4.3, there exists a collection $\{\Gamma_1, \ldots, \Gamma_r\} \subset H_n(\check{\mathcal{X}}; \mathbb{C})$ of linearly independent cycles such that*

$$q_a = \exp\left(-\int_{\Gamma_a} \check{\Omega}_{\mathcal{F}^{\mathrm{SYZ}}(q,\tau)}\right), \quad a = 1, \ldots, r',$$

$$\tau_{\boldsymbol{b}_j} = \int_{\Gamma_{j-m+r'+1}} \check{\Omega}_{\mathcal{F}^{\mathrm{SYZ}}(q,\tau)}, \quad j = m, \ldots, m'-1,$$

where q_a's and $\tau_{\boldsymbol{b}_j}$'s are the Kähler and orbifold parameters in the extended complexified Kähler moduli space of $\mathcal{X}$.

In particular, we have the following relation between disk invariants and period integrals in the manifold case:

Corollary 1.7 (See Corollary 7.4)**.** *For a semi-projective toric Calabi-Yau manifold $\mathcal{X}$, there exists a collection $\{\Gamma_1, \ldots, \Gamma_r\} \subset H_n(\check{\mathcal{X}}; \mathbb{C})$ of linearly independent cycles such that*

$$q_a = \exp\left(-\int_{\Gamma_a} \check{\Omega}_{\mathcal{F}^{\mathrm{SYZ}}(q,\tau)}\right), \quad a = 1, \ldots, r,$$

where q_a's are the Kähler parameters in the complexified Kähler moduli space of $\mathcal{X}$.

Our results provide an enumerative meaning to period integrals, as conjectured by Gross and Siebert in [67, Conjecture 0.2 and Remark 5.1]. One difference between our results and their conjecture is that we use holomorphic disks while they considered *tropical* disks. On the other hand, their conjecture is much more general and is expected to hold even when $\mathcal{X}$ is a *compact* Calabi-Yau manifold. A more precise formulation of the Gross-Siebert conjecture in the case of toric Calabi-Yau manifolds can be found in [20, Conjecture 1.1] (see also [23, Conjecture 2]).

Corollary 1.7 proves a weaker form of [20, Conjecture 1.1], which concerns periods over *integral* cycles in $\check{\mathcal{X}}$ (while here the cycles $\Gamma_1, \ldots, \Gamma_r$ are allowed to have complex coefficients), for *all* semi-projective toric Calabi-Yau manifolds. The case when $\mathcal{X}$ is the total space of the canonical line bundle of a toric Fano manifold was previous proved in [23].[1]

1.4.2. *Open crepant resolution conjecture.* The second main application concerns how genus 0 open (orbifold) Gromov-Witten invariants change under crepant birational maps. String theoretic considerations suggest that Gromov-Witten theory should remain unchanged as the target space changes under a crepant birational map. This is known as the *crepant*

[1] As explained in [23, Section 5.2], to prove the original stronger form of the conjecture, what we need are *integral* cycles whose periods have specific logarithmic terms. It turns out that such cycles have already been constructed by Doran and Kerr in [40, Section 5.3 and Theorem 5.1], at least when $\mathcal{X}$ is a toric Calabi-Yau 3-fold of the form K_Y where Y is a toric del Pezzo surface. Doran suggested to us that it should not be difficult to extend their construction to general toric Calabi-Yau varieties. Hence the stronger form of the conjecture should follow from Corollary 1.7 and their construction; cf. the discussion in [39, Section 4]. We thank Charles Doran for pointing this out.

resolution conjecture and has been intensively studied in closed Gromov-Witten theory; see e.g. [96, 13, 34, 30, 36] and references therein. In [18], a conjecture on how generating functions of genus 0 open Gromov-Witten invariants behave under crepant resolutions was formulated and studied for compact Gorenstein toric orbifolds. In this paper, we apply our calculations to prove an analogous result for toric Calabi-Yau orbifolds:

Theorem 1.8 (See Theorem 8.1). *Let $\mathcal{X}$ be a toric Calabi-Yau orbifold as in Setting 4.3, and let $\mathcal{X}'$ be a toric orbifold which is a toric crepant partial resolution of $\mathcal{X}$ (such an $\mathcal{X}'$ will automatically be as in Setting 4.3). Then we have*

$$\mathcal{F}^{\mathrm{SYZ}}_{\mathcal{X}} = \mathcal{F}^{\mathrm{SYZ}}_{\mathcal{X}'},$$

after analytic continuation and a change of variables.

See Section 8 for more details.

We shall mention that there are recent works of Brini, Cavalieri and Ross [16, 12] on open versions of the crepant resolution conjecture for Aganagic-Vafa type Lagrangian branes in 3-dimensional toric Calabi-Yau orbifolds. Ke and Zhou also informed us that they have proved the quantum McKay correspondence for disk invariants of outer Aganagic-Vafa branes in semi-projective toric Calabi-Yau 3-orbifolds [77].

1.5. **Organization.** The rest of the paper is organized as follows. Section 2 contains a review on the basic materials about toric orbifolds that we need. The mirror theorem for toric orbifolds is discussed in Section 2.6. In Section 3 we give a summary on the theory of genus 0 open orbifold Gromov-Witten invariants for toric orbifolds. In Section 4 we define and study the Gross fibration of a toric Calabi-Yau orbifold. In Section 5 we construct the instanton-corrected mirror of a toric Calabi-Yau orbifold by applying the SYZ recipe to the Gross fibration of a suitable toric modification. The genus 0 open orbifold Gromov-Witten invariants which are relevant to the SYZ mirror construction are computed in Section 6 via an open/closed equality and an equivariant toric mirror theorem applied to various toric (partial) compactifications. In Section 7 we apply our calculation of these invariants to deduce the open mirror theorems which relate various mirror maps associated to a toric Calabi-Yau orbifold. Our calculation is also applied in Section 8 to prove a relationship between genus 0 open orbifold Gromov-Witten invariants of a toric Calabi-Yau orbifold and those of its toric crepant (partial) resolutions. Appendix A discusses some useful facts about Maslov indices. Appendix B contains the technical discussions on the analytic continuations of mirror maps.

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2. Preliminaries on toric orbifolds

In this section we briefly review the construction and basic properties of toric orbifolds. We also describe the closed mirror theorem for toric orbifolds due to [31]. We refer the readers to [9, 74] for more details on the essential ingredients of toric orbifolds, and to [72, 44, 31] for more details on mirror theorems for toric orbifolds.

2.1. **Construction.** A *toric orbifold*, as introduced in [9], is associated to a set of combinatorial data called a *stacky fan*:

$$(\Sigma, \boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1}),$$

where Σ is a simplicial fan contained in the $\mathbb{R}$-vector space $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ associated to a lattice N of rank n, and $\{\boldsymbol{b}_i \mid 0 \leq i \leq m-1\}$ are integral generators of 1-dimensional cones (or rays) in Σ. We call $\boldsymbol{b}_i$ the *stacky vectors*. We denote by $|\Sigma| \subset N_{\mathbb{R}}$ the support of Σ.

Let $\boldsymbol{b}_m, \ldots, \boldsymbol{b}_{m'-1} \in N \cap |\Sigma|$ be additional vectors such that the set $\{\boldsymbol{b}_i\}_{i=0}^{m-1} \cup \{\boldsymbol{b}_j\}_{j=m}^{m'-1}$ generates N over $\mathbb{Z}$. Following [74], the data

$$(\Sigma, \{\boldsymbol{b}_i\}_{i=0}^{m-1} \cup \{\boldsymbol{b}_j\}_{j=m}^{m'-1})$$

is called an *extended stacky fan*, and $\{\boldsymbol{b}_j\}_{j=m}^{m'-1}$ are called *extra vectors*. We describe the construction of toric orbifolds from extended stacky fans. The flexibility of choosing extra vectors is important in the toric mirror theorem, see Section 2.6.

Consider the surjective group homomorphism, the *fan map*,

$$\phi : \widetilde{N} := \bigoplus_{i=0}^{m'-1} \mathbb{Z} e_i \to N, \quad \phi(e_i) := \boldsymbol{b}_i \text{ for } i = 0, \ldots, m'-1.$$

This gives an exact sequence (the "fan sequence")

$$0 \longrightarrow \mathbb{L} := \operatorname{Ker}(\phi) \xrightarrow{\psi} \widetilde{N} \xrightarrow{\phi} N \longrightarrow 0. \tag{2.1}$$

Note that $\mathbb{L} \simeq \mathbb{Z}^{m'-n}$. Tensoring with $\mathbb{C}^\times$ gives the following exact sequence:

$$0 \longrightarrow G := \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{C}^\times \longrightarrow \widetilde{N} \otimes_{\mathbb{Z}} \mathbb{C}^\times \simeq (\mathbb{C}^\times)^{m'} \xrightarrow{\phi_{\mathbb{C}^\times}} \mathbb{T} := N \otimes_{\mathbb{Z}} \mathbb{C}^\times \to 0. \tag{2.2}$$

Consider the set of "anti-cones",

$$\mathcal{A} := \left\{ I \subset \{0, 1, \ldots, m'-1\} \mid \sum_{i \notin I} \mathbb{R}_{\geq 0} \boldsymbol{b}_i \text{ is a cone in } \Sigma \right\}. \tag{2.3}$$

For $I \in \mathcal{A}$, let $\mathbb{C}^I \subset \mathbb{C}^{m'}$ be the subvariety defined by the ideal in $\mathbb{C}[Z_0, \ldots, Z_{m'-1}]$ generated by $\{Z_i \mid i \in I\}$. Put

$$U_{\mathcal{A}} := \mathbb{C}^{m'} \setminus \bigcup_{I \notin \mathcal{A}} \mathbb{C}^I.$$

The algebraic torus G acts on $\mathbb{C}^{m'}$ via the map $G \to (\mathbb{C}^\times)^{m'}$ in (2.2). Since N is torsion-free, the induced G-action on $U_\mathcal{A}$ is effective and has finite isotropy groups. The global quotient

$$\mathcal{X}_\Sigma := [U_\mathcal{A}/G]$$

is called the *toric orbifold* associated to $(\Sigma, \{\boldsymbol{b}_i\}_{i=0}^{m-1} \cup \{\boldsymbol{b}_j\}_{j=m}^{m'-1})$.

By construction, the standard $(\mathbb{C}^\times)^{m'}$-action on $U_\mathcal{A}$ induces a $\mathbb{T}$-action on $\mathcal{X}_\Sigma$.

Remark 2.1. *Let $\{v_i \in N \mid i = 0, \ldots, m-1\}$ be the collection of primitive generators of the rays in Σ. In general, for $0 \le i \le m-1$, we have $\boldsymbol{b}_i = c_i v_i$ for some positive integer $c_i \in \mathbb{Z}_{>0}$. If $c_i = 1$ for all $0 \le i \le m-1$, then the coarse moduli space of $\mathcal{X}_\Sigma$ is a simplicial toric variety and in this case we call $\mathcal{X}_\Sigma$ a simplicial toric orbifold. Such a toric orbifold has orbifold structures in at least codimension 2.*

Definition 2.2. *Let X_Σ be the toric variety which is the coarse moduli space of a toric orbifold $\mathcal{X}_\Sigma$. We say that X_Σ is* semi-projective *if X_Σ admits a $\mathbb{T}$-fixed point, and the natural map $X_\Sigma \to \mathrm{Spec}\, H^0(X_\Sigma, \mathcal{O}_{X_\Sigma})$ is projective.*

As we will see, toric orbifolds we consider in this paper all have semi-projective coarse moduli spaces. We refer to [38, Section 7.2] for more detailed discussions on semi-projective toric varieties.

Remark 2.3. *By abuse of terminology, we say that the toric orbifold $\mathcal{X}_\Sigma$ is (semi-)projective when the underlying toric variety X_Σ is (semi-)projective.*

2.2. **Twisted sectors.** For a d-dimensional cone σ in Σ generated by $\boldsymbol{b}_\sigma = (\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_d})$, we define

$$\mathrm{Box}_{\boldsymbol{b}_\sigma} := \left\{ \nu \in N \mid \nu = \sum_{k=1}^{d} t_k \boldsymbol{b}_{i_k},\ t_k \in [0,1) \cap \mathbb{Q} \right\}.$$

Let $N_{\boldsymbol{b}_\sigma}$ be the submodule of N generated by lattice vectors $\{\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_d}\}$. Then $\mathrm{Box}_{\boldsymbol{b}_\sigma}$ is in a one-to-one correspondence with the finite group $G_{\boldsymbol{b}_\sigma} = N/N_{\boldsymbol{b}_\sigma}$. It is easy to see that if $\tau \prec \sigma$, then we have $\mathrm{Box}_{\boldsymbol{b}_\tau} \subset \mathrm{Box}_{\boldsymbol{b}_\sigma}$. Define

$$\mathrm{Box}^\circ_{\boldsymbol{b}_\sigma} := \mathrm{Box}_{\boldsymbol{b}_\sigma} - \bigcup_{\tau \precneqq \sigma} \mathrm{Box}_{\boldsymbol{b}_\tau},$$

and

$$\mathrm{Box}(\Sigma) := \bigcup_{\sigma \in \Sigma^{(n)}} \mathrm{Box}_{\boldsymbol{b}_\sigma} = \bigsqcup_{\sigma \in \Sigma} \mathrm{Box}^\circ_{\boldsymbol{b}_\sigma}$$

where $\Sigma^{(n)}$ is the set of n-dimensional cones in Σ. We set $\mathrm{Box}'(\Sigma) = \mathrm{Box}(\Sigma) \setminus \{0\}$.

According to [9], $\mathrm{Box}'(\Sigma)$ is in a one-to-one correspondence with the *twisted sectors*, i.e. non-trivial connected components of the inertia orbifold of $\mathcal{X}_\Sigma$. For $\nu \in \mathrm{Box}(\Sigma)$, we denote by $\mathcal{X}_\nu$ the corresponding twisted sector of $\mathcal{X}$. Note that $\mathcal{X}_0 = \mathcal{X}$ as topological spaces. See Figure 1a for an example illustrating $\mathrm{Box}'(\Sigma)$.

For the toric orbifold $\mathcal{X}$, the *Chen-Ruan orbifold cohomology* $H^*_{\mathrm{CR}}(\mathcal{X}; \mathbb{Q})$, as defined in [26], is given by

$$H^d_{\mathrm{CR}}(\mathcal{X}; \mathbb{Q}) = \bigoplus_{\nu \in \mathrm{Box}} H^{d - 2\mathrm{age}(\nu)}(\mathcal{X}_\nu; \mathbb{Q}),$$

where $\mathrm{age}(\nu)$ is the *degree shifting number* or *age* of the twisted sector $\mathcal{X}_\nu$ and the cohomology groups on the right hand side are singular cohomology groups. If we write $\nu = \sum_{k=1}^{d} t_k \boldsymbol{b}_{i_k} \in \mathrm{Box}(\Sigma)$ where $\{\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_d}\}$ generates a cone in Σ, then

$$\mathrm{age}(\nu) = \sum_{k=1}^{d} t_k \in \mathbb{Q}_{\geq 0}.$$

The $\mathbb{T}$-action on $\mathcal{X}$ induces $\mathbb{T}$-actions on twisted sectors. This allows one to define the $\mathbb{T}$-*equivariant Chen-Ruan orbifold cohomology* $H^*_{\mathrm{CR},\mathbb{T}}(\mathcal{X};\mathbb{Q})$ as

$$H^d_{\mathrm{CR},\mathbb{T}}(\mathcal{X};\mathbb{Q}) = \bigoplus_{\nu\in\mathrm{Box}} H_{\mathbb{T}}^{d-2\mathrm{age}(\nu)}(\mathcal{X}_\nu;\mathbb{Q}),$$

where $H^*_{\mathbb{T}}(-)$ denotes $\mathbb{T}$-equivariant cohomology.

The trivial $\mathbb{T}$-bundle over a point pt defines a map $\mathrm{pt} \to B\mathbb{T}$. This induces a map $H^*_{\mathbb{T}}(\mathrm{pt},\mathbb{Q}) = H^*(B\mathbb{T},\mathbb{Q}) \to H^*(\mathrm{pt})$. Let Y be a space with a $\mathbb{T}$-action. By construction the $\mathbb{T}$-equvariant cohomology of Y admits a map $H^*_{\mathbb{T}}(\mathrm{pt}) \to H^*_{\mathbb{T}}(Y,\mathbb{Q})$. This defines a natural map

$$H^*_{\mathbb{T}}(Y,\mathbb{Q}) \to H^*_{\mathbb{T}}(Y,\mathbb{Q}) \otimes_{H^*_{\mathbb{T}}(\mathrm{pt})} H^*(\mathrm{pt}) \simeq H^*(Y,\mathbb{Q}).$$

For a class $C \in H^*_{\mathbb{T}}(Y,\mathbb{Q})$, its image under this map, which is a class in $H^*(Y,\mathbb{Q})$, is called the *non-equivariant limit* of C. In Section 6, we will need to consider non-equivariant limits of certain classes in $H^*_{\mathrm{CR},\mathbb{T}}(\mathcal{X};\mathbb{Q})$.

2.3. **Toric divisors, Kähler cones, and Mori cones.** Let $\mathcal{X}$ be a toric orbifold defined by an extended stacky fan $(\Sigma, \{\boldsymbol{b}_i\}_{i=0}^{m-1} \cup \{\boldsymbol{b}_j\}_{j=m}^{m'-1})$. Let $\mathcal{A}$ be the set of anticones given in (2.3). Applying $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ to the fan sequence (2.1) gives the following exact sequence (the "divisor sequence"):

$$0 \longrightarrow M \xrightarrow{\phi^\vee} \widetilde{M} \xrightarrow{\psi^\vee} \mathbb{L}^\vee \longrightarrow 0.$$

Here $M := N^\vee = \mathrm{Hom}(N,\mathbb{Z})$, $\widetilde{M} := \widetilde{N}^\vee = \mathrm{Hom}(\widetilde{N},\mathbb{Z})$ and $\mathbb{L}^\vee = \mathrm{Hom}(\mathbb{L},\mathbb{Z})$ are dual lattices. The map $\psi^\vee : \widetilde{M} \to \mathbb{L}^\vee$ is surjective since N is torsion-free.

By construction, line bundles on $\mathcal{X}$ correspond to G-equivariant line bundles on $\mathcal{U}_{\mathcal{A}}$. Because of (2.2), $\mathbb{T}$-equivariant line bundles on $\mathcal{X}$ correspond to $(\mathbb{C}^\times)^{m'}$-equivariant line bundles on $\mathcal{U}_{\mathcal{A}}$. Because $\cup_{I\notin\mathcal{A}}\mathbb{C}^I \subset \mathbb{C}^{m'}$ is of codimension at least 2, we have the following descriptions of the Picard groups:

$$Pic(\mathcal{X}) \simeq Hom(G,\mathbb{C}^\times) \simeq \mathbb{L}^\vee, Pic_{\mathbb{T}}(\mathcal{X}) \simeq Hom((\mathbb{C}^\times)^{m'},\mathbb{C}^\times) \simeq \widetilde{N}^\vee = \widetilde{M}.$$

Moreover, the natural map $Pic_{\mathbb{T}}(\mathcal{X}) \to Pic(\mathcal{X})$ is identified with $\psi^\vee : \widetilde{M} \to \mathbb{L}^\vee$.

Let $\{e_i^\vee | i = 0,1,\ldots,m'-1\} \subset \widetilde{M}$ be the basis dual to $\{e_i | i = 0,1,\ldots,m'-1\} \subset \widetilde{N}$. For $i = 0,1,\ldots,m'-1$, we denote by $D_i^{\mathbb{T}}$ the $\mathbb{T}$-equivariant line bundle on $\mathcal{X}$ corresponding to $e_i^\vee$ under the identification $Pic_{\mathbb{T}}(\mathcal{X}) \simeq \widetilde{M}$. Also put

$$D_i := \psi^\vee(e_i^\vee) \in \mathbb{L}^\vee.$$

The collection $\{D_i \mid 0 \leq i \leq m-1\}$ are toric prime divisors corresponding to the generators $\{\boldsymbol{b}_i \mid 0 \leq i \leq m-1\}$ of rays in Σ, and $\{D_i^{\mathbb{T}} \mid 0 \leq i \leq m-1\}$ are their $\mathbb{T}$-equivariant lifts.

There is a natural commutative diagram

$$\begin{array}{ccc} \widetilde{M}\otimes\mathbb{Q} & \xrightarrow{\psi^\vee\otimes\mathbb{Q}} & \mathbb{L}^\vee\otimes\mathbb{Q} \\ \downarrow & & \downarrow \\ H^2_{\mathbb{T}}(\mathcal{X},\mathbb{Q}) & \longrightarrow & H^2(\mathcal{X},\mathbb{Q}). \end{array}$$

Also, there are isomorphisms

$$H^2(\mathcal{X};\mathbb{Q}) \simeq (\mathbb{L}^\vee\otimes\mathbb{Q}) \Big/ \left(\sum_{j=m}^{m'-1}\mathbb{Q}D_j\right), \quad H^2_{\mathbb{T}}(\mathcal{X};\mathbb{Q}) \simeq \left(\widetilde{M}\otimes\mathbb{Q}\right) \Big/ \left(\sum_{j=m}^{m'-1}\mathbb{Q}D_j^{\mathbb{T}}\right).$$

Remark 2.4. *By (standard) abuse of notation, we use D_i (respective $D_i^{\mathbb{T}}$) to denote*

1. *an element in $\mathbb{L}^\vee$ (respectively $\widetilde{M}$);*
2. *the corresponding line bundle on $\mathcal{X}$ (respectively $\mathbb{T}$-equivariant line bundle on $\mathcal{X}$);*
3. *the corresponding divisor in $\mathcal{X}$ (respectively $\mathbb{T}$-equivariant divisor in $\mathcal{X}$);*
4. *the first Chern class of the line bundle on $\mathcal{X}$, taking values in $H^2(\mathcal{X})$ (respectively $H^2_{\mathbb{T}}(\mathcal{X})$).*

As explained in [72, Section 3.1.2], there is a canonical splitting of the quotient map $\mathbb{L}^\vee\otimes\mathbb{Q}\to H^2(\mathcal{X};\mathbb{Q})$, which we now describe. For $m\le j\le m'-1$, $\boldsymbol{b}_j$ is contained in a cone in Σ. Let $I_j\in\mathcal{A}$ be the anticone of the cone containing $\boldsymbol{b}_j$. Then we can write the following equation in $N\otimes\mathbb{Q}$:

$$\boldsymbol{b}_j = \sum_{i\notin I_j} c_{ji}\boldsymbol{b}_i, \quad c_{ji}\in\mathbb{Q}_{\geq 0}.$$

By the fan sequence (2.1) tensored with $\mathbb{Q}$, there exists a unique $D_j^\vee\in\mathbb{L}\otimes\mathbb{Q}$ such that

$$\langle D_i, D_j^\vee\rangle = \begin{cases} 1 & \text{if } i=j, \\ -c_{ji} & \text{if } i\notin I_j, \\ 0 & \text{if } i\in I_j\setminus\{j\}. \end{cases} \tag{2.4}$$

Here and henceforth $\langle -,-\rangle$ denotes the natural pairing between $\mathbb{L}^\vee$ and $\mathbb{L}$ (or relevant extensions of scalars). This defines a decomposition

$$\mathbb{L}^\vee\otimes\mathbb{Q} = \operatorname{Ker}\left(\left(D_m^\vee,\dots,D_{m'-1}^\vee\right):\mathbb{L}^\vee\otimes\mathbb{Q}\to\mathbb{Q}^{m'-m}\right)\oplus\bigoplus_{j=m}^{m'-1}\mathbb{Q}D_j. \tag{2.5}$$

Moreover, the term $\operatorname{Ker}\left(\left(D_m^\vee,\dots,D_{m'-1}^\vee\right):\mathbb{L}^\vee\otimes\mathbb{Q}\to\mathbb{Q}^{m'-m}\right)$ is naturally identified with $H^2(\mathcal{X};\mathbb{Q})$ via the quotient map $\mathbb{L}^\vee\otimes\mathbb{Q}\to H^2(\mathcal{X};\mathbb{Q})$, which allows us to regard $H^2(\mathcal{X};\mathbb{Q})$ as a subspace of $\mathbb{L}^\vee\otimes\mathbb{Q}$.

The *extended Kähler cone* is defined to be

$$\widetilde{C}_{\mathcal{X}} := \bigcap_{I\in\mathcal{A}}\left(\sum_{i\in I}\mathbb{R}_{>0}D_i\right)\subset\mathbb{L}^\vee\otimes\mathbb{R}.$$

The genuine Kähler cone $C_{\mathcal{X}}$ is the image of $\widetilde{C}_{\mathcal{X}}$ under the quotient map $\mathbb{L}^\vee \otimes \mathbb{R} \to H^2(\mathcal{X};\mathbb{R})$. The splitting of $\mathbb{L}^\vee \otimes \mathbb{Q}$ (2.5) induces a splitting of the extended Kähler cone:

$$\widetilde{C}_{\mathcal{X}} = C_{\mathcal{X}} + \sum_{j=m}^{m'-1} \mathbb{R}_{>0} D_j$$

in $\mathbb{L}^\vee \otimes \mathbb{R}$.

Recall that the rank of $\mathbb{L}^\vee$ is $r := m' - n$ while the rank of $H_2(\mathcal{X};\mathbb{Z})$ is given by $r' := r - (m' - m) = m - n$. We choose an integral basis

$$\{p_1, \dots, p_r\} \subset \mathbb{L}^\vee$$

such that p_a is in the closure of $\widetilde{C}_{\mathcal{X}}$ for all a and $p_{r'+1}, \dots, p_r \in \sum_{i=m}^{m'-1} \mathbb{R}_{\geq 0} D_i$. Then the images $\{\bar{p}_1, \dots, \bar{p}_{r'}\}$ of $\{p_1, \dots, p_{r'}\}$ under the quotient map $\mathbb{L}^\vee \otimes \mathbb{Q} \to H^2(\mathcal{X};\mathbb{Q})$ gives a nef basis for $H^2(\mathcal{X};\mathbb{Q})$ and $\bar{p}_a = 0$ for $r' + 1 \leq a \leq r$.

We choose $\{p_1^{\mathbb{T}}, \dots, p_r^{\mathbb{T}}\} \subset \widetilde{M} \otimes \mathbb{Q}$ such that $\psi^\vee(p_a^{\mathbb{T}}) = p_a$ for each a, and $\bar{p}_a^{\mathbb{T}} = 0$ for $a = r'+1, ..., r$. Here, for $p \in \widetilde{M} \otimes \mathbb{Q}$, we denote by $\bar{p} \in H^2_{\mathbb{T}}(\mathcal{X}, \mathbb{Q})$ the image of p under the natural map $\widetilde{M} \otimes \mathbb{Q} \to H^2_{\mathbb{T}}(\mathcal{X}, \mathbb{Q})$. By construction, for $a = 1, ..., r'$, $\bar{p}_a$ is the non-equivariant limit of $\bar{p}_a^{\mathbb{T}}$.

We define a matrix (Q_{ia}) by

$$D_i = \sum_{a=1}^{r} Q_{ia} p_a, \quad Q_{ia} \in \mathbb{Z}.$$

Denote by $\bar{D}_i$ the image of D_i under $\mathbb{L}^\vee \otimes \mathbb{Q} \to H^2(\mathcal{X};\mathbb{Q})$. Then for $i = 0, \dots, m-1$, the class $\bar{D}_i$ of the toric prime divisor D_i is given by

$$\bar{D}_i = \sum_{a=1}^{r'} Q_{ia} \bar{p}_a;$$

and for $i = m, \dots, m'-1$, $\bar{D}_i = 0$ in $H^2(\mathcal{X};\mathbb{R})$. Likewise, for $i = 0, \dots, m-1$, we have

$$\bar{D}_i^{\mathbb{T}} = \sum_{a=1}^{r'} Q_{ia} \bar{p}_a^{\mathbb{T}} + \lambda_i,$$

where $\lambda_i \in H^2(B\mathbb{T};\mathbb{Q})$. For $i = m, \dots, m'-1$, $\bar{D}_i^{\mathbb{T}} = 0$.

Let $\mathbf{1} \in H^0(\mathcal{X}, \mathbb{Q})$ be the fundamental class. For $\nu \in \mathrm{Box}$ with $\mathrm{age}(\nu) = 1$, let $\mathbf{1}_\nu \in H^0(\mathcal{X}_\nu, \mathbb{Q})$ be the fundamental class. It is then straightforward to see that

$$H^0_{\mathrm{CR},\mathbb{T}}(\mathcal{X}, K_{\mathbb{T}}) = K_{\mathbb{T}} \mathbf{1}, \quad H^2_{\mathrm{CR},\mathbb{T}}(\mathcal{X}, K_{\mathbb{T}}) = \bigoplus_{a=1}^{r'} K_{\mathbb{T}} \bar{p}_a^{\mathbb{T}} \oplus \bigoplus_{\nu \in \mathrm{Box}, \mathrm{age}(\nu)=1} K_{\mathbb{T}} \mathbf{1}_\nu,$$

where $K_{\mathbb{T}}$ is the field of fractions of $H^*_{\mathbb{T}}(\mathrm{pt}, \mathbb{Q})$, and $H^*_{\mathbb{T}}(-, K_{\mathbb{T}}) := H^*_{\mathbb{T}}(-, \mathbb{Q}) \otimes_{H^*_{\mathbb{T}}(\mathrm{pt},\mathbb{Q})} K_{\mathbb{T}}$.

The dual basis of $\{p_1, \dots, p_r\} \subset \mathbb{L}^\vee$ is given by $\{\gamma_1, \dots, \gamma_r\} \subset \mathbb{L}$ where

$$\gamma_a = \sum_{i=0}^{m'-1} Q_{ia} e_i \in \widetilde{N}.$$

Then $\{\gamma_1, \ldots, \gamma_{r'}\}$ provides a basis of $H_2^{\text{eff}}(\mathcal{X};\mathbb{Q})$. In particular, we have $Q_{ia} = 0$ when $m \leq i \leq m'-1$ and $1 \leq a \leq r'$.

We set

$$\mathbb{K} := \{d \in \mathbb{L}\otimes\mathbb{Q} \mid \{j \in \{0,1,\ldots,m'-1\} \mid \langle D_j, d\rangle \in \mathbb{Z}\} \in \mathcal{A}\},$$
$$\mathbb{K}_{\text{eff}} := \{d \in \mathbb{L}\otimes\mathbb{Q} \mid \{j \in \{0,1,\ldots,m'-1\} \mid \langle D_j, d\rangle \in \mathbb{Z}_{\geq 0}\} \in \mathcal{A}\},$$

Roughly speaking $\mathbb{K}_{\text{eff}}$ is the set of effective curve classes. In particular, the intersection $\mathbb{K}_{\text{eff}} \cap H_2(\mathcal{X};\mathbb{R})$ consists of classes of stable maps $\mathbb{P}(1,m) \to \mathcal{X}$ for some $m \in \mathbb{Z}_{\geq 0}$. See e.g. [72, Section 3.1] for more details.

For a real number $\lambda \in \mathbb{R}$, we let $\lceil\lambda\rceil$, $\lfloor\lambda\rfloor$ and $\{\lambda\}$ denote the ceiling, floor and fractional part of λ respectively. Now for $d \in \mathbb{K}$, we define

$$\nu(d) := \sum_{i=0}^{m'-1} \lceil\langle D_i, d\rangle\rceil \boldsymbol{b}_i \in N, \tag{2.6}$$

and let $I_d := \{j \in \{0,1,\ldots,m'-1\} \mid \langle D_j, d\rangle \in \mathbb{Z}\} \in \mathcal{A}$. Then since we can rewrite

$$\nu(d) = \sum_{i=0}^{m'-1}(\{-\langle D_i,d\rangle\} + \langle D_i,d\rangle)\boldsymbol{b}_i = \sum_{i=0}^{m'-1}\{-\langle D_i,d\rangle\}\boldsymbol{b}_i = \sum_{i\notin I_d}\{-\langle D_i,d\rangle\}\boldsymbol{b}_i,$$

we have $\nu(d) \in \text{Box}$, and hence $\nu(d)$, if nonzero, corresponds to a twisted sector $\mathcal{X}_{\nu(d)}$ of $\mathcal{X}$.

2.4. **The I-function.**

In this subsection we define a combinatorial object called the (equivariant) I-function of $\mathcal{X}$.

Definition 2.5. *The* $\mathbb{T}$-equivariant I-function *of a toric orbifold $\mathcal{X}$ is an $H^*_{\text{CR},\mathbb{T}}(\mathcal{X})$-valued power series defined by*

$$I_{\mathcal{X},\mathbb{T}}(y,z) = \mathbf{e}^{\sum_{a=1}^r \bar{p}_a^{\mathbb{T}} \log y_a/z}\left(\sum_{d\in\mathbb{K}_{\text{eff}}} y^d \prod_{i=0}^{m'-1} \frac{\prod_{k=\lceil\langle D_i,d\rangle\rceil}^{\infty}(\bar{D}_i^{\mathbb{T}} + (\langle D_i,d\rangle - k)z)}{\prod_{k=0}^{\infty}(\bar{D}_i^{\mathbb{T}} + (\langle D_i,d\rangle - k)z)}\mathbf{1}_{\nu(d)}\right),$$

where $y^d = y_1^{\langle p_1,d\rangle}\cdots y_r^{\langle p_r,d\rangle}$ and $\mathbf{1}_{\nu(d)} \in H^0(\mathcal{X}_{\nu(d)}) \subset H^{2\text{age}(\nu(d))}_{\text{CR}}(\mathcal{X})$ is the fundamental class of the twisted sector $\mathcal{X}_{\nu(d)}$.

Definition 2.6. *The* I-function *of a toric orbifold $\mathcal{X}$ is an $H^*_{\text{CR}}(\mathcal{X})$-valued power series defined by*

$$I_{\mathcal{X}}(y,z) = \mathbf{e}^{\sum_{a=1}^r \bar{p}_a \log y_a/z}\left(\sum_{d\in\mathbb{K}_{\text{eff}}} y^d \prod_{i=0}^{m'-1} \frac{\prod_{k=\lceil\langle D_i,d\rangle\rceil}^{\infty}(\bar{D}_i + (\langle D_i,d\rangle - k)z)}{\prod_{k=0}^{\infty}(\bar{D}_i + (\langle D_i,d\rangle - k)z)}\mathbf{1}_{\nu(d)}\right),$$

where $y^d = y_1^{\langle p_1,d\rangle}\cdots y_r^{\langle p_r,d\rangle}$ and $\mathbf{1}_{\nu(d)} \in H^0(\mathcal{X}_{\nu(d)}) \subset H^{2\text{age}(\nu(d))}_{\text{CR}}(\mathcal{X})$ is the fundamental class of the twisted sector $\mathcal{X}_{\nu(d)}$.

Remark 2.7. *It is clear from definitions that the non-equivariant limit of $I_{\mathcal{X},\mathbb{T}}$ is $I_{\mathcal{X}}$.*

Definition 2.8. *A toric orbifold $\mathcal{X}$ is said to be semi-Fano if $\hat{\rho}(\mathcal{X}) := \sum_{i=0}^{m'-1} D_i$ is contained in the closure of the extended Kähler cone $\widetilde{C}_{\mathcal{X}}$ in $\mathbb{L}^\vee\otimes\mathbb{R}$.*

We remark that this condition *depends* on the choice of the extra vectors $\boldsymbol{b}_m, \ldots, \boldsymbol{b}_{m'-1}$. It holds if and only if the first class $c_1(\mathcal{X}) \in H^2(\mathcal{X};\mathbb{Q})$ of $\mathcal{X}$ is contained in the closure of the Kähler cone $C_\mathcal{X}$ (i.e. the anticanonical divisor $-K_\mathcal{X}$ is nef) and $\mathrm{age}(\boldsymbol{b}_j) := \sum_{i \notin I_j} c_{ji} \leq 1$ for $m \leq j \leq m'-1$, because we have

$$\hat{\rho}(\mathcal{X}) = c_1(\mathcal{X}) + \sum_{j=m}^{m'-1} (1 - \mathrm{age}(\boldsymbol{b}_j)) D_j;$$

see [72, Lemma 3.3]. In particular, when $\mathcal{X}$ is a toric manifold, the condition is equivalent to requiring the anticanonical divisor $-K_\mathcal{X}$ to be nef.

As we will see, the examples we consider in this paper will all satisfy the following assumption.

Assumption 2.9. *The set* $\{\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1}\} \cup \{\nu \in \mathrm{Box}(\Sigma) \mid age(\nu) \leq 1\}$ *generates the lattice* N *over* $\mathbb{Z}$.

Under this assumption, we choose the extra vectors $\boldsymbol{b}_m, \ldots, \boldsymbol{b}_{m'-1} \in \{\nu \in \mathrm{Box}(\Sigma) \mid \mathrm{age}(\nu) \leq 1\}$ so that $\{\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m'-1}\}$ generates N over $\mathbb{Z}$. Then the fan sequence (2.1) determines the elements $D_0, \ldots, D_{m'-1}$ and $\hat{\rho}(\mathcal{X}) = D_0 + \cdots + D_{m'-1}$ holds (see [72, Remark 3.4]). Furthermore, we can then identify $\mathbb{L}^\vee \otimes \mathbb{C}$ with the subspace

$$H^2(\mathcal{X}) \oplus \bigoplus_{j=m}^{m'-1} H^0(\mathcal{X}_{\boldsymbol{b}_j}) \subset H^{\leq 2}_{\mathrm{CR}}(\mathcal{X}).$$

If $\mathcal{X}$ is semi-Fano, then its I-function is a convergent power series in $y_1, \ldots, y_r$ by [72, Lemma 4.2]. Moreover, it can be expanded as

$$I_\mathcal{X}(y,z) = 1 + \frac{\tau(y)}{z} + O(z^{-2}),$$

where $\tau(y)$ is a (multi-valued) function with values in $H^{\leq 2}_{\mathrm{CR}}(\mathcal{X})$ which expands as

$$\tau(y) = \sum_{a=1}^{r'} \bar{p}_a \log y_a + \sum_{j=m}^{m'-1} y^{D_j^\vee} \mathbf{1}_{\boldsymbol{b}_j} + \text{higher order terms}.$$

We call $q(y) = \exp \tau(y)$ the *toric mirror map*, and it defines a local embedding near $y = 0$ (it is a local embedding if we further assume that $\{\boldsymbol{b}_m, \ldots, \boldsymbol{b}_{m'-1}\} = \{\nu \in \mathrm{Box}(\Sigma) \mid \mathrm{age}(\nu) \leq 1\}$); see [72, Section 4.1] for more details. Similar discussion is valid for equivariant I-functions.

2.5. **Equivariant Gromov-Witten invariants.** In this subsection we discuss the construction of equivariant Gromov-Witten invariants. We refer to [25] and [2] for the basics of Gromov-Witten theory of orbifolds, and to e.g. [55] and [92] for generalities on equivariant Gromov-Witten theory.

The $\mathbb{T}$-action on $\mathcal{X}$ induces $\mathbb{T}$-actions on moduli spaces of stable maps to $\mathcal{X}$. It is well-known that in this situation we can define $\mathbb{T}$-*equivariant Gromov-Witten invariants* of $\mathcal{X}$ as integrals against $\mathbb{T}$-equivariant virtual fundamental classes of these moduli spaces.

Let $\mathcal{M}^{cl}_n(\mathcal{X}, d)$ be the moduli space of n-pointed genus 0 orbifold stable maps to $\mathcal{X}$ of degree $d \in H_2(\mathcal{X};\mathbb{Q})$. For $i = 1, ..., n$, there is an evaluation maps $ev_i : \mathcal{M}^{cl}_n(\mathcal{X}, d) \to I\mathcal{X}$, and a complex line bundle $L_i \to \mathcal{M}^{cl}_n(\mathcal{X}, d)$ whose fibers are cotangent lines at the i-th marked point of the coarse domain curves.

Suppose $\mathcal{M}^{cl}_n(\mathcal{X}, d)$ is compact. Then there is a virtual fundamental class $[\mathcal{M}^{cl}_n(\mathcal{X}, d)]_{virt} \in H_*(\mathcal{M}^{cl}_n(\mathcal{X}, d), \mathbb{Q})$. Genus 0 closed orbifold Gromov-Witten invariants of $\mathcal{X}$ can be defined as follows. For cohomology classes $\phi_1, ..., \phi_n \in H^*_{\mathrm{CR}}(\mathcal{X}, \mathbb{Q})$ and integers $k_1, ..., k_n \geq 0$, we define

$$\left\langle \phi_1\psi_1^{k_1}, ..., \phi_n\psi_n^{k_n} \right\rangle^{\mathcal{X}}_{0,n,d} := \int_{[\mathcal{M}^{cl}_n(\mathcal{X},d)]_{virt}} \prod_{i=1}^n (ev_i^*\phi_i \cup \psi_i^{k_i}) \in \mathbb{Q}, \tag{2.7}$$

where $\psi_i := c_1(L_i) \in H^2(\mathcal{M}^{cl}_n(\mathcal{X}, d), \mathbb{Q})$.

The $\mathbb{T}$-action on $\mathcal{X}$ induces a $\mathbb{T}$-action on $\mathcal{M}^{cl}_n(\mathcal{X}, d)$. When $\mathcal{M}^{cl}_n(\mathcal{X}, d)$ is compact, there is a $\mathbb{T}$-equivariant virtual fundamental class $[\mathcal{M}^{cl}_n(\mathcal{X}, d)]_{virt,\mathbb{T}} \in H_{*,\mathbb{T}}(\mathcal{M}^{cl}_n(\mathcal{X}, d), \mathbb{Q})$. $\mathbb{T}$-equivariant genus 0 closed orbifold Gromov-Witten invariants of $\mathcal{X}$ can be defined as follows. For cohomology classes $\phi_{1,\mathbb{T}}, ..., \phi_{n,\mathbb{T}} \in H^*_{\mathrm{CR},\mathbb{T}}(\mathcal{X}, \mathbb{Q})$ and integers $k_1, ..., k_n \geq 0$, we define

$$\left\langle \phi_{1,\mathbb{T}}\psi_1^{k_1}, ..., \phi_n\psi_{n,\mathbb{T}}^{k_n} \right\rangle^{\mathcal{X},\mathbb{T}}_{0,n,d} := \int_{[\mathcal{M}^{cl}_n(\mathcal{X},d)]_{virt,\mathbb{T}}} \prod_{i=1}^n (ev_i^*\phi_{i,\mathbb{T}} \cup \psi_i^{k_i}) \in H^*_{\mathbb{T}}(\mathrm{pt}, \mathbb{Q}), \tag{2.8}$$

where $\psi_i := c_1^{\mathbb{T}}(L_i) \in H^2_{\mathbb{T}}(\mathcal{M}^{cl}_n(\mathcal{X}, d), \mathbb{Q})$ are $\mathbb{T}$-equivariant first Chern classes.

Suppose again that $\mathcal{M}^{cl}_n(\mathcal{X}, d)$ is compact. Suppose that $\phi_1, ..., \phi_n \in H^*_{\mathrm{CR}}(\mathcal{X}, \mathbb{Q})$ are non-equivariant limits of $\phi_{1,\mathbb{T}}, ..., \phi_{n,\mathbb{T}} \in H^*_{\mathrm{CR},\mathbb{T}}(\mathcal{X}, \mathbb{Q})$. Then by construction of virtual fundamental classes, the non-equivariant limit of $\left\langle \phi_{1,\mathbb{T}}\psi_1^{k_1}, ..., \phi_n\psi_{n,\mathbb{T}}^{k_n} \right\rangle^{\mathcal{X},\mathbb{T}}_{0,n,d}$, i.e. its image under the natural map $H^*_{\mathbb{T}}(\mathrm{pt}, \mathbb{Q}) \to H^*(\mathrm{pt}) = \mathbb{Q}$, is equal to $\left\langle \phi_1\psi_1^{k_1}, ..., \phi_n\psi_n^{k_n} \right\rangle^{\mathcal{X}}_{0,n,d}$.

