ON SYZ MIRROR TRANSFORMATIONS

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Abstract. In this expository paper, we discuss how Fourier-Mukai-type transformations, which we call SYZ mirror transformations, can be applied to provide a geometric understanding of the mirror symmetry phenomena for semi-flat Calabi-Yau manifolds and toric Fano manifolds. We also speculate the possible applications of these transformations to other more general settings.

Contents

1. Introduction

In 1996, Strominger, Yau and Zaslow suggested, in their ground-breaking work [40], a geometric approach to the mirror symmetry for Calabi-Yau manifolds. Roughly speaking, the *Strominger-Yau-Zaslow (SYZ) Conjecture* asserts that any Calabi-Yau manifold X should admit a fibration by special Lagrangian tori and the mirror of X, which is another Calabi-Yau manifold Y, can be obtained by T-duality, i.e. dualizing the special Lagrangian torus fibration of X. Moreover, the symplectic geometry (A-model) of X should be interchanged with the complex geometry (B-model) of Y, and vice versa, through fiberwise Fourier-Mukai-type transformations, suitably modified by quantum corrections. These transformations are called *SYZ mirror transformations* and they will be the theme in this article.

Much work has been done on the SYZ Conjecture. Following the work of Hitchin [24], Leung-Yau-Zaslow [32] and Leung [31] explained successfully and neatly the mirror symmetry for semi-flat Calabi-Yau manifolds by using *semi-flat*

SYZ mirror transformations. These are honest fiberwise real Fourier-Mukai transformations. The advantage in this case is the absence of quantum corrections by holomorphic curves and discs. This is due to the fact that the special Lagrangian torus fibrations on semi-flat Calabi-Yau manifolds do not admit singularities, and, accordingly, the bases are smooth affine manifolds.

To deal with general compact Calabi-Yau manifolds, however, one cannot avoid singularities in Lagrangian torus fibrations, and hence singularities in the base affine manifolds. Consequently, quantum corrections will come into play. This necessitates the study of moduli spaces of special Lagrangian submanifolds and affine manifolds with singularities, which makes the subject much more sophisticated and difficult. Nevertheless, the recent progress made by Gross and Siebert [21], after earlier works of Fukaya [13] and Kontsevich-Soibelman [30], was doubtlessly a significant step towards establishing the SYZ Conjecture for general compact Calabi-Yau manifolds.[1]

On the other hand, mirror symmetry phenomena have also been observed for Fano manifolds (and other classes of manifolds or orbifolds as well). The mirror of a Fano manifold $\bar{X}$ is predicted by Physicists to be given by a *Landau-Ginzburg model*, which is a pair (Y, W), consisting of a non-compact Kähler manifold Y and a holomorphic function $W : Y \to \mathbb{C}$ called the *superpotential*. A very important class of examples is provided by toric Fano manifolds. In this case, the mirror manifold Y is biholomorphic to (a bounded domain of) $(\mathbb{C}^*)^n$ and the superpotential W is a Laurent polynomial which can be written down explicitly. Ample evidences have been found in this toric Fano case; in particular, Cho and Oh [9] proved that the superpotential can be computed in terms of the counting of Maslov index two holomorphic discs in $\bar{X}$ with boundary in Lagrangian torus fibers. In [4], Auroux applied the SYZ philosophy to the study of the mirror symmetry for a compact Kähler manifold equipped with an anticanonical divisor. This is a generalization of the mirror symmetry for Fano manifolds, and, again, the mirror is given by a Landau-Ginzburg model. Auroux also made an attempt to compute the superpotential in terms of the counting of holomorphic discs, and analyzed the resulting wall-crossing phenomena. In [7], we studied the mirror symmetry for toric Fano manifolds, again through the SYZ approach, and we constructed and applied *SYZ mirror transformations for toric Fano manifolds* to explain various geometric results implied by mirror symmetry.

A brief explanation of the results in [7] is now in order; for more details, see Section 3. Let $\bar{X}$ be a toric Fano manifold, i.e. a smooth projective toric variety such that the anticanonical line bundle $K_{\bar{X}}$ is ample. Let $\omega_{\bar{X}}$ be a toric Kähler structure on $\bar{X}$. The moment map $\mu_{\bar{X}} : \bar{X} \to \bar{P}$ of the Hamiltonian T^n-action on $(\bar{X}, \omega_{\bar{X}})$ is a natural Lagrangian torus fibration. Here $\bar{P} \subset \mathbb{R}^n$ is a polytope defining $(\bar{X}, \omega_{\bar{X}})$. The restriction of the moment map to the open dense T^n-orbit $X \cong (\mathbb{C}^*)^n \subset \bar{X}$ is a Lagrangian torus bundle $\mu_X = \mu_{\bar{X}}|_X : X \to P$, where P denotes the interior of the polytope $\bar{P}$. Our first result in [7] showed that the mirror manifold Y is nothing but the *SYZ mirror manifold* of X, i.e. the total space

[1]We should mention that the Gross-Siebert program is expected to work for non-Calabi-Yau manifolds (e.g. Fano manifolds) as well.

of the torus bundle dual to $\mu_X : X \to P$ (see Proposition 3.1).[2] Furthermore, the semi-flat SYZ transformation $\mathcal{F}^{\text{sf}}$ takes the exponential of ($\sqrt{-1}$ times) the symplectic structure $\omega_X = \omega_{\bar{X}}|_X$ on X to the holomorphic volume form Ω_Y on Y.[3] Note that Ω_Y determines a complex structure on Y by declaring that a 1-form α is a $(1,0)$-form if and only if $\alpha \lrcorner \Omega_Y = 0$. This part of the mirror symmetry does not involve quantum corrections.