Suppose $\mathcal{M}^{cl}_n(\mathcal{X}, d)$ is not compact, but the locus $\mathcal{M}^{cl}_n(\mathcal{X}, d)^{\mathbb{T}} \subset \mathcal{M}^{cl}_n(\mathcal{X}, d)$ of $\mathbb{T}$-fixed points is compact. Then the $\mathbb{T}$-equivariant invariant $\left\langle \phi_{1,\mathbb{T}}\psi_1^{k_1}, ..., \phi_n\psi_{n,\mathbb{T}}^{k_n} \right\rangle^{\mathcal{X},\mathbb{T}}_{0,n,d}$ can still be defined by (2.8), with the integration $\int_{[\mathcal{M}^{cl}_n(\mathcal{X},d)]_{virt,\mathbb{T}}}$ defined by the virtual localization formula [59]. Namely,

$$\int_{[\mathcal{M}^{cl}_n(\mathcal{X},d)]_{virt,\mathbb{T}}} (-) := \sum_{F \subset \mathcal{M}^{cl}_n(\mathcal{X},d)^{\mathbb{T}}} \int_{[F]_{virt}} \frac{\iota_F^*(-)}{e_{\mathbb{T}}(N_F^{virt})} \in K_{\mathbb{T}},$$

where F runs through connected components of $\mathcal{M}^{cl}_n(\mathcal{X}, d)^{\mathbb{T}}$, $\iota_F : F \to \mathcal{M}^{cl}_n(\mathcal{X}, d)^{\mathbb{T}}$ is the inclusion, $[F]_{virt}$ is the natural virtual fundamental class on F, and $e_{\mathbb{T}}(N_F^{virt})$ is the $\mathbb{T}$-equivariant Euler class of the virtual normal bundle N_F^{virt} of $F \subset \mathcal{M}^{cl}_n(\mathcal{X}, d)$. It follows easily from virtual localization formula that if both $\mathcal{M}^{cl}_n(\mathcal{X}, d)^{\mathbb{T}}$ and $\mathcal{M}^{cl}_n(\mathcal{X}, d)$ are compact, the two definitions of $\mathbb{T}$-equivariant invariants agree.

Remark 2.10. *If the toric orbifold $\mathcal{X}$ is projective, then $\mathcal{M}^{cl}_n(\mathcal{X}, d)$ is compact. If $\mathcal{X}$ is not projective but semi-projective, then it is straightforward to show that the locus $\mathcal{M}^{cl}_n(\mathcal{X}, d)^{\mathbb{T}} \subset \mathcal{M}^{cl}_n(\mathcal{X}, d)$ of $\mathbb{T}$-fixed points is compact. In this case, $\mathbb{T}$-equvariant Gromov-Witten invariants are still defined.*

2.6. **Toric mirror theorem.** We give a review of the mirror theorem for toric orbifolds proven in [31] in the case of semi-Fano toric orbifolds. Our exposition follows [72] and [44].

Let $\mathcal{X}$ be a toric orbifold as in Section 2.1.

Definition 2.11. *The* $\mathbb{T}$-equivariant (small) J-function *of a toric orbifold* $\mathcal{X}$ *is an* $H^*_{\mathrm{CR},\mathbb{T}}(\mathcal{X})$*-valued power series defined by*

$$J_{\mathcal{X}}(q,z) = \mathbf{e}^{\tau_{0,2}/z}\left(1+\sum_{\alpha}\sum_{\substack{(d,l)\neq(0,0)\\ d\in H_2^{\mathrm{eff}}(\mathcal{X})}}\frac{q^d}{l!}\left\langle 1,\tau_{\mathrm{tw}},\ldots,\tau_{\mathrm{tw}},\frac{\phi_\alpha}{z-\psi}\right\rangle_{0,l+2,d}^{\mathcal{X},\mathbb{T}}\phi^\alpha\right),$$

where $\tau_{0,2}=\sum_{a=1}^{r'}\bar{p}_a^{\mathbb{T}}\log q_a\in H^2_{\mathbb{T}}(\mathcal{X})$*,* $\tau_{\mathrm{tw}}=\sum_{j=m}^{m'-1}\tau_{\boldsymbol{b}_j}\mathbf{1}_{\boldsymbol{b}_j}\in\bigoplus_{j=m}^{m'-1}H^0_{\mathbb{T}}(\mathcal{X}_{\boldsymbol{b}_j})$*,* $q^d=\mathbf{e}^{\langle\tau_{0,2},d\rangle}=q_1^{\langle\bar{p}_1,d\rangle}\cdots q_{r'}^{\langle\bar{p}_{r'},d\rangle}$*,* $\{\phi_\alpha\}$*,* $\{\phi^\alpha\}$ *are dual basis of* $H^*_{\mathrm{CR},\mathbb{T}}(\mathcal{X})$*.*

Definition 2.12. *The* (small) J-function *of a toric orbifold* $\mathcal{X}$ *is an* $H^*_{\mathrm{CR}}(\mathcal{X})$*-valued power series defined by*

$$J_{\mathcal{X}}(q,z) = \mathbf{e}^{\tau_{0,2}/z}\left(1+\sum_{\alpha}\sum_{\substack{(d,l)\neq(0,0)\\ d\in H_2^{\mathrm{eff}}(\mathcal{X})}}\frac{q^d}{l!}\left\langle 1,\tau_{\mathrm{tw}},\ldots,\tau_{\mathrm{tw}},\frac{\phi_\alpha}{z-\psi}\right\rangle_{0,l+2,d}^{\mathcal{X}}\phi^\alpha\right),$$

where $\tau_{0,2}=\sum_{a=1}^{r'}\bar{p}_a\log q_a\in H^2(\mathcal{X})$*,* $\tau_{\mathrm{tw}}=\sum_{j=m}^{m'-1}\tau_{\boldsymbol{b}_j}\mathbf{1}_{\boldsymbol{b}_j}\in\bigoplus_{j=m}^{m'-1}H^0(\mathcal{X}_{\boldsymbol{b}_j})$*,* $q^d=\mathbf{e}^{\langle\tau_{0,2},d\rangle}=q_1^{\langle\bar{p}_1,d\rangle}\cdots q_{r'}^{\langle\bar{p}_{r'},d\rangle}$*,* $\{\phi_\alpha\}$*,* $\{\phi^\alpha\}$ *are dual basis of* $H^*_{\mathrm{CR}}(\mathcal{X})$*.*

Remark 2.13. *It is clear from definitions that the non-equivariant limit of* $J_{\mathcal{X},\mathbb{T}}$ *is* $J_{\mathcal{X}}$*.*

Roughly speaking, the (equivariant) mirror theorem for the toric orbifold $\mathcal{X}$ states that the (equivariant) J-function coincides with the (equivariant) I-function via the mirror map.

Theorem 2.14 (Equivariant mirror theorem for toric orbifolds [31]; see also [44], Conjecture 4.1)**.** *Let* $\mathcal{X}$ *be a semi-projective toric Kähler orbifold which is semi-Fano, i.e.* $\hat{\rho}(\mathcal{X})$ *is contained in the closure of the extended Kähler cone* $\widetilde{C}_{\mathcal{X}}$*. Then we have*

$$e^{q_0(y)\mathbf{1}/z}J_{\mathcal{X},\mathbb{T}}(q,z)=I_{\mathcal{X},\mathbb{T}}(y(q,\tau),z),$$

where $y=y(q,\tau)$ *is the inverse of the toric mirror map* $q=q(y)$*,* $\tau=\tau(y)$ *determined by the expansion of the equivariant* I*-function:*

$$I_{\mathcal{X},\mathbb{T}}(y,z)=1+\frac{q_0(y)\mathbf{1}+\tau(y)}{z}+O(z^{-2}),\quad \tau(y)\in H^2_{\mathrm{CR},\mathbb{T}}(\mathcal{X}).$$

Taking non-equivariant limits gives the following

Theorem 2.15 (Closed mirror theorem for toric orbifolds [31]; see also [72], Conjecture 4.3)**.** *Let* $\mathcal{X}$ *be a compact toric Kähler orbifold which is semi-Fano, i.e.* $\hat{\rho}(\mathcal{X})$ *is contained in the closure of the extended Kähler cone* $\widetilde{C}_{\mathcal{X}}$*. Then we have*

$$J_{\mathcal{X}}(q,z)=I_{\mathcal{X}}(y(q,\tau),z),$$

where $y=y(q,\tau)$ *is the inverse of the toric mirror map* $q=q(y)$*,* $\tau=\tau(y)$*.*

Remark 2.16. *The non-equivariant limit of* $q_0(y)$ *is* 0*.*

3. Orbi-disk invariants

In this section we briefly review the construction of genus 0 open orbifold Gromov-Witten invariants of toric orbifolds carried out in [28].

Let $(\mathcal{X}, \omega)$ be a toric Kähler orbifold of complex dimension n, equipped with the standard toric complex structure J_0 and a toric Kähler structure ω. Suppose that $\mathcal{X}$ is associated to the stacky fan $(\Sigma, \boldsymbol{b})$, where $\boldsymbol{b} = (\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1})$ and $\boldsymbol{b}_i = c_i v_i$. As before, we let D_i $(i = 0, \ldots, m-1)$ be the toric prime divisor associated to $\boldsymbol{b}_i$.

Let $L \subset \mathcal{X}$ be a Lagrangian torus fiber of the moment map $\mu_0 : \mathcal{X} \to M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, and consider a relative homotopy class $\beta \in \pi_2(\mathcal{X}, L) = H_2(\mathcal{X}, L; \mathbb{Z})$. We are interested in holomorphic orbi-disks in $\mathcal{X}$ bounded by L and representing the class β.

3.1. Holomorphic orbi-disks and their moduli spaces.

A *holomorphic orbi-disk* in $\mathcal{X}$ with boundary in L is a continuous map

$$w : (\mathcal{D}, \partial\mathcal{D}) \to (\mathcal{X}, L)$$

such that the following conditions are satisfied:

(1) $(\mathcal{D}, z_1^+, \ldots, z_l^+)$ is an orbi-disk with interior orbifold marked points $z_1^+, \ldots, z_l^+$. Namely $\mathcal{D}$ is analytically the disk $D^2 \subset \mathbb{C}$, together with orbifold structure at each marked point z_j^+ for $j = 1, \ldots, l$. For each j, the orbifold structure at z_j^+ is given by a disk neighborhood of z_j^+ which is uniformized by a branched covering map $br : z \to z^{m_j}$ for some[2] $m_j \in \mathbb{Z}_{>0}$.

(2) For any $z_0 \in \mathcal{D}$, there is a disk neighborhood of z_0 with a branched covering map $br : z \to z^m$, and there is a local chart $(V_{w(z_0)}, G_{w(z_0)}, \pi_{w(z_0)})$ of $\mathcal{X}$ at $w(z_0)$ and a local holomorphic lifting $\widetilde{w}_{z_0}$ of w satisfying

$$w \circ br = \pi_{w(z_0)} \circ \widetilde{w}_{z_0}.$$

(3) The map w is *good* (in the sense of Chen-Ruan [25]) and *representable*. In particular, for each marked point z_j^+, the associated homomorphism

$$h_p : \mathbb{Z}_{m_j} \to G_{w(z_j^+)} \tag{3.1}$$

between local groups which makes $\widetilde{w}_{z_j^+}$ equivariant, is injective.

Denote by $\nu_j \in \mathrm{Box}(\Sigma)$ the image of the generator $1 \in \mathbb{Z}_{m_j}$ under h_j and let $\mathcal{X}_{\nu_j}$ be the twisted sector of $\mathcal{X}$ corresponding to ν_j. Such a map w is said to be of *type* $\boldsymbol{x} := (\mathcal{X}_{\nu_1}, \ldots, \mathcal{X}_{\nu_l})$.

We recall the following classification result of orbi-disks:

Theorem 3.1 ([28], Theorem 6.2)**.** *Let $\mathcal{X}$ be a symplectic toric orbifold corresponding to a stacky fan $(\Sigma(P), \boldsymbol{b})$ and $L \subset \mathcal{X}$ a Lagrangian torus fiber of the moment map. Consider a fixed orbit $\widetilde{L} \subset \mathbb{C}^m \setminus Z(\Sigma)$ of the real m-torus T^m which projects to L. Suppose*

$$w : (\mathcal{D}, \partial\mathcal{D}) \to (\mathcal{X}, L)$$

is a holomorphic map with orbifold singularities at interior marked points $z_1^+, \ldots, z_l^+ \in \mathcal{D}$. Then

[2] If $m_j = 1$, z_j^+ is a smooth interior marked point.

(1) *For each orbifold marked point* z_j^+, *we have a twisted sector* $\nu_j = \sum_{i \notin I_j} t_{ji} \boldsymbol{b}_i \in \mathrm{Box}^\circ_{\boldsymbol{b}_{\sigma_j}}$ *where* σ_j *is a cone in* Σ *and* $I_j \in \mathcal{A}$ *is the anticone of* σ_j, *obtained via* (3.1). *(See Section 2.2 for the definition of* $\mathrm{Box}^\circ_{\boldsymbol{b}_\sigma}$.*)*

(2) *For an analytic coordinate* z *on* $D^2 = |\mathcal{D}|$, *the map* w *can be lifted to a holomorphic map*

$$\widetilde{w} : (D^2, \partial D^2) \to ((\mathbb{C}^m \setminus Z(\Sigma))/K_{\mathbb{C}}, \widetilde{L}/(K_{\mathbb{C}} \cap T^m)),$$

so that the homogeneous coordinate functions (modulo $K_{\mathbb{C}}$*-action)* $\widetilde{w} = (\widetilde{w}_0, \ldots, \widetilde{w}_{m-1})$ *are given by*

$$\widetilde{w}_i = a_i \cdot \prod_{s=1}^{d_i} \frac{z - \alpha_{i,s}}{1 - \overline{\alpha}_{i,s} z} \prod_{j=1}^{l} \left(\frac{z - z_j^+}{1 - \overline{z_j^+} z} \right)^{t_{ji}} \tag{3.2}$$

for $d_i \in \mathbb{Z}_{\geq 0}, (i = 0, \ldots, m-1)$ *and* $\alpha_{i,s} \in \mathrm{int}(D^2)$, $a_i \in \mathbb{C}^\times$. *Here* K *is defined by the following exact sequence*

$$0 \to K \to T^m \to T^n \to 0$$

where $T^m \to T^n$ *is induced by the map* $\bigoplus_{i=0}^{m-1} \mathbb{Z} e_i \to N$ *by sending* e_i *to* $\boldsymbol{b}_i$ *for* $i = 0, \ldots, m-1$. *(We remark that* K *may have non-trivial torsion part.)*

(3) *The Chern-Weil Maslov index (see Appendix A) of the map* w *whose lift is given as in* (3.2) *satisfies*

$$\mu_{CW}(w) = \sum_{i=0}^{m-1} 2d_i + \sum_{j=1}^{l} 2\mathrm{age}(\nu_j).$$

Setting $l = 0$ and $d_i = 0$ for all i except for one i_0 where $d_{i_0} = 1$ in the above theorem gives a holomorphic disk which is smooth and intersects the associated toric prime divisor $D_{i_0} \subset \mathcal{X}$ with multiplicity one; its homotopy class is denoted as β_{i_0}. Given $\nu \in \mathrm{Box}'(\Sigma)$, setting $l = 1$ and $d_i = 0$ for all i gives a holomorphic orbi-disk, whose homotopy class is denoted as β_ν.

Lemma 3.2 ([28], Lemma 9.1)**.** *For* $\mathcal{X}$ *and* L *as above, the relative homotopy group* $\pi_2(\mathcal{X}, L)$ *is generated by the classes* β_i *for* $i = 0, \ldots, m-1$ *together with* β_ν *for* $\nu \in \mathrm{Box}'(\Sigma)$.

We call these generators of $\pi_2(\mathcal{X}, L)$ the *basic disk classes*. They are the analogue of Maslov index two disk classes in toric manifolds. Basic disk classes were used in [28] to define the leading order bulk orbi-potential, and it can be used to determine Floer homology of torus fibers with suitable bulk deformations. Basic disks are classified as follows:

Corollary 3.3 ([28], Corollaries 6.3 and 6.4)**.**

(1) *The smooth holomorphic disks of Maslov index two (modulo* T^n*-action and automorphisms of the domain) are in a one-to-one correspondence with the stacky vectors* $\{\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1}\}$.

(2) *The holomorphic orbi-disks with one interior orbifold marked point and desingularized Maslov index zero (modulo* T^n*-action and automorphisms of the domain) are in a one-to-one correspondence with the twisted sectors* $\nu \in \mathrm{Box}'(\Sigma)$ *of the toric orbifold* $\mathcal{X}$.

Let

$$\mathcal{M}^{main}_{k+1,l}(L, \beta, \boldsymbol{x})$$

be the moduli space of good representable stable maps from bordered orbifold Riemann surfaces of genus zero with $k+1$ boundary marked points $z_0, z_1 \dots, z_k$ and l interior (orbifold) marked points $z_1^+, \dots, z_l^+$ in the homotopy class β of type $\boldsymbol{x} = (\mathcal{X}_{\nu_1}, \dots, \mathcal{X}_{\nu_l})$. Here, the superscript "*main*" indicates that we have chosen a connected component on which the boundary marked points respect the cyclic order of $S^1 = \partial D^2$. Let

$$\mathcal{M}^{main,reg}_{k+1,l}(L, \beta, \boldsymbol{x}) \subset \mathcal{M}^{main}_{k+1,l}(L, \beta, \boldsymbol{x})$$

be the subset consisting of all maps from an (orbi-)disk (i.e. without (orbi-)sphere/disk bubbles). It was shown in [28] that $\mathcal{M}^{main}_{k+1,l}(L, \beta, \boldsymbol{x})$ has a Kuranishi structure of real virtual dimension

$$n + \mu_{CW}(\beta) + k + 1 + 2l - 3 - 2\sum_{j=1}^{l} \mathrm{age}(\nu_j). \tag{3.3}$$

According to [28, Proposition 9.4], if $\mathcal{M}^{main}_{1,1}(L, \beta)$ is non-empty and if $\partial\beta$ is not in the sublattice generated by $\boldsymbol{b}_0, \dots, \boldsymbol{b}_{m-1}$, then there exist $\nu \in \mathrm{Box}'(\Sigma)$, $k_i \in \mathbb{N}$ $(i = 0, \dots, m-1)$ and $\alpha \in H_2^{\mathrm{eff}}(\mathcal{X})$ such that

$$\beta = \beta_\nu + \sum_{i=0}^{m-1} k_i \beta_i + \alpha,$$

where α is realized by a union of holomorphic (orbi-)spheres. The Chern-Weil Maslov index of β written in this way is given by

$$\mu_{CW}(\beta) = 2\mathrm{age}(\nu) + 2\sum_{i=0}^{m-1} k_i + 2c_1(\mathcal{X}) \cdot \alpha.$$

3.2. **The invariants.** Let $\mathcal{X}_{\nu_1}, \dots, \mathcal{X}_{\nu_l}$ be twisted sectors of the toric orbifold $\mathcal{X}$. Consider the moduli space $\mathcal{M}^{main}_{1,l}(L, \beta, \boldsymbol{x})$ of good representable stable maps from bordered orbifold Riemann surfaces of genus zero with one boundary marked point and l interior orbifold marked points of type $\boldsymbol{x} = (\mathcal{X}_{\nu_1}, \dots, \mathcal{X}_{\nu_l})$ representing the class $\beta \in \pi_2(\mathcal{X}, L)$. According to [28], the moduli space $\mathcal{M}^{main}_{1,l}(L, \beta, \boldsymbol{x})$ carries a virtual fundamental chain, which vanishes unless the following equality holds:

$$\mu_{CW}(\beta) = 2 + \sum_{j=1}^{l} (2\mathrm{age}(\nu_j) - 2). \tag{3.4}$$

Definition 3.4. *An orbifold $\mathcal{X}$ is called* Gorenstein *if its canonical divisor $K_{\mathcal{X}}$ is Cartier.*

For a Gorenstein orbifold, the age of every twisted sector is a non-negative integer. Now we assume that the toric orbifold $\mathcal{X}$ is semi-Fano (see Definition 2.8) and Gorenstein. Then a basic orbi-disk class β_ν has Maslov index $2\mathrm{age}(\nu) \geq 2$ (see Lemma 4.13), and hence every non-constant stable disk class has at least Maslov index two.

Let us further restrict to the case where all the interior orbifold marked points are mapped to age-one twisted sectors, i.e. the type $\boldsymbol{x}$ consists of twisted sectors with age $= 1$. This will be enough for our purpose of constructing the mirror over $H^2_{\mathrm{CR}}(\mathcal{X})$. In this case, the virtual fundamental chain $[\mathcal{M}^{main}_{1,l}(L, \beta, \boldsymbol{x})]^{\mathrm{vir}}$ is non-zero only when $\mu_{CW}(\beta) = 2$, and in fact we get even a virtual fundamental *cycle* because β attains the minimal Maslov index and thus disk bubbling does not occur. Therefore the following definition of *genus 0 open orbifold Gromov-Witten invariants* (also termed *orbi-disk invariants*) is independent of the choice of perturbations of the Kuranishi structures (in the general case one may restrict to torus-equivariant perturbations to make sense of the following definition following the works of Fukaya-Oh-Ohta-Ono [49, 50, 45]):

Definition 3.5 (Orbi-disk invariants)**.** *Let* $\beta \in \pi_2(\mathcal{X}, L)$ *be a relative homotopy class with Maslov index given by* (3.4)*. Suppose that the moduli space* $\mathcal{M}_{1,l}(L, \beta, \boldsymbol{x})$ *is compact. Then we define* $n^{\mathcal{X}}_{1,l,\beta}([\mathrm{pt}]_L; \mathbf{1}_{\nu_1}, \ldots, \mathbf{1}_{\nu_l}) \in \mathbb{Q}$ *to be the push-forward*

$$n^{\mathcal{X}}_{1,l,\beta}([\mathrm{pt}]_L; \mathbf{1}_{\nu_1}, \ldots, \mathbf{1}_{\nu_l}) := ev_{0*}\left([\mathcal{M}_{1,l}(L, \beta, \boldsymbol{x})]^{\mathrm{vir}}\right) \in H_n(L; \mathbb{Q}) \cong \mathbb{Q},$$

where $ev_0 : \mathcal{M}^{main}_{1,l}(L, \beta, \boldsymbol{x}) \to L$ *is evaluation at the boundary marked point,* $[\mathrm{pt}]_L \in H^n(L; \mathbb{Q})$ *is the point class of the Lagrangian torus fiber* L*, and* $\mathbf{1}_{\nu_j} \in H^0(\mathcal{X}_{\nu_j}; \mathbb{Q}) \subset H^{2\mathrm{age}(\nu_j)}_{\mathrm{CR}}(\mathcal{X}; \mathbb{Q})$ *is the fundamental class of the twisted sector* $\mathcal{X}_{\nu_j}$*.*

Remark 3.6. *For the cases we need in this paper, the required compactness of the disk moduli space* $\mathcal{M}_{1,l}(L, \beta, \boldsymbol{x})$ *will be proved in Proposition 6.10 and Corollary 6.11.*

Remark 3.7. *The Kuranishi structures in this paper are the same as those defined in* [49, 50] *(we refer the readers to* [47, 48, Appendix] *and* [46] *for the detailed construction, and also to* [93] *(and its forthcoming sequels) for a different approach). But we remark that the moduli spaces we consider here are in fact much simpler than those in* [49, 50] *(and* [47, 48]*) because we only need to consider stable disks with just one disk component which is minimal, and hence disk bubbling does not occur. Also, we consider only disk counting invariants, but not the whole* A_∞ *structure; this reduces the problem to studying moduli spaces of virtual dimensions 0 or 1, which simplifies several issues involved.*

For a basic (orbi-)disk with at most one interior orbifold marked point, the corresponding moduli space $\mathcal{M}_{1,0}(L, \beta_i)$ (or $\mathcal{M}_{1,1}(L, \beta_\nu, \nu)$ when β_ν is a basic orbi-disk class) is regular and can be identified with L. Thus the associated invariants are evaluated as follows [28]:

(1) For $\nu \in \mathrm{Box}'$, we have $n^{\mathcal{X}}_{1,1,\beta_\nu}([\mathrm{pt}]_L; \mathbf{1}_\nu) = 1$.
(2) For $i \in \{0, \ldots, m-1\}$, we have $n^{\mathcal{X}}_{1,0,\beta_i}([\mathrm{pt}]_L) = 1$.

When there are more interior orbifold marked points or when the disk class is not basic, the corresponding moduli space is in general non-regular and virtual theory is involved in the definition, making the invariant much more difficult to compute. One main aim of this paper is to compute all these invariants for toric Calabi-Yau orbifolds.

4. Gross fibrations for toric Calabi-Yau orbifolds

In order to carry out the SYZ construction, the first ingredient we need is a Lagrangian torus fibration. For a toric Calabi-Yau manifold, such fibrations were constructed by Gross [62] and Goldstein [57] independently. In this section we generalize their constructions to toric Calabi-Yau orbifolds; cf. the manifold case as discussed in [20, Sections 4.1-4.5].

4.1. **Toric Calabi-Yau orbifolds.**

Definition 4.1. *A Gorenstein toric orbifold $\mathcal{X}$ is called* Calabi-Yau *if there exists a dual vector $\underline{\nu} \in M = N^\vee = \mathrm{Hom}(N, \mathbb{Z})$ such that $(\underline{\nu}, \boldsymbol{b}_i) = 1$ for all stacky vectors $\boldsymbol{b}_i$.*

Let $\mathcal{X}$ be a toric Calabi-Yau orbifold associated to a stacky fan $(\Sigma, \boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1})$. Since $\boldsymbol{b}_i = c_i v_i$ for some primitive vector $v_i \in N$ and $(\underline{\nu}, v_i) \in \mathbb{Z}$, we have $c_i = 1$ for all $i = 0, \ldots, m-1$. Therefore toric Calabi-Yau orbifolds are always simplicial.

Example 4.2. *For a compact toric orbifold $\mathcal{X}$, the total space of the canonical line bundle of $\mathcal{X}$ is a toric Calabi-Yau orbifold. Namely, if $\mathcal{X}$ is given by a fan Σ in the lattice N of rank $n-1$ with stacky vectors $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1}$, then the total space of the canonical line bundle of $\mathcal{X}$ is given by a fan Σ' in the lattice $N \oplus \mathbb{Z}$ of rank n, whose rays are generated by $(0,1), (\boldsymbol{b}_0, 1), \ldots, (\boldsymbol{b}_{m-1}, 1) \in N \oplus \mathbb{Z}$. If $\sigma \in \Sigma$ is a cone generated by $\{\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_k}\}$, then there is a corresponding cone $\sigma' \in \Sigma'$ generated by $\{(0,1), (\boldsymbol{b}_{i_1}, 1), \ldots, (\boldsymbol{b}_{i_k}, 1)\}$. In this case we can take $\underline{\nu} = (0,1) \in (N \oplus \mathbb{Z})^\vee \simeq N^\vee \oplus \mathbb{Z}$.*

For the purpose of this paper, we will always assume that the coarse moduli space of the toric Calabi-Yau orbifold $\mathcal{X}$ is *semi-projective* (Definition 2.2).

Setting 4.3 (Partial resolutions of toric Gorenstein canonical singularities)**.** *Let $\sigma \subset N_\mathbb{R}$ be a strongly convex rational polyhedral Gorenstein canonical cone with primitive generators $\{\tilde{\boldsymbol{b}}_i\} \subset N$. Here,* strongly convex *means that the cone σ is convex in $N_\mathbb{R}$ and does not contain any whole straight line; while* Gorenstein canonical *means that there exists $\underline{\nu} \in M$ such that $\left(\underline{\nu}\,,\,\tilde{\boldsymbol{b}}_i\right) = 1$ for all i, and $(\underline{\nu}\,,\,v) \geq 1$ for all $v \in \sigma \cap (N \setminus \{0\})$. We denote by $\mathcal{P} \subset N_\mathbb{R}$ the convex hull of $\{\tilde{\boldsymbol{b}}_i\} \subset N$ in the hyperplane $\{v \in N_\mathbb{R} \mid (\underline{\nu}, v) = 1\} \subset N_\mathbb{R}$. $\mathcal{P}$ is an $(n-1)$-dimensional lattice polytope.*

Let $\Sigma \subset N_\mathbb{R}$ be a simplicial refinement of σ obtained by taking the cones over a triangulation of $\mathcal{P}$ (where all vertices of the triangulation belong to $\mathcal{P} \cap N$). Then Σ together with the collection

$$\{\boldsymbol{b}_i \mid i = 0, \ldots, m-1\} \subset N$$

of primitive generators of rays in Σ is a stacky fan. The associated toric orbifold $\mathcal{X} = \mathcal{X}_\Sigma$ is Gorenstein and Calabi-Yau.

By relabeling the $\boldsymbol{b}_i$'s if necessary, we assume that $\{\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{n-1}\}$ generates a top-dimensional cone in Σ and hence forms a rational basis of $N_\mathbb{Q} := N \otimes_\mathbb{Z} \mathbb{Q}$.

Proposition 4.4. *The coarse moduli space of a toric Calabi-Yau orbifold $\mathcal{X}$ is semi-projective if and only if $\mathcal{X}$ satisfies Setting 4.3.*

Proof. If $\mathcal{X}$ satisfies Setting 4.3, it is clear that its fan has full-dimensional convex support. Moreover, $\mathcal{X}$ can be constructed by using its moment map polytope, so its coarse moduli space is quasi-projective.

Conversely, suppose that the coarse moduli space of $\mathcal{X}$ is semi-projective. Since $\mathcal{X}$ is Gorenstein, there exists $\underline{\nu} \in M$ such that $(\underline{\nu}\,,\,\boldsymbol{b}_i) = 1$ for all primitive generators $\boldsymbol{b}_i$ of rays in Σ. Then the convex hull of $\boldsymbol{b}_i$'s in the hyperplane $\{(\underline{\nu}\,,\,\cdot) = 1\} \subset N_\mathbb{R}$ defines a lattice polytope $\mathcal{P}$, and the support of the fan is equal to the cone σ over this lattice polytope by convexity of the fan. Obviously, the cone σ is strongly convex and Gorenstein. Also the fan of $\mathcal{X}$ is obtained by a triangulation of the lattice polytope $\mathcal{P}$. $\square$

For the rest of this paper, we will assume that $\mathcal{X}$ is a toric Calabi-Yau orbifold $\mathcal{X}$ as in Setting 4.3. This implies Assumption 2.9 is satisfied: If $\mathcal{P}$ has no interior lattice point, then clearly $\{0\} \cup (\mathcal{P} \cap N)$ generates the lattice N. Otherwise we can inductively find a minimal simplex contained in $\mathcal{P}$ which does not contain any interior lattice point, and it follows that $\{0\} \cup (\mathcal{P} \cap N)$ generates the lattice N.

Without loss of generality we may assume that $\underline{\nu} = (0,1) \in M \simeq \mathbb{Z}^{n-1} \oplus \mathbb{Z}$ so that $\mathcal{P}$ is contained in the hyperplane $\{v \in N_{\mathbb{R}} \mid ((0,1)\,,\, v) = 1\}$. We enumerate

$$\mathrm{Box}'(\Sigma)^{\mathrm{age}=1} := \{\nu \in \mathrm{Box}'(\Sigma) \mid \mathrm{age}(\nu) = 1\} = \{\boldsymbol{b}_m, \ldots, \boldsymbol{b}_{m'-1}\}$$

and choose $\boldsymbol{b}_m, \ldots, \boldsymbol{b}_{m'-1}$ to be the extra vectors so that

$$\mathcal{P} \cap N = \{\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1}, \boldsymbol{b}_m, \ldots, \boldsymbol{b}_{m'-1}\}.$$

4.2. **The Gross fibration.** In this section we construct a special Lagrangian torus fibration on a toric Calabi-Yau orbifold $\mathcal{X}$. This is a fairly straightforward generalization of the constructions of Gross [62] and Goldstein [57] to the orbifold setting.

To begin with, notice that the vector $\underline{\nu} \in M$ corresponds to a holomorphic function on $\mathcal{X}$ which we denote by $w : \mathcal{X} \to \mathbb{C}$. The following two lemmas are easy generalizations of the corresponding statements for toric Calabi-Yau manifolds [20].

Lemma 4.5 (cf. [20], Proposition 4.2)**.** *The function w on $\mathcal{X}$ corresponding to $\underline{\nu} \in M$ is holomorphic, and its zero divisor (w) is precisely given by the anticanonical divisor $-K_{\mathcal{X}} = \sum_{i=0}^{m-1} D_i$.*

Proof. Let $\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_n}$ be the primitive generators of a top-dimensional cone σ in Σ, which span a sublattice $N_\sigma \subset N$ of rank n. Consider the dual basis $\{u_j\}_{j=1}^n$ of $M_{\mathbb{Q}}$ which gives rise to coordinate functions $\{\zeta_j\}_{j=1}^n$ on the uniformizing cover $\widetilde{U}_\sigma$, with an action of finite abelian group $G_\sigma = N/N_\sigma$.

Then the corresponding function w is given by the product of coordinate functions

$$w = \prod_{j=1}^n \zeta_j$$

which is regular. We need to show that this function is invariant under N/N_σ action. The group action defined for the coordinate functions on the uniformizing cover

$$g \cdot \zeta_i = \exp(2\pi\sqrt{-1}\langle u_i, g\rangle)\zeta_i \tag{4.1}$$

is based on the pairing

$$N/N_\sigma \times M_\sigma/M \to \mathbb{Q}/\mathbb{Z}.$$

Since $\underline{\nu} \in M$, $(g\,,\,\underline{\nu}) \in \mathbb{Z}$ for all $g \in N$. Thus $g \cdot w = w$ for all $g \in N/N_\sigma$. This proves our claim. $\square$

Lemma 4.6 (cf. [20], Proposition 4.3)**.** *For the dual basis* $\{u_0, \ldots, u_{n-1}\} \subset M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ *of the basis* $\{\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{n-1}\}$*, denote by* ζ_j *the corresponding meromorphic function to* u_j*. Then*

$$d\zeta_0 \wedge \cdots \wedge d\zeta_{n-1}$$

extends to a nowhere-zero holomorphic n*-form* Ω *on* $\mathcal{X}$*.*

Proof. Notice that

$$d\zeta_0 \wedge \cdots \wedge d\zeta_{n-1} = w d\log\zeta_0 \wedge \cdots \wedge d\log\zeta_{n-1}.$$

w is invariant under N/N_σ (see the proof of Lemma 4.5). Moreover N/N_σ acts on $\log\zeta_i$ by adding constants, and hence $d\log\zeta_i$ are also invariant under the action. It is easy to see that $w d\log\zeta_0 \wedge \cdots \wedge d\log\zeta_{n-1}$ extends to be nowhere-zero holomorphic n-form in all other charts. ☐

Next, we equip $\mathcal{X}$ with a toric Kähler structure ω and consider the associated moment map $\mu_0 : \mathcal{X} \to P$, where P is the moment polytope defined by a system of inequalities:

$$(\boldsymbol{b}_i, \cdot) \geq c_i, \quad i = 0, \ldots, m-1.$$

Consider the subtorus $T^{\perp\underline{\nu}} := N_{\mathbb{R}}^{\perp\underline{\nu}}/N^{\perp\underline{\nu}} \subset N_{\mathbb{R}}/N$. The moment map of the $T^{\perp\underline{\nu}}$ action is given by composing μ_0 with the natural quotient map:

$$[\mu_0] : \mathcal{X} \xrightarrow{\mu_0} M_{\mathbb{R}} \to M_{\mathbb{R}}/\mathbb{R}\langle\underline{\nu}\rangle.$$

The following is a generalization of the Gross fibration for toric Calabi-Yau manifolds [57, 62], which gives a Lagrangian torus fibration (SYZ fibration).

Definition 4.7. *Fix* $K_2 > 0$*. A* Gross fibration *of* $\mathcal{X}$ *is defined to be*

$$\begin{aligned} \mu : \mathcal{X} &\to (M_{\mathbb{R}}/\mathbb{R}\langle\underline{\nu}\rangle) \times \mathbb{R}_{\geq -K_2^2} \\ x &\mapsto ([\mu_0(x)], |w(x) - K_2|^2 - K_2^2). \end{aligned}$$

We denote by $B := (M_{\mathbb{R}}/\mathbb{R}\langle\underline{\nu}\rangle) \times \mathbb{R}_{\geq -K_2^2}$ *the base of the Gross fibration* μ*.*

Since the holomorphic function w vanishes on the toric prime divisors $D_i \subset \mathcal{X}$, the images of $D_i \subset \mathcal{X}$ under the map μ have second coordinate zero. Moreover, the hypersurface defined by $w(x) = K_2$ maps to the boundary of the image of μ.

The following proposition can be proved in exactly the same way as in the manifold case (cf. [62, Theorem 2.4] or [20, Proposition 4.7]). It follows from the construction of symplectic reduction: the function w descends to the symplectic reduction $\mathcal{X}//T^{\perp\underline{\nu}} \to \mathbb{C}$; since the circles centered at K_2 are special Lagrangian with respect to the volume form $\mathrm{d}\log(w - K_2)$, it follows that their preimages are also special Lagrangian in $\mathcal{X}$ with respect to the holomorphic volume form $\Omega/(w - K_2)$.

Proposition 4.8. *With respect to the holomorphic volume form* $\Omega/(w - K_2)$ *defined on* $\mu^{-1}(B^{int})$ *and the toric Kähler form* ω*, the map* μ *is a special Lagrangian torus fibration.*

4.2.1. *Discriminant locus and local trivialization.* For each $\emptyset \neq I \subset \{0, \dots, m-1\}$ such that $\{\boldsymbol{b}_i \mid i \in I\}$ generates a cone in Σ, we define

$$T_I := \{\xi \in P \mid (\boldsymbol{b}_i, \xi) = c_i, \ i \in I\} \subset \partial P. \tag{4.2}$$

T_I is a codimension-$(|I|-1)$ face of ∂P. Let $[T_I] := [\mu_0](T_I)$.

Let $\Gamma := \{r \in B \mid r \text{ is a critical value of } \mu\} \subset B$ be the discriminant locus of μ. Put $B_0 := B \setminus \Gamma$.

Proposition 4.9. *The discriminant locus of the Gross fibration μ is given by*

$$\Gamma = \partial B \cup \left(\left(\bigcup_{|I|=2} [T_I] \right) \times \{0\} \right).$$

Proof. This is similar to the manifold case ([20, Proposition 4.9]). A fiber degenerates when the $T^{\perp \underline{\nu}}$-orbit degenerates or $|w - K_2| = 0$. An $T^{\perp \underline{\nu}}$-orbit degenerates if and only if $w = 0$ and $[\mu_0] \in \left(\bigcup_{|I|=2} [T_I]\right)$; $|w - K_2| = 0$ implies that the base point is located in ∂B. It follows that the discriminant locus is of the above form. □

By the arguments in [20, Section 2.1], the restriction $\mu : \mathcal{X}_0 := \mu^{-1}(B_0) \to B_0$ is a torus bundle. For facets $T_0, \dots, T_{m-1}$ of P, consider the following open subsets of B_0:

$$U_i := B_0 \setminus \bigcup_{k \neq i} ([T_k] \times \{0\}).$$

The torus bundle μ over each U_i can be explicitly trivialized. Without loss of generality we describe this explicit trivialization over U_0.

Definition 4.10. *We choose $\underline{v}_1, \dots, \underline{v}_{n-1} \in N$ such that*

(1) $\{\boldsymbol{b}_0\} \cup \{\underline{v}_1, \dots, \underline{v}_{n-1}\}$ is an integral basis of N;
(2) $(\underline{v}_i, \underline{\nu}) = 0$ for $1 \leq i \leq n-1$.

Let $\{\nu_0, \dots, \nu_{n-1}\} \subset M$ be the dual basis of $\{\boldsymbol{b}_0\} \cup \{\underline{v}_1, \dots, \underline{v}_{n-1}\}$.

Definition 4.11. *Denote*

$$T^{\perp \boldsymbol{b}_0} := \frac{N_{\mathbb{R}}/\mathbb{R}\langle \boldsymbol{b}_0 \rangle}{N/\mathbb{Z}\langle \boldsymbol{b}_0 \rangle}.$$

Then, over U_0, we have a trivialization

$$\mu^{-1}(U_0) \cong U_0 \times T^{\perp \boldsymbol{b}_0} \times (\mathbb{R}/2\pi\mathbb{Z}).$$

Here the first map is given by μ, the last map is given by arg$(w - K_2)$, and the second map is given by the argument over 2π of the meromorphic functions corresponding to $\nu_1, \dots, \nu_{n-1}$.

4.2.2. *Generators of homotopy groups.* Fix $r_0 := (q_1, q_2) \in U_0$ with $q_2 > 0$. Consider the fiber $F_{r_0} := \mu^{-1}(q_1, q_2)$. By the trivialization in Definition 4.11, we have $F_{r_0} \simeq T^{\perp \boldsymbol{b}_0} \times (\mathbb{R}/2\pi\mathbb{Z})$. Hence $\pi_1(F_{r_0}) \simeq N/\mathbb{Z}\langle \boldsymbol{b}_0 \rangle \times \mathbb{Z}$ has the following basis (over $\mathbb{Q}$)

$$\{\lambda_i \mid 0 \leq i \leq n-1\},$$

where $\lambda_0 = (0, 1)$ and $\lambda_i = ([\underline{v}_i], 0)$ for $1 \leq i \leq n-1$.

As mentioned in Section 3.1, for a regular Lagrangian torus fiber L of the moment map $\mathcal{X} \to P$, the basic disk classes form a natural basis of $\pi_2(\mathcal{X}, L)$. We now construct a basis for $\pi_2(\mathcal{X}, F_{r_0})$ by exhibiting a Lagrangian isotopy between F_{r_0} and L and using this natural basis of $\pi_2(\mathcal{X}, L)$. The following is an explicit Lagrangian isotopy between F_{r_0} and L:

$$L_t := \{x \in \mathcal{X} \mid [\mu_0(x)] = q_1,\ |w(x) - t|^2 = K_2^2 + q_2\}, \quad t \in [0, K_2]. \tag{4.3}$$

This allows us to identify $\pi_2(\mathcal{X}, F_{r_0})$ with $\pi_2(\mathcal{X}, L)$ and view the basic disk classes in $\pi_2(\mathcal{X}, L)$ as classes in $\pi_2(\mathcal{X}, F_{r_0})$. By abuse of notation, we still denote these classes by $\beta_0, \ldots, \beta_{m-1}$ and $\{\beta_\nu \mid \nu \in \mathrm{Box}'(\Sigma)\}$.

For a general $r \in U_0$, a basis for $\pi_2(\mathcal{X}, F_r)$ may be obtained by identifying F_r with F_{r_0} using the trivialization in Definition 4.11.

The boundaries of the classes $\beta_0, \ldots, \beta_{m-1}$ and $\{\beta_\nu \mid \nu \in \mathrm{Box}'(\Sigma)\}$ can be described as follows.

Lemma 4.12. *For a fiber F_r of π^K where $r \in U_0$, the boundary of the disk classes are described as follows:*

$$\partial\beta_j = \lambda_0 + \sum_{i=1}^{n-1} (\nu_i, \boldsymbol{b}_j)\lambda_i, \quad 0 \le j \le m-1$$

$$\partial\beta_\nu = \lambda_0 + \sum_{i=1}^{n-1} (\nu_i, \nu)\lambda_i, \quad \nu = \sum_{i=1}^{n-1} (\nu_i, \nu)\underline{v}_i \in \mathrm{Box}'(\Sigma).$$

Proof. Under the Lagrangian isotopy given by Equation (4.7) and identification between F_r and F_{r_0} using the trivialization over U_0, $\lambda_0 \in \pi_1(F_r)$ is identified with $\partial\beta_0 \in \pi_1(T)$ of a toric fiber, and $\lambda_i = ([\underline{v}_i], 0)$ has the same expression under such identification. We have the required equalities for a toric fiber, and these equalities are preserved under Lagrangian isotopy. □

The intersection numbers of these basic disk classes with toric prime divisors can be described as follows.

Lemma 4.13. *Consider $\beta_i \in \pi_2(X, F_r)$ for $r \in U_0$ defined as above. We have*

$$\beta_0 \cdot D_j = 0, \quad 1 \le j \le m-1$$
$$\beta_i \cdot D_j = \delta_{ij}, \quad 1 \le i \le m-1, 1 \le j \le m-1$$
$$\beta_i \cdot \tilde{D}_0 = 1, \quad 0 \le i \le m-1,$$

where $\tilde{D}_0 := \{w(x) = K_2\} \subset \mathcal{X}$. For a twisted sector $\nu \in Box^\circ_{\mathbf{b}_\sigma}$, $\nu = \sum_k t_k \boldsymbol{b}_{i_k}$ where $t_k \in \mathbb{Q} \cap [0,1)$ and $\boldsymbol{b}_{i_k}$'s are the primitive generators of σ. Then the intersection number of a basic orbi-disk class β_ν with a divisor can be expressed in terms of that of $\beta_0, \ldots, \beta_{m-1}$:

$$\beta_\nu \cdot D = \sum_k t_k (\beta_{i_k} \cdot D)$$

for any divisor D. In particular, we have

$$\beta_\nu \cdot \tilde{D}_0 = \mathrm{age}(\nu)$$

and so $\mu(\beta_\nu) = 2\,\mathrm{age}(\nu)$.

Proof. The proof is similar to that of Lemma 4.12: we use Lagrangian isotopy to reduce the calculations for F_r to that for a toric fiber. Since the Lagrangian submanifolds in the isotopy given by Equation (4.7) never intersect the divisors D_j for $j = 1, \ldots, m-1$ and $\tilde{D}_0$, the intersection numbers of the disk classes with these divisors remain unchanged under the isotopy. Moreover, Lagrangians over U_0 also never hit these divisors (notice that this is not true for D_0), and hence the inersection numbers are independent of the base point $r \in U_0$. □

4.2.3. *Wall-crossing of orbi-disk invariants.* Like the manifold case, the behavior of disk invariants with boundary conditions on a fiber F_r depends on the location of the fiber. In this section we examine this behavior for orbi-disks in the Gross fibration $\mu : \mathcal{X} \to B$ of a toric Calabi-Yau orbifold.

Let $\beta \in \pi_2(\mathcal{X}, F_r)$ be a class represented by a stable disk. Then it must be of the form $\beta = \sum_i u_i + \alpha$ where u_i's are disk classes and α is the class of a rational curve. So we have $\mu_{CW}(\beta) = \sum_i \mu_{CW}(u_i) + 2c_1(\mathcal{X}) \cdot \alpha$. Since $\mathcal{X}$ is Calabi-Yau, we have $c_1(\mathcal{X}) \cdot \alpha = 0$. The fiber $F_r \subset \mathcal{X}$ is a special Lagrangian submanifold with respect to the meromorphic form $\Omega/(w - K_2)$. Since the pole divisor of $\Omega/(w - K_2)$ is $\tilde{D}_0 := \{w(x) = K_2\} \subset \mathcal{X}$, Lemma A.3 implies that $\mu_{CW}(u_i) = 2u_i \cdot \tilde{D}_0 \geq 0$. Thus we have

Lemma 4.14. *If a class $\beta \in \pi_2(\mathcal{X}, F_r)$ is represented by a stable disk, then $\mu_{CW}(\beta) \geq 0$.*

The following result describes when the minimal Maslov index 0 can be achieved.

Lemma 4.15. *Let $r = (q_1, q_2) \in B_0$.*

(1) The fiber F_r bounds a non-constant stable disk of Chern-Weil Maslov index 0 if and only if $q_2 = 0$.
(2) If $q_2 \neq 0$, then the fiber F_r has minimal Chern-Weil Maslov index at least 2, i.e. F_r does not bound any non-constant stable disks with Chern-Weil Maslov index less than 2.

Proof. The proof of the corresponding result in the manifold case (see [20, Lemma 4.27 and Corollary 4.28]) applies, provided that we make the following observation: given a holomorphic orbi-disk $u : \mathcal{D} \to \mathcal{X}$, the composition $w \circ u : \mathcal{D} \to \mathbb{C}$ is a holomorphic function on every local chart of $\mathcal{D}$ and is invariant under the action of the local groups. Therefore $w \circ u$ descends to a holomorphic function $\overline{w \circ u} : |\mathcal{D}| \to \mathbb{C}$ on the smooth disk $|\mathcal{D}|$ underlying $\mathcal{D}$.

Then we can apply maximal principle on $\overline{w \circ u} - K$ as in the manifold case: Since u has Maslov index zero, it never intersects the boundary divisor D_0 by Lemma A.3. Thus $\overline{w \circ u} - K$ is never zero, and hence $\overline{w \circ u}$ is constant. Thus the image of u lies in a level set of w, and for topological reason this forces $w = 0$. Thus $q_2 = 0$. Thus if $q_2 \neq 0$, F_r has minimal Maslov index two. □

By definition, the *wall* of a Lagrangian fibration $\mu : \mathcal{X} \to B$ is the locus $H \subset B_0$ of all $r \in B_0$ such that the Lagrangian fiber F_r bounds a non-constant stable disk of Chern-Weil Maslov index 0. The above lemma shows that

$$H = M_{\mathbb{R}}/\mathbb{R}\langle \underline{\nu} \rangle \times \{0\}.$$

The complement $B_0 \setminus H$ is the union of two connected components

$$B_+ := M_{\mathbb{R}}/\mathbb{R}\langle \underline{\nu} \rangle \times (0, +\infty), \quad B_- := M_{\mathbb{R}}/\mathbb{R}\langle \underline{\nu} \rangle \times (-K_2^2, 0).$$

For $r \in B_0 \setminus H$, orbi-disk invariants with arbitrary numbers of age-one insertions are well-defined for relative homotopy classes with Chern-Weil Maslov index 2. We need to consider the two possibilities, namely $r \in B_+$ and $r \in B_-$.

Case 1: $r \in B_+$. Let $r = (q_1, q_2) \in B_+$, namely $q_2 > 0$. Then (4.3) gives a Lagrangian isotopy between the fiber F_r and a regular Lagrangian torus fiber L. Furthermore, since $q_2 > 0$, for each $t \in [0, K_2]$, w is never 0 on L_t. It follows that the Lagrangians L_t in the isotopy do not bound non-constant disks of Chern-Weil Maslov index 0. Hence for $r \in B_+$, the orbi-disk invariants of $(\mathcal{X}, F_r)$ with arbitrary numbers of age-one insertions and Chern-Weil Maslov index 2 coincide with those of $(\mathcal{X}, L)$, which are reviewed in Section 3.2.

Case 2: $r \in B_-$. In this case we have the following

Proposition 4.16. *Let $r = (q_1, q_2) \in B_-$, namely $q_2 < 0$. Let $\beta \in \pi_2(\mathcal{X}, F_r)$. Suppose $\mathbf{1}_{\nu_1}, \ldots, \mathbf{1}_{\nu_l} \in H^*_{\mathrm{CR}}(\mathcal{X})$ are fundamental classes of twisted sectors $\mathcal{X}_{\nu_1}, \ldots, \mathcal{X}_{\nu_l}$ such that* $\mathrm{age}(\nu_1) = \cdots = \mathrm{age}(\nu_l) = 1$. *Then we have*

$$n^{\mathcal{X}}_{1,l,\beta}([\mathrm{pt}]_{F_r}; \mathbf{1}_{\nu_1}, \ldots, \mathbf{1}_{\nu_l}) = \begin{cases} 1 & \text{if } \beta = \beta_0 \text{ and } l = 0 \\ 0 & \text{otherwise .} \end{cases}$$

Proof. By dimension reason, we may assume that $\mu_{CW}(\beta) = 2$.

Let $u : (\mathcal{D}, \partial\mathcal{D}) \to (\mathcal{X}, F_r)$ be a non-constant holomorphic orbi-disk. Then the composition $(w - K_2) \circ u$ descends to a holomorphic function $\overline{(w - K_2) \circ u} : |\mathcal{D}| \to \mathbb{C}$ on the smooth disk $|\mathcal{D}|$ underlying $\mathcal{D}$. Since $r \in B_-$, $|w - K_2|$ is constant on F_r with value less than K_2. Since $u(\partial|\mathcal{D}|) = u(\partial\mathcal{D}) \subset F_r$, we have $|\overline{(w - K_2) \circ u}| < K_2$ on $\partial|\mathcal{D}|$. By maximal principle, $|\overline{(w - K_2) \circ u}| < K_2$ on the whole $|\mathcal{D}|$. Hence the image of u is contained in $S_- := \mu^{-1}(\{(q_1, q_2) \in B \mid q_2 < 0\})$. Also observe that $u(\mathcal{D})$ must intersect $\tilde{D}_0 := \{w(x) = K_2\} \subset \mathcal{X}$. Since the hypersurface $w(x) = K_2$ does not contain orbifold points, we have $u(\mathcal{D}) \cdot \tilde{D}_0 \in \mathbb{Z}_{>0}$. By Lemma A.3, this implies that the Chern-Weil Maslov index of u is at least 2.

Let $h : \mathcal{C} \to \mathcal{X}$ be a non-constant holomorphic map from an orbifold sphere $\mathcal{C}$. Then $\overline{h(\mathcal{C})} \cap S_- = \emptyset$. To see this, we consider $w \circ h$, which descends to a holomorphic function $\overline{w \circ h}$ on the $\mathbb{P}^1$ underlying $\mathcal{C}$. Since $\overline{w \circ h}$ must be a constant function, the image $h(\mathcal{C})$ is contained in a level set $w^{-1}(c)$ for some $c \in \mathbb{C}$. For $c \neq 0$, we have $w^{-1}(c) \simeq (\mathbb{C}^\times)^{n-1}$ which does not support non-constant holomorphic spheres, so $c = 0$. Now we conclude by noting that $w^{-1}(0) \cap S_- = \emptyset$.

Now let $v \in \mathcal{M}^{main}_{1,l}(F_r, \beta, (\mathcal{X}_{\nu_1}, \ldots, \mathcal{X}_{\nu_l}))$ be a stable orbi-disk of Chern-Weil Maslov index 2. As explained above, each orbi-disk component contributes at least 2 to the Maslov index. Hence v only has one orbi-disk component. Also by above discussion, a non-constant holomorphic orbi-sphere in $\mathcal{X}$ cannot meet an orbi-disk. Therefore v does not have any orbi-sphere components. This shows that for any $\beta \in \pi_2(\mathcal{X}, F_r)$ of Maslov index 2, the moduli space $\mathcal{M}^{main}_{1,l}(F_r, \beta, (\mathcal{X}_{\nu_1}, \ldots, \mathcal{X}_{\nu_l}))$ parametrizes only orbi-disks. Also, all these orbi-disks are contained in S_- and do not meet the toric divisors $D_1, \ldots, D_{m-1}$. Since each orbifold point on the orbi-disk of type $\nu \in \mathrm{Box}'(\Sigma)$ contributes $2\mathrm{age}(\nu)$ to the Chern-Weil Maslov index

$\mu_{CW}(\beta)$, and since we assume $\mathrm{age}(\nu) = 1$ and $\mu_{CW}(\beta) = 2$, we cannot have any orbifold marked points on the disk.

Recall that relative homotopy classes β_ν can be written as (fractional) linear combinations of $\beta_0, \ldots, \beta_{m-1}$ with non-negative coefficients. Thus, the class β of any orbi-disk can be written as a linear combination of $\beta_0, \ldots, \beta_{m-1}$ with non-negative coefficients. Hence, from the fact that intersection numbers of β with the divisors $D_1, \ldots, D_{m-1}$ are zero, we may conclude that $\beta = k\beta_0$ for some $k \geq 0$, and $\mu(\beta) = 2$ implies that $k = 1$ and $\beta = \beta_0$. Holomorphic smooth disks representing the class β_0 are confined in an affine toric chart. The argument analogous to that in [20, Proof of Proposition 4.32] then shows that the invariant is 1 in this case. This concludes the proof. □

4.3. **Toric modification.** In this section we describe a toric modification of $\mathcal{X}$. As explained in [20, Section 4.3], considering certain toric modification provides a way to construct sufficiently many coordinate functions on the mirror of $\mathcal{X}$ by disk counting.

Let $\mathcal{X}$ be a toric Calabi-Yau orbifold as in Setting 4.3. Pick a top-dimensional cone in Σ with primitive generators $\{\boldsymbol{b}_i \mid i = 0, \ldots, n-1\} \subset N$. Let $\{\underline{v}_1, \ldots, \underline{v}_{n-1}\} \subset N$ and $\{\nu_0, \ldots, \nu_{n-1}\} \subset M$ be as in Definition 4.10.

Definition 4.17. *Fix $K_1 > 0$. Define*

$$P^{(K_1)} := \{\xi \in P \mid (\underline{v}_j, \xi) \geq -K_1 \text{ for all } j = 1, \ldots, n-1\} \subset P.$$

We assume that K_1 is sufficiently large so that none of the defining equations is redundant. Let $\Sigma^{(K_1)} \subset N$ be the inward normal fan to $P^{(K_1)}$ which consists of rays generated by $\{\boldsymbol{b}_i \mid i = 0, \ldots, m-1\} \cup \{\underline{v}_j \mid j = 1, \ldots, n-1\}$. This gives a stacky fan. Let $\mathcal{X}^{(K_1)}$ be the corresponding toric orbifold with moment map

$$\mu_0^{(K_1)} : \mathcal{X}^{(K_1)} \to P^{(K_1)}.$$

To simplify notation, we denote the above moment map by $\mu_0' : \mathcal{X}' \to P'$.

We now describe various properties of the toric modification $\mathcal{X}'$, whose proofs are similar to those of the corresponding statements in the manifold case (cf. [20, Sections 4.3–4.4]) and are omitted.

The element $\underline{\nu} \in M = N^\vee$ corresponds to a holomorphic function denoted by $w' : \mathcal{X}' \to \mathbb{C}$.

For $0 \leq i \leq m-1$, let

$$D_i \subset \mathcal{X}'$$

be the toric prime divisor corresponding to $\boldsymbol{b}_i$. For $1 \leq j \leq n-1$, let

$$D_j' \subset \mathcal{X}'$$

be the toric prime divisor corresponding to $\underline{v}_j$. We have the following result analogous to its counterpart in toric Calabi-Yau case:

Lemma 4.18. *The zero divisor of the function w' is given by*

$$(w') = \sum_{i=0}^{m-1} D_i.$$

In particular, w' is non-zero on $D_j', 1 \leq j \leq n-1$.

We observe that $\mathcal{X}'$ is no longer Calabi-Yau. But $\mathcal{X}'$ still admits a natural meromorphic n-form:

Lemma 4.19. *For the dual basis* $\{u_0, \ldots, u_{n-1}\} \subset M_{\mathbb{Q}}$ *of the basis* $\{\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{n-1}\}$*, denote by* ζ_j *the corresponding meromorphic function to* u_j*.Then*

$$d\zeta_0 \wedge \cdots \wedge d\zeta_{n-1}$$

extends to a meromorphic n*-form* Ω' *on* $\mathcal{X}'$*. Moreover, we have*

$$(\Omega') = -\sum_{j=1}^{n-1} D'_j.$$

We now define the Gross fibration for $\mathcal{X}'$.

Definition 4.20. *Consider*

$$E^{(K_1)} := \{q \in M_{\mathbb{R}}/\mathbb{R}\langle \underline{\nu} \rangle \mid (\underline{v}_j, q) \geq -K_1 \text{ for all } 1 \leq j \leq n-1\}.$$

Define the Gross fibration to be the following map

$$\begin{aligned} \mu^{(K_1)} : \mathcal{X}^{(K_1)} &\to B^{(K_1)} := E^{(K_1)} \times \mathbb{R}_{\geq -K_2} \\ x &\mapsto ([\mu_0^{(K_1)}(x)], |w'(x) - K_2|^2 - K_2^2). \end{aligned}$$

For simplicity, we omit (K_1) *in the notation and write* E *and* $\mu' : \mathcal{X}' \to B'$ *instead.*

The base B' is a manifold with the following n connected codimension-1 boundary strata:

$$\begin{aligned} \Psi_0 &:= \{(q_1, q_2) \in B' \mid q_2 = -K_2\}, \text{ and} \\ \Psi_j &:= \{(q_1, q_2) \in B' \mid (\underline{v}_j, q_1) = -K_1\}, \quad 1 \leq j \leq n-1. \end{aligned}$$

Their pre-images

$$\tilde{D}_j := (\mu')^{-1}(\Psi_j), \quad 0 \leq j \leq n-1$$

are divisors in $\mathcal{X}'$.

Proposition 4.21.

(a) *The quotient map* $M_{\mathbb{R}} \to M_{\mathbb{R}}/\mathbb{R}\langle \underline{\nu} \rangle$ *gives a homeomorphism from*

$$\{\xi \in \partial P' \mid (\underline{v}_j, \xi) > -K_1,\ 1 \leq j \leq n-1\} \tag{4.4}$$

to

$$E^{int} = \{q \in M_{\mathbb{R}}/\mathbb{R}\langle \underline{\nu} \rangle \mid (\underline{v}_j, q) > -K_1,\ 1 \leq j \leq n-1\}. \tag{4.5}$$

Consequently $\mu' : \mathcal{X}' \to B'$ *is surjective.*

(b) $\mu' : \mathcal{X}' \to B'$ *is a special Lagrangian torus fibration with respect to the toric Kähler form and the holomorphic volume form* $\Omega'/(w' - K_2)$ *defined on* $\mathcal{X}' \setminus \bigcup_{j=0}^{n-1} \tilde{D}_j$*.*

One observes that as $K_1 \to +\infty$, the divisors $\tilde{D}_j, 1 \leq j \leq n-1$ tend to infinity. Hence as $K_1 \to +\infty$, μ' tends to μ.

4.3.1. *Discriminant locus and local trivialization.*

Definition 4.22. *Let $\emptyset \neq I \subset \{0, \ldots, m-1\}$ such that $\{\boldsymbol{b}_i \mid i \in I\}$ generates a cone in Σ'. Define*

$$T'_I := T_I \cap \{\xi \in P' \mid (\underline{v}_j, \xi) > -K_1, 1 \leq j \leq n-1\}.$$

Here T_I is a face of P defined in (4.2)*. T'_I is a codimension-$(|I|-1)$ face of the set given by* (4.4)*.*

Proposition 4.23. *The discriminant locus of μ' is*

$$\Gamma' = \partial B' \cup \left(\left(\bigcup_{|I|=2} [T'_I] \right) \times \{0\} \right).$$

The restriction of μ' to $B'_0 := B' \setminus \Gamma'$ is a Lagrangian fibration $\mu' : \mathcal{X}'_0 := (\mu')^{-1}(B'_0) \to B'_0$. We may trivialize the fibration over each of the following open sets

$$U'_i := B'_0 \setminus \bigcup_{k \neq i} ([T'_k] \times \{0\}).$$

Without loss of generality we describe this explicit trivialization over U'_0. One can check that

$$[T'_0] = \{q \in E^{int} | (\underline{v}_j, q) \geq c_j - c_0, 1 \leq j \leq m-1\}.$$

So U'_0 can be described as

$$U'_0 = \{(q_1, q_2) \in E^{int} \times \mathbb{R}_{>-K_2} \mid q_2 \neq 0 \text{ or } (\underline{v}_j, q_1) > c_j - c_0,\ 1 \leq j \leq m-1\}. \tag{4.6}$$

A trivialization of μ' over U'_0 may be given in a way similar to Definition 4.11:

$$(\mu')^{-1}(U'_0) \cong U'_0 \times T^{\perp \boldsymbol{b}_0} \times (\mathbb{R}/2\pi\mathbb{Z}).$$

4.3.2. *Generators of homotopy groups.* Fix $r := (q_1, q_2) \in U'_0$ with $q_2 > 0$. Consider the fiber $F_r := (\mu')^{-1}(q_1, q_2)$. By the trivialization discussed above, we have $F_r \simeq T^{\perp \boldsymbol{b}_0} \times (\mathbb{R}/2\pi\mathbb{Z})$. Hence $\pi_1(F_r) \simeq N/\mathbb{Z}\langle \boldsymbol{b}_0 \rangle \times \mathbb{Z}$ has the following basis (over $\mathbb{Q}$)

$$\{\lambda_i \mid 0 \leq i \leq n-1\},$$

where $\lambda_0 = (0, 1)$ and $\lambda_i = ([\underline{v}_i], 0)$ for $1 \leq i \leq n-1$.

As mentioned in Section 3.1, for a regular Lagrangian torus fiber L of the moment map $\mathcal{X}' \to P'$, basic disk classes for a natural basis of $\pi_2(\mathcal{X}', L)$. We construct basis for $\pi_2(\mathcal{X}', F_r)$ by exhibiting a Lagrangian isotopy between F_r and L and using this natural basis of $\pi_2(\mathcal{X}', L)$. The following is an explicit Lagrangian isotopy between F_r and L:

$$L_t := \{x \in \mathcal{X}' \mid [\mu'_0(x)] = q_1,\ |w'(x) - t|^2 = K_2^2 + q_2\}. \tag{4.7}$$

This allows us to identify $\pi_2(\mathcal{X}', F_r)$ with $\pi_2(\mathcal{X}', L)$ and view basic disk classes in $\pi_2(\mathcal{X}, L)$ as classes in $\pi_2(\mathcal{X}, F_r)$. By abuse of notations, we denote these classes by $\beta_0, \ldots, \beta_{m-1}$, $\beta'_1, \ldots, \beta'_{n-1}$ and $\{\beta'_\nu \mid \nu \in \text{Box}'(\Sigma')\}$.

Remark 4.24. *For a general $r' \in B'_0$, a basis for $\pi_2(\mathcal{X}', F_{r'})$ may be obtained by identifying $F_{r'}$ with F_r using the trivialization mentioned above.*

The boundaries of the classes $\beta_0, \ldots, \beta_{m-1}$, $\beta'_1, \ldots, \beta'_{n-1}$ and $\{\beta'_\nu \mid \nu \in \text{Box}'(\Sigma')\}$ can be described as follows.

Lemma 4.25.

$$\begin{aligned}
\partial\beta_j &= \lambda_0 + \sum_{i=1}^{n-1} (\nu_i, \boldsymbol{b}_j)\lambda_i, \quad 0 \le j \le m-1 \\
\partial\beta'_k &= \lambda_k, \quad 1 \le k \le n-1 \\
\partial\beta'_\nu &= \lambda_0 + \sum_{i=1}^{n-1} c'_{\nu i}\lambda_i, \quad \nu = \sum_{i=0}^{n-1} c'_{\nu i}\underline{v}_i \in \mathrm{Box}'(\Sigma').
\end{aligned}$$

The intersection numbers of these basic disk classes with toric divisors can be described as follows.

Lemma 4.26.

$$\begin{aligned}
&\beta_0 \cdot D_j = 0, \quad 1 \le j \le m-1 \\
&\beta_i \cdot D_j = \delta_{ij}, \quad 1 \le i \le m-1, 1 \le j \le m-1 \\
&\beta_i \cdot D'_k = 0, \quad 1 \le i \le m-1, 1 \le k \le n-1 \\
&\beta_i \cdot \tilde{D}_0 = 1, \quad 0 \le i \le m-1 \\
&\beta'_l \cdot \tilde{D}_0 = 0, \quad 1 \le l \le n-1 \\
&\beta'_l \cdot \tilde{D}_k = \delta_{lk}, \quad 1 \le l \le n-1, 1 \le k \le n-1.
\end{aligned}$$

The intersection number of a basic orbi-disk class β'_ν with the above divisors can be computed from the above by expressing β'_ν as a linear combination of $\beta_0, \ldots, \beta_{m-1}$ and $\beta'_1, \ldots, \beta'_{n-1}$ with rational coefficients.

4.3.3. *Wall-crossing of orbi-disk invariants after modification.* The discussion of orbi-disk invariants of $(\mathcal{X}', F_r)$ is similar to the manifold case. Observe that the fiber $F_r \subset \mathcal{X}'$ is a special Lagrangian submanifold with respect to the meromorphic form $\Omega'/(w' - K_2)$, and the pole divisor of $\Omega'/(w' - K_2)$ is $\sum_{j=0}^{n-1} \tilde{D}_j$.

Lemma 4.27. *Let $r = (q_1, q_2) \in B'_0$.*

(1) The fiber F_r of μ' bounds a non-constant stable disk of Chern-Weil Maslov index 0 in $\mathcal{X}'$ if and only if $q_2 = 0$.
(2) If $q_2 \neq 0$, then the fiber F_r has minimal Chern-Weil Maslov index at least 2.

The wall of the fibration μ', which is the locus $H' \subset B'_0$ of all $r \in B'_0$ such that the fiber F_r bounds a non-constant stable disk of Chern-Weil Maslov index 0, may be described by

$$H' = E^{int} \times \{0\},$$

where E^{int} is given in (4.5). The complement $B'_0 \backslash H'$ is the union of two connected components

$$B'_+ := E^{int} \times (0, +\infty), \quad B'_- := E^{int} \times (-K_2, 0).$$

For $r \in B'_0 \backslash H'$, orbi-disk invariants with arbitrary numbers of age 1 insertions are well-defined for classes with Chern-Weil Maslov index 2. We need to consider the two possibilities, namely $r \in B'_+$ and $r \in B'_-$.

Case 1: $r \in B'_+$. Let $r = (q_1, q_2) \in B'_+$, namely $q_2 > 0$. Then (4.7) gives a Lagrangian isotopy between the fiber F_r and a regular Lagrangian torus fiber L. Furthermore, since $q_2 > 0$, for each $t \in [0, K_2]$, w is never 0 on L_t. It follows that L_t does not bound non-constant disks of Chern-Weil Maslov index 0. Hence for $r \in B'_+$, the orbi-disk invariants of $(\mathcal{X}', F_r)$ with arbitrary numbers of age-one insertions and Chern-Weil Maslov index 2 coincide with those of $(\mathcal{X}', L)$, which are reviewed in Section 3.2.

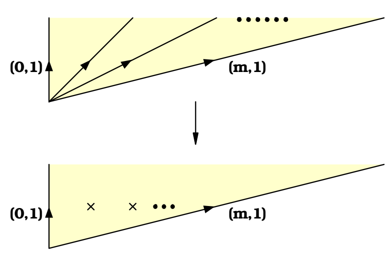
Case 2: $r \in B'_-$. In this case we have

Proposition 4.28. *Let $r = (q_1, q_2) \in B'_-$, namely $q_2 < 0$. Let $\beta \in \pi_2(\mathcal{X}', F_r)$. Suppose $\mathbf{1}_{\nu_1}, \ldots, \mathbf{1}_{\nu_l} \in H^*_{\mathrm{CR}}(\mathcal{X}')$ are fundamental classes of twisted sectors $\mathcal{X}'_{\nu_1}, \ldots, \mathcal{X}'_{\nu_l}$ such that* $\mathrm{age}(\nu_1) = \cdots = \mathrm{age}(\nu_l) = 1$. *Then we have*

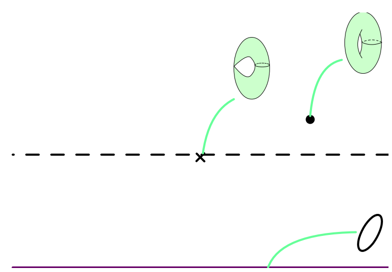
$$n^{\mathcal{X}'}_{1,l,\beta}([\mathrm{pt}]_{F_r}; \mathbf{1}_{\nu_1}, \ldots, \mathbf{1}_{\nu_l}) = \begin{cases} 1 & \text{if } \beta \in \{\beta_0, \beta'_1, \ldots, \beta'_{n-1}\} \text{ and } l = 0 \\ 0 & \text{otherwise .} \end{cases}$$

4.4. **Examples.**

(1) $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_m]$. This is known as the 2-dimensional A_{m-1} singularity. The stacky fan is a cone generated by $(0,1)$ and $(m,1)$ in $N = \mathbb{Z}^2$. See Figure 1a. By subdividing the cone by the rays generated by $(k,1)$ for $k = 1, \ldots, m-1$, one obtains a resolution of the singularity. The age-one twisted sectors of $\mathcal{X}$ are in a one-to-one correspondence with the lattice points $(k,1) \in \mathrm{Box}'$ for $k = 1, \ldots, n-1$. The Gross fibration and the wall of this orbifold is depicted in Figure 1b.



(A) Fan picture for $[\mathbb{C}^2/\mathbb{Z}_m]$ and its resolution. The crosses represent the twisted sectors.



(B) Gross fibration on $[\mathbb{C}^2/\mathbb{Z}_m]$. The dotted line is the wall at which fibers bound stable disks of Maslov index zero, and the cross is the discriminant locus at which the fiber degenerates.

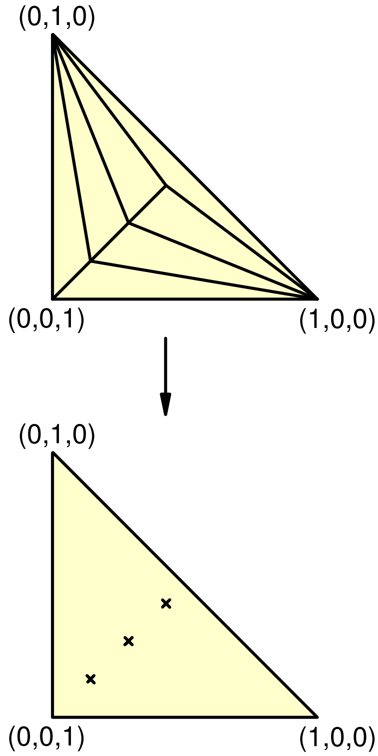
FIGURE 1. $[\mathbb{C}^2/\mathbb{Z}_m]$.

(2) $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_{2g+1}]$ for $g \in \mathbb{N}$. Let N be the lattice

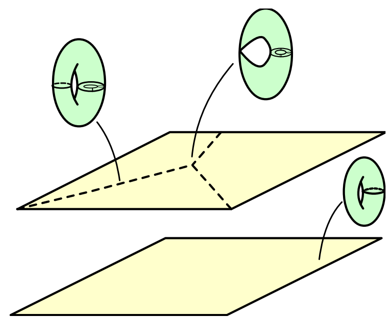
$$\mathbb{Z}^3 + \mathbb{Z}\left\langle \frac{(1,1,2g-1)}{2g+1} \right\rangle.$$

The stacky fan is a cone generated by $(1,0,0), (0,1,0), (0,0,1) \in N$, which is a cone over the convex hull of these 3 vectors in the hyperplane $\{(a,b,c) \in N_{\mathbb{R}} : a+b+c = 1\}$.

Using the triangulation of the polygon by the lattice points $(k, k, 2g+1-2k)/(2g+1)$ as depicted in Figure 2a, one obtains a resolution of the orbifold singularity, which is the mirror manifold of a Riemann surface of genus g (see [75, 41]).[3] The age-one twisted sectors of $\mathcal{X}$ are in a one-to-one correspondence with the lattice points $(k, k, 2g+1-2k)/(2g+1) \in \mathrm{Box}'$ for $k = 1, \ldots, g$. The Gross fibration and the wall of this orbifold is depicted in Figure 2b.



(A) $[\mathbb{C}^3/\mathbb{Z}_{2g+1}]$ and its resolution. Cone over the polytopes give the corresponding fans. The crosses represents the twisted sectors. This figure is for $g = 3$.



(B) Gross fibration on $[\mathbb{C}^3/\mathbb{Z}_{2g+1}]$. The base is an upper-half-space. The plane in the middle is the wall at which fibers bound stable disks of Maslov index zero. The dotted line and the plane in the bottom are the discriminant loci, with the singular fibers as shown in the diagram.

Figure 2. $[\mathbb{C}^3/\mathbb{Z}_{2g+1}]$.

(3) $\mathcal{X} = [\mathbb{C}^n/\mathbb{Z}_n]$ for $n \in \mathbb{Z}$. This gives an example in any dimension. The stacky fan is a cone generated by $(e_1, 1), \ldots, (e_n, 1), (-e_1 - \cdots - e_n, 1) \in N = \mathbb{Z}^n \times \mathbb{Z}$, where $\{e_i\}$ denotes the standard basis of $\mathbb{Z}^n$. One obtains a resolution of the orbifold singularity by subdividing the cone using the ray generated by $(0, 1) \in N$, and the resulting manifold is the total space of canonical line bundle over $\mathbb{P}^n$. There is only one age-one twisted sector, namely the lattice point $(0, 1) \in \mathrm{Box}'$. The Gross fibration and the wall of this orbifold is similar to that depicted in Figure 2b in dimension 3.

[3] The mirror of a Riemann surface of genus g is a Landau-Ginzburg model, which is a holomorphic function defined on the manifold described here [75, 41].

5. SYZ mirror construction

In this section we carry out the SYZ mirror construction for toric Calabi-Yau orbifolds. The procedure may be summarized as follows. Let $\mathcal{X}$ be a toric Calabi-Yau orbifold as in Setting 4.3, and let $\mathcal{X}'$ be its toric modification introduced in Definition 4.17. Let $\mu : \mathcal{X} \to B$ and $\mu' : \mathcal{X}' \to B'$ be the Gross fibrations introduced in Definition 4.7 and Definition 4.20 respectively.

Step 1. Consider the torus bundle $\mu' : \mathcal{X}'_0 \to B'_0$. Take the dual torus bundle $\check{\mu}' : \check{\mathcal{X}}'_0 \to B'_0$. The total space $\check{\mathcal{X}}'_0$ together with its canonical complex structure is called the *semi-flat mirror* of $\mathcal{X}$. The problem with the semi-flat mirror is that its complex structure is not extendable to any partial compactification of $\check{\mathcal{X}}'_0$ because monodromy of the integral affine structure around the discriminant loci in B'_0 is nontrivial.

Step 2. Construct instanton corrections to the semi-flat complex coordinates by taking family Fourier transforms of generating functions of genus 0 open orbifold Gromov-Witten invariants which count (virtually) orbi-disks with the minimal Chern-Weil Maslov index (which is 2). The wall-crossing of orbi-disk counting we discuss in the previous section modifies the gluing between charts in $\check{\mathcal{X}}'_0$ and resolves the nontrivial monodromy of the affine structure so that the complex structure becomes extendable.

Step 3. (Partially) compactifying the resulting geometry to obtain the mirror.

This procedure was pioneered by Auroux in [3, 4], and was generalized to all toric Calabi-Yau manifolds in [20]; see also the recent work of Abouzaid-Auroux-Katzarkov [1]. We are going to carry out this construction for toric Calabi-Yau orbifolds in the remainder of this section.

5.1. **The semi-flat mirror.**

We construct the semi-flat mirror of $\mathcal{X}$ as follows. Consider the torus bundle $\mu' : \mathcal{X}'_0 := (\mu')^{-1}(B'_0) \to B'_0$. Let $\check{\mathcal{X}}'_0$ be the space of pairs (F_r, ∇), where $F_r := (\mu')^{-1}(r), r \in B'_0$ and ∇ is a flat $U(1)$-connection on the trivial complex line bundle over F_r up to gauge. There is a natural projection map $\check{\mu}' : \check{\mathcal{X}}'_0 \to B'_0$. We write $\check{F}_r := \check{\mu}'^{-1}(r)$ for $r \in B'_0$. According to [20, Proposition 2.5], $\check{\mu}' : \check{\mathcal{X}}'_0 \to B'_0$ is a torus bundle.

Recall that B'_0 has an open cover $\{U'_i\}$. Let $U' := U'_0 \subset B'_0$ be the open set described in (4.6). We describe the semi-flat complex coordinates on the chart $\check{\mu}'^{-1}(U')$. Fix a base point $r_0 \in U'$. For $r \in U'$, consider the class $\lambda_i \in \pi_1(F_r)$ defined in Section 4.3.2. Define cylinder classes

$$[h_i(r)] \in \pi_2((\mu')^{-1}(U'), F_{r_0}, F_r)$$

as follows. Recall the following trivialization defined in Section 4.3.1:

$$(\mu')^{-1}(U') \cong U' \times T^{\perp \boldsymbol{b}_0} \times (\mathbb{R}/2\pi\mathbb{Z}).$$

Pick a path $\gamma : [0,1] \to U'$ with $\gamma(0) = r_0$ and $\gamma(1) = r$. For $j = 1, \ldots, n-1$, define

$$h_j : [0,1] \times \mathbb{R}/\mathbb{Z} \to U' \times T^{\perp \boldsymbol{b}_0} \times (\mathbb{R}/2\pi\mathbb{Z}), \quad h_j(R, \Theta) := \left(\gamma(R), \frac{\Theta}{2\pi}[\underline{v}_j], 0\right),$$

also define

$$h_0 : [0,1] \times \mathbb{R}/\mathbb{Z} \to U' \times T^{\perp \boldsymbol{b}_0} \times (\mathbb{R}/2\pi\mathbb{Z}), \quad h_0(R,\Theta) := (\gamma(R), 0, 2\pi\Theta).$$

The classes $[h_i(r)]$ are independent of the choice of γ. Now the semi-flat complex coordinates of $(\mu')^{-1}(U')$ are $z_0, z_1, \dots, z_{n-1}$ where

$$z_i(F_r, \nabla) := \exp(\rho_i + 2\pi\sqrt{-1}\check{\theta}_i), \tag{5.1}$$

where $\exp(2\pi\sqrt{-1}\check{\theta}_i) := \mathrm{Hol}_\nabla(\lambda_i(r))$ and $\rho_i := -\int_{[h_i(r)]} w$. The semi-flat holomorphic volume form is the following nowhere vanishing form on $(\mu')^{-1}(U')$:

$$dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n-1} \wedge dz_0.$$

Semi-flat complex coordinates on the other charts $\check{\mu}'^{-1}(U'_j)$ can be similarly described.

5.2. **Instanton corrections.** Let $0 \le i \le n-1$. The instanton corrections of the semi-flat complex coordinate z_i are obtained by taking a family version of Fourier transformations of generating functions of genus 0 open orbifold Gromov-Witten invariants which count orbi-disks with Chern-Weil Maslov index 2. The result is a complex-valued function

$$\tilde{z}_i : (\check{\mu}')^{-1}(B'_0 \setminus H') \to \mathbb{C}.$$

For $(F_r, \nabla) \in (\check{\mu}')^{-1}(B'_0 \setminus H')$, the value of $\tilde{z}_i$ is schematically given by

$$\tilde{z}_i = \sum_{\beta \in \pi_2(\mathcal{X}', F_r)} \sum_{l \ge 0} \frac{1}{l!} (\beta \cdot \tilde{D}_i) n^{\mathcal{X}'}_{1,l,\beta}([\mathrm{pt}]_{F_r}; \tau, \dots, \tau) \exp\left(-\int_\beta \omega\right) \mathrm{Hol}_\nabla(\partial\beta) \tag{5.2}$$

where $\tau \in H^*_{\mathrm{CR}}(\mathcal{X}) \subset H^*_{\mathrm{CR}}(\mathcal{X}')$ and $\mu_{CW}(\beta) = 2$.

We consider the class

$$\tau = \sum_i \tau_{\nu_i} \mathbf{1}_{\nu_i} \in H^2_{\mathrm{CR}}(\mathcal{X}) \subset H^2_{\mathrm{CR}}(\mathcal{X}'),$$

which is a linear combination of fundamental classes of age-one twisted sectors ν_i of $\mathcal{X}$. By the discussion in Section 4.3.3, we know that the above genus 0 open orbifold Gromov-Witten invariants $n^{\mathcal{X}'}_{1,l,\beta}([\mathrm{pt}]_{F_r}; \tau, \dots, \tau)$ vanish except in one of the following situations:

(1) $\beta = \beta'_j$ for some $1 \le j \le n-1$;
(2) $\beta = \beta_k + \alpha$ for some $0 \le k \le m-1$ and $\alpha \in H_2(\mathcal{X}')$ has Chern number 0 (which implies $\alpha \in H_2(\mathcal{X})$);
(3) $\beta = \beta_\nu + \alpha$ for some $\nu \in \mathrm{Box}'(\Sigma)$ of age 1 and $\alpha \in H_2(\mathcal{X}')$ has Chern number 0.

First we consider $\tilde{z}_i$, $1 \le i \le n-1$. For each $1 \le i \le n-1$, we have the following observations:

(1) $\beta'_j \cdot \tilde{D}_i = \delta_{ji}$ for any $1 \le j \le n-1$;
(2) $(\beta_k + \alpha) \cdot \tilde{D}_i = 0$ for $0 \le k \le m-1$ and $\alpha \in H_2(\mathcal{X})$ with Chern number 0, by Lemma 4.26;
(3) $(\beta_\nu + \alpha) \cdot \tilde{D}_i = 0$ for $\nu \in \mathrm{Box}'(\Sigma)$ of age 1 and $\alpha \in H_2(\mathcal{X})$ with Chern number 0, because β_ν can be written as a linear combination of $\beta_0, \dots, \beta_{m-1}$ with coefficients in $\mathbb{Q}$.

Therefore, only the class β'_i contributes to $\tilde{z}_i$ and (5.2) becomes

$$\begin{aligned}
\tilde{z}_i =&(\beta'_i \cdot \tilde{D}_i) n_{1,0,\beta'_i} \exp\left(-\int_{\beta'_i}\omega\right)\mathrm{Hol}_\nabla(\partial\beta'_i)\\
=&\exp\left(-\int_{\beta'_i}\omega\right)\mathrm{Hol}_\nabla(\partial\beta'_i) \qquad (\text{because } \beta'_i\cdot\tilde{D}_i = 1, n_{\beta'_i}=1)\\
=&\exp\left(-\int_{\beta'_i(r)}\omega\right)\mathrm{Hol}_\nabla(\lambda_i(r)) \qquad (\text{because } \partial\beta'_i = \lambda_i)\\
=&\exp\left(-\int_{\beta'_i(r)}\omega\right)\exp\left(\int_{[h_i(r)]}\omega\right) z_i \qquad (\text{by the definition of } z_i)\\
=&\exp\left(-\int_{\beta'_i(r_0)}\omega\right) z_i \qquad (\text{because } [h_i(r)] = \beta'_i(r) - \beta'_i(r_0)).
\end{aligned}$$

To simplify notations, we put $C'_i := \exp\left(-\int_{\beta'_i(r_0)}\omega\right)$.

The situation for $\tilde{z}_0$ is more complicated, as it depends on the chamber in the decomposition $B'_0 \setminus H' = B'_+ \cup B'_-$ to which the image of the Lagrangian torus fiber belongs.

When $r \in B'_-$, Proposition 4.28 shows that the only non-vanishing genus 0 open Gromov-Witten invariants are $n_{1,0,\beta} = 1$ where $\beta = \beta_0$ or $\beta'_1, \ldots, \beta'_{n-1}$. On the other hand, we have $\beta_0 \cdot \tilde{D}_0 = 1$, $\beta'_i \cdot \tilde{D}_0 = 0$ for $i = 1, \ldots, n-1$. Therefore again (5.2) only has one term:

$$\begin{aligned}
\tilde{z}_0 =&(\beta_0 \cdot \tilde{D}_0) n_{1,0,\beta_0} \exp\left(-\int_{\beta_0(r)}\omega\right)\mathrm{Hol}_\nabla(\partial\beta_0)\\
=&\exp\left(-\int_{\beta_0(r)}\omega\right)\mathrm{Hol}_\nabla(\partial\beta_0) \qquad (\text{because } \beta_0\cdot\tilde{D}_0 = 1, n_{1,0,\beta_0}=1)\\
=&\exp\left(-\int_{\beta_0(r)}\omega\right)\mathrm{Hol}_\nabla(\lambda_0(r)) \qquad (\text{because } \partial\beta_0 = \lambda_0)\\
=&\exp\left(-\int_{\beta_0(r)}\omega\right)\exp\left(\int_{[h_0(r)]}\omega\right) z_0 \qquad (\text{by the definition of } z_0)\\
=&\exp\left(-\int_{\beta_0(r_0)}\omega\right) z_0 \qquad (\text{because } [h_0(r)] = \beta_0(r) - \beta_0(r_0)).
\end{aligned}$$

Again, to simplify notation, we put $C_0 := \exp\left(-\int_{\beta_0(r_0)}\omega\right)$.

We then consider the case when $r \in B'_+$. Since $\beta'_l \cdot \tilde{D}_0 = 0$ for $1 \le l \le n-1$, open orbifold Gromov-Witten invariants in class β'_l do not contribute to (5.2). On the other hand, given $\alpha \in H_2(\mathcal{X}')$ with Chern number 0, we have $(\beta_i + \alpha) \cdot \tilde{D}_0 = 1$ for $0 \le i \le m-1$ and

$(\beta'_\nu + \alpha) \cdot \tilde{D}_0 = \mathrm{age}(\nu) = 1$ for $\nu \in \mathrm{Box}'(\Sigma)$ with $\mathrm{age}(\nu) = 1$. Therefore (5.2) reads

$$
\begin{aligned}
\tilde{z}_0 =& \sum_{j=0}^{m-1} \sum_{\alpha} \sum_{l \geq 0} \sum_{\nu_1, \ldots, \nu_l \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} \frac{\prod_{i=1}^{l} \tau_{\nu_i}}{l!} n_{1,l,\beta_j(r)+\alpha}([\mathrm{pt}]_{F_r}; \prod_{i=1}^{l} \mathbf{1}_{\nu_i}) \\
& \qquad \times \exp\left(-\int_{\beta_j(r)+\alpha} \omega\right) \mathrm{Hol}_\nabla(\partial \beta_j(r)) \\
& + \sum_{\nu \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} \sum_{\alpha} \sum_{l \geq 0} \sum_{\nu_1, \ldots, \nu_l \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} \frac{\prod_{i=1}^{l} \tau_{\nu_i}}{l!} n_{1,l,\beta'_\nu(r)+\alpha}([\mathrm{pt}]_{F_r}; \prod_{i=1}^{l} \mathbf{1}_{\nu_i}) \\
& \qquad \times \exp\left(-\int_{\beta'_\nu(r)+\alpha} \omega\right) \mathrm{Hol}_\nabla(\partial \beta'_\nu(r)) \\
=& \sum_{j=0}^{m-1} (1+\delta_j) \exp\left(-\int_{\beta_j(r_0)} \omega - \int_{[h_0(r)]} \omega - \sum_{i=1}^{n-1} (\nu_i, \boldsymbol{b}_j) \int_{[h_i(r)]} \omega\right) \\
& \qquad \times \mathrm{Hol}_\nabla\left(\lambda_0 + \sum_{i=1}^{n-1} (\nu_i, \boldsymbol{b}_j) \lambda_i\right) \\
& + \sum_{\nu \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} (\tau_\nu + \delta_\nu) \exp\left(-\int_{\beta_\nu(r_0)} \omega - \int_{[h_0(r)]} \omega - \sum_{i=1}^{n-1} (\nu_i, \nu) \int_{[h_i(r)]} \omega\right) \\
& \qquad \times \mathrm{Hol}_\nabla\left(\lambda_0 + \sum_{i=1}^{n-1} (\nu_i, \nu) \lambda_i\right) \\
=& z_0 \sum_{j=0}^{m-1} C_j (1+\delta_j) \prod_{i=1}^{n-1} z_i^{(\nu_i, \boldsymbol{b}_j)} + z_0 \sum_{\nu \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} C_\nu (\tau_\nu + \delta_\nu) \prod_{i=1}^{n-1} z_i^{(\nu_i, \nu)},
\end{aligned}
$$

where

$$
\begin{aligned}
C_j &:= \exp\left(-\int_{\beta_j(r_0)} \omega\right), \quad 0 \leq j \leq m-1, \\
C_\nu &:= \exp\left(-\int_{\beta_\nu(r_0)} \omega\right), \quad \nu \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1},
\end{aligned}
$$

and

$$
\begin{aligned}
& 1+\delta_j := \sum_{\alpha} \sum_{l \geq 0} \sum_{\nu_1, \ldots, \nu_l \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} \frac{\prod_{i=1}^{l} \tau_{\nu_i}}{l!} n_{1,l,\beta_j(r)+\alpha}([\mathrm{pt}]_L; \prod_{i=1}^{l} \mathbf{1}_{\nu_i}) \exp\left(-\int_\alpha \omega\right), \\
& (0 \leq j \leq m-1), \\
& \tau_\nu + \delta_\nu := \sum_{\alpha} \sum_{l \geq 0} \sum_{\nu_1, \ldots, \nu_l \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} \frac{\prod_{i=1}^{l} \tau_{\nu_i}}{l!} n_{1,l,\beta_\nu(r)+\alpha}([\mathrm{pt}]_L; \prod_{i=1}^{l} \mathbf{1}_{\nu_i}) \exp\left(-\int_\alpha \omega\right), \\
& (\nu \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1})
\end{aligned} \tag{5.3}
$$

are generating functions of genus 0 open orbifold Gromov-Witten invariants. Here we use the relation

$$-\beta_j(r) = -\beta_j(r_0) - [h_0(r)] - \sum_{i=1}^{n-1} (\nu_i, \boldsymbol{b}_j)[h_i(r)].$$

Also, the generating functions can be written as in the left-hand-sides of (5.3) because

$$n_{1,0,\beta_j(r)}([\mathrm{pt}]_L) = n_{1,1,\beta_\nu(r)}([\mathrm{pt}]_L; \mathbf{1}_\nu) = 1$$

for any j and ν. Notice that $n_{1,l,\beta_\nu(r)+\alpha}([\mathrm{pt}]_L; \prod_{i=1}^l \mathbf{1}_{\nu_i})$ is nonzero only when $l \geq 1$, so the generating function $\tau_\nu + \delta_\nu$ has no constant term.

The above discussion may be summarized as follows. For $0 \leq j \leq m-1$ and $\nu \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}$ we put $z^{\boldsymbol{b}_j} := \prod_{i=1}^{n-1} z_i^{(\nu_i, \boldsymbol{b}_j)}$ and $z^\nu := \prod_{i=1}^{n-1} z_i^{(\nu_i,\nu)}$.

Proposition 5.1.

(1) For $1 \leq i \leq n-1$, we have

$$\tilde{z}_i = C_i' z_i,$$

(2) For $r \in B_+'$, we have

$$\tilde{z}_0 = z_0 \sum_{j=0}^{m-1} C_j(1+\delta_j) z^{\boldsymbol{b}_j} + z_0 \sum_{\nu \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} C_\nu(\tau_\nu + \delta_\nu) z^\nu,$$

and for $r \in B_-'$, we have

$$\tilde{z}_0 = C_0 z_0.$$

5.3. **The mirror.** Let $\mathbb{C}[[q,\tau]]$ be the ring of formal power series in the variables

$$\{q_1, \ldots, q_{r'}\} \cup \{\tau_\nu \mid \nu \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}\},$$

which are parameters in the complexified extended Kähler moduli space of $\mathcal{X}$ (see Section 7.1.1 for precise definitions of these parameters) with coefficients in $\mathbb{C}$. Consider $R_+ = R_- := \mathbb{C}[[q,\tau]][z_0^\pm, \ldots, z_{n-1}^\pm]$. Let R_0 be the localization of $\mathbb{C}[[q,\tau]][z_0^\pm, \ldots, z_{n-1}^\pm]$ at

$$g := \sum_{j=0}^{m-1} C_j(1+\delta_j) z^{\boldsymbol{b}_j} + \sum_{\nu \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} C_\nu(\tau_\nu + \delta_\nu) z^\nu.$$

Let $[Id] : R_- \to R_0$ be the localization map. Also define $R_+ \to R_0$ by $z_k \mapsto [z_k]$ for $k = 1, \ldots, n-1$ and $z_0 \mapsto [g^{-1} z_0]$.

Using these two maps, we define

$$R := R_- \times_{R_0} R_+.$$

We identify $\tilde{z}_0$ with $u := (C_0 z_0, z_0 g) \in R$. For $j = 1, \ldots, n-1$, we identify $\tilde{z}_j$ with $(C_j' z_j, C_j' z_j) \in R$. Put

$$v := (C_0^{-1} z_0^{-1} g, z_0^{-1}) \in R.$$

Then we have

$$R \simeq \mathbb{C}[[q,\tau]][u^\pm, v^\pm, z_1^\pm, \ldots, z_{n-1}^\pm]/\langle uv - g \rangle.$$

The relative spectrum $\mathrm{Spec}(R)$ over $\mathbb{C}[[q,\tau]]$ can be described as

$$\{(u,v,z_1,\ldots,z_{n-1}) \in (\mathrm{Spec}(\mathbb{C}[[q,\tau]][u^{\pm},v^{\pm}]))^2 \times (\mathrm{Spec}(\mathbb{C}[[q,\tau]][z_1,\ldots,z_{n-1}]))^{n-1}) \mid uv = g(z_1,\ldots,z_{n-1})\},$$

which admits an obvious partial compactification

$$\check{\mathcal{X}} := \{(u,v,z_1,\ldots,z_{n-1}) \in (\mathrm{Spec}(\mathbb{C}[[q,\tau]][u,v]))^2 \times (\mathrm{Spec}(\mathbb{C}[[q,\tau]][z_1,\ldots,z_{n-1}]))^{n-1}) \mid uv = g(z_1,\ldots,z_{n-1})\}.$$

This gives the SYZ mirror of *the complement of the hypersurface* $\{w(x) = K_2\}$ *in* $\mathcal{X}$. The SYZ mirror of the toric Calabi-Yau orbifold $\mathcal{X}$ itself is given by the *Landau-Ginzburg model* $(\check{\mathcal{X}}, W)$, where $W : \check{\mathcal{X}} \to \mathbb{C}$ is the Fourier transformation of the generating function orbi-disk invariants for classes with Chern-Weil Maslov index 2, which is simply the holomorphic function defined by $W := u$; see Chan-Lau-Leung [20, Section 4.6] and Abouzaid-Auroux-Katzarkov [1, Section 7] for related discussions in the manifold case.

There is a canonical map

$$\rho_0 : \check{\mu}^{-1}(B_0 \setminus H) \to \check{\mathcal{X}} \tag{5.4}$$

given by

$$u := \begin{cases} C_0 z_0 & \text{on } (\check{\mu}')^{-1}(B_-) \\ z_0 g & \text{on } (\check{\mu}')^{-1}(B_+). \end{cases}$$

$$v := \begin{cases} C_0^{-1} z_0^{-1} g & \text{on } (\check{\mu}')^{-1}(B_-) \\ z_0^{-1} & \text{on } (\check{\mu}')^{-1}(B_+). \end{cases}$$

Proposition 5.2. *There exists a coordinate change such that under the new coordinates the defining equation* $uv = g$ *of* $\check{\mathcal{X}}$ *can be written as*

$$uv = (1+\delta_0) + \sum_{j=1}^{n-1}(1+\delta_j)z_j + \sum_{j=n}^{m-1}(1+\delta_j)q_j z^{\boldsymbol{b}_j} + \sum_{\nu \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} (\tau_\nu + \delta_\nu) q^{-D_\nu^\vee} z^\nu,$$

where for $j = n,\ldots,m-1$, $q_j := q^{\xi_j}$ *and* $\xi_j \in H_2(\mathcal{X};\mathbb{Q})$ *is the class defined by* $\boldsymbol{b}_j = \sum_{i=0}^{n-1} a_{ji}\boldsymbol{b}_i$, *while* $q^{-D_\nu^\vee} := \prod_{a=1}^{r'} q_a^{-\langle p_a, D_\nu^\vee\rangle}$ *for* $\nu \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}$.

Proof. We need to introduce a new set of coordinates $\hat{z}_0,\ldots,\hat{z}_{n-1}$ such that

$$C_j z^{\boldsymbol{b}_j} = C_0 \hat{z}_j, \quad j = 0,\ldots,n-1,$$

where $z^{\boldsymbol{b}_j} = \prod_{i=0}^{n-1} z_i^{(\nu_i,\boldsymbol{b}_j)}$. Since $\boldsymbol{b}_0,\ldots,\boldsymbol{b}_{n-1}$ is a basis of $N_{\mathbb{Q}}$, the $n\times n$ matrix with entries $(\nu_i,\boldsymbol{b}_j)$ is invertible. Hence the system

$$\log C_0 + \log \hat{z}_j = \log C_j + \sum_{i=0}^{n-1}(\nu_i,\boldsymbol{b}_j)\log z_i, \quad j = 0,\ldots,n-1$$

may be solved to express $\{\log z_0,\ldots,\log z_{n-1}\}$ in terms of $\{\log \hat{z}_0,\ldots,\log \hat{z}_{n-1}\}$. Hence the coordinates $\hat{z}_0,\ldots,\hat{z}_{n-1}$ exist.

For $j = n, \dots, m-1$, we can write $\boldsymbol{b}_j = \sum_{i=0}^{n-1} a_{ji} \boldsymbol{b}_i$. Then we have

$$z^{\boldsymbol{b}_j} = z^{\sum_{i=0}^{n-1} a_{ji} \boldsymbol{b}_i} = \prod_{i=0}^{n-1} \left(\frac{C_0}{C_i} \hat{z}_i \right)^{a_{ji}} = C_0^{\sum_{i=0}^{n-1} a_{ji}} \prod_{i=0}^{n-1} \hat{z}_i^{a_{ji}} \left(\prod_{i=0}^{n-1} C_i^{a_{ji}} \right)^{-1}.$$

We put $\hat{z}^{\boldsymbol{b}_j} := \prod_{i=0}^{n-1} \hat{z}_i^{a_{ji}}$. Applying $(-, \underline{v})$ to $\boldsymbol{b}_j = \sum_{i=0}^{n-1} a_{ji} \boldsymbol{b}_i$ gives $\sum_{i=0}^{n-1} a_{ji} = 1$. Also,

$$\prod_{i=0}^{n-1} C_i^{a_{ji}} = \exp \left(- \int_{\sum_{i=0}^{n-1} a_{ji} \beta_i(r_0)} \omega \right).$$

Therefore

$$C_j z^{\boldsymbol{b}_j} = C_0 q_j \hat{z}^{\boldsymbol{b}_j}, \quad \text{where } q_j = \exp \left(- \int_{\beta_j(r_0) - \sum_{i=0}^{n-1} a_{ji} \beta_i(r_0)} \omega \right).$$

For $\nu = \sum_{j=0}^{n-1} c_{\nu j} \boldsymbol{b}_j \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}$, we have

$$\begin{aligned} z^{\nu} &= z^{\sum_{j=0}^{n-1} c_{\nu j} \boldsymbol{b}_j} = \prod_{j=0}^{n-1} (z^{\boldsymbol{b}_j})^{c_{\nu j}} \\ &= \left(\prod_{j=0}^{n-1} (C_0 \hat{z}_j)^{c_{\nu j}} \right) \left(\prod_{j=0}^{n-1} C_j^{c_{\nu j}} \right)^{-1} \\ &= C_0^{\sum_{j=0}^{n-1} c_{\nu j}} \prod_{j=0}^{n-1} \hat{z}_j^{c_{\nu j}} \left(\prod_{j=0}^{n-1} C_j^{c_{\nu j}} \right)^{-1} \\ &= C_0 C_\nu^{-1} q^{-D_\nu^\vee} \hat{z}^{\nu}, \end{aligned}$$

where we define $\hat{z}^{\nu} := \prod_{j=0}^{n-1} \hat{z}_j^{c_{\nu j}}$ and use the following calculations and notations:

$$\sum_{j=0}^{n-1} c_{\nu j} = 1, \ \prod_{j=0}^{n-1} C_j^{c_{\nu j}} = \exp \left(- \int_{\sum_{j=0}^{n-1} c_{\nu j} \beta_j(r_0)} \omega \right) = C_\nu q^{-D_\nu^\vee - 1},$$

$$q^{-D_\nu^\vee} = \exp \left(- \int_{\beta_\nu(r_0) - \sum_{j=0}^{n-1} c_{\nu j} \beta_j(r_0)} \omega \right).$$

Therefore we have

$$C_\nu z^{\nu} = C_0 q^{-D_\nu^\vee} \hat{z}^{\nu}, \quad \nu = \sum_{j=0}^{n-1} c_{\nu j} \boldsymbol{b}_j \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}.$$

Now put $\hat{u} := u / C_0$. Then $uv = g$ is transformed into

$$\hat{u} v = (1 + \delta_0) + \sum_{j=1}^{n-1} (1 + \delta_j) \hat{z}_j + \sum_{j=n}^{m-1} (1 + \delta_j) q_j \hat{z}^{\boldsymbol{b}_j} + \sum_{\nu \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} (\tau_\nu + \delta_\nu) q^{-D_\nu^\vee} \hat{z}^{\nu}.$$

□

Composing the canonical map ρ_0 in (5.4) with the coordinate change in Proposition 5.2 yields a map

$$\rho : \check{\mu}^{-1}(B_0 \setminus H) \to \check{\mathcal{X}}$$

given by

$$u := \begin{cases} z_0 & \text{on } (\check{\mu}')^{-1}(B_-) \\ z_0 G & \text{on } (\check{\mu}')^{-1}(B_+). \end{cases}$$

$$v := \begin{cases} z_0^{-1} G & \text{on } (\check{\mu}')^{-1}(B_-) \\ z_0^{-1} & \text{on } (\check{\mu}')^{-1}(B_+), \end{cases}$$

where

$$G(z_1, \ldots, z_{n-1}) := (1+\delta_0) + \sum_{j=1}^{n-1}(1+\delta_j) z_j + \sum_{j=n}^{m-1}(1+\delta_j) q_j z^{\boldsymbol{b}_j} + \sum_{\nu \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} (\tau_\nu + \delta_\nu) q^{-D_\nu^\vee} z^\nu.$$

Proposition 5.3. *There exists a holomorphic volume form $\check{\Omega}$ on $\check{\mathcal{X}}$ such that*

$$\rho^* \check{\Omega} = d\log z_0 \wedge \cdots \wedge d\log z_{n-1} \wedge du \wedge dv.$$

More precisely, in coordinates, we have

$$\check{\Omega} = \mathrm{Res}\left(\frac{1}{uv - G(z_1, \ldots, z_{n-1})} d\log z_0 \wedge \cdots \wedge d\log z_{n-1} \wedge du \wedge dv \right).$$

Proof. The proof is similar to the proof of the analogous statement in the manifold case [20, Proposition 4.44] and is omitted. □

Remark 5.4 (Dependence on choices)**.** *The construction of the mirror $\check{\mathcal{X}}$ depends on the choice of an integral basis in Definition 4.10. By arguments similar to those in* [20, Section 4.6.5] *it is straightforward to check that different choices yield the same mirror manifold $\check{\mathcal{X}}$ up to biholomorphisms which preserve the holomorphic volume form $\check{\Omega}$. We omit the details.*

Remark 5.5 (Convergence)**.** *A priori the Kähler parameters q_a's and the variables τ_ν's keeping track of stacky insertions in the generating functions* (5.3) *are only formal. However in our case they are not formal, since the generating functions can be shown to be convergent, see Corollary 6.22 below.*

5.4. **Examples.**

(1) $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_m]$. The stacky fan and Gross fibration are shown in Figure 1a and 1b respectively. It has $m-1$ twisted sectors of age one which are in one-to-one correspondence with the vectors $\nu_i = (i, 1)$ for $i = 1, \ldots, m-1$. Each twisted sector ν_i has a corresponding basic orbi-disk class β_{ν_i}.

The SYZ mirror constructed in this section is

$$uv = 1 + z^m + \sum_{j=1}^{m-1} (\tau_j + \delta_{\nu_j}(\tau)) z^j \tag{5.5}$$

where

$$\tau_j + \delta_{\nu_j}(\tau) = \sum_{k_1, \ldots, k_{m-1} \geq 0} \frac{\tau_1^{k_1} \cdots \tau_{m-1}^{k_{m-1}}}{(k_1 + \ldots + k_{m-1})!} n_{1,l,\beta_{\nu_j}}([\mathrm{pt}]_L; (\mathbf{1}_{\nu_1})^{k_1} \times \ldots \times (\mathbf{1}_{\nu_{m-1}})^{k_{m-1}}),$$

$l = k_1 + \ldots + k_g$ and $\tau = \sum_{i=1}^{m-1} \tau_i \mathbf{1}_{\nu_i} \in H^2_{\mathrm{CR}}(\mathcal{X})$. All Kähler parameters τ_i are contributed from twisted sectors in this case, and the non-triviality of the orbi-disk invariants is also due to the presence of twisted sectors.

The A_{m-1} singularity $X = \mathbb{C}^2/\mathbb{Z}_m$ has a resolution $\tilde{X}$ whose fan and Gross fibration are shown in Figure 1a and 1b. It has $m-1$ irreducible (-2) curves l_i's which have Chern number zero, and they are in one-to-one correspondence with the primitive generators $(i,1)$, $i = 1, \ldots, m-1$.

The SYZ mirror of the resolution $\tilde{X}$ is

$$uv = 1 + z^m + \sum_{j=1}^{m-1} (1 + \delta_j(q)) z^j \tag{5.6}$$

where

$$1 + \delta_j(q) = \sum_{k_1,\ldots,k_{m-1} \geq 0} n_{1,0,\beta_j+\alpha_k} q^{\alpha_k}$$

and $\alpha_k = \sum_{i=1}^{m-1} k_i l_i$ in the above expression. The Kähler parameters q^{l_i}'s are given by $\exp(-\int_{l_i} \omega)$, and the non-triviality of the disk invariants is due to the presence of rational curves of Chern number zero. The SYZ mirror construction for toric Calabi-Yau surfaces $\tilde{X}$ has been studied in [84], where δ_j has been computed explicitly.

(2) $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_{2g+1}]$ for $g \in \mathbb{N}$. See Figure 2a and 2b for the fan and Gross fibration. It has g twisted sectors of age one which are in one-to-one correspondence with the vectors $\nu_i = (i, i, 2g+1-2i)/(2g+1) \in N$ for $i = 1, \ldots, g$.

Let z_1 be the affine complex coordinate corresponding to the vector $(1, 0, -1) \in N$, z_2 to $(1, 1, -2)/(2g+1)$ and u to $(0, 0, 1)$. Then the SYZ mirror of $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_{2g+1}]$ is

$$uv = 1 + z_1 + z_1^{-1} z_2^{2g+1} + \sum_{j=1}^{g} (\tau_j + \delta_{\nu_j}(\tau)) z_2^j$$

where

$$\tau_j + \delta_{\nu_j}(\tau) = \sum_{k_1,\ldots,k_g \geq 0} \frac{\tau_1^{k_1} \ldots \tau_g^{k_g}}{(k_1 + \ldots + k_g)!} n_{1,l,\beta_{\nu_j}}([\mathrm{pt}]_L; (\mathbf{1}_{\nu_1})^{k_1} \times \ldots \times (\mathbf{1}_{\nu_g})^{k_g}),$$

$l = k_1 + \ldots + k_g$ and $\tau = \sum_{i=1}^{g} \tau_i \mathbf{1}_{\nu_i} \in H^2_{\mathrm{CR}}(\mathcal{X})$.

The orbifold $X = \mathbb{C}^3/\mathbb{Z}_{2g+1}$ has a toric resolution $\tilde{X}$. Figure 3 shows the codimension-two skeleta of its moment map polytope, which is also the discriminant locus of Gross fibration. Its Mori cone of effective curve classes is generated by $C_1, \ldots, C_g$ as shown in Figure 3. The SYZ mirror of the resolution $\tilde{X}$ is

$$uv = 1 + z_1 + q^{\sum_{i=1}^{g}(2i-1)C_i} z_1^{-1} z_2^{2g+1} + \sum_{j=1}^{g} (1 + \delta_j(q)) q^{\sum_{i=0}^{j-2}(j-1-i)C_{g-i}} z_2^j$$

where

$$1 + \delta_j(q) = \sum_{k_1,\ldots,k_g \geq 0} n_{1,0,\beta_j+\alpha_k} q^{\alpha_k},$$

$\alpha_k = \sum_{i=1}^{g} k_i C_i$, and β_j is the basic disk class corresponding to the toric divisor D_j as shown in Figure 3.

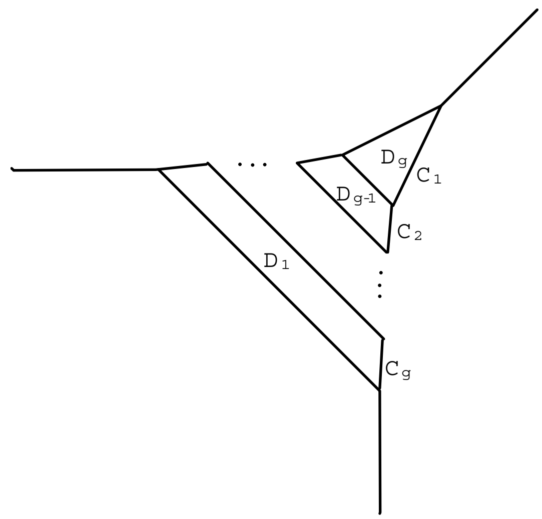


FIGURE 3. A toric resolution of $\mathbb{C}^3/\mathbb{Z}_{2g+1}$. The diagram shows the 1-strata of its moment map polytope. C_i's are labelling the holomorphic spheres which are mapped to the corresponding edges by the moment map. D_i's are labelling the toric divisors which are mapped to the corresponding facets.

(3) $\mathcal{X} = [\mathbb{C}^n/\mathbb{Z}_n]$ for $n \in \mathbb{Z}$. Its fan has been described in Section 4.4. It has a twisted sector of age one, which corresponds to $\nu = (0,1) \in \mathbb{Z}^n \times \mathbb{Z}$. Its SYZ mirror is

$$uv = (\tau + \delta_\nu(\tau)) + z_1 + \ldots + z_n + z_1^{-1} \ldots z_n^{-1}$$

where

$$\tau + \delta_\nu(\tau) = \sum_{k \geq 1} \frac{\tau^k}{k!} n_{1,k,\beta_\nu}([\mathrm{pt}]_L; (\mathbf{1}_\nu)^k).$$

The total space of the canonical line bundle $K_{\mathbb{P}^{n-1}}$ of the projective space $\mathbb{P}^{n-1}$ gives its crepant resolution, whose SYZ mirror is

$$uv = (1 + \delta) + z_1 + \ldots + z_n + q z_1^{-1} \ldots z_n^{-1}$$

where

$$1 + \delta = \sum_{k \geq 0} q^k n_{1,k,\beta_0 + kl}$$

where l is the line class in $K_{\mathbb{P}^{n-1}}$ and its corresponding Kähler parameter is q. When $n = 3$, this serves as one of the first nontrivial examples for the SYZ mirror construction for toric Calabi-Yau 3-folds in [20].

We note that in all the above examples, the mirror of $\mathcal{X}$ and its crepant resolution almost have the same expressions, except that they have different coefficients. This motivates the Open Crepant Resolution Theorem 8.1 in Section 8 which gives a precise relation between their mirrors.

6. Computation of orbi-disk invariants

In this section we compute the orbi-disk invariants of a toric Calabi-Yau orbifold relative to a Lagrangian torus fiber of the moment map.

Let $\mathcal{X}$ be a toric Calabi-Yau orbifold as in Setting 4.3. Let $L \subset \mathcal{X}$ be a Lagrangian torus fiber of the moment map. Let $\beta \in \pi_2(\mathcal{X}, L)$ be such that $\mu_{CW}(\beta) = 2$. Let $\boldsymbol{x} = (\mathcal{X}_{\nu_1}, \ldots, \mathcal{X}_{\nu_l})$ be a collection of twisted sectors of $\mathcal{X}$ such that $\nu_i \in \mathrm{Box}'$ satisfies $\mathrm{age}(\nu_i) = 1$ for all i. Suppose that the moduli space $\mathcal{M}^{main}_{1,l}(L, \beta, \boldsymbol{x})$ is non-empty. We would like to compute the corresponding orbi-disk invariant or genus 0 open orbifold Gromov-Witten invariant

$$n^{\mathcal{X}}_{1,l,\beta}([\mathrm{pt}]_L; \mathbf{1}_{\nu_1}, \ldots, \mathbf{1}_{\nu_l})$$

defined in Definition 3.5.

The approach we take here is to construct a suitable toric partial compactification $\bar{\mathcal{X}}$ of $\mathcal{X}$ for each $\beta \in \pi_2(\mathcal{X}, L)$ with $\mu_{CW}(\beta) = 2$, and prove that the above invariants are equal to certain genus 0 *closed* orbifold Gromov-Witten invariants of $\bar{\mathcal{X}}$, which can then be evaluated by toric mirror theorems; this generalizes the approach in [23]. The proof of this *open/closed equality*, which is geometric in nature, is by comparing moduli spaces of stable (orbi-)disks to $\mathcal{X}$ with moduli spaces of stable orbi-maps to $\bar{\mathcal{X}}$, as Kuranishi spaces. The key geometric idea, namely, "capping off" the disk component to form a genus 0 closed Riemann surface, was first employed in [17, 83] and subsequently in [84] (for toric Calabi-Yau surfaces) and [19, 22] (for compact semi-Fano toric manifolds). It was also applied in [18] to calculate orbi-disk invariants for certain compact toric orbifolds.

6.1. Toric (partial) compactifications.

We begin with the construction of the toric (partial) compactification $\bar{\mathcal{X}}$. According to our discussion in Section 3.1, the class $\beta \in \pi_2(\mathcal{X}, L)$ must be of the form

$$\beta = \beta' + \alpha,$$

where $\beta' \in \pi_2(\mathcal{X}, L)$ is a basic (orbi) disk class with Chern-Weil Maslov index 2 and $\alpha \in H_2^{\mathrm{eff}}(\mathcal{X})$ is an effective curve class such that $c_1(\mathcal{X}) \cdot \alpha = 0$. We have $\partial\beta' = \boldsymbol{b}_{i_0} \in N$ for some $i_0 \in \{0, 1, \ldots, m'-1\}$.

Construction 6.1. *Let*

$$\boldsymbol{b}_\infty := -\boldsymbol{b}_{i_0} \in N.$$

Let $\bar{\Sigma} \subset N_{\mathbb{R}}$ be the smallest complete simplicial fan that contains Σ and the ray $\mathbb{R}_{\geq 0}\boldsymbol{b}_\infty \subset N_{\mathbb{R}}$. More concretely, the fan $\bar{\Sigma}$ consists of cones in Σ together with additional cones, each is spanned by the ray $\mathbb{R}_{\geq 0}\boldsymbol{b}_\infty$ together with a cone over a face of the polytope $\mathcal{P}$ (recall the definition of $\mathcal{P}$ in Setting 4.3). The data

$$(\bar{\Sigma}, \{\boldsymbol{b}_i\}_{i=0}^{m-1} \cup \{\boldsymbol{b}_\infty\})$$

gives a stacky fan. Let

$$\bar{\mathcal{X}} := \mathcal{X}_{\bar{\Sigma}}$$

be the associated toric orbifold. We choose the extra vectors to be the same as that for $\mathcal{X}$, namely, $\{\boldsymbol{b}_m, \ldots, \boldsymbol{b}_{m'-1}\} \subset N$.

Remark 6.2. *We emphasize that, although not reflected in the notation, the toric (partial) compactification $\bar{\mathcal{X}}$ depends on the class $\beta \in \pi_2(\mathcal{X}, L)$.*

Since Σ satisfies the Assumption 2.9, the stacky fan $\bar{\Sigma}$ satisfies it as well. The fan $\bar{\Sigma}$ has more primitive generators than Σ. We also have $\mathcal{X} \subset \bar{\mathcal{X}}$ and the toric prime divisor $D_\infty := \bar{\mathcal{X}} \setminus \mathcal{X}$ corresponding to $\boldsymbol{b}_\infty$.

The inclusion $\mathcal{X} \subset \bar{\mathcal{X}}$ divides the toric prime divisors of $\bar{\mathcal{X}}$ into two kinds: the set of generators $\{\boldsymbol{b}_i\}_{i=0}^{m-1}$ is a disjoint union $\{\boldsymbol{b}_i\} = I \coprod J$, where for $\boldsymbol{b}_i \in I$ the corresponding toric prime divisor $D_i \subset \bar{\mathcal{X}}$ is contained entirely in $\mathcal{X}$ (these correspond to the compact toric prime divisors in $\mathcal{X}$), and for $\boldsymbol{b}_j \in J$ the corresponding toric prime divisor $D_j \subset \bar{\mathcal{X}}$ has non-empty intersection with D_∞ (these correspond to the non-compact toric prime divisors in $\mathcal{X}$).

Let $\beta_\infty \in \pi_2(\bar{\mathcal{X}}, L)$ be the basic disk class corresponding to $\boldsymbol{b}_\infty$. Then since $\partial(\beta' + \beta_\infty) = \boldsymbol{b}_{i_0} + \boldsymbol{b}_\infty = 0 \in N$, the class $\bar{\beta}' := \beta' + \beta_\infty$ belongs to $H_2(\bar{\mathcal{X}}; \mathbb{Q})$ (see [28, Section 9.1]), and we have $c_1(\bar{\mathcal{X}}) \cdot \bar{\beta}' = 2$. Moreover we have the decompositions

$$H_2(\bar{\mathcal{X}}; \mathbb{Q}) = H_2(\mathcal{X}; \mathbb{Q}) \oplus \mathbb{Q}\bar{\beta}' \text{ and } H_2^{\mathrm{eff}}(\bar{\mathcal{X}}) = \mathbb{Z}_{\geq 0}\bar{\beta}' \oplus H_2^{\mathrm{eff}}(\mathcal{X}).$$

Denote by $\bar{\mathbb{L}}$, $\bar{\mathbb{K}}$ and $\bar{\mathbb{K}}_{\mathrm{eff}}$ respectively the counterparts for $\bar{\mathcal{X}}$ of the spaces $\mathbb{L}$, $\mathbb{K}$ and $\mathbb{K}_{\mathrm{eff}}$ for $\mathcal{X}$. Then we have the corresponding decompositions

$$\bar{\mathbb{L}} = \mathbb{L} \oplus \mathbb{Z}d_\infty, \quad \bar{\mathbb{K}} = \mathbb{K} \oplus \mathbb{Z}d_\infty, \quad \bar{\mathbb{K}}_{\mathrm{eff}} = \mathbb{K}_{\mathrm{eff}} \oplus \mathbb{Z}_{\geq 0}d_\infty,$$

where $d_\infty = e_{i_0} + e_\infty \in \widetilde{N} \oplus \mathbb{Z}e_\infty = \bigoplus_{i=0}^{m'-1} \mathbb{Z}e_i \oplus \mathbb{Z}e_\infty$.

Since the class α can be represented by a holomorphic map to $\bar{\mathcal{X}}$ whose image is contained entirely in $\mathcal{X}$ and misses $D_\infty = \bar{\mathcal{X}} \setminus \mathcal{X}$, we have $D_\infty \cdot \alpha = 0$ and hence $c_1(\bar{\mathcal{X}}) \cdot \alpha = 0$. Moreover, each $\nu_i \in \mathrm{Box}'(\Sigma)$ with $\mathrm{age}(\nu_i) = 1$ determines uniquely an element $\bar{\nu}_i \in \mathrm{Box}'(\bar{\Sigma})$ with $\mathrm{age}(\bar{\nu}_i) = 1$.

We make some important observations about $\bar{\mathcal{X}}$.

Proposition 6.3. *The toric orbifold $\bar{\mathcal{X}}$ with the extra vectors $\{\boldsymbol{b}_m, \ldots, \boldsymbol{b}_{m'-1}\}$ is semi-Fano in the sense of Definition 2.8.*

Proof. To show that $\bar{\mathcal{X}}$ is semi-Fano, we need to prove that

$$c_1(\bar{\mathcal{X}}) = \sum_{i=0}^{m-1} D_i + D_\infty$$

is nef (since $\mathrm{age}(\boldsymbol{b}_j) = 1$ for $j = m, \ldots, m' - 1$). In other words, every rational orbi-curve C satisfies

$$(D_0 + \ldots + D_{m-1} + D_\infty) \cdot C \geq 0.$$

Let $C \cdot D_\infty = k \in \mathbb{Z}$. We must have $k \geq 0$. Otherwise, C has a component contained in D_∞ whose intersection with D_∞ is negative. Now $D_\infty = \{\underline{\nu} = \infty\}$ is linearly equivalent to the divisor $\tilde{D} = \{\underline{\nu} = c\}$ for any $c \neq 0$.[4] A rational curve in D_∞ has transverse intersections with $\tilde{D}$, and hence the intersection number is non-negative. Since intersection number is topological, this implies D_∞ has non-negative intersection with any curve contained in D_∞ itself. Thus k cannot be negative.

[4]Two divisors D_1 and D_2 are said to be linear equivalent if there exists a meromorphic function ϕ such that D_1 and D_2 are the zero and pole divisors of ϕ respectively. In such a case given a rational curve C, the intersection number of C with D_1 is the same as that with D_2. In our situation we take the meromorphic function ϕ to be $\underline{\nu} - c$ for a fixed complex number c.

Now consider $C - kC_0$, where C_0 is a holomorphic sphere representing the class $\beta' + \beta_\infty$ which has Chern number $c_1(\bar{\mathcal{X}}) \cdot C_0 = 2$. $C - kC_0$ has zero intersection with the divisor D_∞. Moreover it can be written as a linear combination of one-dimensional toric strata of $\mathcal{X}$. Since $\mathcal{X}$ is Calabi-Yau, $(C - kC_0) \cdot (D_0 + \ldots + D_{m-1}) = 0$. Then

$$(D_0 + \ldots + D_{m-1} + D_\infty) \cdot C = (D_0 + \ldots + D_{m-1} + D_\infty) \cdot (C - kC_0) + 2k = 2k \geq 0.$$

This completes the proof. □

Proposition 6.4. *The toric variety $\bar{X}$ underlying $\bar{\mathcal{X}}$ is semi-projective.*

Proof. By [38, Proposition 7.2.9], the toric variety X is semi-projective, as its moment map image is a full-dimensional lattice polyhedron P. The toric variety $\bar{X}$ corresponds to intersecting P with a half space normal to $\boldsymbol{b}_\infty$. The result is still a full dimensional lattice polyhedron. Hence $\bar{X}$ is semi-projective again by [38, Proposition 7.2.9]. □

If $\boldsymbol{b}_{i_0} \in N$ lies in the interior of the support $|\Sigma|$, then in fact $\bar{\mathcal{X}}$ is projective:

Proposition 6.5. *Suppose that $\boldsymbol{b}_{i_0} \in N$ lies in the interior of the support $|\Sigma|$. Then the fan $\bar{\Sigma}$ is complete, and hence the toric variety $\bar{X}$ underlying $\bar{\mathcal{X}}$ is projective.*

Proof. To prove that $\bar{\Sigma}$ is complete, it suffices to see that any vector $v \in N_\mathbb{R}$ can be written as a non-negative linear combination of generators of the fan $\bar{\Sigma}$. Since $\boldsymbol{b}_{i_0}$ lies in the interior of the support $|\Sigma|$, there exists $t \in \mathbb{R}_{>0}$ large enough such that $v + t\boldsymbol{b}_{i_0} \in |\Sigma|$. Thus $v + t\boldsymbol{b}_{i_0} = \sum_{i=0}^{m-1} a_i \boldsymbol{b}_i$ for $a_i \in \mathbb{R}_{\geq 0}$. Then

$$v = \sum_{i=0}^{m-1} a_i \boldsymbol{b}_i - t\boldsymbol{b}_{i_0} = \sum_{i=0}^{m-1} a_i \boldsymbol{b}_i + t\boldsymbol{b}_\infty.$$

□

Remark 6.6. *Suppose that $\boldsymbol{b}_{i_0} \in N$ lies on the boundary of the support $|\Sigma|$. In this case, the fan $\bar{\Sigma}$ in Construction 6.1 is not complete, and hence the toric orbifold $\bar{\mathcal{X}}$ is not projective, but only semi-projective.*

We will need the following lemma when we analyze the curve moduli.

Lemma 6.7. *Given a generic point in $\bar{\mathcal{X}}$, there exists a unique non-constant holomorphic sphere of Chern number two passing through the point.*

Proof. Choose local toric coordinates $(\underline{\nu}, z_1, \ldots, z_{n-1})$ such that $z_1, \ldots, z_{n-1}$ are not identically zero when restricted on D_{i_0}. We take the point to be in the open toric orbit $(\mathbb{C}^\times)^n \subset \bar{\mathcal{X}}$. Suppose it has coordinates $(c_0, c_1, \ldots, c_{n-1})$, where $c_i \neq 0$ for all $i = 0, \ldots, n-1$. Then the holomorphic sphere defined by $z_i = c_i$ for all $i = 1, \ldots, n-1$ passes through the point, and it only intersects D_{i_0} and D_∞ once but not any other divisors. Thus it intersects with the anti-canonical divisor (which is the sum over all toric prime divisors) twice and hence has Chern number two.

To show uniqueness, suppose we have a non-constant holomorphic sphere of Chern number two passing through a point in the open toric orbit. It must intersect D_∞, since otherwise, it will be entirely contained in the toric Calabi-Yau $\mathcal{X}$, and by the maximal principle applied to the holomorphic function $\underline{\nu}$ on the sphere, the sphere must lie entirely in the toric divisors of $\mathcal{X}$, and hence cannot pass through a point in the open toric orbit. Since it has Maslov index two, it intersects D_∞ at most two times (counted with multiplicity). The meromorphic function $\underline{\nu}$ on the sphere must have both zeroes and poles, and thus it must have one zero and one pole. This means that the sphere intersects both D_0 and D_∞ once, and that it cannot intersect other divisors since it only has Maslov index two. Thus the functions z_i's on the sphere have neither poles nor zeroes, and hence can only be constants. We conclude that it is precisely the holomorphic sphere defined by $z_i = c_i$ for all $i = 1, \ldots, n-1$. □

Example 6.8. *The fan of the Hirzebruch surface $\mathbb{F}_2$ has primitive generators $(-1,1)$, $(0,1)$, $(1,1)$, $(0,-1)$. The total space of the canonical line bundle $\mathcal{X} = K_{\mathbb{F}_2}$ is again a toric manifold, whose fan has primitive generators $\boldsymbol{b}_0 = (0,0,1)$, $\boldsymbol{b}_1 = (-1,1,1)$, $\boldsymbol{b}_2 = (0,1,1)$, $\boldsymbol{b}_3 = (1,1,1)$ and $\boldsymbol{b}_4 = (0,-1,1)$. The polytope $\mathcal{P}$ is the convex hull of $(-1,1), (1,1), (0,-1)$ in the plane $\mathbb{R}^2$. The generator $(0,1)$ lies in the boundary of $\mathcal{P}$ but is* not *a vertex of $\mathcal{P}$. The toric compactification $\bar{\mathcal{X}}$ corresponding to $\boldsymbol{b}_2$ (see Construction 6.1) is not compact because $\boldsymbol{b}_2$ lies in the boundary of the support $|\Sigma|$.*

The toric prime divisor D_2 in $\mathcal{X}$ corresponding to $\boldsymbol{b}_2$ is non-compact and is biholomorphic to $\mathbb{P}^1 \times \mathbb{C}$. The inclusion $(z,c) : \mathbb{P}^1 \hookrightarrow \mathbb{P}^1 \times \mathbb{C} \cong D_2$ for any constant $c \in \mathbb{C}$ gives a $(0,-2)$ rational curve in $\mathcal{X} = K_{\mathbb{F}_2}$, whose class is denoted by $l \in H_2(\mathcal{X};\mathbb{Z})$. It has Chern number zero and does *contribute to sphere bubbling. Thus the open Gromov-Witten invariants $n^{\mathcal{X}}_{\beta_2+kl}$ for $k \in \mathbb{Z}_{\geq 0}$ are non-trivial. We will see in Section 6.5 that in fact $n^{\mathcal{X}}_{\beta_2+kl} = 1$ when $k = 0, 1$ and zero otherwise. Hence*

$$n^{\mathcal{X}}_{\beta_2+kl} = n^{\mathbb{F}_2}_{\beta_2+kl}$$

where β_2 and l on the right hand side of the equality denotes the basic disk class corresponding to $D_2 \subset \mathbb{F}_2$ and the class of the (-2) curve in $\mathbb{F}_2$ respectively.

6.2. **An open/closed equality.** We now consider three moduli spaces. We first let $\iota : \{p\} \to L$ be the inclusion of a point.

Definition 6.9. *Let $\mathcal{X}$ and $\bar{\mathcal{X}}$ be as in Construction 6.1.*

(1) Let $\mathcal{M}^{op}_{1,l}(\mathcal{X},\beta,\boldsymbol{x}) := \mathcal{M}^{main}_{1,l}(L,\beta,\boldsymbol{x})$ be the moduli space of stable maps from genus 0 bordered orbifold Riemann surfaces with one boundary component to $(\mathcal{X}, L)$ of class $\beta = \beta' + \alpha$ such that there is one boundary marked point and l interior marked points of type $\boldsymbol{x} = (\mathcal{X}_{\nu_1}, \ldots, \mathcal{X}_{\nu_l})$. Let $ev_0 : \mathcal{M}^{op}_{1,l}(\mathcal{X},\beta,\boldsymbol{x}) \to L$ denote the evaluation map at the boundary marked point. Consider the fiber product

$$\mathcal{M}^{op}_{1,l}(\mathcal{X},\beta,\boldsymbol{x},p) := \mathcal{M}^{op}_{1,l}(\mathcal{X},\beta,\boldsymbol{x}) \times_{ev_0,\iota} \{p\}.$$

(2) Let $\mathcal{M}^{op}_{1,l}(\bar{\mathcal{X}},\beta,\boldsymbol{x}') := \mathcal{M}^{main}_{1,l}(L,\beta,\boldsymbol{x}')$ be the moduli space of stable maps from genus 0 bordered orbifold Riemann surfaces with one boundary component to $(\bar{\mathcal{X}}, L)$ of class β such that there is one boundary marked point and l interior marked points of type $\boldsymbol{x}' = (\bar{\mathcal{X}}_{\nu_1}, \ldots, \bar{\mathcal{X}}_{\nu_l})$. Let $ev_0 : \mathcal{M}^{op}_{1,l}(\bar{\mathcal{X}},\beta,\boldsymbol{x}') \to L$ denote the evaluation map at the boundary marked point. Consider the fiber product

$$\mathcal{M}^{op}_{1,l}(\bar{\mathcal{X}},\beta,\boldsymbol{x}',p) := \mathcal{M}^{op}_{1,l}(\bar{\mathcal{X}},\beta,\boldsymbol{x}') \times_{ev_0,\iota} \{p\}.$$

(3) Let $\mathcal{M}^{cl}_{1+l}(\bar{\mathcal{X}},\bar{\beta},\bar{\boldsymbol{x}})$ be the moduli space of stable maps from genus 0 orbifold Riemann surfaces to $\bar{\mathcal{X}}$ of class $\bar{\beta} := \bar{\beta}' + \alpha$ such that the $1+l$ interior marked points of are

type $\bar{\boldsymbol{x}} = (\bar{\mathcal{X}}, \bar{\mathcal{X}}_{\nu_1}, \ldots, \bar{\mathcal{X}}_{\nu_l})$. *Let* $ev_0 : \mathcal{M}^{cl}_{1+l}(\bar{\mathcal{X}}, \bar{\beta}, \bar{\boldsymbol{x}}) \to \bar{\mathcal{X}}$ *denote the evaluation map at the first marked point. Consider the fiber product*

$$\mathcal{M}^{cl}_{1+l}(\bar{\mathcal{X}}, \bar{\beta}, \bar{\boldsymbol{x}}, p) := \mathcal{M}^{cl}_{1+l}(\bar{\mathcal{X}}, \bar{\beta}, \bar{\boldsymbol{x}}) \times_{ev_0,\iota} \{p\}.$$

We need the following compactness result.

Proposition 6.10.

(a) *Let* D *be a toric prime divisor of the toric Calabi-Yau orbifold* $\mathcal{X}$, $\alpha \in H_2(D;\mathbb{Z})$ *and* $p \in D$. *Then the moduli space of rational curves in* D *representing* α *with one marked point passing through* p *is compact.*

(b) *Let* $\alpha \in H_2(\mathcal{X};\mathbb{Z})$ *and* $p \in \mathcal{X}$. *Then the moduli space of rational curves in* $\mathcal{X}$ *representing* α *with one marked point passing through* p *is compact.*

(c) *The disk moduli* $\mathcal{M}^{op}_{1,l}(\mathcal{X}, \beta_i + \alpha, \boldsymbol{x})$ *for every* $i = 0, \ldots, m'-1$ *and* $\alpha \in H_2(\mathcal{X};\mathbb{Z})$ *is compact.*

Proof.

(a) The statement certainly holds when the divisor D is compact. Now suppose that D is a non-compact divisor. We are going to prove that all rational curves representing α with one marked point passing through p must lie in a compact subvariety of D, and hence the moduli space is compact.

The toric divisor $D \subset \mathcal{X}$ itself is a toric orbifold, whose fan Σ_D is given by the quotient of Σ in the v-direction and localization at zero, where v is the primitive generator of Σ corresponding to D. Since D is non-compact, v lies in the boundary of the polytope $\mathcal{P}$. Thus there exists a half space defined by $\{\nu \geq 0\} \subset (N/\langle v \rangle)_{\mathbb{R}}$ for some $\nu \in M^{\perp v}$ containing $|\Sigma_D|$. Then the function on D corresponding to ν is holomorphic, and by abuse of notation we also denote it by ν. By the maximal principle, ν is constant on each sphere component of a rational curve in D. Since the rational curve is connected, ν takes the same constant on the whole rational curve. Let $\nu(p) = c \in \mathbb{C}$. Then any rational curve with one marked point passing through p lies in the level set $\{\nu = c\} \subset D$.

The above is true for all $\nu \in M^{\perp v}$ such that the corresponding half space $\{\nu \geq 0\}$ contains $|\Sigma_D|$. Let $\nu_1, \ldots, \nu_k$ be the extremal ones, meaning that each of the corresponding half spaces contains $|\Sigma_D|$ and a codimension-one face of $|\Sigma_D|$. Then there exist $c_1, \ldots, c_k \in \mathbb{C}$ such that any rational curve with one marked point passing through p lies in $\{\nu_i = c_i \text{ for all } i = 1, \ldots, k\}$, which is a compact subvariety of D. Hence the moduli space of rational curves representing α with one marked point and passing through p is compact.

(b) We may assume that p lies in a toric divisor of $\mathcal{X}$, or otherwise the moduli space is empty since $\mathcal{X}$ is a toric Calabi-Yau orbifold. All rational curves in $\mathcal{X}$ lie in toric divisors of $\mathcal{X}$. Thus the moduli space can be written as a fiber product of moduli spaces of rational curves in prime divisors of $\mathcal{X}$. By part (a) the moduli space of rational curves in a toric prime divisor passing through a fixed target point is compact. Hence the fiber product is also compact.

(c) The disk moduli $\mathcal{M}^{op}_{1,l}(\mathcal{X}, \beta_i + \alpha, \boldsymbol{x})$ is equal to the fiber product $\mathcal{M}^{op}_{1,1}(\mathcal{X}, \beta_i) \times_{ev} \mathcal{M}^{cl}_{\bullet+l}(\mathcal{X}, \alpha, \boldsymbol{x})$, where $\mathcal{M}^{op}_{1,1}(\mathcal{X}, \beta_i)$ is the moduli space of stable disks in $\mathcal{X}$ representing the basic disks class β_i with one interior marked point and one boundary marked point, $\mathcal{M}^{cl}_{\bullet+l}(\mathcal{X},\alpha,\boldsymbol{x})$ is the moduli space of rational curves in $\mathcal{X}$ representing α with one marked point $\bullet$ and l other marked points of type $\boldsymbol{x}$, and the fiber product is over evaluation maps at the interior marked point of the disk and the marked point $\bullet$ of the rational curve. Now, the moduli space $\mathcal{M}^{op}_{1,1}(\mathcal{X},\beta_i)$ is known to be compact by the classification result of Cho-Poddar [28]. By part (b), $\mathcal{M}^{cl}_{\bullet+l}(\mathcal{X},\alpha,\boldsymbol{x})\times_{ev}\{\mathrm{pt}\}$ is compact. Thus the fiber product $\mathcal{M}^{op}_{1,1}(\mathcal{X},\beta_i)\times_{ev}\mathcal{M}^{cl}_{\bullet+l}(\mathcal{X},\alpha,\boldsymbol{x})$ is also compact.

□

Corollary 6.11. *The moduli space $\mathcal{M}^{op}_{1,l}(\mathcal{X},\beta,\boldsymbol{x},p)$ in Definition 6.9 is compact. Hence, the open orbifold Gromov-Witten invariant $n^{\mathcal{X}}_{1,l,\beta}([\mathrm{pt}]_L;\mathbf{1}_{\nu_1},\ldots,\mathbf{1}_{\nu_l})$ in Definition 3.5 is well-defined.*

The following is the main result of this subsection.

Theorem 6.12.

(a) The moduli spaces $\mathcal{M}^{op}_{1,l}(\mathcal{X},\beta,\boldsymbol{x},p)$ and $\mathcal{M}^{op}_{1,l}(\bar{\mathcal{X}},\beta,\boldsymbol{x}',p)$ are isomorphic as Kuranishi spaces. Hence we have the following equality between genus 0 open orbifold Gromov-Witten invariants:

$$n^{\mathcal{X}}_{1,l,\beta}([\mathrm{pt}]_L;\mathbf{1}_{\nu_1},\ldots,\mathbf{1}_{\nu_l})=n^{\bar{\mathcal{X}}}_{1,l,\beta}([\mathrm{pt}]_L;\mathbf{1}_{\bar{\nu}_1},\ldots,\mathbf{1}_{\bar{\nu}_l}).$$

(b) The moduli spaces $\mathcal{M}^{op}_{1,l}(\bar{\mathcal{X}},\beta,\boldsymbol{x}',p)$ and $\mathcal{M}^{cl}_{1+l}(\bar{\mathcal{X}},\bar{\beta},\bar{\boldsymbol{x}},p)$ are isomorphic as Kuranishi spaces. Hence we have the following equality between genus 0 open and closed orbifold Gromov-Witten invariants, called the open/closed equality*:*

$$n^{\mathcal{X}}_{1,l,\beta}([\mathrm{pt}]_L;\mathbf{1}_{\nu_1},\ldots,\mathbf{1}_{\nu_l})=\langle[\mathrm{pt}],\mathbf{1}_{\bar{\nu}_1},\ldots,\mathbf{1}_{\bar{\nu}_l}\rangle^{\bar{\mathcal{X}}}_{0,1+l,\bar{\beta}}. \tag{6.1}$$

Proof. We begin with part (a). The inclusion $\mathcal{X}\subset\bar{\mathcal{X}}$ gives a natural map

$$\mathcal{M}^{op}_{1,l}(\mathcal{X},\beta,\boldsymbol{x},p)\to\mathcal{M}^{op}_{1,l}(\bar{\mathcal{X}},\beta,\boldsymbol{x}',p),$$

which is clearly injective. To show that this map is surjective, we need to prove that a stable disk in $\mathcal{M}^{op}_{1,l}(\bar{\mathcal{X}},\beta,\boldsymbol{x}',p)$ is indeed contained in $\mathcal{X}$. This means there are no stable disk maps $f:(\mathcal{C},\partial\mathcal{C})\to(\bar{\mathcal{X}},L)$ of class $\beta=\beta'+\alpha$ such that

$$\mathcal{C}=\mathcal{D}\cup\mathcal{C}_0\cup\mathcal{C}_\infty$$

is a union where $\mathcal{D}$ is the disk component; $\mathcal{C}_0$ is a closed (orbifold) Riemann surface whose components are contained in $\bigcup_{\boldsymbol{b}_i\in I}D_i$; and $\mathcal{C}_\infty$ is a non-empty closed (orbifold) Riemann surface whose components are contained in $D_\infty\cup\bigcup_{\boldsymbol{b}_j\in J}D_j$ and have non-negative intersections with divisors D_i, $\boldsymbol{b}_i\in I$ (via f).

Suppose there is such a stable disk map. Let $A:=f_*[\mathcal{C}_0]$ and $B:=f_*[\mathcal{C}_\infty]$. Then $\alpha=A+B$. Since $c_1(\bar{\mathcal{X}})\cdot\alpha=0$ and $-K_{\bar{\mathcal{X}}}$ is nef, we have

$$c_1(\bar{\mathcal{X}})\cdot A=0=c_1(\bar{\mathcal{X}})\cdot B.$$

Writing $B=\sum_k b_kB_k$ as an effective linear combination of the classes B_k of irreducible 1-dimensional torus-invariant orbits in $\bar{\mathcal{X}}$, we have $c_1(\bar{\mathcal{X}})\cdot(b_kB_k)=0$ for all k (again using the fact that $-K_{\bar{\mathcal{X}}}$ is nef). Each B_k corresponds to an $(n-1)$-dimensional cone $\sigma_k\in\bar{\Sigma}$, and by our construction, either σ_k contains $\boldsymbol{b}_\infty$, or σ_k and $\boldsymbol{b}_\infty$ together span an n-dimensional cone in $\bar{\Sigma}$.

Since $f(\mathcal{C}_\infty) \subset D_\infty \cup \bigcup_{\boldsymbol{b}_j \in J} D_j$, we see that if $\boldsymbol{b}_i \in I$ then $\boldsymbol{b}_i \notin \sigma_k$. Also, since $D \cdot (b_k B_k) \geq 0$ for every toric prime divisor of $\bar{\mathcal{X}}$ not corresponding to a ray in σ_k, we have by[5] [58, Lemma 4.5] that $D \cdot (b_k B_k) = 0$ for every toric prime divisor D corresponding to an element in $(\{\boldsymbol{b}_i\} \cup \{\boldsymbol{b}_\infty\}) \setminus F(\sigma_k)$; here $F(\sigma_k)$ is the minimal face in the fan polytope of $\bar{\Sigma}$ that contains rays in σ_k. As the divisors D corresponding to $(\{\boldsymbol{b}_i\} \cup \{\boldsymbol{b}_\infty\}) \setminus F(\sigma_k)$ span $H^2(\bar{\mathcal{X}})$, we must have $b_k B_k = 0$. We conclude that $B = 0$.

Therefore we have a bijection between moduli spaces

$$\mathcal{M}^{op}_{1,l}(\mathcal{X}, \beta, \boldsymbol{x}, p) \cong \mathcal{M}^{op}_{1,l}(\bar{\mathcal{X}}, \beta, \boldsymbol{x}', p).$$

Since every stable disk in $\mathcal{M}^{op}_{1,l}(\bar{\mathcal{X}}, \beta, \boldsymbol{x}', p)$ is supported in (a compact region of) $\mathcal{X}$, it is clear that it has the same deformations and obstructions as the corresponding stable disk in $\mathcal{M}^{op}_{1,l}(\mathcal{X}, \beta, \boldsymbol{x}, p)$. By the same arguments as in Part(C) of the proof of [22, Propostion 5.6] (which can be adapted to the orbifold setting here in a straightforward way), it follows that the above bijection gives an isomorphism of Kuranishi structures. This proves (a).

The proof of part (b) is basically the same as that of [18, Theorem 35]. First of all, for a stable disk map in $\mathcal{M}^{op}_{1,l}(\bar{\mathcal{X}}, \beta, \boldsymbol{x}', p)$, it consists of a unique disk component u_0 and a rational curve component C'. We denote such a stable disk by $u_0 + C'$. The disk component represents a basic (orbi-)disk class and hence is regular by [28, Propositions 8.3 and 8.6]. Thus the obstruction merely comes from the rational curve component.

On the other hand, by Lemma 6.7, there is a unique holomorphic sphere C_0 with Chern number two in $\bar{\mathcal{X}}$ passing through a generic point $p \in \bar{\mathcal{X}}$. So for a stable curve in $\mathcal{M}^{cl}_{1+l}(\bar{\mathcal{X}}, \bar{\beta}, \bar{\boldsymbol{x}}, p)$, since it passes through p and it has Chern number two, it has C_0 as one of its components, and the rest is a rational curve C' with Chern number zero contained in the toric divisors. We denote such a rational curve by $C_0 + C'$. Since C_0 is a holomorphic sphere whose normal bundle is trivial, it is unobstructed. Thus the obstruction of $C_0 + C'$ merely comes from C'. A bijective map between $\mathcal{M}^{op}_{1,l}(\bar{\mathcal{X}}, \beta, \boldsymbol{x}', p)$ and $\mathcal{M}^{cl}_{1+l}(\bar{\mathcal{X}}, \bar{\beta}, \bar{\boldsymbol{x}}, p)$ is given by sending $u_0 + C'$ to $C_0 + C'$ and vice versa. They have the same deformations and obstructions (which are contributed from the same rational curve component C'), and hence

$$\mathcal{M}^{op}_{1,l}(\bar{\mathcal{X}}, \beta, \boldsymbol{x}', p) \cong \mathcal{M}^{cl}_{1+l}(\bar{\mathcal{X}}, \bar{\beta}, \bar{\boldsymbol{x}}, p)$$

as Kuranishi structures.

The identification of the two Kuranishi structures can be done as explained in Step 3 of the proof of [18, Theorem 35], except that the choices of obstruction bundles have to be suitably modified in order to obtain smoothly compatible Kuranishi charts which can be glued together to obtain a global structure (see [93, 46]).

Recall that in the general scheme developed by Fukaya, Oh, Ohta and Ono in constructing Kuranishi structures of a moduli space, one first constructs a Kuranishi neighborhood for each point of the moduli space. To obtain a global Kuranishi structure which is smoothly compatible, one then chooses a sufficiently dense finite set of points in the moduli space, and redefines the Kuranishi neighborhood by considering a new obstruction bundle obtained as

[5]Their argument extends to the simplicial cases needed here.

the direct sum of parallel transports of the obstruction bundles over the finite set of points. When the domain of the stable map is not stable, however, one has to further consider a stabilization of the domain and extra care is needed in choosing the obstruction bundles. We refer the readers to [46, Section 3.2] for a brief description and to [46, Sections 15-18] for the detailed construction.

The construction of Kuranishi neighborhoods given in the proof [18, Theorem 35] corresponds to the case where the domain of a stable map is also stable, in which the above description of the obstruction bundles already suffices. But for the moduli spaces we consider here, the domain of a stable map may not be stable, so we need the general construction as described in [46, Part 4]. Nevertheless, we emphasize that all these (or any such) constructions can be carried out in the same way for the open and closed moduli spaces because the obstruction bundles on the disk component u_0 and the sphere component C_0 both vanish, and therefore the Kuranishi structures are naturally identified with each other. □

Remark 6.13. *The proof of Theorem 6.12 identifies the moduli space $\mathcal{M}^{cl}_{1+l}(\bar{\mathcal{X}}, \bar{\beta}, \bar{\boldsymbol{x}}, p)$ with $\mathcal{M}^{op}_{1,l}(\mathcal{X}, \beta, \boldsymbol{x}, p)$, which is compact by Corollary 6.11. Therefore $\mathcal{M}^{cl}_{1+l}(\bar{\mathcal{X}}, \bar{\beta}, \bar{\boldsymbol{x}}, p)$ is also compact and hence the closed orbifold Gromov-Witten invariant $\langle [\mathrm{pt}], \mathbf{1}_{\bar{\nu}_1}, \ldots, \mathbf{1}_{\bar{\nu}_l}\rangle^{\bar{\mathcal{X}}}_{0,1+l,\bar{\beta}}$ is well-defined, even when $\bar{\mathcal{X}}$ is not compact.*

6.3. **Calculation via mirror theorem.** By the open/closed equality (6.1), the open orbifold Gromov-Witten invariants of $\mathcal{X}$ we need may be computed by evaluating the genus 0 closed orbifold Gromov-Witten invariants of $\bar{\mathcal{X}}$:

$$\langle [\mathrm{pt}], \mathbf{1}_{\bar{\nu}_1}, \ldots, \mathbf{1}_{\bar{\nu}_l}\rangle^{\bar{\mathcal{X}}}_{0,1+l,\bar{\beta}}.$$

These closed orbifold Gromov-Witten invariants are certain coefficients in the J-function of $\bar{\mathcal{X}}$. We evaluate these invariants by extending the approach developed in [23] to the orbifold setting.

The idea is to use closed mirror theorem for toric orbifolds to explicitly compute these coefficients using the combinatorially defined I-function of $\bar{\mathcal{X}}$. However, since $\bar{\mathcal{X}}$ may not be compact, we cannot directly apply the closed mirror theorem (Theorem 2.15) to $\bar{\mathcal{X}}$ as in [23]. We get around this by first applying the *equivariant* mirror theorem (Theorem 2.14) to evaluate the genus 0 equivariant closed orbifold Gromov-Witten invariants of $\bar{\mathcal{X}}$:

$$\langle [\mathrm{pt}]_{\mathbb{T}}, \mathbf{1}_{\bar{\nu}_1}, \ldots, \mathbf{1}_{\bar{\nu}_l}\rangle^{\bar{\mathcal{X}},\mathbb{T}}_{0,1+l,\bar{\beta}},$$

where $[\mathrm{pt}]_{\mathbb{T}} \in H^*_{\mathbb{T}}(\bar{\mathcal{X}})$ is the equivariant lift of $[\mathrm{pt}] \in H^*(\bar{\mathcal{X}})$ represented by a $\mathbb{T}$-fixed point, and then evaluating $\langle [\mathrm{pt}], \mathbf{1}_{\bar{\nu}_1}, \ldots, \mathbf{1}_{\bar{\nu}_l}\rangle^{\bar{\mathcal{X}}}_{0,1+l,\bar{\beta}}$ by taking non-equivariant limits.

6.3.1. *Identifying the invariants.* We now begin the computation of the relevant equivariant orbifold Gromov-Witten invariants.

The $\mathbb{T}$-equivariant J-function of $\bar{\mathcal{X}}$ (cf. Definition 2.11) expands as a series in $1/z$ as follows:

$$\begin{aligned} J_{\bar{\mathcal{X}},\mathbb{T}}(q,z) =& \mathbf{e}^{\tau_{0,2}/z}\left(1+\sum_{\alpha}\sum_{\substack{(d,l)\neq(0,0)\\ d\in H_2^{\mathrm{eff}}(\bar{\mathcal{X}})}}\frac{q^d}{l!}\frac{1}{z}\sum_{k\geq 0}\left\langle 1,\tau_{\mathrm{tw}},\ldots,\tau_{\mathrm{tw}},\phi_\alpha\psi^k\right\rangle_{0,l+2,d}^{\bar{\mathcal{X}},\mathbb{T}}\frac{\phi^\alpha}{z^k}\right)\\ =&\left(1+\frac{\tau_{0,2}}{z}+O\left(\frac{1}{z^2}\right)\right)\left(1+\sum_{\alpha}\sum_{\substack{(d,l)\neq(0,0)\\ d\in H_2^{\mathrm{eff}}(\bar{\mathcal{X}})}}\frac{q^d}{l!}\frac{1}{z}\sum_{k\geq 0}\left\langle \tau_{\mathrm{tw}},\ldots,\tau_{\mathrm{tw}},\phi_\alpha\psi^{k-1}\right\rangle_{0,l+1,d}^{\bar{\mathcal{X}},\mathbb{T}}\frac{\phi^\alpha}{z^k}\right), \end{aligned}$$

where we use the string equation in the second equality. Note that $\tau_{0,2}\in H^2_{\mathbb{T}}(\bar{\mathcal{X}})$. Also note that $\phi_\alpha=[\mathrm{pt}]_{\mathbb{T}}$ if and only if $\phi^\alpha=\mathbf{1}\in H^0(\bar{\mathcal{X}})$. If we consider

$$\tau_{\mathrm{tw}}=\sum_{\nu\in\mathrm{Box}'(\Sigma)^{\mathrm{age}=1}}\tau_\nu\mathbf{1}_{\bar{\nu}},$$

then the closed equivariant orbifold Gromov-Witten invariants $\langle[\mathrm{pt}]_{\mathbb{T}},\mathbf{1}_{\bar{\nu}_1},\ldots,\mathbf{1}_{\bar{\nu}_l}\rangle_{0,1+l,\bar{\beta}}^{\bar{\mathcal{X}},\mathbb{T}}$ occur as the coefficients of $q^{\bar{\beta}}\tau_{\nu_1}\cdots\tau_{\nu_l}$ in the $1/z^2$-term of $J_{\bar{\mathcal{X}},\mathbb{T}}(q,z)$ that takes values in $H^0(\bar{\mathcal{X}})$.

Since $\bar{\mathcal{X}}$ is semi-Fano (by Proposition 6.3) and semi-projective (by Proposition 6.4), we can apply the equivariant toric mirror theorem (Theorem 2.14) which says that

$$e^{q_0(y)/z}J_{\bar{\mathcal{X}},\mathbb{T}}(q,z)=I_{\bar{\mathcal{X}},\mathbb{T}}(y(q,\tau),z)$$

via the inverse $y=y(q,\tau)$ of the toric mirror map. Recall that the equivariant I-function here is the one defined using the extended stacky fan

$$(\bar{\Sigma},\{\boldsymbol{b}_i\mid 0\leq i\leq m-1\}\cup\{\boldsymbol{b}_\infty\}\cup\{\boldsymbol{b}_j\mid m\leq j\leq m'-1\}),$$

where

$$\{\boldsymbol{b}_j\mid m\leq j\leq m'-1\}=\{\nu\in\mathrm{Box}'(\Sigma)\mid \mathrm{age}(\nu)=1\}.$$

Therefore our next task is to explicitly identify the part of the $1/z^2$-term of the equivariant I-function of $\bar{\mathcal{X}}$ that takes values in $H^0(\bar{\mathcal{X}})$. According to the definition of the equivariant I-function in Definition 2.6, the part taking values in $H^0(\bar{\mathcal{X}})$ arises from terms with $d\in\bar{\mathbb{K}}_{\mathrm{eff}}$ such that

$$\nu(d)=0,\ \text{i.e. } \mathbf{1}_{\nu(d)}=\mathbf{1}\in H^0(\bar{\mathcal{X}}). \tag{6.2}$$

And for $d\in\bar{\mathbb{K}}_{\mathrm{eff}}$ to satisfy (6.2), we must have

$$\langle D_i,d\rangle\in\mathbb{Z},\ \text{for } i\in\{0,\ldots,m'-1\}\cup\{\infty\}.$$

This follows from the definition of $\nu(d)$.

Let $d\in\bar{\mathbb{K}}_{\mathrm{eff}}$ be such that $\nu(d)=0$. We examine the $(1/z)$-series expansion of the corresponding term in the equivariant I-function of $\bar{\mathcal{X}}$:

$$y^d\prod_{i\in\{0,\ldots,m'-1\}\cup\{\infty\}}\frac{\prod_{k=\lceil\langle D_i,d\rangle\rceil}^{\infty}(\bar{D}_i^{\mathbb{T}}+(\langle D_i,d\rangle-k)z)}{\prod_{k=0}^{\infty}(\bar{D}_i^{\mathbb{T}}+(\langle D_i,d\rangle-k)z)}. \tag{6.3}$$

Recall that $\bar{D}_0^{\mathbb{T}}, \ldots, \bar{D}_{m-1}^{\mathbb{T}}, \bar{D}_\infty^{\mathbb{T}} \in H^2(\bar{\mathcal{X}})$ are $\mathbb{T}$-divisor classes corresponding to $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1}, \boldsymbol{b}_\infty$, and $\bar{D}_j^{\mathbb{T}} = 0$ in $H^2_{\mathbb{T}}(\bar{\mathcal{X}})$ for $m \leq j \leq m'-1$. We may factor out copies of z to rewrite (6.3) as

$$\frac{y^d}{z^{\langle \hat{\rho}(\bar{\mathcal{X}}), d\rangle}} \prod_{i \in \{0,\ldots,m'-1\} \cup \{\infty\}} \frac{\prod_{k=\lceil \langle D_i, d\rangle \rceil}^{\infty} (\bar{D}_i^{\mathbb{T}}/z + (\langle D_i, d\rangle - k))}{\prod_{k=0}^{\infty} (\bar{D}_i^{\mathbb{T}}/z + (\langle D_i, d\rangle - k))}. \tag{6.4}$$

where $\hat{\rho}(\bar{\mathcal{X}}) = \sum_{i=0}^{m-1} D_i + D_\infty + \sum_{j=m}^{m'-1} D_j$. So we need

$$\langle \hat{\rho}(\bar{\mathcal{X}}), d\rangle = \sum_{i=0}^{m-1} \langle D_i, d\rangle + \langle D_\infty, d\rangle + \sum_{j=m}^{m'-1} \langle D_j, d\rangle \leq 2. \tag{6.5}$$

Since we need the part taking values in $H^0(\bar{\mathcal{X}})$, we need the terms in (6.4) in which the divisor classes $\bar{D}_0^{\mathbb{T}}, \ldots, \bar{D}_{m-1}^{\mathbb{T}}, \bar{D}_\infty^{\mathbb{T}}$ do not occur. For $0 \leq i \leq m-1$ or $i = \infty$, the fraction

$$\frac{\prod_{k=\lceil \langle D_i, d\rangle \rceil}^{\infty} (\bar{D}_i^{\mathbb{T}}/z + (\langle D_i, d\rangle - k))}{\prod_{k=0}^{\infty} (\bar{D}_i^{\mathbb{T}}/z + (\langle D_i, d\rangle - k))}$$

is proportional to $\bar{D}_j^{\mathbb{T}}$ if $\langle D_j, d\rangle = \lceil \langle D_j, d\rangle \rceil < 0$. Thus we need

$$\langle D_i, d\rangle \geq 0, \quad i \in \{0, \ldots, m-1\} \cup \{\infty\}. \tag{6.6}$$

Also observe that since $d \in \bar{\mathbb{K}}_{\mathrm{eff}}$, $\langle D_j, d\rangle \geq 0$ for $m \leq j \leq m'-1$. So there are only two possible cases: either

- there is exactly one j such that $\langle D_j, d\rangle = 2$ in (6.5) and $\langle D_i, d\rangle = 0$ for $i \neq j$; or
- there are j_1, j_2 such that $\langle D_{j_1}, d\rangle = \langle D_{j_2}, d\rangle = 1$ in (6.5) and $\langle D_i, d\rangle = 0$ for $i \neq j_1, j_2$.

By the fan sequence (2.1), an element $d \in \bar{\mathbb{K}}_{\mathrm{eff}}$ corresponds to an element

$$\sum_{0 \leq i \leq m-1} \langle D_i, d\rangle e_i + \langle D_\infty, d\rangle e_\infty + \sum_{m \leq j \leq m'-1} \langle D_j, d\rangle e_j \in \bigoplus_{0 \leq j \leq m-1} \mathbb{Z} e_j \oplus \mathbb{Z} e_\infty \oplus \bigoplus_{m \leq j \leq m'-1} \mathbb{Z} e_j$$

such that

$$\sum_{0 \leq i \leq m-1} \langle D_i, d\rangle \boldsymbol{b}_i + \langle D_\infty, d\rangle \boldsymbol{b}_\infty + \sum_{m \leq j \leq m'-1} \langle D_j, d\rangle \boldsymbol{b}_j = 0.$$

In order for this equality to hold, we cannot have $\langle D_i, d\rangle = 0$ for all but one i. So we must be in the other case, namely, there are exactly two indices j_1, j_2 such that $\langle D_{j_1}, d\rangle = \langle D_{j_2}, d\rangle = 1$, and $\langle D_i, d\rangle = 0$ for $i \neq j_1, j_2$. Since the vectors $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1}, \boldsymbol{b}_m, \ldots, \boldsymbol{b}_{m'-1}$ belong to the half-space in $N_{\mathbb{R}} \oplus \mathbb{R}$ opposite to the half-space containing $\boldsymbol{b}_\infty$, we must have $\infty \in \{j_1, j_2\}$. As noted in Remark 6.2, the fan $\bar{\Sigma}$ depends on the disk class $\beta \in \pi_2(\mathcal{X}, L)$ in question. There are two possibilities:

- **Case 1: β is a smooth disk class.** This means that $\beta = \beta' + \alpha$ with $\alpha \in H_2(\mathcal{X})$ and $\beta' \in \pi_2(\mathcal{X}, L)$ is the class of a basic smooth disk. In this case $\partial\beta' = \boldsymbol{b}_{i_0}$ for some $0 \leq i_0 \leq m-1$ and $\boldsymbol{b}_\infty = -\boldsymbol{b}_{i_0}$. So the only possible $d \in \bar{\mathbb{K}}_{\mathrm{eff}}$ comes from the relation $\boldsymbol{b}_{i_0} + \boldsymbol{b}_\infty = 0$. In this case the necessary term in the equivariant I-function of $\bar{\mathcal{X}}$ is y^{d_∞}, where $d_\infty = e_{i_0} + e_\infty = \bar{\beta}' \in H_2(\bar{\mathcal{X}}; \mathbb{Q})$.

- **Case 2: β is an orbi-disk class.** This means that $\beta = \beta' + \alpha$ with $\alpha \in H_2(\mathcal{X})$ and $\beta' = \beta_{\nu_{j_0}} \in \pi_2(\mathcal{X}, L)$ is the class of a basic orbi-disk corresponding to $\boldsymbol{b}_{j_0} \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}$ for some $m \leq j_0 \leq m'-1$. In this case $\partial\beta' = \boldsymbol{b}_{j_0}$ and $\boldsymbol{b}_\infty = -\boldsymbol{b}_{j_0}$. So the only possible $d \in \bar{\mathbb{K}}_{\mathrm{eff}}$ comes from the relation $\boldsymbol{b}_{j_0} + \boldsymbol{b}_\infty = 0$. In this case the necessary term in the equivariant I-function of $\bar{\mathcal{X}}$ is y^{d_∞}, where $d_\infty = e_{j_0} + e_\infty$. Note that in this case, d_∞ is *not* a class in $H_2(\bar{\mathcal{X}}; \mathbb{Q})$.

Equating the relevant $1/z^2$-terms in the equivariant I-function and equivariant J-function yields the following

$$y^{d_\infty} = \frac{q_0(y)^2}{2} + \sum_{d \in H_2^{\mathrm{eff}}(\bar{\mathcal{X}})} \sum_{l \geq 0} \sum_{\nu_1, \ldots, \nu_l \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} \frac{\prod_{i=1}^l \tau_{\nu_i}}{l!} \langle [\mathrm{pt}]_{\mathbb{T}}, \prod_{i=1}^l \mathbf{1}_{\bar{\nu}_i} \rangle_{0,l+1,d}^{\bar{\mathcal{X}},\mathbb{T}} q^d. \tag{6.7}$$

6.3.2. *Computing toric mirror maps.* In order to explicitly evaluate (6.7), we will compute the toric mirror map for $\bar{\mathcal{X}}$, which is part of the $1/z$-term in the expansion of the equivariant I-function.

Let $d \in \bar{\mathbb{K}}_{\mathrm{eff}}$. Similar to the calculations in the previous section, we first examine the $(1/z)$-series expansion of the corresponding term in the equivariant I-function of $\bar{\mathcal{X}}$:

$$\begin{aligned} & y^d \prod_{i \in \{0,\ldots,m'-1\} \cup \{\infty\}} \frac{\prod_{k=\lceil \langle D_i, d \rangle \rceil}^{\infty} (\bar{D}_i^{\mathbb{T}} + (\langle D_i, d \rangle - k)z)}{\prod_{k=0}^{\infty} (\bar{D}_i^{\mathbb{T}} + (\langle D_i, d \rangle - k)z)} \mathbf{1}_{\nu(d)} \\ =& \frac{y^d}{z^{\langle \hat{\rho}(\bar{\mathcal{X}}), d \rangle + \mathrm{age}(\nu(d))}} \prod_{i \in \{0,\ldots,m'-1\} \cup \{\infty\}} \frac{\prod_{k=\lceil \langle D_i, d \rangle \rceil}^{\infty} (\bar{D}_i^{\mathbb{T}}/z + (\langle D_i, d \rangle - k))}{\prod_{k=0}^{\infty} (\bar{D}_i^{\mathbb{T}}/z + (\langle D_i, d \rangle - k))} \mathbf{1}_{\nu(d)}. \end{aligned}$$

What we need is the $1/z$-term that takes value in $H^{\leq 2}_{\mathrm{CR},\mathbb{T}}(\bar{\mathcal{X}})$. There are three types.

- **degree 0 term:** This requires that $\nu(d) = 0$. As noted above, this implies $\langle D_i, d \rangle \in \mathbb{Z}$ for all i. Furthermore, we must have $\langle D_i, d \rangle \geq 0$ for all i in order for the term to be of cohomological degree 0. Also, we need $1/z^{\langle \hat{\rho}(\bar{\mathcal{X}}), d \rangle + \mathrm{age}(\nu(d))} = 1/z$, which means that $\langle \hat{\rho}(\bar{\mathcal{X}}), d \rangle = 1$. All together this implies that $\langle D_i, d \rangle = 1$ for exactly one D_i and $= 0$ otherwise. As we have seen, such a class $d \in \bar{\mathbb{K}}_{\mathrm{eff}}$ does not exist. So there is no $H^0(\bar{\mathcal{X}})$-term.

- **degree 2 term from untwisted sector:** This means terms proportional to $\mathbb{T}$-divisors $\bar{D}_i^{\mathbb{T}}$. Again this requires that $\nu(d) = 0$, which implies $\langle D_i, d \rangle \in \mathbb{Z}$ for all i. Furthermore, we must have exactly one $\bar{D}_j^{\mathbb{T}}/z$, which requires $\langle D_j, d \rangle < 0$ for this j and $\langle D_i, d \rangle \geq 0$ for all $i \neq j$. To get the $1/z$-term, we need $\langle \hat{\rho}(\bar{\mathcal{X}}), d \rangle + \mathrm{age}(\nu(d)) = 0$, so we should have $\langle \hat{\rho}(\bar{\mathcal{X}}), d \rangle = 0$.

  For each $j \in \{0, 1, \ldots, m-1\} \cup \{\infty\}$, we define

$$\Omega_j^{\bar{\mathcal{X}}} := \{d \in \bar{\mathbb{K}}_{\mathrm{eff}} \mid \langle \hat{\rho}(\bar{\mathcal{X}}), d \rangle = 0, \nu(d) = 0, \langle D_j, d \rangle \in \mathbb{Z}_{<0} \text{ and } \langle D_i, d \rangle \in \mathbb{Z}_{\geq 0}\ \forall i \neq j\},$$

  and set

$$A_j^{\bar{\mathcal{X}}}(y) := \sum_{d \in \Omega_j^{\bar{\mathcal{X}}}} y^d \frac{(-1)^{-\langle D_j, d \rangle - 1} (-\langle D_j, d \rangle - 1)!}{\prod_{i \neq j} \langle D_i, d \rangle !}.$$

Then the degree 2 term from untwisted sector is given by

$$\sum_{j=0}^{m-1} A_j^{\bar{\mathcal{X}}}(y)\bar{D}_j^{\mathbb{T}}/z + A_\infty^{\bar{\mathcal{X}}}(y)\bar{D}_\infty^{\mathbb{T}}/z.$$

- **degree 2 term from twisted sectors:** This requires that $\nu(d)=\nu$. Since $\mathrm{age}(\nu)=1$, we must have $\langle\hat{\rho}(\bar{\mathcal{X}}),d\rangle=0$. In order to avoid being proportional to a $\mathbb{T}$-divisor, $\langle D_i,d\rangle$ cannot be a negative integer for any i.

  For each $j\in\{m,m+1,\dots,m'-1\}$, we define

  $$\Omega_j^{\bar{\mathcal{X}}} := \{d\in\bar{\mathbb{K}}_{\mathrm{eff}} \mid \langle\hat{\rho}(\bar{\mathcal{X}}),d\rangle=0, \nu(d)=\boldsymbol{b}_j \text{ and } \langle D_i,d\rangle\notin\mathbb{Z}_{<0}\ \forall i\},$$

  and set

  $$A_j^{\bar{\mathcal{X}}}(y) := \sum_{d\in\Omega_j^{\bar{\mathcal{X}}}} y^d \prod_{i\in\{0,\dots,m'-1\}\cup\{\infty\}} \frac{\prod_{k=\lceil\langle D_i,d\rangle\rceil}^{\infty}(\langle D_i,d\rangle-k)}{\prod_{k=0}^{\infty}(\langle D_i,d\rangle-k)}.$$

  Then the degree 2 term from twisted sectors is

  $$\sum_{j=m}^{m'-1} A_j^{\bar{\mathcal{X}}}(y)\mathbf{1}_{\boldsymbol{b}_j}/z.$$

The fan sequence of $\bar{\mathcal{X}}$ is given by

$$0\to\ker\to\widetilde{N}^- := \widetilde{N}\oplus\mathbb{Z}\to N\to 0,$$

and the divisor sequence of $\bar{\mathcal{X}}$ is given by

$$0\to M\to\widetilde{M}^- := (\widetilde{N}^-)^\vee\to\bar{\mathbb{L}}^\vee\to 0.$$

Observe that $\mathrm{rk}(\bar{\mathbb{L}}^\vee)=\mathrm{rk}(\mathbb{L}^\vee)+1=r+1=m'+1-n$ and $\mathrm{rk}(H^2(\bar{\mathcal{X}}))=\mathrm{rk}(H^2(\mathcal{X}))+1=r'+1=m+1-n$. We choose an integral basis

$$\{p_1,\dots,p_r,p_\infty\}\subset\bar{\mathbb{L}}^\vee$$

such that p_a is in the closure of $\widetilde{C}_{\bar{\mathcal{X}}}$ for all a and $p_{r'+1},\dots,p_r\in\sum_{i=m}^{m'-1}\mathbb{R}_{\geq 0}D_i$ so that the images $\{\bar{p}_1,\dots,\bar{p}_{r'},\bar{p}_\infty\}$ of $\{p_1,\dots,p_{r'},p_\infty\}$ under the quotient $\bar{\mathbb{L}}^\vee\otimes\mathbb{Q}\to H^2(\bar{\mathcal{X}};\mathbb{Q})$ form a nef basis of $H^2(\bar{\mathcal{X}};\mathbb{Q})$ and $\bar{p}_a=0$ for $a=r'+1,\dots,r$. And we pick $\{p_1^{\mathbb{T}},...,p_r^{\mathbb{T}},p_\infty^{\mathbb{T}}\}\subset\widetilde{M}^-$ in the way described in Section 2.3. We further assume that $\{p_1,\dots,p_r\}$ gives the original basis of $\mathbb{L}^\vee$ which we chose for $\mathcal{X}$.

Expressing D_i in terms of the basis $\{p_a\}$ defines an integral matrix (Q_{ia}) by

$$D_i=\sum_{a\in\{1,\dots,r\}\cup\{\infty\}} Q_{ia}p_a, \quad Q_{ia}\in\mathbb{Z}.$$

As above, the image of D_i under the quotient $\bar{\mathbb{L}}^\vee\otimes\mathbb{Q}\to H^2(\bar{\mathcal{X}};\mathbb{Q})$ is denoted by $\bar{D}_i$. Then for $i\in\{0,\dots,m-1\}\cup\{\infty\}$, the class $\bar{D}_i^{\mathbb{T}}$ of the toric prime $\mathbb{T}$-divisor $D_i^{\mathbb{T}}$ is given by

$$\bar{D}_i^{\mathbb{T}}=\lambda_i+\sum_{a\in\{1,\dots,r'\}\cup\{\infty\}} Q_{ia}\bar{p}_a^{\mathbb{T}}, \quad \lambda_i\in H^2_{\mathbb{T}}(\mathrm{pt});$$

and for $i=m,\dots,m'-1$, $\bar{D}_i^{\mathbb{T}}=0$ in $H^2(\mathcal{X};\mathbb{R})$.

Hence the coefficient of the $1/z$-term in the equivariant I-function can be expressed as

(6.8)
$$\begin{aligned}
&\sum_{a\in\{1,\dots,r'\}\cup\{\infty\}} \bar{p}_a^{\mathbb{T}} \log y_a + \sum_{j\in\{0,\dots,m-1\}\cup\{\infty\}} A_j^{\bar{\mathcal{X}}}(y)\bar{D}_j^{\mathbb{T}} + \sum_{j=m}^{m'-1} A_j^{\bar{\mathcal{X}}}(y)\mathbf{1}_{\boldsymbol{b}_j} \\
=& \sum_{a\in\{1,\dots,r'\}\cup\{\infty\}} \bar{p}_a^{\mathbb{T}} \log y_a + \sum_{j\in\{0,\dots,m-1\}\cup\{\infty\}} A_j^{\bar{\mathcal{X}}}(y)\left(\lambda_j + \sum_{a\in\{1,\dots,r'\}\cup\{\infty\}} Q_{ja}\bar{p}_a^{\mathbb{T}}\right) + \sum_{j=m}^{m'-1} A_j^{\bar{\mathcal{X}}}(y)\mathbf{1}_{\boldsymbol{b}_j} \\
=& \sum_{a\in\{1,\dots,r'\}\cup\{\infty\}} \left(\log y_a + \sum_{j\in\{0,\dots,m-1\}\cup\{\infty\}} Q_{ja}A_j^{\bar{\mathcal{X}}}(y)\right)\bar{p}_a^{\mathbb{T}} + \sum_{j=m}^{m'-1} A_j^{\bar{\mathcal{X}}}(y)\mathbf{1}_{\boldsymbol{b}_j} + \sum_{j\in\{0,\dots,m-1\}\cup\{\infty\}} \lambda_j A_j^{\bar{\mathcal{X}}}(y).
\end{aligned}$$

On the other hand, the coefficient of the $1/z$-term in the J-function is given by

(6.9)
$$\sum_{a\in\{1,\dots,r'\}\cup\{\infty\}} \bar{p}_a^{\mathbb{T}} \log q_a + \tau_{\mathrm{tw}} = \sum_{a=1}^{r} \bar{p}_a^{\mathbb{T}} \log q_a + \sum_{j=m}^{m'-1} \tau_{\boldsymbol{b}_j}\mathbf{1}_{\boldsymbol{b}_j}.$$

The toric mirror map for $\bar{\mathcal{X}}$ is obtained by comparing (6.8) and (6.9):

(6.10)
$$\begin{aligned}
\log q_a &= \log y_a + \sum_{j\in\{0,\dots,m-1\}\cup\{\infty\}} Q_{ja}A_j^{\bar{\mathcal{X}}}(y), \quad a\in\{1,\dots,r'\}\cup\{\infty\}, \\
\tau_{\boldsymbol{b}_j} &= A_j^{\bar{\mathcal{X}}}(y), \quad j = m,\dots,m'-1,
\end{aligned}$$

and set $q_0(y) := \sum_{j\in\{0,\dots,m-1\}\cup\{\infty\}} \lambda_j A_j^{\bar{\mathcal{X}}}(y)$.

Let us have a closer look at the toric mirror map (6.10) for $\bar{\mathcal{X}}$. First of all, recall that $\bar{\mathbb{K}}_{\mathrm{eff}} = \mathbb{K}_{\mathrm{eff}} \oplus \mathbb{Z}_{\geq 0}d_\infty$, so we can decompose any $d \in \bar{\mathbb{K}}_{\mathrm{eff}}$ as

$$d = d' + kd_\infty,$$

where $d' \in \mathbb{K}_{\mathrm{eff}}$ and $k \in \mathbb{Z}_{\geq 0}$. Suppose that $\langle \hat{\rho}(\bar{\mathcal{X}}), d\rangle = 0$. Then we have

$$0 = \sum_{i=0}^{m'-1} \langle D_i, d'\rangle + \langle D_\infty, d\rangle = \langle \hat{\rho}(\mathcal{X}), d'\rangle + k.$$

But $\mathcal{X}$ is semi-Fano, so $\langle \hat{\rho}(\mathcal{X}), d'\rangle \geq 0$. This implies that $\langle D_\infty, d\rangle = k = 0$, and hence $d = d' \in \mathbb{K}_{\mathrm{eff}}$.

As an immediate consequence, we have $A_\infty^{\bar{\mathcal{X}}} = 0$, since $d \in \Omega_\infty^{\bar{\mathcal{X}}}$ implies that $\langle \hat{\rho}(\bar{\mathcal{X}}), d\rangle = 0$ and $\langle D_\infty, d\rangle < 0$ which is impossible and so $\Omega^{\bar{\mathcal{X}}} = \emptyset$. Also for $j \in \{0, 1, \dots, m-1, m, \dots, m'-1\}$, $d \in \Omega_j^{\bar{\mathcal{X}}}$ implies that $\langle \hat{\rho}(\bar{\mathcal{X}}), d\rangle = 0$, so d lies in $\mathbb{K}_{\mathrm{eff}}$ and hence we have $\Omega_j^{\bar{\mathcal{X}}} = \Omega_j^{\mathcal{X}}$, where

$$\Omega_j^{\mathcal{X}} := \{d \in \mathbb{K}_{\mathrm{eff}} \mid \nu(d) = 0, \langle D_j, d\rangle \in \mathbb{Z}_{<0} \text{ and } \langle D_i, d\rangle \in \mathbb{Z}_{\geq 0}\ \forall i \neq j\}$$

for $j = 0, 1, \dots, m-1$, and

$$\Omega_j^{\mathcal{X}} := \{d \in \mathbb{K}_{\mathrm{eff}} \mid \nu(d) = \boldsymbol{b}_j \text{ and } \langle D_i, d\rangle \notin \mathbb{Z}_{<0}\ \forall i\}$$

for $j = m, m+1, \dots, m'-1$. Here we have made use of the fact that $\hat{\rho}(\mathcal{X}) = 0$.

Proposition 6.14. *The toric mirror map of the toric compactification* $\bar{\mathcal{X}}$ *is of the form*

$$\begin{aligned} \log q_a &= \log y_a + \sum_{j=0}^{m-1} Q_{ja} A_j^{\mathcal{X}}(y), \quad a = 1, \ldots, r', \\ \log q_\infty &= \log y_\infty + A_{i_0}^{\mathcal{X}}(y), \\ \tau_{\boldsymbol{b}_j} &= A_j^{\mathcal{X}}(y), \quad j = m, \ldots, m'-1, \end{aligned} \tag{6.11}$$

when $\beta = \beta_{i_0} + \alpha$ *is a smooth disk class, and of the form*

$$\begin{aligned} \log q_a &= \log y_a + \sum_{j=0}^{m-1} Q_{ja} A_j^{\mathcal{X}}(y), \quad a = 1, \ldots, r', \\ \log q_\infty &= \log y_\infty, \\ \tau_{\boldsymbol{b}_j} &= A_j^{\mathcal{X}}(y), \quad j = m, \ldots, m'-1, \end{aligned} \tag{6.12}$$

when $\beta = \beta_{\nu_{j_0}} + \alpha$ *is an orbi-disk class, where*

$$A_j^{\mathcal{X}}(y) := \sum_{d \in \Omega_j^{\mathcal{X}}} y^d \frac{(-1)^{-\langle D_j, d\rangle - 1}(-\langle D_j, d\rangle - 1)!}{\prod_{i \neq j} \langle D_i, d\rangle!} \tag{6.13}$$

for $j = 0, 1, \ldots, m-1$*, and*

$$A_j^{\mathcal{X}}(y) := \sum_{d \in \Omega_j^{\mathcal{X}}} y^d \prod_{i=0}^{m'-1} \frac{\prod_{k=\lceil \langle D_i, d\rangle \rceil}^{\infty} (\langle D_i, d\rangle - k)}{\prod_{k=0}^{\infty} (\langle D_i, d\rangle - k)} \tag{6.14}$$

for $j = m, m+1, \ldots, m'-1$.

Proof. We already have $\Omega_\infty^{\bar{\mathcal{X}}} = \emptyset$ and $\Omega_j^{\bar{\mathcal{X}}} = \Omega_j^{\mathcal{X}}$ for $j = 0, \ldots, m'-1$. Also, $d \in \Omega_j^{\bar{\mathcal{X}}} = \Omega_j^{\mathcal{X}}$ implies that $\langle D_\infty, d\rangle = 0$. Thus we have $A_\infty^{\bar{\mathcal{X}}} = 0$ and $A_j^{\bar{\mathcal{X}}} = A_j^{\mathcal{X}}$ for $j = 0, \ldots, m'-1$. Finally, when $\beta = \beta_{i_0} + \alpha$ is a smooth disk class, we have $Q_{j\infty} = 1$ for $j \in \{i_0, \infty\}$ and $Q_{j\infty} = 0$ for $j \notin \{i_0, \infty\}$; whereas when $\beta = \beta_{\nu_{j_0}} + \alpha$ is an orbi-disk class, we have $Q_{j\infty} = 1$ for $j \in \{j_0, \infty\}$ and $Q_{j\infty} = 0$ for $j \notin \{j_0, \infty\}$, and in particular, $Q_{j\infty} = 0$ for all $j = 0, \ldots, m-1$. The result now follows from the formula (6.10). □

A key observation is that in both cases (6.11) and (6.12), the toric mirror map of $\bar{\mathcal{X}}$ contains parts which depend only on $\mathcal{X}$:

Proposition 6.15. *The toric mirror map for the toric Calabi-Yau orbifold* $\mathcal{X}$ *is given by*

$$\begin{aligned} \log q_a &= \log y_a + \sum_{j=0}^{m-1} Q_{ja} A_j^{\mathcal{X}}(y), \quad a = 1, \ldots, r', \\ \tau_{\boldsymbol{b}_j} &= A_j^{\mathcal{X}}(y), \quad j = m, \ldots, m'-1, \end{aligned} \tag{6.15}$$

where the functions $A_j^{\mathcal{X}}(y)$ *are defined in* (6.13) *and* (6.14) *in Proposition 6.14.*

Proof. This can be seen by exactly the same calculations as in this subsection applied to the equivariant I-function of $\mathcal{X}$; see also [44, Section 4.1]. □

Remark 6.16.

(1) In the non-equivariant limit $H^*_{\mathbb{T}}(pt) \to H^*(pt)$, we have $\lambda_i \to 0$. Hence $q_0(y) \to 0$ in the non-equivariant limit.
(2) It is clear from the description that (6.11), (6.12), (6.15) do not depend on $\mathbb{T}$-actions, and remain unchanged in the non-equivariant limit $H^*_{\mathbb{T}}(pt) \to H^*(pt)$.
(3) Also note that, for $j = m, m+1, \dots, m'-1$,

$$A_j^{\mathcal{X}}(y) = y^{D_j^\vee} + \textit{higher order terms},$$

where $D_j^\vee \in \mathbb{K}_{\mathit{eff}}$ is the class described in (2.4).

6.4. **Explicit formulas.** In this subsection we put together previous discussions to derive explicit formulas for generating functions of genus 0 open orbifold Gromov-Witten invariants of $\mathcal{X}$.

First we discuss non-equivariant limits.

Proposition 6.17. *The non-equivariant limit of $\langle [\mathrm{pt}]_{\mathbb{T}}, \prod_{i=1}^l \mathbf{1}_{\bar{\nu}_i}\rangle^{\bar{\mathcal{X}},\mathbb{T}}_{0,l+1,d}$ is $\langle [\mathrm{pt}], \prod_{i=1}^l \mathbf{1}_{\bar{\nu}_i}\rangle^{\bar{\mathcal{X}}}_{0,l+1,d}$.*

Proof. If $\bar{\mathcal{X}}$ is projective (this is the case when $\boldsymbol{b}_{i_0} \in N$ lies in the interior of the support $|\Sigma|$ by Proposition 6.5), then moduli spaces of stable maps to $\bar{\mathcal{X}}$ of fixed genus, degree, and number of marked points is compact. In this case the result follows by the discussion in Section 2.5.

Suppose that $\bar{\mathcal{X}}$ is semi-projective but not projective. As noted in Remark 6.13, the moduli space $\mathcal{M}^{cl}_{1+l}(\bar{\mathcal{X}}, \bar{\beta}, \bar{\boldsymbol{x}}, p)$ used to define the invariant $\langle [\mathrm{pt}], \prod_{i=1}^l \mathbf{1}_{\bar{\nu}_i}\rangle^{\bar{\mathcal{X}}}_{0,l+1,d}$ is compact for $p \in L$. In fact it is straightforward to check that $\mathcal{M}^{cl}_{1+l}(\bar{\mathcal{X}}, \bar{\beta}, \bar{\boldsymbol{x}}, p)$ is compact for any p, using the arguments in the proof of Proposition 6.10. A standard cobordism argument shows that the invariant $\langle [\mathrm{pt}], \prod_{i=1}^l \mathbf{1}_{\bar{\nu}_i}\rangle^{\bar{\mathcal{X}}}_{0,l+1,d}$ does not depend on the choice of p. If $p \in \bar{\mathcal{X}}$ is a $\mathbb{T}$-fixed point, then $\mathbb{T}$ acts on $\mathcal{M}^{cl}_{1+l}(\bar{\mathcal{X}}, \bar{\beta}, \bar{\boldsymbol{x}}, p)$ and for such p the moduli space $\mathcal{M}^{cl}_{1+l}(\bar{\mathcal{X}}, \bar{\beta}, \bar{\boldsymbol{x}}, p)$ can be used to define $\mathbb{T}$-equivariant Gromov-Witten invariant $\langle [\mathrm{pt}]_{\mathbb{T}}, \prod_{i=1}^l \mathbf{1}_{\bar{\nu}_i}\rangle^{\bar{\mathcal{X}},\mathbb{T}}_{0,l+1,d}$. Choose $p \in \bar{\mathcal{X}}$ to be a $\mathbb{T}$-fixed point and argue as in Section 2.5, the result follows. ☐

This proposition allows us to obtain the following

Proposition 6.18. *Using the notations in Section 6.2, we have*

$$y^{d_\infty} = q^{\bar{\beta}'} \sum_{\alpha \in H_2^{\mathit{eff}}(\mathcal{X})} \sum_{l \geq 0} \sum_{\nu_1,\dots,\nu_l \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} \frac{\prod_{i=1}^l \tau_{\nu_i}}{l!} n^{\mathcal{X}}_{1,l,\beta'+\alpha}([\mathrm{pt}]_L; \prod_{i=1}^l \mathbf{1}_{\nu_i}) q^{\alpha}. \tag{6.16}$$

Proof. In view of Remark 6.16 and Proposition 6.17, the non-equivariant limit of (6.7) gives

$$y^{d_\infty} = \sum_{d \in H_2^{\mathrm{eff}}(\bar{\mathcal{X}})} \sum_{l \geq 0} \sum_{\nu_1,\dots,\nu_l \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} \frac{\prod_{i=1}^l \tau_{\nu_i}}{l!} \langle [\mathrm{pt}], \prod_{i=1}^l \mathbf{1}_{\bar{\nu}_i}\rangle^{\bar{\mathcal{X}}}_{0,l+1,d} q^d. \tag{6.17}$$

By dimension reason, the invariant $\langle [\mathrm{pt}], \prod_{i=1}^l \mathbf{1}_{\bar{\nu}_i}\rangle^{\bar{\mathcal{X}}}_{0,l+1,d}$ vanishes unless $c_1(\bar{\mathcal{X}}) \cdot d = 2$. Now we have $H_2^{\mathrm{eff}}(\bar{\mathcal{X}}) = \mathbb{Z}_{\geq 0}\bar{\beta}' \oplus H_2^{\mathrm{eff}}(\mathcal{X})$. Also $\bar{\mathcal{X}}$ is semi-Fano and $c_1(\bar{\mathcal{X}}) \cdot \bar{\beta}' = 2$. So $c_1(\bar{\mathcal{X}}) \cdot d = 2$ implies that d must be of the form $\bar{\beta}' + \alpha$ where $\alpha \in H_2^{\mathrm{eff}}(\mathcal{X})$ has Chern number $c_1(\bar{\mathcal{X}}) \cdot \alpha = 0$. The formula (6.16) then follows from the open/closed equality (6.1). ☐

The formula (6.16) can also be written in a more succinct way as

$$y^{d_\infty} = q^{\bar{\beta}'} \sum_{\alpha \in H_2^{\text{eff}}(\mathcal{X})} \sum_{l \geq 0} \frac{1}{l!} n^{\mathcal{X}}_{1,l,\beta'+\alpha}([\text{pt}]_L; \prod_{i=1}^{l} \tau_{\text{tw}}) q^\alpha,$$

where $\tau_{\text{tw}} = \sum_{\nu \in \text{Box}'(\Sigma)^{\text{age}(\nu)=1}} \tau_\nu \mathbf{1}_{\bar{\nu}}$.

Recall that (4.3) gives a Lagrangian isotopy between a moment map fiber L and a fiber F_r of the Gross fibration when r lies in the chamber B_+. Hence the formula (6.16) also gives a computation of the generating functions of genus 0 open orbifold Gromov-Witten invariants defined in (5.3):

$$y^{d_\infty} = q^{\bar{\beta}'}(1 + \delta_j),$$

when β' corresponds to $\beta_j(r)$ under the isotopy (4.3), and

$$y^{d_\infty} = q^{\bar{\beta}'}\tau_\nu(1 + \delta_\nu),$$

when β' corresponds to $\beta_\nu(r)$ under the isotopy (4.3).

The formula (6.16) identifies the generating function of genus 0 open orbifold Gromov-Witten invariants with $y^{d_\infty} q^{-\bar{\beta}'}$. We can now derive an even more explicit formula for computing the orbi-disk invariants using our results in the previous subsection.

Theorem 6.19. *If $\beta' = \beta_{i_0}$ is a basic smooth disk class corresponding to the ray generated by $\boldsymbol{b}_{i_0}$ for some $i_0 \in \{0, 1, \ldots, m-1\}$, then we have*
(6.18)

$$\sum_{\alpha \in H_2^{\text{eff}}(\mathcal{X})} \sum_{l \geq 0} \sum_{\nu_1, \ldots, \nu_l \in \text{Box}'(\Sigma)^{\text{age}=1}} \frac{\prod_{i=1}^{l} \tau_{\nu_i}}{l!} n^{\mathcal{X}}_{1,l,\beta_{i_0}+\alpha}([\text{pt}]_L; \prod_{i=1}^{l} \mathbf{1}_{\nu_i}) q^\alpha = \exp\left(-A^{\mathcal{X}}_{i_0}(y(q,\tau))\right),$$

via the inverse $y = y(q, \tau)$ of the toric mirror map (6.15) of $\mathcal{X}$.

Proof. Recall that in this case, we have $d_\infty = \bar{\beta}'$. Also, $D_\infty = p_\infty$. So $\langle p_\infty, d_\infty \rangle = 1$. On the other hand, since $d_\infty \in H_2(\bar{\mathcal{X}}; \mathbb{Q})$, we have $\langle \bar{D}_i, d_\infty \rangle = \langle D_i, d_\infty \rangle$ for any i and $\langle \bar{p}_a, d_\infty \rangle = \langle p_a, d_\infty \rangle$ for any a. Using the toric mirror map (6.11) for $\bar{\mathcal{X}}$, we have

$$\begin{aligned}
\log q^{d_\infty} &= \sum_{a=1}^{r'} \langle \bar{p}_a, d_\infty \rangle \log q_a + \langle \bar{p}_\infty, d_\infty \rangle \log q_\infty \\
&= \sum_{a=1}^{r'} \langle \bar{p}_a, d_\infty \rangle \left(\log y_a + \sum_{i=0}^{m-1} Q_{ia} A^{\mathcal{X}}_i(y) \right) + \left(\log y_\infty + A^{\mathcal{X}}_{i_0}(y) \right) \\
&= \sum_{a=1}^{r'} \langle \bar{p}_a, d_\infty \rangle \log y_a + \log y_\infty + \sum_{i=0}^{m-1} \left(\sum_{a=1}^{r'} Q_{ia} \langle \bar{p}_a, d_\infty \rangle \right) A^{\mathcal{X}}_i(y) + A^{\mathcal{X}}_{i_0}(y) \\
&= \log y^{d_\infty} + A^{\mathcal{X}}_{i_0}(y) + \sum_{i=0}^{m-1} \left(\langle D_i, d_\infty \rangle - Q_{i\infty} \right) A^{\mathcal{X}}_i(y).
\end{aligned}$$

But $\langle D_i, d_\infty \rangle = Q_{i\infty}$ for $i = 0, \ldots, m-1$, so we arrive at the desired formula. □

Theorem 6.20. *If $\beta' = \beta_{\nu_{j_0}}$ is a basic orbi-disk class corresponding to $\nu_{j_0} \in Box'(\Sigma)^{age=1}$ for some $j_0 \in \{m, m+1, \ldots, m'-1\}$, then we have*
(6.19)

$$\sum_{\alpha \in H_2^{\text{eff}}(\mathcal{X})} \sum_{l \geq 0} \sum_{\nu_1,\ldots,\nu_l \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}} \frac{\prod_{i=1}^l \tau_{\nu_i}}{l!} n^{\mathcal{X}}_{1,l,\beta_{\nu_{j_0}}+\alpha}([\mathrm{pt}]_L; \prod_{i=1}^l \mathbf{1}_{\nu_i}) q^\alpha = y^{D^\vee_{j_0}} \exp\left(-\sum_{i \notin I_{j_0}} c_{j_0 i} A_i^{\mathcal{X}}(y(q,\tau))\right),$$

via the inverse $y = y(q,\tau)$ of the toric mirror map (6.15) of $\mathcal{X}$, where $D^\vee_{j_0} \in \mathbb{K}_{\mathit{eff}}$ is the class defined in (2.4), $I_{j_0} \in \mathcal{A}$ is the anticone of the minimal cone containing $\boldsymbol{b}_{j_0} = \nu_{j_0}$ and $c_{j_0 i} \in \mathbb{Q} \cap [0,1)$ are rational numbers such that $\boldsymbol{b}_{j_0} = \sum_{i \notin I_{j_0}} c_{j_0 i} \boldsymbol{b}_i$.

Proof. In this case, the class $\bar{\beta}' \in H_2(\bar{\mathcal{X}}; \mathbb{Q})$ is given by

$$\bar{\beta}' = \left(\sum_{i \notin I_{j_0}} c_{j_0 i} e_i\right) + e_\infty \in \widetilde{N} \oplus \mathbb{Z}e_\infty = \bigoplus_{i=0}^{m'-1} \mathbb{Z}e_i \oplus \mathbb{Z}e_\infty;$$

while $d_\infty = e_{j_0} + e_\infty$ (recall that this d_∞ is *not* a class in $H_2(\bar{\mathcal{X}}; \mathbb{Q})$). Hence $d_\infty - \bar{\beta}'$ is precisely the class $D^\vee_{j_0} \in \mathbb{K}_{\text{eff}}$. So we can write $y^{d_\infty} q^{-\bar{\beta}'} = y^{D^\vee_{j_0}} y^{\bar{\beta}'} q^{-\bar{\beta}'}$.

Now,

$$\log y^{\bar{\beta}'} = \sum_{a=1}^r \langle p_a, \bar{\beta}' \rangle \log y_a + \langle p_\infty, \bar{\beta}' \rangle \log y_\infty,$$

and using the toric mirror map (6.12) for $\bar{\mathcal{X}}$, we have

$$\begin{aligned} \log q^{\bar{\beta}'} &= \sum_{a=1}^{r'} \langle \bar{p}_a, \bar{\beta}' \rangle \log q_a + \langle \bar{p}_\infty, \bar{\beta}' \rangle \log q_\infty \\ &= \sum_{a=1}^{r'} \langle \bar{p}_a, \bar{\beta}' \rangle \left(\log y_a + \sum_{i=0}^{m-1} Q_{ia} A_i^{\mathcal{X}}(y)\right) + \langle \bar{p}_\infty, \bar{\beta}' \rangle \log y_\infty \\ &= \sum_{a=1}^{r'} \langle \bar{p}_a, \bar{\beta}' \rangle \log y_a + \sum_{i=0}^{m-1} \left(\sum_{a=1}^{r'} Q_{ia} \langle \bar{p}_a, \bar{\beta}' \rangle\right) A_i^{\mathcal{X}}(y) + \langle \bar{p}_\infty, \bar{\beta}' \rangle \log y_\infty. \end{aligned}$$

Since $Q_{i\infty} = 0$ for $i = 0, \ldots, m-1$, we have $\sum_{a=1}^{r'} Q_{ia} \langle \bar{p}_a, \bar{\beta}' \rangle = \langle \bar{D}_i, \bar{\beta}' \rangle$. Also, since $\bar{\beta}' \in H_2(\bar{\mathcal{X}}; \mathbb{Q})$, we have $\langle \bar{D}_i, \bar{\beta}' \rangle = \langle D_i, \bar{\beta}' \rangle$ for any i (and $\langle \bar{p}_a, \bar{\beta}' \rangle = \langle p_a, \bar{\beta}' \rangle$ for any a), so

$$\sum_{i=0}^{m-1} \left(\sum_{a=1}^{r'} Q_{ia} \langle \bar{p}_a, \bar{\beta}' \rangle\right) A_i^{\mathcal{X}}(y) = \sum_{i \notin I_{j_0}} c_{j_0 i} A_i^{\mathcal{X}}(y),$$

and hence

$$\log y^{\bar{\beta}'} - \log q^{\bar{\beta}'} = -\sum_{i \notin I_{j_0}} c_{j_0 i} A_i^{\mathcal{X}}(y).$$

The formula follows. □

Corollary 6.21. *Let F_r be a Lagrangian torus fiber of the Gross fibration over a point r in the chamber B_+. Then we have the following formulas for the generating functions of genus 0 open orbifold Gromov-Witten invariants defined in* (5.3)*:*

$$1 + \delta_i = \exp\left(-A_i^{\mathcal{X}}(y(q,\tau))\right), \tag{6.20}$$

for $i = 0, 1, \ldots, m-1$ when β' is a basic smooth disk class corresponding to $\beta_i(r)$ under the isotopy (4.3)*, and*

$$\tau_{\nu_j} + \delta_{\nu_j} = y^{D_j^\vee} \exp\left(-\sum_{i \notin I_j} c_{ji} A_i^{\mathcal{X}}(y(q,\tau))\right) \tag{6.21}$$

for $j = m, m+1, \ldots, m'-1$ when β' is a basic orbi-disk class corresponding to $\beta_{\nu_j}(r)$ under the isotopy (4.3).

As a by-product of our calculations, we also obtain the following convergence result:

Corollary 6.22. *The generating series of genus 0 open orbifold Gromov-Witten invariants*

$$\sum_{\alpha \in H_2^{\text{eff}}(\mathcal{X})} \sum_{l \geq 0} \sum_{\nu_1, \ldots, \nu_l \in \text{Box}'(\Sigma)^{\text{age}=1}} \frac{\prod_{i=1}^l \tau_{\nu_i}}{l!} n_{1,l,\beta'+\alpha}^{\mathcal{X}}([\text{pt}]_L; \prod_{i=1}^l \mathbf{1}_{\nu_i}) q^\alpha.$$

appearing in (6.16) *and hence those in* (5.3) *are convergent power series in the variables q_a's and τ_{ν_i}'s.*

Proof. As already noted in [72, Section 4.1], the toric mirror map (6.15) is a local isomorphism near $y = 0$. The inverse of the toric mirror map is therefore also analytic near $q = 0$, which allows us to express the variables y_a's as convergent power series in the variables q_a'a and τ_{ν_i}'s. Also note that the expressions in (6.18) and (6.19) are convergent power series in the variables y_a. The result follows. ☐

6.5. **Examples.** In this subsection we present some examples.

(1) $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_m]$. See Example (1) of Section 5.4. There are $m-1$ twisted sectors ν_j, $j = 1, \ldots, m-1$, and each corresponds to a basic orbi-disk class β_{ν_j}. The generating functions of genus 0 open orbifold Gromov-Witten invariants are

$$\tau_j + \delta_{\nu_j}(\tau) = \sum_{k_1, \ldots, k_{m-1} \geq 0} \frac{\tau_1^{k_1} \ldots \tau_{m-1}^{k_{m-1}}}{(k_1 + \ldots + k_{m-1})!} n_{1,l,\beta_{\nu_j}}([\text{pt}]_L; (\mathbf{1}_{\nu_1})^{k_1} \times \ldots \times (\mathbf{1}_{\nu_{m-1}})^{k_{m-1}})$$

where $l = k_1 + \ldots + k_g$ and $\tau = \sum_{i=1}^{m-1} \tau_i \mathbf{1}_{\nu_i} \in H^2_{\text{CR}}(\mathcal{X})$ for $j = 1, \ldots, m-1$. By Theorem 6.20, this is equal to the inverse of the toric mirror map. The toric mirror map for $\mathcal{X}$ was computed explicitly in [32]:

$$\tau_r = g_r(y)$$

where

$$g_r(y) = \sum_{\substack{k_1,\ldots,k_{m-1}\geq 0\\ \langle b(k)\rangle = r/m}} \frac{y_1^{k_1}\ldots y_{n-1}^{k_{m-1}}}{k_1!\ldots k_{m-1}!}\frac{\Gamma(\langle D_0(k)\rangle)}{\Gamma(1+D_0(k))}\frac{\Gamma(\langle D_m(k)\rangle)}{\Gamma(1+D_m(k))},$$

$$b(k) = \sum_{i=1}^{m-1}\frac{i}{n}k_i,\ D_0(k) = -\frac{1}{m}\sum_{i=1}^{m-1}(m-i)k_i,\ D_m(k) = -\frac{1}{m}\sum_{i=1}^{m-1} ik_i,$$

and $\langle r\rangle$ denotes the fractional part of a rational number r. Denote the inverse of $(g_1(y),\ldots,g_{m-1}(y))$ by $(f_1(\tau),\ldots,f_{m-1}(\tau))$. Then

$$f_j(\tau) = \tau_j + \delta_{\nu_j}(\tau), \quad j = 1,...,m-1.$$

Furthermore, the inverse mirror maps $(f_1(\tau),\ldots,f_{m-1}(\tau))$ have been computed in [32, Proposition 6.2]:

$$f_j(\tau) = (-1)^{m-j}e_{m-j}(\kappa_0,...,\kappa_{m-1}), \quad j = 1,...,m-1,$$

where e_j is the j-th elementary symmetric polynomial in m variables, $\zeta := \exp(\pi\sqrt{-1}/m)$, and

$$\kappa_k(\tau_1,...,\tau_{m-1}) = \zeta^{2k+1}\prod_{r=1}^{m-1}\exp\left(\frac{1}{m}\zeta^{(2k+1)r}\tau_r\right). \tag{6.22}$$

From these calculations, we find that quantum corrected mirror of $\mathbb{C}^2/\mathbb{Z}_m$ can be written in the following nice form. Recall that the mirror curve is given by (5.5)

$$uv = 1 + z^m + \sum_{j=1}^{m-1}(\tau_j + \delta_{\nu_j}(\tau))z^j$$

As we have

$$\tau_j + \delta_{\nu_j}(\tau) = f_j(\tau) = (-1)^{m-j}e_{m-j}(\kappa_0,...,\kappa_{m-1}),$$

and also it is easy to check that

$$1 = (-1)^m\kappa_0\cdots\kappa_{m-1}.$$

Hence, SYZ mirror of $[\mathbb{C}^2/\mathbb{Z}_m]$ from Gross fibration is given as

$$uv = \prod_{j=0}^{m-1}(z-\kappa_j). \tag{6.23}$$

For the crepant resolution Y of $X = \mathbb{C}^2/\mathbb{Z}_m$, its genus 0 open Gromov-Witten invariants have been computed in [84]. The result can be stated as follows. Let $D_0,\ldots,D_m$ be the toric prime divisors corresponding to the primitive generators $(0,1),\ldots,(m,1)$ of the fan, $\beta_1,\ldots,\beta_m$ be the corresponding basic disks, and q_i for $i = 1,\ldots,m-1$ be the Kähler parameters corresponding to the (-2)-curves D_i. It turns out that the generating functions of genus 0 open Gromov-Witten invariants

$$q_{j-1}q_{j-2}^2\ldots q_1^{j-1}(1+\delta_j(q)) = q_{j-1}q_{j-2}^2\ldots q_1^{j-1}\left(\sum_\alpha n_{\beta_j+\alpha}q^\alpha\right)$$

are equal to the coefficients of z^j of the following polynomial

$$(1+z)(1+q_1 z)(1+q_1 q_2 z)\dots(1+q_1\dots q_{m-1}z).$$

(2) $\mathcal{X}=[\mathbb{C}^3/\mathbb{Z}_{2g+1}]$. See Example (2) of Section 5.4. In this case $[\mathbb{C}^3/\mathbb{Z}_{2g+1}]$ is obtained as the quotient orbifold of $\mathbb{C}^3$ by the action of $\mathbb{Z}_{2g+1}$ with weights $(1,1,2g-1)$. The standard $(\mathbb{C}^*)^3$ action on $\mathbb{C}^3$ commutes with this $\mathbb{Z}_{2g+1}$ action and induces a $(\mathbb{C}^*)^3$-action on the quotient $[\mathbb{C}^3/\mathbb{Z}_{2g+1}]$.

There is an alternative route to derive the mirror map of $[\mathbb{C}^3/\mathbb{Z}_{2g+1}]$, as follows. The J-function of $(\mathbb{C}^*)^3$-equivariant Gromov-Witten theory of $[\mathbb{C}^3/\mathbb{Z}_{2g+1}]$ coincides with a suitable *twisted* J-function of the orbifold $B\mathbb{Z}_{2g+1}$, considered in [101] and [32]. The J-function of $B\mathbb{Z}_{2g+1}$ has been computed in [73] (see also [32, Proposition 6.1], and the answer is

$$J^{B\mathbb{Z}_{2g+1}}(y,z)=\sum_{k_0,\dots,k_{2g}\geq 0}\frac{1}{z^{k_0+\dots+k_{2g}}}\frac{y_0^{k_0}\dots y_{2g}^{k_{2g}}}{k_0!\dots k_{2g}!}\mathbf{1}_{\langle\sum_{i=0}^{2g} i\frac{k_i}{2g+1}\rangle}.$$

The twisted Gromov-Witten theory we need is the Gromov-Witten theory of $B\mathbb{Z}_{2g+1}$ twisted by the inverse $(\mathbb{C}^*)^3$-equivariant Euler class and the vector bundle $L_1\oplus L_1\oplus L_{2g-1}$, where L_k is the line bundle on $B\mathbb{Z}_{2g+1}$ defined by the 1-dimensional representation $\mathbb{C}_k$ of $\mathbb{Z}_{2g+1}$ on which $1\in\mathbb{Z}_{2g+1}$ acts with eigenvalue $\exp(\frac{2\pi\sqrt{-1}k}{2g+1})$. The generalities of twisted Gromov-Witten theory are developed in [101]. The J-function of the twisted Gromov-Witten theory can be computed by applying [32, Theorem 4.8]. The answer is

$$I^{tw}(y,z)=\sum_{k_0,\dots,k_{2g}\geq 0}\frac{M_{1,k}M_{2,k}M_{3,k}}{z^{k_0+\dots+k_{2g}}}\frac{y_0^{k_0}\dots y_{2g}^{k_{2g}}}{k_0!\dots k_{2g}!}\mathbf{1}_{\langle\sum_{i=0}^{2g} i\frac{k_i}{2g+1}\rangle},$$

where

$$M_{1,k}:=\prod_{m=0}^{\lfloor b(k)\rfloor-1}\left(\lambda_1-\left(\langle b(k)\rangle+m\right)z\right),$$
$$M_{2,k}:=\prod_{m=0}^{\lfloor b(k)\rfloor-1}\left(\lambda_2-\left(\langle b(k)\rangle+m\right)z\right),$$
$$M_{3,k}:=\prod_{N(k)+1\leq m\leq 0}\left(\lambda_3+\left(m-\left(1-\langle c(k)\rangle\right)\right)z\right),$$

and

$$b(k):=\sum_{i=1}^{2g}\frac{ik_i}{2g+1},\quad c(k):=-\sum_{i=1}^{2g}\frac{ik_i}{2g+1}(2g-1),$$
$$N(k):=1+\sum_{i=1}^{2g}\lfloor\frac{i(2g-1)}{2g+1}\rfloor k_i+\lfloor c(k)\rfloor.$$

Here $\lambda_k, k=1,2,3$ is the weight of the k-th factor of $(\mathbb{C}^*)^3$ acting on the k-th factor of $\mathbb{C}^3$.

By [32, Theorem 4.8] it is then straightforward to extract the J-function of $[\mathbb{C}^3/\mathbb{Z}_{2g+1}]$, the mirror map, and generating functions of orbi-disk invariants from $I^{tw}(y,z)$. We leave the details to the readers.

(3) $\mathcal{X} = [\mathbb{C}^n/\mathbb{Z}_n]$. See Example (3) of Section 5.4. In this case there is only one twisted sector ν of age one, and let τ be the corresponding orbifold parameter. The toric mirror map has been computed explicitly in [18], which is

$$\tau = g(y) = \sum_{k=0}^{\infty} \frac{((-\frac{1}{n})\dots(1-k-\frac{1}{n}))^n}{(kn+1)!} y^{kn+1}.$$

Then Theorem 6.20 tells us that the generating function of genus 0 open orbifold Gromov-Witten invariants

$$\tau + \delta_\nu(\tau) = \sum_{k\geq 1} \frac{\tau^k}{k!} n_{1,k,\beta_\nu}([\mathrm{pt}]_L; (\mathbf{1}_\nu)^k)$$

is equal to the inverse series of $g(y)$.

The crepant resolution of $X = \mathbb{C}^n/\mathbb{Z}_n$ is $Y = -K_{\mathbb{P}^{n-1}}$ is the total space of the canonical line bundle over $\mathbb{P}^{n-1}$. Its cohomology is generated by the line class l of $\mathbb{P}^{n-1}$, and let q denote the corresponding Kähler parameter. Let β_0 be the basic disk class corresponding to the zero-section divisor. The generating function of genus 0 open Gromov-Witten invariants

$$1 + \delta(q) = \sum_{k\geq 0} n_{\beta_0 + kl} q^k$$

equals to $\exp g(y)$, where

$$g(y) = \sum_{k>0} (-1)^{nk} \frac{(nk-1)!}{(k!)^n} y^k,$$

and q and y are related by the mirror map

$$q = y \exp(-ng(y)).$$

(4) $\mathcal{X} = K_{\mathbb{F}_2}$. See Example 6.8. $\mathcal{X}$ is a smooth toric manifold, whose fan has primitive generators $\boldsymbol{b}_0 = (0,0,1)$, $\boldsymbol{b}_1 = (-1,1,1)$, $\boldsymbol{b}_2 = (0,1,1)$, $\boldsymbol{b}_3 = (1,1,1)$ and $\boldsymbol{b}_4 = (0,-1,1)$. Note that the Hirzebruch $\mathbb{F}_2$ is not Fano (it is semi-Fano). $\mathcal{X} = K_{\mathbb{F}_2}$ is a new example whose open Gromov-Witten invariants were not computed in previous works.

The primitive generators which are not vertices of $\mathcal{P}$ (the convex hull of $\boldsymbol{b}_1$, $\boldsymbol{b}_3$ and $\boldsymbol{b}_4$) are $\boldsymbol{b}_0$ and $\boldsymbol{b}_2$. Hence

$$n_{\beta_i+\alpha} = 0$$

for $i = 1,3,4$ and $\alpha \neq 0$. Also $n_{\beta_i} = 1$ for $i = 0,\dots,4$. Only the open Gromov-Witten invariants $n_{\beta_0+\alpha}$ and $n_{\beta_2+\alpha}$ for $\alpha \neq 0$ are non-trivial, and we will compute them below.

Take $p_1 = D_0, p_2 = D_2$ to be the basis of $H^2(\mathcal{X},\mathbb{Q})$, and let C_1, C_2 be the dual basis. Denote the (-2) exceptional curve class of $\mathbb{F}_2$ by e, and denote the fiber curve class of $\mathbb{F}_2$ by f. e and f form a basis of $H_2(\mathcal{X};\mathbb{Z})$. By computing the intersection numbers of e and f with p_1 and p_2, we obtain the relation

$$\begin{aligned} f &= C_2 - 2C_1, \\ e &= -2C_2. \end{aligned}$$

The Kähler parameters of C_1 and C_2 are denoted as q_1 and q_2 respectively, while that of e and f are denoted as q^e and q^f respectively. we have

$$\begin{aligned} q^f &= q_2 q_1^{-2}, \\ q^e &= q_2^{-2}. \end{aligned}$$

The corresponding parameters of the complex moduli of the mirror are denoted by (y_1, y_2). One has

$$\begin{aligned} y^f &= y_2 y_1^{-2}, \\ y^e &= y_2^{-2}. \end{aligned}$$

The mirror map is given by

$$\begin{aligned} q_1 &= y_1 \exp(A_1^{\mathcal{X}}(y_1, y_2)) \\ q_2 &= y_2 \exp(A_2^{\mathcal{X}}(y_1, y_2)) \end{aligned}$$

where

$$A_j^{\mathcal{X}}(y) := \sum_{d \in \Omega_j^{\mathcal{X}}} y^d \frac{(-1)^{-\langle D_j, d\rangle - 1}(-\langle D_j, d\rangle - 1)!}{\prod_{i \neq j} \langle D_i, d\rangle !}$$

by Equation 6.13, and

$$\Omega_j^{\mathcal{X}} := \{d \in \mathbb{K}_{\text{eff}} \mid \langle D_j, d\rangle \in \mathbb{Z}_{<0} \text{ and } \langle D_i, d\rangle \in \mathbb{Z}_{\geq 0}\ \forall i \neq j\}.$$

First consider $A_2^{\mathcal{X}}$. For $C = ae + bf$ where $a, b \in \mathbb{Z}$,

$$C \cdot D_2 = -2a + b < 0, C \cdot D_0 = -2b \geq 0$$

implies $b = 0$ and $a \geq 0$. Also $C \cdot D_i \geq 0$ for $i \neq 2$. Hence $\Omega_2^{\mathcal{X}} = \{ke : k \in \mathbb{N}\}$, and

$$A_2^{\mathcal{X}}(y_1, y_2) = \sum_{k=1}^{\infty} y^{ke} \frac{(-1)^{2k-1}(2k-1)!}{(k!)^2} = -\log 2 + \log(1 + \sqrt{1 - 4y^e}).$$

Thus

$$q^e = q_2^{-2} = y^e \exp(-2A_2^{\mathcal{X}}(y_1, y_2)) = \frac{4y^e}{(1 + \sqrt{1 - 4y^e})^2}.$$

Taking the inverse, we obtain

$$y^e = \frac{q^e}{(1 + q^e)^2}$$

and so

$$y_2 = y^{-e/2} = (1 + q^e) q_2.$$

Comparing with $y_2 = q_2 \exp(-A_2^{\mathcal{X}}(y_1, y_2))$, this implies

$$\exp(-A_2^{\mathcal{X}}(y_1, y_2)) = 1 + q^e$$

under the mirror map. By Theorem 6.19, this is the generating function of open Gromov-Witten invariants:

$$\sum_{\alpha} n_{\beta_2+\alpha} q^{\alpha} = 1 + q^{e}.$$

Thus $n_{\beta_2+\alpha} = 1$ when $\alpha = 0, e$, and zero for all other classes α. The hypergeometric series $A_2^{\mathcal{X}}$ above also gives the mirror map of $\mathbb{F}_2$. This is the analytic reason why the open Gromov-Witten invariants above are the same as those of $\mathbb{F}_2$:

$$n^{\mathcal{X}}_{\beta_2+\alpha} = n^{\mathbb{F}_2}_{\beta_2+\alpha}.$$

It is geometrically intuitive: the bubbling contributions of the curve class e to β_2 in $\mathbb{F}_2$ are the same as that in $K_{\mathbb{F}_2}$, because D_2 in $K_{\mathbb{F}_2}$ is just the product of the corresponding divisor in $\mathbb{F}_2$ with the complex line $\mathbb{C}$.

Now consider $A_1^{\mathcal{X}}$. For $C = ae + bf$ where $a, b \in \mathbb{Z}$,

$$C \cdot D_2 = -2a + b \geq 0, C \cdot D_0 = -2b < 0$$

implies $b \geq 2a > 0$. Also $C \cdot D_i \geq 0$ for $i \neq 2$. Hence $\Omega_1^{\mathcal{X}} = \{kf + a(e+2f) : a \in \mathbb{N}, k \in \mathbb{Z}_{\geq 0}\}$. We have

$$A_1^{\mathcal{X}}(y_1, y_2) = \sum_{a=1}^{\infty} \sum_{k=0}^{\infty} y^{kf+a(e+2f)} \frac{(-1)^{2(2a+k)-1}(2(2a+k)-1)!}{(k!)(a!)^2(2a+k)!}.$$

By Theorem 6.19, this gives the generating function of open Gromov-Witten invariants via the inverse mirror map $y(q)$:

$$\sum_{\alpha} n_{\beta_0+\alpha} q^{\alpha} = \exp(-A_1^{\mathcal{X}}(y_1(q), y_2(q)))$$

where the mirror map $q(y)$ is given by

$$q^f = y^f \exp(-2A_1(y^e, y^f) + A_2(y^e))$$
$$q^e = y^e \exp(-2A_2(y^e)).$$

The following table can be obtained by inverting the mirror map using computers:

$n_{\beta_0+ae+bf}$	$a=0$	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$
$b=0$	1	0	0	0	0	0	0
$b=1$	0	0	0	0	0	0	0
$b=2$	0	-3	0	0	0	0	0
$b=3$	0	-20	-20	0	0	0	0
$b=4$	0	-105	-294	-105	0	0	0
$b=5$	0	-504	-2808	-2808	-504	0	0
$b=6$	0	-2310	-21835	-42867	-21835	-2310	0

7. Open mirror theorems

In this section we define the SYZ map, and prove an open mirror theorem which says that the SYZ map coincides with the inverse of the toric mirror map. In the case of toric Calabi-Yau manifolds, this theorem implies that the inverse of a mirror map defined using period integrals (so this is *not* the toric mirror map) can be expressed explicitly in terms of generating functions of genus 0 open Gromov-Witten invariants defined by Fukaya-Oh-Ohta-Ono [49]. This confirms in the affirmative a conjecture of Gross-Siebert [67, Conjecture 0.2], which was later made precise in [20, Conjecture 1.1] in the toric Calabi-Yau case.

7.1. **The SYZ map.**

7.1.1. *Kähler moduli.* As before, $\mathcal{X}$ is a toric Calabi-Yau orbifold as in Setting 4.3. Let $\widetilde{C}_{\mathcal{X}} \subset \mathbb{L}^\vee \otimes \mathbb{R}$ be the extended Kähler cone of $\mathcal{X}$ as defined in Section 2.6. Recall that there is a splitting $\widetilde{C}_{\mathcal{X}} = C_{\mathcal{X}} + \sum_{j=m}^{m'-1} \mathbb{R}_{>0} D_j \subset \mathbb{L}^\vee \otimes \mathbb{R}$, where $C_{\mathcal{X}} \subset H^2(\mathcal{X};\mathbb{R})$ is the Kähler cone of $\mathcal{X}$. We define the complexified (extended) Kähler moduli space of $\mathcal{X}$ as

$$\mathcal{M}_K(\mathcal{X}) := \left(\widetilde{C}_{\mathcal{X}} + \sqrt{-1}H^2(\mathcal{X},\mathbb{R})\right)/H^2(\mathcal{X},\mathbb{Z}) + \sum_{j=m}^{m'-1} \mathbb{C}D_j.$$

Elements of $\mathcal{M}_K(\mathcal{X})$ are represented by complexified (extended) Kähler class

$$\omega^{\mathbb{C}} = \omega + \sqrt{-1}B + \sum_{j=m}^{m'-1} \tau_j D_j,$$

where $\omega \in C_{\mathcal{X}}$, $B \in H^2(\mathcal{X},\mathbb{R})$ and $\tau_j \in \mathbb{C}$.

We identify $\mathcal{M}_K(\mathcal{X})$ with $(\Delta^*)^{r'} \times \mathbb{C}^{r-r'}$, where Δ^* is the punctured unit disk, via the following coordinates:

$$q_a = \exp\left(-2\pi \int_{\gamma_a} \left(\omega + \sqrt{-1}B\right)\right), \quad a = 1, \ldots, r',$$
$$\tau_j \in \mathbb{C}, \quad j = m, \ldots, m'-1,$$

where $\{\gamma_1, \ldots, \gamma_{r'}\}$ is the integral basis of $H_2(\mathcal{X};\mathbb{Z})$ we chose in Section 2.6. A partial compactification of $\mathcal{M}_K(\mathcal{X})$ is given by $(\Delta^*)^{r'} \times \mathbb{C}^{r-r'} \subset \Delta^{r'} \times \mathbb{C}^{r-r'}$.

Recall that the SYZ mirror of $\mathcal{X}$ equipped with a Gross fibration $\mu : \mathcal{X} \to B$ is given by

$$\check{\mathcal{X}}_{q,\tau} = \left\{(u, v, z_1, \ldots, z_{n-1}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} \mid uv = G_{(q,\tau)}(z_1, \ldots, z_{n-1})\right\},$$

where

$$G_{(q,\tau)}(z_1, \ldots, z_{n-1}) = \sum_{i=0}^{m-1} C_i(1+\delta_i) z^{\boldsymbol{b}_i} + \sum_{j=m}^{m'-1} C_{\nu_j}(\tau_{\nu_j} + \delta_{\nu_j}) z^{\nu_j},$$

and the coefficients $C_i, C_{\nu_j} \in \mathbb{C}$ are subject to the following constraints:

$$\prod_{i=0}^{m-1} C_i^{Q_{ia}} = q_a, \quad a = 1, \ldots, r',$$
$$\prod_{i=0}^{m-1} C_i^{Q_{ia}} \prod_{j=m}^{m'-1} C_{\nu_j}^{Q_{ja}} = \prod_{j=m}^{m'-1} \left(q^{D_j^\vee}\right)^{-Q_{ja}}, \quad a = r'+1, \ldots, r,$$

where $q^{D_j^\vee} = \prod_{a=1}^{r'} q_a^{\langle p_a, D_j^\vee\rangle}$.

7.1.2. *Complex moduli.* On the mirror side, recall that

$$\mathcal{P} \cap N = \{\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1}, \boldsymbol{b}_m, \ldots, \boldsymbol{b}_{m'-1}\}$$

and $\mathcal{P}$ is contained in the hyperplane $\{v \in N_{\mathbb{R}} \mid ((0,1)\,,\, v) = 1\}$. Denote by $L(\mathcal{P}) \simeq \mathbb{C}^{m'}$ the space of Laurent polynomials $G \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_{n-1}^{\pm 1}]$ of the form $\sum_{i=0}^{m'-1} C_i z^{\boldsymbol{b}_i}$, i.e. those with Newton polytope $\mathcal{P}$. Let $\mathbb{P}_{\mathcal{P}}$ be the projective toric variety defined by the normal fan of $\mathcal{P}$. In Batyrev [7], a Laurent polynomial $G \in L(\mathcal{P})$ is defined to be $\mathcal{P}$-*regular* if the intersection of the closure $\bar{Z}_f \subset \mathbb{P}_{\mathcal{P}}$, of the associated affine hypersurface $Z_f := \{(z_1, \ldots, z_{n-1}) \in (\mathbb{C}^\times)^{n-1} \mid f(z_1, \ldots, z_{n-1}) = 0\}$ in $(\mathbb{C}^\times)^{n-1}$, with every torus orbit $O \subset \mathbb{P}_{\mathcal{P}}$ is a smooth subvariety of codimension 1 in O. Denote by $L_{\text{reg}}(\mathcal{P})$ the space of all $\mathcal{P}$-regular Laurent polynomials.

Following Batyrev [7] and Konishi-Minabe [78], we define the complex moduli space $\mathcal{M}_{\mathbb{C}}(\check{\mathcal{X}})$ of the mirror $\check{\mathcal{X}}$ to be the GIT quotient of $L_{\text{reg}}(\mathcal{P})$ by a natural $(\mathbb{C}^\times)^n$-action, which is nonempty and has complex dimension $r = m' - n$ [7]. It parametrizes a family of non-compact Calabi-Yau manifolds $\{\check{\mathcal{X}}_y\}$:

$$\check{\mathcal{X}}_y := \left\{(u, v, z_1, \ldots, z_{n-1}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} \mid uv = G_y(z_1, \ldots, z_{n-1})\right\}, \tag{7.1}$$

where

$$G_y(z_1, \ldots, z_{n-1}) = \sum_{i=0}^{m-1} \check{C}_i z^{\boldsymbol{b}_i} + \sum_{j=m}^{m'-1} \check{C}_{\nu_j} z^{\nu_j},$$

and the coefficients $\check{C}_i, \check{C}_{\nu_j} \in \mathbb{C}$ are subject to the following constraints:

$$\prod_{i=0}^{m-1} \check{C}_i^{Q_{ia}} = y_a, \quad a = 1, \ldots, r',$$

$$\prod_{i=0}^{m-1} \check{C}_i^{Q_{ia}} \prod_{j=m}^{m'-1} \check{C}_{\nu_j}^{Q_{ja}} = y_a, \quad a = r'+1, \ldots, r.$$

Note that the non-compact Calabi-Yau manifolds in the family (7.1) may become singular and develop orbifold singularities when some of the y_a's go to zero.

To define period integrals, we let $\check{\Omega}_y$ be the holomorphic volume form on $\check{\mathcal{X}}_y$ defined by (cf. Proposition 5.3)

$$\check{\Omega}_y = \operatorname{Res}\left(\frac{1}{uv - G_y(z_1, \ldots, z_{n-1})} d\log z_0 \wedge \cdots \wedge d\log z_{n-1} \wedge du \wedge dv\right),$$

where $G_y(z_1, \ldots, z_{n-1}) := \sum_{i=0}^{m-1} \check{C}_i z^{\boldsymbol{b}_i} + \sum_{j=m}^{m'-1} \check{C}_{\nu_j} z^{\nu_j}$.

7.1.3. *Two mirror maps.*

Definition 7.1. *We define the* SYZ map *as follows:*

$$
\begin{aligned}
&\mathcal{F}^{\mathrm{SYZ}}: \mathcal{M}_K(\mathcal{X}) \to \mathcal{M}_{\mathbb{C}}(\check{\mathcal{X}}), \quad y \mapsto \mathcal{F}^{\mathrm{SYZ}}(q,\tau) \\
&y_a := q_a \prod_{i=0}^{m-1} (1+\delta_i)^{Q_{ia}}, \quad a = 1,\ldots,r', \\
&y_a := \prod_{i=0}^{m-1} (1+\delta_i)^{Q_{ia}} \prod_{j=m}^{m'-1} \left(q^{-D_j^\vee}\left(\tau_{\nu_j}+\delta_{\nu_j}\right)\right)^{Q_{ja}}, \quad a = r'+1,\ldots,r,
\end{aligned}
\tag{7.2}
$$

where $q^{-D_j^\vee} := \prod_{a=1}^{r'} q_a^{\langle p_a, D_j^\vee\rangle}$*, and* $1+\delta_i$ *and* $\tau_{\nu_j}+\delta_{\nu_j}$ *are the generating functions of genus 0 open orbifold Gromov-Witten invariants in* $\mathcal{X}$ *relative to a Lagrangian torus fiber of a Gross fibration* $\mu: \mathcal{X} \to B$*, defined in* (5.3).

On the other hand, recall that the toric mirror map (6.15) for $\mathcal{X}$ is given by

$$
\begin{aligned}
&\mathcal{F}^{\mathrm{mirror}}: \mathcal{M}_{\mathbb{C}}(\check{\mathcal{X}}) \to \mathcal{M}_K(\mathcal{X}), \quad (q,\tau) \mapsto \mathcal{F}^{\mathrm{mirror}}(y) \\
&q_a = y_a \prod_{j=0}^{m-1} \exp\left(A_j^{\mathcal{X}}(y)\right)^{Q_{ja}}, \quad a = 1,\ldots,r', \\
&\tau_{\boldsymbol{b}_j} = A_j^{\mathcal{X}}(y), \quad j = m,\ldots,m'-1.
\end{aligned}
$$

7.2. **Open mirror theorems.**

7.2.1. *Orbifolds.* We are now ready to prove one of the main results in this paper:

Theorem 7.2 (Open mirror theorem for toric Calabi-Yau orbifolds - Version 1)**.** *Let* $\mathcal{X}$ *be a toric Calabi-Yau orbifold* $\mathcal{X}$ *as in Setting 4.3. Then locally around* $(q,\tau)=0$*, the SYZ map is inverse to the toric mirror map, i.e. we have*

$$
\mathcal{F}^{\mathrm{SYZ}} = \left(\mathcal{F}^{\mathrm{mirror}}\right)^{-1}. \tag{7.3}
$$

In particular, this holds for a semi-projective toric Calabi-Yau manifold.

Proof. Recall that the toric mirror map $\mathcal{F}^{\mathrm{mirror}}$ is a local isomorphism near $y=0$, so we can consider its inverse $\left(\mathcal{F}^{\mathrm{mirror}}\right)^{-1}$ given by $y = y(q,\tau)$ near $(q,\tau)=0$.

For $a = 1,\ldots,r'$, we have, by the formula (6.20),

$$
\log q_a + \sum_{i=0}^{m-1} Q_{ia}(1+\delta_i) = \log q_a - \sum_{i=0}^{m-1} Q_{ia} A_i^{\mathcal{X}}(y(q,\tau)) = \log y_a.
$$

For $a = r'+1, \ldots, r$, we have, by the formulas (6.20) and (6.21),

$$
\begin{aligned}
&\sum_{j=m}^{m'-1} Q_{ja}\left(\log q^{-D_j^\vee} + \log(\tau_{\nu_j}+\delta_{\nu_j})\right)\\
&= \sum_{j=m}^{m'-1} Q_{ja}\left(-\sum_{b=1}^{r'}\langle p_b, D_j^\vee\rangle \log q_b + \sum_{b=1}^{r}\langle p_b, D_j^\vee\rangle \log y_b - \sum_{i\notin I_j} c_{ji} A_i^{\mathcal{X}}(y(q,\tau))\right)\\
&= \sum_{b=r'+1}^{r}\left(\sum_{j=m}^{m'-1} Q_{ja}\langle p_b, D_j^\vee\rangle\right)\log y_b + \sum_{j=m}^{m'-1} Q_{ja}\left(\sum_{b=1}^{r'}\langle p_b, D_j^\vee\rangle \log\left(y_b q_b^{-1}\right)\right)\\
&\qquad - \sum_{j=m}^{m'-1} Q_{ja}\left(\sum_{i\notin I_j} c_{ji} A_i^{\mathcal{X}}(y(q,\tau))\right).
\end{aligned}
\tag{7.4}
$$

Now, the definition of $D_j^\vee$ implies that $\langle D_i, D_j^\vee\rangle = \delta_{ij}$ for $m \le i, j \le m'-1$. Since $D_i = \sum_{a=1}^r Q_{ia} p_a$ and $Q_{ia} = 0$ for $1 \le a \le r'$ and $m \le i \le m'-1$, we have $\sum_{a=r'+1}^r Q_{ia}\langle p_a, D_j^\vee\rangle = \delta_{ij}$ for $m \le i, j \le m'-1$. This shows that the $(r-r')\times(r-r')$ square matrices (Q_{ia}) and $(\langle p_a, D_i^\vee\rangle)$ (where $m \le i \le m'-1$ and $r'+1 \le a \le r$) are inverse to each other (note that $r - r' = m' - m$), so

$$\sum_{j=m}^{m'-1} Q_{ja}\langle p_b, D_j^\vee\rangle = \delta_{ab}$$

for $r'+1 \le a, b \le r$. Hence the first term of the last expression in (7.4) is precisely given by $\log y_a$.

On the other hand, we have

$$
\begin{aligned}
\sum_{b=1}^{r'}\langle p_b, D_j^\vee\rangle \log\left(y_b q_b^{-1}\right) &= \sum_{b=1}^{r'}\langle p_b, D_j^\vee\rangle\left(-\sum_{k=0}^{m-1} Q_{kb} A_k^{\mathcal{X}}(y)\right)\\
&= -\sum_{k=0}^{m-1}\left(\sum_{b=1}^{r'} Q_{kb}\langle p_b, D_j^\vee\rangle\right) A_k^{\mathcal{X}}(y),
\end{aligned}
$$

and using the above formula $\sum_{j=m}^{m'-1} Q_{ja}\langle p_b, D_j^\vee\rangle = \delta_{ab}$ again, we can write

$$
\begin{aligned}
\sum_{k=0}^{m-1} Q_{ka}\log(1+\delta_k) = -\sum_{k=0}^{m-1} Q_{ka} A_k^{\mathcal{X}}(y) &= -\sum_{k=0}^{m-1}\left(\sum_{b=r'+1}^{r} Q_{kb}\left(\sum_{j=m}^{m'-1} Q_{ja}\langle p_b, D_j^\vee\rangle\right)\right) A_k^{\mathcal{X}}(y)\\
&= -\sum_{j=m}^{m'-1} Q_{ja}\left(\sum_{b=r'+1}^{r}\langle p_b, D_j^\vee\rangle\left(\sum_{k=0}^{m-1} Q_{kb} A_k^{\mathcal{X}}(y)\right)\right)
\end{aligned}
$$

We compute the sum

$$
\begin{aligned}
&\sum_{k=0}^{m-1} Q_{ka} \log(1+\delta_k) + \sum_{j=m}^{m'-1} Q_{ja} \left(\sum_{b=1}^{r'} \langle p_b, D_j^\vee \rangle \log\left(y_b q_b^{-1} \right) \right) \\
=& -\sum_{j=m}^{m'-1} Q_{ja} \left(\sum_{b=r'+1}^{r} \langle p_b, D_j^\vee \rangle \left(\sum_{k=0}^{m-1} Q_{kb} A_k^{\mathcal{X}}(y) \right) \right) \\
&\quad - \sum_{j=m}^{m'-1} Q_{ja} \left(\sum_{b=1}^{r'} \langle p_b, D_j^\vee \rangle \left(\sum_{k=0}^{m-1} Q_{kb} A_k^{\mathcal{X}}(y) \right) \right) \\
=& -\sum_{j=m}^{m'-1} Q_{ja} \left(\sum_{k=0}^{m-1} \left(\sum_{b=1}^{r} Q_{kb} \langle p_b, D_j^\vee \rangle \right) A_k^{\mathcal{X}}(y) \right) \\
=& -\sum_{j=m}^{m'-1} Q_{ja} \left(\sum_{k=0}^{m-1} \langle D_k, D_j^\vee \rangle A_k^{\mathcal{X}}(y) \right) \\
=& \sum_{j=m}^{m'-1} Q_{ja} \left(\sum_{k\notin I_j} c_{jk} A_k^{\mathcal{X}}(y) \right),
\end{aligned}
$$

which cancels with the third term of the last expression in (7.4). Hence we conclude that

$$
\sum_{i=0}^{m-1} Q_{ia} \log(1+\delta_i) + \sum_{j=m}^{m'-1} Q_{ja} \left(\log q^{-D_j^\vee} + \log(\tau_{\nu_j} + \delta_{\nu_j}) \right) = \log y_a
$$

for $a = r'+1, \ldots, r$.

This proves the theorem. □

7.2.2. *Connection with period integrals.* Traditionally, mirror maps are defined in terms of period integrals, which are integrals $\int_\Gamma \check{\Omega}_y$ of the holomorphic volume form $\check{\Omega}_y$ over middle-dimensional cycles $\Gamma \in H_n(\check{\mathcal{X}}_y; \mathbb{C})$ (see, e.g. [37, Chapter 6]). The following theorem shows that the inverse of such a mirror map also coincides with the SYZ map:

Theorem 7.3 (Open mirror theorem for toric Calabi-Yau orbifolds - Version 2)**.** *Let $\mathcal{X}$ be a toric Calabi-Yau orbifold $\mathcal{X}$ as in Setting 4.3. Then there exist linearly independent cycles $\Gamma_1, \ldots, \Gamma_r \in H_n(\check{\mathcal{X}}_y; \mathbb{C})$ such that*

$$
\begin{aligned}
q_a &= \exp\left(-\int_{\Gamma_a} \check{\Omega}_{\mathcal{F}^{\mathrm{SYZ}}(q,\tau)} \right), \quad a = 1, \ldots, r', \\
\tau_{\boldsymbol{b}_j} &= \int_{\Gamma_{j-m+r'+1}} \check{\Omega}_{\mathcal{F}^{\mathrm{SYZ}}(q,\tau)}, \quad j = m, \ldots, m'-1.
\end{aligned} \tag{7.5}
$$

where $\mathcal{F}^{\mathrm{SYZ}}(q,\tau)$ is the SYZ map in Definition 7.1.

When $\mathcal{X}$ is a toric Calabi-Yau *manifold*, we do not have extra vectors so that $m' = m$ and $r = r'$, and there are no twisted sectors insertions in the invariants $n^{\mathcal{X}}_{1,l,\beta_i+\alpha}([\mathrm{pt}]_L)$.

Corollary 7.4 (Open mirror theorem for toric Calabi-Yau manifolds)**.** *Let $\mathcal{X}$ be a semi-projective toric Calabi-Yau manifold. Then there exist linearly independent cycles $\Gamma_1, \ldots, \Gamma_r \in H_n(\check{\mathcal{X}}_y; \mathbb{C})$ such that*

$$q_a = \exp\left(-\int_{\Gamma_a} \check{\Omega}_{\mathcal{F}^{\mathrm{SYZ}}(q,\tau)}\right), \quad a = 1, \ldots, r,$$

where $\mathcal{F}^{\mathrm{SYZ}}(q)$ is the SYZ map in Definition 7.1, now defined in terms of the generating functions $1+\delta_i$ of genus 0 open Gromov-Witten invariants $n^{\mathcal{X}}_{1,l,\beta_i+\alpha}([\mathrm{pt}]_L)$.

Theorem 7.3 and Corollary 7.4 give an enumerative meaning to period integrals, which was first envisioned by Gross and Siebert in [67, Conjecture 0.2 and Remark 5.1] where they conjectured that period integrals of the mirror can be interpreted as (virtual) counting of *tropical* disks (instead of holomorphic disks) in the base of an SYZ fibration for a *compact* Calabi-Yau manifold; in [68, Example 5.2], they also observed a precise relation between the so-called *slab functions*, which appeared in their program, and period computations for the toric Calabi-Yau 3-fold $K_{\mathbb{P}^2}$ in [60]. A more precise relation in the case of toric Calabi-Yau manifolds was later formulated in [20, Conjecture 1.1].[6]

We should point out that Corollary 7.4 is weaker than [20, Conjecture 1.1] in the sense that the cycles $\Gamma_1, \ldots, \Gamma_r$ are allowed to have complex coefficients instead of being *integral*. In the special case where $\mathcal{X}$ is the total space of the canonical bundle over a compact toric Fano manifold, Corollary 7.4 was proven in [23]. As discussed in [23, Section 5.2], to enhance Corollary 7.4 to [20, Conjecture 1.1], one needs to study the monodromy of $H_n(\check{\mathcal{X}}_y; \mathbb{Z})$ around the limit points in the complex moduli space $\mathcal{M}_{\mathbb{C}}(\check{\mathcal{X}})$.

Theorem 7.3 is essentially a consequence of Theorem 7.2 and the analysis of the relationships between period integrals over n-cycles of the mirror and GKZ hypergeometric systems in [23, Section 4]. Recall that the *Gel'fand-Kapranov-Zelevinsky (GKZ) system* [52, 53] of differential equations (also called A-hypergeometric system) associated to $\mathcal{X}$, or to the set of lattice points $\Sigma(1) = \{\boldsymbol{b}_0, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_{m-1}\}$, is the following system of partial differential equations on functions $\Phi(\check{C})$ of $\check{C} = (\check{C}_0, \check{C}_1, \ldots, \check{C}_{m-1}) \in \mathbb{C}^m$:

$$\begin{gathered}\left(\sum_{i=0}^{m-1} \boldsymbol{b}_i \check{C}_i \partial_i\right)\Phi(\check{C}) = 0,\\ \left(\prod_{i:\langle D_i, d\rangle > 0} \partial_i^{\langle D_i, d\rangle} - \prod_{i:\langle D_i, d\rangle < 0} \partial_i^{-\langle D_i, d\rangle}\right)\Phi(\check{C}) = 0, \quad d \in \mathbb{L},\end{gathered} \tag{7.6}$$

where $\partial_i = \partial/\partial\check{C}_i$ for $i = 0, 1, \ldots, m-1$. Notice that the first equation in (7.6) consists of n equations, so altogether there are $n + r = m$ equations. By [23, Proposition 14], the period integrals

$$\int_\Gamma \check{\Omega}_y, \quad \Gamma \in H_n(\check{\mathcal{X}}_y; \mathbb{Z}),$$

provide a $\mathbb{C}$-basis of solutions to the GKZ hypergeometric system (7.6); see also [71] and [78, Corollary A.16].

[6] It was wrongly asserted that the cycles $\Gamma_1, \ldots, \Gamma_r$ form a basis of $H_n(\check{\mathcal{X}}_y; \mathbb{C})$ in [20, Conjecture 1.1] while they should just be linearly independent cycles; see [23, Conjecture 2] for the correct version.

Now Theorem 7.3 follows from the following

Lemma 7.5. *The components*

$$\log q_a = \log y_a + \sum_{j=0}^{m-1} Q_{ja} A_j^{\mathcal{X}}(y), \quad a = 1, \ldots, r',$$

$$\tau_{\boldsymbol{b}_j} = A_j^{\mathcal{X}}(y), \quad j = m, \ldots, m' - 1,$$

of the toric mirror map (6.15) *of a toric Calabi-Yau orbifold* $\mathcal{X}$ *are solutions to the GKZ hypergeometric system* (7.6)*.*

Proof. The proof is more or less the same as that of [23, Theorem 12], which in turn is basically a corollary of a result of Iritani [72, Lemma 4.6]. We first fix $i_0 \in \{0, \ldots, m' - 1\}$, and consider the corresponding toric compactification $\bar{\mathcal{X}}$. For $i \in \{0, \ldots, m - 1\} \cup \{\infty\}$, we set

$$\mathcal{D}_i = \sum_{a \in \{1, \ldots, r\} \cup \{\infty\}} Q_{ia} y_a \frac{\partial}{\partial y_a},$$

and, for $d \in \bar{\mathbb{L}}$, we define a differential operator

$$\Box_d := \prod_{i:\langle D_i, d\rangle > 0} \prod_{k=0}^{\langle D_i, d\rangle - 1} (\mathcal{D}_i - k) - y^d \prod_{i:\langle D_i, d\rangle < 0} \prod_{k=0}^{-\langle D_i, d\rangle - 1} (\mathcal{D}_i - k).$$

Now [72, Lemma 4.6] says that the I-function $I_{\bar{\mathcal{X}}}(y, z)$ satisfy the following system of GKZ-type differential equations:

$$\Box_d \Psi = 0, \quad d \in \bar{\mathbb{L}}. \tag{7.7}$$

In particular, the components

$$\log q_a = \log y_a + \sum_{j=0}^{m-1} Q_{ja} A_j^{\mathcal{X}}(y), \quad a = 1, \ldots, r',$$

$$\tau_{\boldsymbol{b}_j} = A_j^{\mathcal{X}}(y), \quad j = m, \ldots, m' - 1,$$

of the toric mirror map of $\mathcal{X}$, which are contained in the toric mirror map (6.11) of $\bar{\mathcal{X}}$, are solutions to the above system.

Hence, it suffices to show that solutions to the above system also satisfy the GKZ hypergeometric system (7.6). This was shown in the proof of [23, Theorem 12], so we will just describe the argument briefly. First of all, we have $\sum_{i=0}^{m'-1} Q_{ia} = 0$ for $a = 1, \ldots, r$. Together with the fact that $y_a = \prod_{i=0}^{m-1} \check{C}_i^{Q_{ia}}$ for $a = 1, \ldots, r$, one can see that the first n equations in (7.6) are satisfied by any solution of (7.7). On the other hand, it is not hard to compute, using the fact that $\langle D_\infty, d\rangle = 0$ for $d \in \mathbb{L} \oplus 0 \subset \bar{\mathbb{L}}$, that

$$\prod_{i:\langle D_i, d\rangle > 0} \partial_i^{\langle D_i, d\rangle} - \prod_{i:\langle D_i, d\rangle < 0} \partial_i^{-\langle D_i, d\rangle} = \left(\prod_{i:\langle D_i, d\rangle > 0} \check{C}_i^{-\langle D_i, d\rangle} \right) \Box_d$$

for $d \in \mathbb{L}$. Hence the other set of equations in (7.6) are also satisfied.

This finishes the proof of the lemma. ∎

8. Application to crepant resolutions

Let $\mathcal{Z}$ be a compact Gorenstein toric orbifold. Suppose the underlying simplicial toric variety Z admits a toric crepant resolution $\widetilde{Z}$. In [18], a conjecture on the relationship between genus 0 open Gromov-Witten invariants of $\widetilde{Z}$ and $\mathcal{Z}$ was formulated and studied. In this section we consider the case of toric Calabi-Yau orbifolds, which are non-compact.

We consider the following setting. Let $\mathcal{X}$ be a toric Calabi-Yau orbifold as in Setting 4.3. It is well-known (see e.g. [51]) that toric crepant birational maps to the coarse moduli space X of $\mathcal{X}$ can be obtained from regular subdivisions of the fan Σ satisfying certain conditions. More precisely, let $\mathcal{X}' = \mathcal{X}_{\Sigma'}$ be the toric orbifold obtained from the fan Σ', where Σ' is a regular subdivision of Σ. Then the morphism $X' \to X$ between the coarse moduli spaces is crepant if and only if for each ray of Σ' with minimal lattice generator u, we have $(\underline{\nu}, u) = 1$.

In this section we prove the following:

Theorem 8.1 (Open crepant resolution theorem). *Let $\mathcal{X}$ be a toric Calabi-Yau orbifold as in Setting 4.3. Let $\mathcal{X}'$ be a toric orbifold obtained by a regular subdivision of the fan Σ, and suppose the natural map $X' \to X$ between the coarse moduli spaces is crepant. The flat coordinates on the Kähler moduli of $\mathcal{X}$ and $\mathcal{X}'$ are denoted as (q, τ) and $(Q, \mathcal{T})$ respectively, and r is the dimension of the extended complexified Kähler moduli space of $\mathcal{X}$ (which is equal to that of $\mathcal{X}'$).*

Then there exists

(1) $\epsilon > 0$;
(2) a coordinate change $(Q(q, \tau), \mathcal{T}(q, \tau))$, which is a holomorphic map $(\Delta(\epsilon) - \mathbb{R}_{\leq 0})^r \to (\mathbb{C}^\times)^r$, and $\Delta(\epsilon)$ is an open disk of radius ϵ in the complex plane;
(3) a choice of an analytic continuation of the SYZ map $\mathcal{F}^{\mathrm{SYZ}}_{\mathcal{X}'}(Q, \mathcal{T})$ to the target of the holomorphic map $(Q(q, \tau), \mathcal{T}(q, \tau))$,

such that

$$\mathcal{F}^{\mathrm{SYZ}}_{\mathcal{X}}(q, \tau) = \mathcal{F}^{\mathrm{SYZ}}_{\mathcal{X}'}(Q(q, \tau), \mathcal{T}(q, \tau)).$$

Theorem 8.1 may be interpreted as saying that generating functions of genus 0 open Gromov-Witten invariants of $\mathcal{X}'$ coincide with those of $\mathcal{X}$ after analytical continuations and changes of variables. See [18, Conjecture 1, Theorem 3] for related statements for compact toric orbifolds.

Our proof of Theorem 8.1 employs the general strategy described in [18]. Namely we use the open mirror theorem (Theorem 7.2) to relate genus 0 open (orbifold) Gromov-Witten invariants of $\mathcal{X}$ and $\mathcal{X}'$ to their toric mirror maps. These toric mirror maps are explicit hypergeometric series and their analytic continuations can be done by using Mellin-Barnes integrals techniques. See Appendix B.

Proof of Theorem 8.1. The proof adapts the strategy used in [18] for proving related results for compact toric orbifolds. First, by Theorem 7.2, we may replace $\mathcal{F}^{\mathrm{SYZ}}$ by $\left(\mathcal{F}^{\mathrm{mirror}}\right)^{-1}$, which are given by the toric mirror maps (6.15). It suffices to show that an analytical continuation of the toric mirror map exists. Then the necessary change of variables is given by composing the inverse of the (analytically continued) toric mirror map of $\mathcal{X}'$ with the toric mirror map of $\mathcal{X}$.

Now the crepant birational map $X' \to X$ may be decomposed into a sequence of crepant birational maps each of which is obtained by a regular subdivision that introduces only one new ray. If we can construct an analytical continuation of the toric mirror map for each of these simpler crepant birational maps, then we would obtain the necessary analytical continuation of the toric mirror map of $\mathcal{X}'$ by composition. Therefore we may assume that the fan Σ' is obtained by a regular subdivision of Σ which introduces only one new ray. In terms of secondary fans, this means that $X' \to X$ is obtained by crossing a single wall. Therefore it remains to construct an analytic continuation of the mirror map in case of a crepant birational map corresponding to crossing a single wall in the secondary fan. This is done in Appendix B. □

Example 8.2. *In the case when $\mathcal{X} = [\mathbb{C}^2/\mathbb{Z}_m]$ (see Example (1) of Section 5.4), and $\mathcal{X}'$ the minimal resolution of $\mathcal{X}$, an analytic continuation of the inverse mirror map was explicitly constructed in* [32]. *We reproduce the result here. Denote by $g^0_{\mathcal{X}'}(y'), \ldots, g^{m-1}_{\mathcal{X}'}(y')$ the inverse mirror map of $\mathcal{X}'$, and denote by $g_0(y), \ldots, g_{m-1}(y)$ the inverse mirror map of $\mathcal{X}$. Then according to* [32, Proposition A.7], *for $1 \leq i \leq m-1$, there is an analytic continuation of $g^i_{\mathcal{X}'}(y')$ such that*

$$g^i_{\mathcal{X}'}(y') = -\frac{2\pi\sqrt{-1}}{m} + \frac{1}{m}\sum_{k=1}^{m-1} \zeta^{2ki}(\zeta^{-k} - \zeta^k) g_k(y),$$

where $\zeta = \exp(\frac{\pi\sqrt{-1}}{m})$.

It may be checked that this yields an identification between the mirrors of $\mathcal{X}$ and $\mathcal{X}'$.

Remark 8.3. *In the case when $\mathcal{X} = [\mathbb{C}^n/\mathbb{Z}_n]$ (see Example (3) of Section 5.4), and $\mathcal{X}' = \mathcal{O}_{\mathbb{P}^{n-1}}(-n)$, an analytic continuation of the inverse mirror map was explicitly carried out in* [18]. *We refer the readers to* [18, Section 6.2] *for more details.*

Appendix A. Maslov index

Let $\mathcal{E}$ be a real $2n$-dimensional symplectic vector bundle over a Riemann surface Σ and $\mathcal{L}$ a Lagrangian subbundle over the boundary $\partial\Sigma$. The Maslov index of the bundle pair $(\mathcal{E}, \mathcal{L})$ is defined to be the rotation number of $\mathcal{L}$ in a symplectic trivialization $\mathcal{E} \cong \Sigma \times \mathbb{R}^{2n}$. The Chern-Weil definition of Maslov index, due to Cho-Shin [29], is described as follows. Let J be a compatible complex structure of $\mathcal{E}$. A unitary connection ∇ of $\mathcal{E}$ is called $\mathcal{L}$-orthogonal ([29, Definition 2.3]) if $\mathcal{L}$ is preserved by the parallel transport via ∇ along the boundary $\partial\Sigma$.

Definition A.1 ([29], Definition 2.8)**.** *The* Chern-Weil Maslov index *of the bundle pair $(\mathcal{E}, \mathcal{L})$ is defined by*

$$\mu_{CW}(\mathcal{E}, \mathcal{L}) = \frac{\sqrt{-1}}{\pi}\int_\Sigma tr(F_\nabla)$$

where $F_\nabla \in \Omega^2(\Sigma, End(\mathcal{E}))$ is the curvature induced by an $\mathcal{L}$-orthogonal connection ∇.

It was proved in [29, Section 3] that the Chern-Weil definition agrees with the usual one.

The Chern-Weil definition of Maslov index is easily extended to the orbifold setting. Let Σ be a bordered orbifold Riemann surface with interior orbifold marked points $z_1^+, \ldots, z_l^+ \in \Sigma$ such that the orbifold structure at each marked point z_j^+ is given by a branched covering map $z \mapsto z^{m_j}$ for some positive integer m_j. According to [29, Definition 6.4], for an orbifold vector bundle $\mathcal{E}$ over Σ and a Lagrangian subbundle $\mathcal{L} \to \partial\Sigma$, the *Chern-Weil Maslov index* $\mu_{CW}(\mathcal{E}, \mathcal{L})$ of the pair $(\mathcal{E}, \mathcal{L})$ is defined by Definition A.1 using an $\mathcal{L}$-orthogonal connection ∇ invariant under the local group action. It was shown in [29, Proposition 6.5] that the Maslov index $\mu_{CW}(\mathcal{E}, \mathcal{L})$ is independent of both the choice of the orthogonal unitary connection ∇ and the choice of a compatible complex structure.

Another orbifold Maslov index, the so-called *desingularized Maslov index* μ^{de}, is defined in [28, Section 3] via the desingularization process introduced by Chen-Ruan [25]. The following result relates the Chern-Weil and the desingularized Maslov indices:

Proposition A.2 ([29], Proposition 6.10)**.**

$$\mu_{CW}(\mathcal{E}, \mathcal{L}) = \mu^{de}(\mathcal{E}, \mathcal{L}) + 2\sum_{j=1}^{l} \mathrm{age}(\mathcal{E}; z_j^+), \tag{A.1}$$

where $\mathrm{age}(\mathcal{E}; z_j^+)$ *is the degree shifting number associated to the* $\mathbb{Z}_{m_j}$*-action on* $\mathcal{E}$ *at the* j*-th marked point* $z_j^+ \in \Sigma$.

In this paper we are mainly concerned with Maslov index arising from holomorphic maps. Let $w : (\Sigma, \partial\Sigma) \to (\mathcal{X}, L)$ be a holomorphic map from a boarded orbifold Riemann surface Σ to a symplectic orbifold $\mathcal{X}$ such that $w(\partial\Sigma)$ is contained in the Lagrangian submanifold L. Then we put $\mu_{CW}(w) := \mu_{CW}(w^*T\mathcal{X}, w^*TL)$. If $\beta \in \pi_2(\mathcal{X}, L)$ is represented by a holomorphic map w, then we put $\mu_{CW}(\beta) := \mu_{CW}(w)$.

The following lemma, which generalizes results in [27, 3, 28], can be used to compute the Maslov index of disks.

Lemma A.3. *Let* $(\mathcal{X}, \omega, J)$ *be a Kähler orbifold of complex dimension* n*, equipped with a non-zero meromorphic* n*-form* Ω *on* $\mathcal{X}$ *which has at worst simple poles. Let* $D \subset \mathcal{X}$ *be the pole divisor of* Ω*. Suppose also that the generic points of* D *are smooth. Then for a special Lagrangian submanifold* $L \subset \mathcal{X}$*, the Chern-Weil Maslov index of a class* $\beta \in \pi_2(\mathcal{X}, L)$ *is given by*

$$\mu_{CW}(\beta) = 2\beta \cdot D.$$

Proof. Suppose β is a homotopy class of a smooth disk. Given a smooth disk representative $u : D^2 \to \mathcal{X}$ of β, note that the pull-back of the canonical line bundle $u^*(K_{\mathcal{X}})$ is an honest vector bundle over D^2, and hence, the proof in [3] applies to this case. Also since the Chern-Weil Maslov index is topological, we can write any class β which is represented by an orbi-disk as a (fractional) linear combination of homotopy classes of smooth disks. Hence the statement for an orbi-disk class β also follows. □

Appendix B. Analytic continuation of mirror maps

In this Appendix we explicitly construct analytic continuations of the toric mirror maps in case of crepant partial resolutions obtained by crossing a single wall in the secondary fan, which are needed in the proof of Theorem 8.1. The technique of constructing analytical continuations using Mellin-Barnes integrals is well-known and has been used in e.g. [14], [10] and [34].

B.1. **Toric basics.** In this subsection we describe the geometric and combinatorial set-up that we are going to consider. Much of the toric geometry needed here is discussed in Section 2 and repeated here in order to properly set up the notations.

Let $\mathcal{X}_1$ be a toric Calabi-Yau orbifold given by the stacky fan

$$(\Sigma_1 \subset N_{\mathbb{R}}, \{\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1}\} \cup \{\boldsymbol{b}_m, \ldots, \boldsymbol{b}_{m'-1}\}) \tag{B.1}$$

where N is a lattice of rank n, $\Sigma_1 \subset N_{\mathbb{R}}$ is a simplicial fan, $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1} \in N$ are primitive generators of the rays of Σ_1, and $\boldsymbol{b}_m, \ldots, \boldsymbol{b}_{m'-1}$ are extra vectors chosen from $\mathrm{Box}(\Sigma_1)^{\mathrm{age}=1}$. The Calabi-Yau condition means that there exists $\underline{\nu} \in M := N^\vee = \mathrm{Hom}(N, \mathbb{Z})$ such that $(\underline{\nu}, \boldsymbol{b}_i) = 1$ for $i = 0, \ldots, m-1$. We also assume that $\mathcal{X}_1$ is as in Setting 4.3 so that it satisfies Assumption 2.9.

The fan sequence of this stacky fan reads

$$0 \longrightarrow \mathbb{L}_1 := \mathrm{Ker}(\phi_1) \xrightarrow{\psi_1} \bigoplus_{i=0}^{m'-1} \mathbb{Z}e_i \xrightarrow{\phi_1} N \longrightarrow 0.$$

Tensoring with $\mathbb{C}^\times$ yields

$$0 \longrightarrow G_1 := \mathbb{L}_1 \otimes_{\mathbb{Z}} \mathbb{C}^\times \longrightarrow (\mathbb{C}^\times)^{m'} \longrightarrow N \otimes_{\mathbb{Z}} \mathbb{C}^\times \to 0.$$

The set of anti-cones of the stacky fan (B.1) is given by

$$\mathcal{A}_1 := \left\{ I \subset \{0, \ldots, m'-1\} \mid \sum_{i \notin I} \mathbb{R}_{\geq 0} \boldsymbol{b}_i \text{ is a cone in } \Sigma_1 \right\}.$$

Note that $\{0, \ldots, m'-1\} \setminus \{i\} \in \mathcal{A}_1$ if and only if $i \in \{0, \ldots, m-1\}$. Hence if $I \in \mathcal{A}_1$, then $\{m, \ldots, m'-1\} \subset I$. Therefore we may define the following

$$\mathcal{A}_1' := \{I' \subset \{0, \ldots, m-1\} \mid I' \cup \{m, \ldots, m'-1\} \in \mathcal{A}_1\}.$$

The divisor sequence is obtained by dualizing the fan sequence:

$$0 \longrightarrow M \xrightarrow{\phi_1^\vee} \bigoplus_{i=0}^{m-1} \mathbb{Z}e_i^\vee \xrightarrow{\psi_1^\vee} \mathbb{L}_1^\vee \longrightarrow 0.$$

For each $i = 0, \ldots, m'-1$, we put $D_i := \psi_1^\vee(e_i^\vee) \in \mathbb{L}_1^\vee$. The extended Kähler cone of $\mathcal{X}_1$ is defined to be

$$\widetilde{C}_{\mathcal{X}_1} := \bigcap_{I \in \mathcal{A}_1} \left(\sum_{i \in I} \mathbb{R}_{>0} D_i \right) \subset \mathbb{L}_1^\vee \otimes \mathbb{R},$$

where $C_{\mathcal{X}_1}$ is the Kähler cone of $\mathcal{X}_1$:

$$C_{\mathcal{X}_1} := \bigcap_{I' \in \mathcal{A}_1'} \left(\sum_{i \in I} \mathbb{R}_{>0} \bar{D}_i \right) \subset H^2(\mathcal{X}_1, \mathbb{R}).$$

We understood that $C_{\mathcal{X}_1}$ is the image of $\widetilde{C}_{\mathcal{X}_1}$ under the quotient map

$$\mathbb{L}_1^\vee \otimes \mathbb{R} \to \mathbb{L}_1^\vee \otimes \mathbb{R} / \sum_{i=m}^{m'-1} \mathbb{R} D_i \simeq H^2(\mathcal{X}_1, \mathbb{R}).$$

There is a splitting

$$\mathbb{L}_1^\vee \otimes \mathbb{R} = \operatorname{Ker}\left((D_m^\vee, \ldots, D_{m'-1}^\vee) : \mathbb{L}_1^\vee \otimes \mathbb{R} \to \mathbb{R}^{m'-m} \right) \oplus \bigoplus_{j=m}^{m'-1} \mathbb{R} D_j,$$

and the extended Kähler cone is decomposed accordingly:

$$\widetilde{C}_{\mathcal{X}_1} = C_{\mathcal{X}_1} + \sum_{j=m}^{m'-1} \mathbb{R}_{>0} D_j.$$

Let $\omega_1 \in \widetilde{C}_{\mathcal{X}_1}$ be an extended Kähler class of $\mathcal{X}_1$. According to [72, Section 3.1.1], the defining condition of $\mathcal{A}_1$ may also be formulated as

$$\omega_1 \in \sum_{i \in I} \mathbb{R}_{>0} D_i.$$

The extended canonical class of $\mathcal{X}_1$ is $\hat{\rho}_{\mathcal{X}_1} := \sum_{i=0}^{m'-1} D_i$. By [72, Lemma 3.3], we have

$$\hat{\rho}_{\mathcal{X}_1} = \sum_{i=0}^{m-1} D_i + \sum_{i=m}^{m'-1} (1 - \operatorname{age}(\boldsymbol{b}_i)) D_i.$$

Since we have chosen $\boldsymbol{b}_i, i = m, \ldots, m'-1$ to have age one, we see that $\hat{\rho}_{\mathcal{X}_1} = \sum_{i=0}^{m-1} D_i = c_1(\mathcal{X}_1) = 0$.

B.2. **Geometry of wall-crossing.** As mentioned earlier, we want to consider toric crepant birational maps obtained by introducing a new ray. We now describe this in terms of wall-crossing. We refer to [38, Chapters 14–15] for the basics of wall-crossings in the toric setting.

By definition, a wall is a subspace

$$\widetilde{W} = W \oplus \bigoplus_{j=m}^{m'-1} \mathbb{R} D_j \subset \mathbb{L}_1^\vee \otimes \mathbb{R},$$

where W is a hyperplane given by a linear functional l, such that

1. $C_{\mathcal{X}_1} \subset \{l > 0\}$, and
2. the intersection $\overline{C}_{\mathcal{X}_1} \cap W$ of the closure of $C_{\mathcal{X}_1}$ with W is a top-dimensional cone in W.

Let $C_{\mathcal{X}_1}(W) \subset \overline{C}_{\mathcal{X}_1} \cap W$ be the relative interior and let $\widetilde{C}_{\mathcal{X}_1}(W) := C_{\mathcal{X}_1}(W) \oplus \bigoplus_{j=m}^{m'-1} \mathbb{R} D_j$.

We want to consider a crepant birational map obtained by introducing one new ray. This means that there is exactly one D_i lying outside the Kähler cone $C_{\mathcal{X}_1}$. By relabeling the

1-dimensional cones, we may assume that D_{m-1} lies outside $C_{\mathcal{X}_1}$. More precisely, we assume that

$$\begin{cases} l(D_i) > 0 & \text{for } 0 \leq i \leq a-1, \\ l(D_i) = 0 & \text{for } a \leq i \leq m-2, \\ l(D_{m-1}) < 0 \end{cases} \tag{B.2}$$

Let ω_2 be an extended Kähler class in the chamber[7] adjacent to $(\overline{C}_{\mathcal{X}_1} \cap W) \oplus \bigoplus_{j=m}^{m'-1} \mathbb{R}D_j$. Following [72, Section 3.1.1], we may use ω_2 to define another toric orbifold $\mathcal{X}_2$ as follows. The set of anti-cones is defined to be

$$\mathcal{A}_2 := \left\{ I \subset \{0, \ldots, m'-1\} \mid \omega_2 \in \sum_{i \in I} \mathbb{R}_{>0} D_i \right\}.$$

The toric orbifold $\mathcal{X}_2$ is then defined to be the following stack quotient

$$\mathcal{X}_2 := \left[\left(\mathbb{C}^{m'} \setminus \bigcup_{I \notin \mathcal{A}_2} \mathbb{C}^I \right) / G_1 \right],$$

where $\mathbb{C}^I := \{(z_0, \ldots, z_{m'-1}) \in \mathbb{C}^{m'} \mid z_i = 0 \text{ for } i \notin I\}$. The fan Σ_2 of this toric orbifold is defined from $\mathcal{A}_2$ as follows: $\sum_{i \notin I} \mathbb{R}_{\geq 0} b_i$ is a cone of Σ_2 if and only if $I \in \mathcal{A}_2$. We also define

$$\mathcal{A}'_2 := \{I' \subset \{0, \ldots, m-1\} \mid I' \cup \{m, \ldots, m'-1\} \in \mathcal{A}_2\}.$$

Next we make a few observations about the two sets $\mathcal{A}_1$, $\mathcal{A}_2$ of anti-cones.

Lemma B.1. *Let $I \in \mathcal{A}_1$. Then $I \in \mathcal{A}_2$ if and only if $m-1 \in I$.*

Proof. Suppose $I \in \mathcal{A}_2$. Then $\omega_2 \in \sum_{i \in I} \mathbb{R}_{>0} D_i$. Since $l(D_i) \geq 0$ for all i except $i = m-1$, and $l(\omega_2) < 0$, in order for $\omega_2 \in \sum_{i \in I} \mathbb{R}_{>0} D_i$ we must have $m-1 \in I$.

Suppose that $I \notin \mathcal{A}_2$. Then $\omega_2 \notin \sum_{i \in I} \mathbb{R}_{>0} D_i$. But this means that $\mathbb{R}_{>0}\omega_2 \notin \sum_{i \in I} \mathbb{R}_{>0} D_i$. This implies $m-1 \notin I$. □

We also have

Lemma B.2. *Let $I \in \mathcal{A}_1$ and $I \notin \mathcal{A}_2$. Then*

(1) $(I \cup \{m-1\}) \setminus \{0, \ldots, a-1\} \in \mathcal{A}_2$.
(2) If $|I| = \dim G_1$, then $I \cap \{0, \ldots, a-1\} = \{i_I\}$ is a singleton, so $(I \cup \{m-1\}) \setminus \{i_I\} \in \mathcal{A}_2$.

Proof. The first statement follows from the fact that $l(D_i) \leq 0$ for all $i \in (I \cup \{m-1\}) \setminus \{0, \ldots, a-1\}$. The second statement follows from the fact that the minimal size of an anti-cone is equal to $\dim G_1$. □

Moving the Kähler class ω_1 across the wall W to ω_2 induces a birational map

$$X_1 \to X_2. \tag{B.3}$$

between the toric varieties underlying $\mathcal{X}_1$ and $\mathcal{X}_2$. In the setting of toric GIT, this map is induced from the variation of GIT quotients given by moving the stability parameter from ω_1 to ω_2.

[7] The chamber structure is given by the secondary fan associated to Σ_1.

We may describe the birational map $X_1 \to X_2$ in terms of the fans. By Lemmas B.1 and B.2, If $\sum_{i\notin I} \mathbb{R}_{\geq 0}\boldsymbol{b}_i$ is a cone in Σ_1, then either this cone is also in Σ_2 (in which case $\mathbb{R}_{\geq 0}\boldsymbol{b}_{m-1}$ is not a ray of this cone), or

$$\sum_{i\notin (I\cup\{m-1\})\setminus\{0,\ldots,a-1\}} \mathbb{R}_{\geq 0}\boldsymbol{b}_i$$

is a cone in Σ_2. This shows that the fan Σ_1 is an refinement of Σ_2 obtained by adding a new ray $\mathbb{R}_{\geq 0}\boldsymbol{b}_{m-1}$. The birational map $X_1 \to X_2$ in (B.3) is induced from this refinement, in a manner described more generally in e.g. [51, Section 1.4].

It is easy to see from the fan description that $X_1 \to X_2$ contracts the divisor $\bar{D}_{m-1} \subset X_1$. Furthermore, we have

Lemma B.3. *The birational map $X_1 \to X_2$ in* (B.3) *is crepant.*

Proof. Since $\mathcal{X}_1$ is toric Calabi-Yau, there exists $\underline{\nu} \in N^\vee$ such that $(\underline{\nu}, \boldsymbol{b}_i) = 1$ for $i = 0, ..., m-1$. We conclude that $X_1 \to X_2$ is crepant by applying the criterion for being crepant (see e.g. [51, Section 3.4] and [9, Remark 7.2]) with the support function $(\underline{\nu}, -)$. □

B.3. **Analytic continuations.** Recall that

$$\mathbb{K}_1 := \{d \in \mathbb{L}_1 \otimes \mathbb{Q} \mid \{i \mid \langle D_i, d\rangle \in \mathbb{Z}\} \in \mathcal{A}_1\},$$
$$\mathbb{K}_2 := \{d \in \mathbb{L}_1 \otimes \mathbb{Q} \mid \{i \mid \langle D_i, d\rangle \in \mathbb{Z}\} \in \mathcal{A}_2\}.$$

As defined in (2.6), there are reduction functions

$$\nu : \mathbb{K}_1 \to \mathrm{Box}(\Sigma_1),$$
$$\nu : \mathbb{K}_2 \to \mathrm{Box}(\Sigma_2),$$

which are surjective and have kernels $\mathbb{L}_1$. This gives the identifications

$$\mathbb{K}_1/\mathbb{L}_1 = \mathrm{Box}(\Sigma_1),$$
$$\mathbb{K}_2/\mathbb{L}_1 = \mathrm{Box}(\Sigma_2). \tag{B.4}$$

Next we recall some details about the toric mirror map. As in (6.15), the toric mirror map of $\mathcal{X}_1$ is given by

$$\log q_a = \log y_a + \sum_{j=0}^{m-1} Q_{ja} A_j^{\mathcal{X}}(y), \quad a = 1, \ldots, r',$$
$$\tau_{\boldsymbol{b}_j} = A_j^{\mathcal{X}}(y), \quad j = m, \ldots, m'-1, \tag{B.5}$$

Some explanations are in order. Fix an integral basis $\{p_1, \ldots, p_r\} \subset \mathbb{L}_1^\vee$, where $r = m' - n$. For $d \in \mathbb{L}_1 \otimes \mathbb{Q}$, we write

$$q^d = \prod_{a=1}^{r'} q_a^{\langle \bar{p}_a, d\rangle}, \quad y^d = \prod_{a=1}^{r} y_a^{\langle p_a, d\rangle}$$

which defines q_a and y_a, where $r' = m - n$ and $\{\bar{p}_1, \ldots, \bar{p}_{r'}\}$ are images of $\{p_1, \ldots, p_{r'}\}$ under the quotient map $\mathbb{L}_1^\vee \otimes \mathbb{Q} \to H^2(\mathcal{X}_1; \mathbb{Q})$ and they give a nef basis for $H^2(\mathcal{X}_1; \mathbb{Q})$. Also, Q_{ia}

are chosen so that

$$D_i = \sum_{a=1}^{r} Q_{ia} p_a, \quad i = 0, \dots, m-1. \tag{B.6}$$

For $j = 0, 1, \dots, m-1$, we have

$$\Omega_j^{\mathcal{X}_1} = \{d \in (\mathbb{K}_1)_{\text{eff}} \mid \nu(d) = 0, \langle D_j, d\rangle \in \mathbb{Z}_{<0} \text{ and } \langle D_i, d\rangle \geq 0 \in \mathbb{Z}_{\geq 0}\ \forall i \neq j\},$$

$$A_j^{\mathcal{X}_1}(y) = \sum_{d \in \Omega_j^{\mathcal{X}_1}} y^d \frac{(-1)^{-\langle D_j, d\rangle - 1}(-\langle D_j, d\rangle - 1)!}{\prod_{i \neq j} \langle D_i, d\rangle!}.$$

For $j = m, \dots, m'-1$, we have

$$\Omega_j^{\mathcal{X}_1} = \{d \in (\mathbb{K}_1)_{\text{eff}} \mid \nu(d) = \boldsymbol{b}_j \text{ and } \langle D_i, d\rangle \notin \mathbb{Z}_{<0}\ \forall i\},$$

$$A_j^{\mathcal{X}_1}(y) = \sum_{d \in \Omega_j^{\mathcal{X}_1}} y^d \prod_{i=0}^{m'-1} \frac{\prod_{k=\lceil \langle D_i, d\rangle \rceil}^{\infty}(\langle D_i, d\rangle - k)}{\prod_{k=0}^{\infty}(\langle D_i, d\rangle - k)}.$$

To study the analytic continuation of (B.5), we first need to be more precise about the variables involved. We pick $p_1, \dots, p_r$ such that p_1 is contained in the closure of $\widetilde{C}_{\mathcal{X}_1}$ and $p_2, \dots, p_r \in \widetilde{C}_{\mathcal{X}_1}(W)$. Applying the linear functional $l \oplus 0$ to (B.6) gives

$$l(D_i) = Q_{i1} l(p_1) + \sum_{a \geq 2} Q_{ia} l(p_a).$$

By the choice of $p_1, \dots, p_r$, we have $l(p_1) > 0$ and $l(p_a) = 0$ for $a \geq 2$. The signs of $l(D_j)$ are given in (B.2). This implies that

$$\begin{cases} Q_{i1} > 0 & \text{for } 0 \leq i \leq a-1, \\ Q_{i1} = 0 & \text{for } a \leq i \leq m-2, \\ Q_{m-1,1} < 0 & \end{cases}$$

Since $0 = \sum_{i=0}^{m'-1} D_i = \sum_{i=0}^{m'-1} \sum_{a=1}^{r} Q_{ia} p_a$, we have $\sum_{i=0}^{m'-1} Q_{ia} = 0$ for all $a = 1, \dots, r$. Also note that $Q_{ia} = 0$ for $1 \leq a \leq r'$ and $m \leq i \leq m'-1$.

We now proceed to construct an analytic continuation of $A_j(y)$ where $j \in \{0, \dots, m'-1\}$. We do this in details for $j \in \{m, \dots, m'-1\}$. The case when $j \in \{0, \dots, m-1\}$ is similar and will be omitted.

Let $j \in \{m, \dots, m'-1\}$. The element $\boldsymbol{b}_j \in \text{Box}(\Sigma_1)^{\text{age}=1}$ corresponds to a component $\mathcal{X}_{1,\boldsymbol{b}_j}$ of the inertia orbifold $I\mathcal{X}_1$. According to [9, Lemma 4.6], $\mathcal{X}_{1,\boldsymbol{b}_j}$ is the toric Deligne-Mumford stack associated to the quotient stacky fan $\Sigma_1/\sigma(\boldsymbol{b}_j)$, where $\sigma(\boldsymbol{b}_j)$ is the minimal cone in Σ_1 that contains $\boldsymbol{b}_j$. Let $d_{\boldsymbol{b}_j} \in \mathbb{K}_1$ be the unique element such that $\nu(d_{\boldsymbol{b}_j}) = \boldsymbol{b}_j$ and $\langle p_a, d_{\boldsymbol{b}_j}\rangle \in [0, 1)$. Then by the identification of Box in (B.4), every $d \in \mathbb{K}_1$ with $\nu(d) = \boldsymbol{b}_j$ can be written as

$$d = d_{\boldsymbol{b}_j} + d_0$$

with $d_0 \in \mathbb{L}_1$.

We consider $A_j^{\mathcal{X}_1}(y)$. Put

$$\mathcal{A}_{1,\boldsymbol{b}_j} := \left\{ I \subset \{0, \ldots, m'-1\} \mid \sum_{i \notin I} \mathbb{R}_{\geq 0} \boldsymbol{b}_i \text{ is a cone in } \Sigma_1, \langle D_i, d_{\boldsymbol{b}_j} \rangle \in \mathbb{Z} \text{ for } i \in I \right\} \subset \mathcal{A}_1,$$

and define

$$\widetilde{C}_{\mathcal{X}_{1,\boldsymbol{b}_j}} := \bigcap_{I \in \mathcal{A}_{1,\boldsymbol{b}_j}} \left(\sum_{i \in I} \mathbb{R}_{>0} D_i \right) = C_{\mathcal{X}_{1,\boldsymbol{b}_j}} + \sum_{i=m}^{m'-1} \mathbb{R}_{\geq 0} D_i.$$

Clearly $\widetilde{C}_{\mathcal{X}_1} \subset \widetilde{C}_{\mathcal{X}_{1,\boldsymbol{b}_j}}$. Taking duals gives

$$\overline{NE}(\mathcal{X}_{1,\boldsymbol{b}_j}) := \widetilde{C}^{\vee}_{\mathcal{X}_{1,\boldsymbol{b}_j}} \subset \widetilde{C}^{\vee}_{\mathcal{X}_1} =: \overline{NE}(\mathcal{X}_1).$$

By definition, $A_j(y)$ is a series in y whose exponents are contained in Ω_j. It is straightforward to check that $\Omega_j \subset \overline{NE}(\mathcal{X}_{1,\boldsymbol{b}_j})$. In this way we interpret $A_j(y)$ as a function on $\widetilde{C}_{\mathcal{X}_{1,\boldsymbol{b}_j}}$ and a function on $\widetilde{C}_{\mathcal{X}_1}$ by restriction.

If we also have $\widetilde{C}_{\mathcal{X}_2} \subset \widetilde{C}_{\mathcal{X}_{1,\boldsymbol{b}_j}}$, then $A_j(y)$ can also be interpreted as a function on $\widetilde{C}_{\mathcal{X}_2}$ by restriction. So in this case no analytic continuation is needed.

It remains to consider those $\boldsymbol{b}_j$ such that $\widetilde{C}_{\mathcal{X}_2}$ is not contained in $\widetilde{C}_{\mathcal{X}_{1,\boldsymbol{b}_j}}$. First observe that $A_j(y)$ can be rewritten as follows:

$$A_j(y) = \sum_{d_0 \in \mathbb{L}_1} y^{d_{\boldsymbol{b}_j}} y^{d_0} \prod_{i=0}^{m'-1} \frac{\Gamma(\{\langle D_i, d_{\boldsymbol{b}_j} + d_0 \rangle\} + 1)}{\Gamma(\langle D_i, d_{\boldsymbol{b}_j} + d_0 \rangle + 1)}.$$

We put $\Gamma_{\boldsymbol{b}_j} := \prod_{i=0}^{m'-1} \Gamma(\{\langle D_i, d_{\boldsymbol{b}_j} + d_0 \rangle\} + 1)$ so that we can write

$$A_j(y) = \sum_{d_0 \in \mathbb{L}_1} y^{d_{\boldsymbol{b}_j}} y^{d_0} \Gamma_{\boldsymbol{b}_j} \frac{1}{\Gamma(\langle D_{m-1}, d_{\boldsymbol{b}_j} + d_0 \rangle + 1)} \frac{1}{\prod_{i \neq m-1} \Gamma(\langle D_i, d_{\boldsymbol{b}_j} + d_0 \rangle + 1)}.$$

Since $\Gamma(s)\Gamma(1-s) = \pi / \sin(\pi s)$, we have

$$\frac{1}{\Gamma(\langle D_{m-1}, d_{\boldsymbol{b}_j} + d_0 \rangle + 1)} = -\frac{\sin(\pi \langle D_{m-1}, d_{\boldsymbol{b}_j} + d_0 \rangle)}{\pi} \Gamma(-\langle D_{m-1}, d_{\boldsymbol{b}_j} + d_0 \rangle),$$

and

$$A_j(y) = \sum_{d_0 \in \mathbb{L}_1} y^{d_{\boldsymbol{b}_j}} y^{d_0} \frac{\Gamma_{\boldsymbol{b}_j}}{\pi} \sin(\pi \langle D_{m-1}, d_{\boldsymbol{b}_j} + d_0 \rangle) \frac{-\Gamma(-\langle D_{m-1}, d_{\boldsymbol{b}_j} + d_0 \rangle)}{\prod_{i \neq m-1} \Gamma(\langle D_i, d_{\boldsymbol{b}_j} + d_0 \rangle + 1)}.$$

We put $d_{0a} := \langle p_a, d_0 \rangle$. In view of (B.6), we have

$$\frac{-\Gamma(-\langle D_{m-1}, d_{\boldsymbol{b}_j} + d_0 \rangle)}{\prod_{i \neq m-1} \Gamma(\langle D_i, d_{\boldsymbol{b}_j} + d_0 \rangle + 1)} = \frac{-\Gamma(-\langle D_{m-1}, d_{\boldsymbol{b}_j} \rangle - Q_{m-1,1} d_{01} - \sum_{a \neq 1} Q_{m-1a} d_{0a})}{\prod_{i \neq m-1} \Gamma(\langle D_i, d_{\boldsymbol{b}_j} \rangle + 1 + Q_{m-1,1} d_{01} + \sum_{a \neq 1} Q_{m-1a} d_{0a})}.$$

Since $y^{d_0} = \prod_{a=1}^r y_a^{\langle p_a, d_0\rangle} = \prod_{a=1}^r y_a^{d_{0a}}$, we have

$$\begin{aligned}
&A_j(y)\\
=&\frac{\Gamma_{\boldsymbol{b}_j}}{\pi}\sum_{d_{01},\ldots,d_{0r}\geq 0} y^{d_{\boldsymbol{b}_j}}\left(\prod_{a\geq 2} y_a^{d_{0a}}\right)\sin(\pi\langle D_{m-1}, d_{\boldsymbol{b}_j}+d_0\rangle)\\
&\qquad\times\frac{-\Gamma(-\langle D_{m-1}, d_{\boldsymbol{b}_j}\rangle - Q_{m-1,1}d_{01} - \sum_{a\neq 1} Q_{m-1a}d_{0a})}{\prod_{i\neq m-1}\Gamma(\langle D_i, d_{\boldsymbol{b}_j}\rangle + 1 + Q_{m-1,1}d_{01} + \sum_{a\neq 1} Q_{m-1a}d_{0a})}\\
=&\frac{\Gamma_{\boldsymbol{b}_j}}{\pi}\sum_{d_{02},\ldots,d_{0r}\geq 0} y^{d_{\boldsymbol{b}_j}}\left(\prod_{a\geq 2} y_a^{d_{0a}}\right)\sin\left(\pi\langle D_{m-1}, d_{\boldsymbol{b}_j}\rangle + \sum_{a\neq 1} Q_{m-1,a}d_{0a}\right)\\
&\qquad\times\left(\sum_{d_{01}\geq 0}\left((-1)^{Q_{m-1,1}}y_1\right)^{d_{01}}\frac{-\Gamma(-\langle D_{m-1}, d_{\boldsymbol{b}_j}\rangle - Q_{m-1,1}d_{01} - \sum_{a\neq 1} Q_{m-1a}d_{0a})}{\prod_{i\neq m-1}\Gamma(\langle D_i, d_{\boldsymbol{b}_j}\rangle + 1 + Q_{m-1,1}d_{01} + \sum_{a\neq 1} Q_{m-1a}d_{0a})}\right).
\end{aligned}$$

Now observe that

$$\begin{aligned}
&\sum_{d_{01}\geq 0}\left((-1)^{Q_{m-1,1}}y_1\right)^{d_{01}}\frac{-\Gamma(-\langle D_{m-1}, d_{\boldsymbol{b}_j}\rangle - Q_{m-1,1}d_{01} - \sum_{a\neq 1} Q_{m-1a}d_{0a})}{\prod_{i\neq m-1}\Gamma(\langle D_i, d_{\boldsymbol{b}_j}\rangle + 1 + Q_{m-1,1}d_{01} + \sum_{a\neq 1} Q_{m-1a}d_{0a})}\\
=&\mathrm{Res}_{s\in\mathbb{N}\cup\{0\}}ds\frac{-\Gamma(-s)((-1)^{Q_{m-1,1}}y_1)^s\Gamma(-\langle D_{m-1}, d_{\boldsymbol{b}_j}\rangle - Q_{m-1,1}s - \sum_{a\neq 1} Q_{m-1a}d_{0a})}{\prod_{i\neq m-1}\Gamma(\langle D_i, d_{\boldsymbol{b}_j}\rangle + 1 + Q_{m-1,1}s + \sum_{a\neq 1} Q_{m-1a}d_{0a})}.
\end{aligned}$$

Fix a sign of y_1 so that $(-1)^{Q_{m-1,1}}y_1 \in \mathbb{R}_{>0}$. By using the Mellin-Barnes integral technique (see e.g. [10, Section 4] and [10, Lemma A.6]), we have

$$\begin{aligned}
&\mathrm{Res}_{s\in\mathbb{N}\cup\{0\}}ds\frac{-\Gamma(-s)((-1)^{Q_{m-1,1}}y_1)^s\Gamma(-\langle D_{m-1}, d_{\boldsymbol{b}_j}\rangle - Q_{m-1,1}s - \sum_{a\neq 1} Q_{m-1a}d_{0a})}{\prod_{i\neq m-1}\Gamma(\langle D_i, d_{\boldsymbol{b}_j}\rangle + 1 + Q_{m-1,1}s + \sum_{a\neq 1} Q_{m-1a}d_{0a})}\\
=&\oint_{C_{d_{02},\ldots,d_{0r}}}ds\frac{-\Gamma(-s)((-1)^{Q_{m-1,1}}y_1)^s\Gamma(-\langle D_{m-1}, d_{\boldsymbol{b}_j}\rangle - Q_{m-1,1}s - \sum_{a\neq 1} Q_{m-1a}d_{0a})}{\prod_{i\neq m-1}\Gamma(\langle D_i, d_{\boldsymbol{b}_j}\rangle + 1 + Q_{m-1,1}s + \sum_{a\neq 1} Q_{m-1a}d_{0a})},
\end{aligned}$$

where $C_{d_{02},\ldots,d_{0r}}$ is a contour on the plane with (complex) coordinate s that runs from $s = -\sqrt{-1}\infty$ to $s = +\sqrt{-1}\infty$, dividing the plane into two parts so that $\{0, 1, \ldots\}$ lies on one part and

$$\mathrm{Pole}_L := \left\{\frac{\langle D_{m-1}, d_{\boldsymbol{b}_j}\rangle + \sum_{a\neq 1} Q_{m-1a}d_{0a} - l}{-Q_{m-1,1}} \mid l = 0, 1, \ldots\right\} \tag{B.7}$$

lies on the other part. Note that $-Q_{m-1,1} > 0$.

To analytically continue to the region where $|y_1|$ is large, we close the contour $C_{d_{02},\ldots,d_{0r}}$ to the left to enclose all poles in Pole_L. This gives

$$\begin{aligned}
&\oint_{C_{d_{02},\ldots,d_{0r}}}ds\frac{-\Gamma(-s)((-1)^{Q_{m-1,1}}y_1)^s\Gamma(-\langle D_{m-1}, d_{\boldsymbol{b}_j}\rangle - Q_{m-1,1}s - \sum_{a\neq 1} Q_{m-1a}d_{0a})}{\prod_{i\neq m-1}\Gamma(\langle D_i, d_{\boldsymbol{b}_j}\rangle + 1 + Q_{m-1,1}s + \sum_{a\neq 1} Q_{m-1a}d_{0a})}\\
=&\mathrm{Res}_{s\in\mathrm{Pole}_L}ds\frac{-\Gamma(-s)((-1)^{Q_{m-1,1}}y_1)^s\Gamma(-\langle D_{m-1}, d_{\boldsymbol{b}_j}\rangle - Q_{m-1,1}s - \sum_{a\neq 1} Q_{m-1a}d_{0a})}{\prod_{i\neq m-1}\Gamma(\langle D_i, d_{\boldsymbol{b}_j}\rangle + 1 + Q_{m-1,1}s + \sum_{a\neq 1} Q_{m-1a}d_{0a})},
\end{aligned}$$

which is equal to

$$\sum_{l\geq 0}\frac{(-1)^l}{l!}\frac{\Gamma\left(\frac{\langle D_{m-1},d_{\boldsymbol{b}_j}\rangle+\sum_{a\neq 1}Q_{m-1a}d_{0a}-l}{Q_{m-1,1}}\right)\left((-1)^{Q_{m-1,1}}y_1\right)^{\frac{\langle D_{m-1},d_{\boldsymbol{b}_j}\rangle+\sum_{a\neq 1}Q_{m-1a}d_{0a}-l}{-Q_{m-1,1}}}}{\prod_{i\neq m-1}\Gamma\left(\langle D_i,d_{\boldsymbol{b}_j}\rangle+1+Q_{m-1,1}\times\frac{\langle D_{m-1},d_{\boldsymbol{b}_j}\rangle+\sum_{a\neq 1}Q_{m-1a}d_{0a}-l}{-Q_{m-1,1}}+\sum_{a\neq 1}Q_{m-1a}d_{0a}\right)}$$

$$=\sum_{l\geq 0}\frac{(-1)^l}{l!}\frac{\left((-1)^{Q_{m-1,1}}y_1\right)^{\frac{\langle D_{m-1},d_{\boldsymbol{b}_j}\rangle+\sum_{a\neq 1}Q_{m-1a}d_{0a}-l}{-Q_{m-1,1}}}\frac{\pi}{-Q_{m-1,1}\sin\pi\left(\frac{\langle D_{m-1},d_{\boldsymbol{b}_j}\rangle+\sum_{a\neq 1}Q_{m-1a}d_{0a}-l}{-Q_{m-1,1}}\right)}}{\prod_{i\neq m-1}\Gamma\left(\langle D_i,d_{\boldsymbol{b}_j}\rangle+1+Q_{m-1,1}\times\frac{\langle D_{m-1},d_{\boldsymbol{b}_j}\rangle+\sum_{a\neq 1}Q_{m-1a}d_{0a}-l}{-Q_{m-1,1}}+\sum_{a\neq 1}Q_{m-1a}d_{0a}\right)}\times$$
$$\times\frac{1}{\Gamma\left(1-\frac{\langle D_{m-1},d_{\boldsymbol{b}_j}\rangle+\sum_{a\neq 1}Q_{m-1a}d_{0a}-l}{Q_{m-1,1}}\right)},$$

where we again use $\Gamma(s)\Gamma(1-s)=\pi/\sin(\pi s)$.

This gives an analytic continuation of $A_j(y)$:

(B.8)

$$A_j(y)$$
$$=\frac{\Gamma_{\boldsymbol{b}_j}}{\pi}\sum_{d_{02},\dots,d_{0r}\geq 0}y^{d_{\boldsymbol{b}_j}}\left(\prod_{a\geq 2}y_a^{d_{0a}}\right)\sin\left(\pi\langle D_{m-1},d_{\boldsymbol{b}_j}\rangle+\pi\sum_{a\neq 1}Q_{m-1,a}d_{0a}\right)$$
$$\times\sum_{l\geq 0}\frac{(-1)^l}{l!}\frac{\left((-1)^{Q_{m-1,1}}y_1\right)^{\frac{\langle D_{m-1},d_{\boldsymbol{b}_j}\rangle+\sum_{a\neq 1}Q_{m-1a}d_{0a}-l}{-Q_{m-1,1}}}\frac{\pi}{-Q_{m-1,1}\sin\pi\left(\frac{\langle D_{m-1},d_{\boldsymbol{b}_j}\rangle+\sum_{a\neq 1}Q_{m-1a}d_{0a}-l}{-Q_{m-1,1}}\right)}}{\prod_{i\neq m-1}\Gamma\left(\langle D_i,d_{\boldsymbol{b}_j}\rangle+1+Q_{m-1,1}\times\frac{\langle D_{m-1},d_{\boldsymbol{b}_j}\rangle+\sum_{a\neq 1}Q_{m-1a}d_{0a}-l}{-Q_{m-1,1}}+\sum_{a\neq 1}Q_{m-1a}d_{0a}\right)}$$
$$\times\frac{1}{\Gamma\left(1-\frac{\langle D_{m-1},d_{\boldsymbol{b}_j}\rangle+\sum_{a\neq 1}Q_{m-1a}d_{0a}-l}{Q_{m-1,1}}\right)}.$$

It remains to show that the expression in (B.8) can be interpreted as a function on $\widetilde{C}_{\mathcal{X}_2}$. To do this, we need a new set of variables. Pick another integral basis of $\{\hat{p}_1,\dots,\hat{p}_r\}\subset\mathbb{L}_1^\vee\otimes\mathbb{Q}$ such that

$$\hat{p}_1:=D_{m-1},\quad \hat{p}_a:=p_a,\ \text{for } a=2,\dots,r.$$

Introduce the corresponding variables $\hat{y}_1,\dots,\hat{y}_r$, namely $y^d=\hat{y}^d=\prod_{a=1}^r\hat{y}_a^{\langle\hat{p}_a,d\rangle}$. From this it is easy to see that

$$\hat{y}_1=y_1^{1/Q_{m-1,1}},\quad \hat{y}_a=y_1^{-Q_{m-1,a}/Q_{m-1,1}}y_a,\ \text{for } a=2,\dots,r.$$

We may express D_i in terms of $\hat{p}_1, \ldots, \hat{p}_r$ as follows:

$$\begin{aligned} D_i &= \sum_{a=1}^{r} Q_{ia} p_a = Q_{i1} p_1 + \sum_{a \geq 2} Q_{ia} p_a \\ &= \frac{Q_{i1}}{Q_{m-1,1}} \left(\hat{p}_1 - \sum_{a \geq 2} Q_{m-1,a} \hat{p}_a \right) + \sum_{a \geq 2} Q_{ia} \hat{p}_a \\ &= \frac{Q_{i1}}{Q_{m-1,1}} \hat{p}_1 + \sum_{a \geq 2} \left(Q_{ia} - \frac{Q_{i1} Q_{m-1,a}}{Q_{m-1,1}} \right) \hat{p}_a. \end{aligned}$$

Next we interpret the expression in (B.8) as a series in $\hat{y}$ whose exponents are contained in $\overline{NE}(\mathcal{X}_2) = \widehat{C}^{\vee}_{\mathcal{X}_2}$. Define $\hat{d}_{\boldsymbol{b}_j} \in \mathbb{L}_1 \otimes \mathbb{Q}$ to be the unique class such that

$$\langle \hat{p}_1, \hat{d}_{\boldsymbol{b}_j} \rangle = 0, \quad \langle \hat{p}_a, \hat{d}_{\boldsymbol{b}_j} \rangle = \langle p_a, d_{\boldsymbol{b}_j} \rangle, \text{ for } a = 2, \ldots, r. \tag{B.9}$$

Given $l, d_{02}, \ldots, d_{0r} \geq 0$, define $\hat{d}_0 \in \mathbb{L}_1 \otimes \mathbb{Q}$ to be the unique class such that

$$\langle \hat{p}_1, \hat{d}_0 \rangle = l, \quad \langle \hat{p}_a, \hat{d}_0 \rangle = d_{0a}, \text{ for } a = 2, \ldots, r. \tag{B.10}$$

Lemma B.4. *Given $l, d_{02}, \ldots, d_{0r} \geq 0$. Then $\hat{d} := \hat{d}_{\boldsymbol{b}_j} + \hat{d}_0$ is contained in $\mathbb{K}_2$.*

Proof. First note that $\langle D_{m-1}, \hat{d} \rangle = \langle \hat{p}_1, \hat{d}_{\boldsymbol{b}_j} + \hat{d}_0 \rangle = l \in \mathbb{Z}$.

Let $i \in \{a, \ldots, m-2\}$. We consider $\langle D_i, \hat{d} \rangle$. Let $\hat{p}_1^{\vee}, \ldots, \hat{p}_r^{\vee}$ be such that $\langle \hat{p}_a, \hat{p}_b^{\vee} \rangle = \delta_{ab}$. We calculate $\langle \hat{p}_1, d_0 \rangle = \sum_{a \geq 1} Q_{m-1,a} d_{0a}$ and $\langle \hat{p}_a, d_0 \rangle = d_{0a}$ for $a \geq 2$. So

$$d_0 = \left(\sum_{a \geq 1} Q_{m-1,a} d_{0a} \right) \hat{p}_1^{\vee} + \sum_{a \geq 2} d_{0a} \hat{p}_a^{\vee}.$$

By (B.9) and (B.10), we have

$$\begin{aligned} \hat{d} = \hat{d}_{\boldsymbol{b}_j} + \hat{d}_0 &= d_{\boldsymbol{b}_j} - \langle p_a, d_{\boldsymbol{b}_j} \rangle \hat{p}_1^{\vee} + d_0 + \left(l - \sum_{a \geq 1} Q_{m-1,a} d_{0a} \right) \hat{p}_1^{\vee} \\ &= d_{\boldsymbol{b}_j} + d_0 + \left(l - \langle p_a, d_{\boldsymbol{b}_j} \rangle - \sum_{a \geq 1} Q_{m-1,a} d_{0a} \right) \hat{p}_1^{\vee}. \end{aligned}$$

Since $i \in \{a, \ldots, m-2\}$, we have $D_i \in \widetilde{C}_{\mathcal{X}_1}(W)$. So D_i is a linear combination of $\hat{p}_2, \ldots, \hat{p}_r$. This implies that $\langle D_i, \hat{p}_1^{\vee} \rangle = 0$, and hence

$$\langle D_i, \hat{d} \rangle = \langle D_i, d_{\boldsymbol{b}_j} + d_0 \rangle.$$

We know that $\langle D_i, d_0 \rangle = \sum_{a=1}^{r} Q_{ia} \langle p_a, d_0 \rangle = \sum_{a=1}^{r} Q_{ia} d_{0a} \in \mathbb{Z}$. So $\langle D_i, \hat{d} \rangle = \langle D_i, d_{\boldsymbol{b}_j} + d_0 \rangle \in \mathbb{Z}$ if and only if $\langle D_i, d_{\boldsymbol{b}_j} \rangle \in \mathbb{Z}$.

By assumption, $\widetilde{C}_{\mathcal{X}_2}$ is not contained in $\widetilde{C}_{\mathcal{X}_{1,\boldsymbol{b}_j}}$. It follows easily that

$$\sum_{\substack{i \in \{a, \ldots, m-2\} \\ \langle D_i, d_{\boldsymbol{b}_j} \rangle \in \mathbb{Z}}} \mathbb{R}_{>0} D_i$$

must contain $\overline{C}_{\mathcal{X}_1} \cap W$. Thus

$$\mathbb{R}_{>0} D_{m-1} + \sum_{\substack{i \in \{a,\ldots,m-2\} \\ \langle D_i, d_{\boldsymbol{b}_j} \rangle \in \mathbb{Z}}} \mathbb{R}_{\geq 0} D_i$$

contains the Kähler class ω_2, and $\{m-1\} \cup \{i \in \{a,\ldots,m-2\} \mid \langle D_i, d_{\boldsymbol{b}_j} \rangle \in \mathbb{Z}\}$ is in $\mathcal{A}'_2$. Since $\langle D_i, \hat{d} \rangle \in \mathbb{Z}$ for all $i \in \{m-1\} \cup \{i \in \{a,\ldots,m-2\} \mid \langle D_i, d_{\boldsymbol{b}_j} \rangle \in \mathbb{Z}\}$, we conclude that $\hat{d} \in \mathbb{K}_2$ by the definition of $\mathbb{K}_2$. $\square$

We calculate

$$\begin{aligned}
&\langle D_i, d_{\boldsymbol{b}_j} \rangle + 1 + Q_{m-1,1} \times \frac{\langle D_{m-1}, d_{\boldsymbol{b}_j} \rangle + \sum_{a \neq 1} Q_{m-1a} d_{0a} - l}{-Q_{m-1,1}} + \sum_{a \neq 1} Q_{m-1a} d_{0a} \\
=&\frac{Q_{i1}}{Q_{m-1,1}} l + \sum_{a \neq 1} \left(Q_{ia} - \frac{Q_{i1} Q_{m-1,a}}{Q_{m-1,1}} \right) d_{0a} - \frac{Q_{i1}}{Q_{m-1,1}} \langle D_{m-1}, d_{\boldsymbol{b}_j} \rangle + \langle D_i, d_{\boldsymbol{b}_j} \rangle \\
=&\langle D_i, \hat{d}_0 \rangle + \langle D_i - \frac{Q_{i1}}{Q_{m-1,1}} D_{m-1}, \hat{d}_{b_j} \rangle.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left((-1)^{Q_{m-1,1}} y_1\right)^{\frac{\langle D_{m-1}, d_{\boldsymbol{b}_j} \rangle + \sum_{a \neq 1} Q_{m-1a} d_{0a} - l}{-Q_{m-1,1}}} \\
=&(-1)^{(\langle D_{m-1}, d_{\boldsymbol{b}_j} \rangle + \sum_{a \neq 1} Q_{m-1a} d_{0a} - l)} \hat{y}_1^{-(\langle D_{m-1}, d_{\boldsymbol{b}_j} \rangle + \sum_{a \neq 1} Q_{m-1a} d_{0a} - l)},
\end{aligned}$$

$$y_a^{d_{0a}} = \hat{y}_a^{d_{0a}} \hat{y}_1^{Q_{m-1,a} d_{0a}} \text{ for } a \geq 2,$$

which gives

$$\begin{aligned}
&y^{d_{\boldsymbol{b}_j}} \left(\prod_{a \geq 2} y_a^{d_{0a}} \right) \left((-1)^{Q_{m-1,1}} y_1\right)^{\frac{\langle D_{m-1}, d_{\boldsymbol{b}_j} \rangle + \sum_{a \neq 1} Q_{m-1a} d_{0a} - l}{-Q_{m-1,1}}} \\
=&(-1)^{Q_{m-1,1} \times \frac{\langle D_{m-1}, d_{\boldsymbol{b}_j} \rangle + \sum_{a \neq 1} Q_{m-1a} d_{0a} - l}{Q_{m-1,1}}} \hat{y}^{\hat{d}_{\boldsymbol{b}_j}} \hat{y}^{\hat{d}_0}.
\end{aligned}$$

Also

$$\frac{\langle D_{m-1}, d_{\boldsymbol{b}_j} \rangle + \sum_{a \neq 1} Q_{m-1a} d_{0a} - l}{Q_{m-1,1}} = \langle \frac{D_{m-1}}{Q_{m-1,1}}, d_{\boldsymbol{b}_j} \rangle + \langle \frac{\hat{p}_1 - \sum_{a \neq 1} Q_{m-1,a} \hat{p}_a}{Q_{m-1,1}}, \hat{d}_0 \rangle.$$

From these calculations it is easy to see that the expression in (B.8) can be interpreted as a series in $\hat{y}$ whose exponents are contained in $\overline{NE}(\mathcal{X}_2) = \widehat{C}^{\vee}_{\mathcal{X}_2}$. This completes the construction of the analytic continuation.

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