To get the superpotential W, however, we need to take into account the quantum corrections due to the anticanonical toric divisor $D_\infty = \bar{X} \setminus X$, which we have ignored above. Before doing that, we first take a digression to a well-known construction. For a simply connected symplectic manifold (M, ω), let $\mathcal{L}M$ be the free loop space, i.e. the space of smooth maps $\gamma : S^1 \to M$. The symplectic structure on M induces a symplectic structure on $\mathcal{L}M$ which will also be denoted by ω. The action functional defined by

$$H(\gamma) := \frac{1}{2\pi} \int_{D_\gamma} \omega,$$

where D_γ is a disk contracting γ, becomes a well-defined function on the universal covering $\widetilde{\mathcal{L}M}$ of the free loop space $\mathcal{L}M$. The group of deck transformations is $H_2(M, \mathbb{Z})$. It is not hard to see that H is the moment map for the built-in S^1-action on $\widetilde{\mathcal{L}M}$, and the gradient flow lines of H are (pseudo-)holomorphic cylinders if we fix a compatible (almost) complex structure on M. Tentatively, the quantum cohomology (or Floer cohomology) is the S^1-equivariant Morse-Witten cohomology of the moment map H on $\widetilde{\mathcal{L}M}$. However, the fact that $\widetilde{\mathcal{L}M}$ is infinite dimensional poses severe difficulties in implementing this idea.

One of our discoveries in [7] was that a finite dimensional subspace of $\mathcal{L}M$ is enough to capture the quantum corrections and recover the quantum cohomology, in the case when $M = \bar{X}$ is a toric Fano manifold. Consider the subspace LX of $\mathcal{L}\bar{X}$ consisting of those loops which are *geodesic* in the Lagrangian torus fibers (with respect to the flat metrics) of the moment map $\mu_{\bar{X}} : \bar{X} \to \bar{P}$. We consider the function Ψ on LX defined by $\Psi(\gamma) = \exp(-H(\gamma))$ if γ bounds a Maslov index two holomorphic disc and $\Psi(\gamma) = 0$ otherwise. The function $\Psi : LX \to \mathbb{C}$, as an object in the A-model of $\bar{X}$, turns out to be the mirror of the superpotential W. In [7], we constructed the SYZ mirror transformation $\mathcal{F}$ for the toric Fano manifold $\bar{X}$, and showed that the SYZ mirror transformation of Ψ is precisely the B-model superpotential W. Moreover, by incorporating the symplectic structure ω_X and the holomorphic volume form Ω_Y, we proved that

$$\begin{aligned} \mathcal{F}(e^{\sqrt{-1}\omega_X + \Psi}) &= e^W \Omega_Y, \\ \mathcal{F}^{-1}(e^W \Omega_Y) &= e^{\sqrt{-1}\omega_X + \Psi}, \end{aligned}$$

where $\mathcal{F}^{-1}$ is the inverse SYZ mirror transformation (see Theorem 3.1). Hence, the *corrected* symplectic structure on X and the complex structure on (Y, W) are interchanged by the SYZ mirror transformation. On the other hand, we identified the small quantum cohomology ring $QH^*(\bar{X})$ of $\bar{X}$ with an algebra of functions

[2]More precisely, the SYZ mirror manifold is a bounded domain in the mirror manifold Y predicted by Physicists.

[3]Throughout this paper, we assume that the B-field is zero.

on LX, and realized the quantum product as a *convolution product* (see Proposition 3.2). Then, we showed that the SYZ mirror transformation $\mathcal{F}$ exhibits a natural isomorphism between $QH^*(\bar{X})$ and the Jacobian ring $Jac(W)$ of the superpotential W, which *takes the quantum product (now as a convolution product) to the ordinary product of Laurent polynomials, just as what classical Fourier series do* (see Theorem 3.2). We conclude that the mirror symmetry for toric Fano manifolds is nothing but a Fourier transformation!

The main goal of this article is to popularize the use of SYZ mirror transformations in exploring mirror symmetry phenomena. In Section 2, we review the use of semi-flat SYZ mirror transformations in the study of the mirror symmetry for semi-flat Calabi-Yau manifolds, where quantum corrections are absent. This is the toy case which lays the basis for subsequent development in the investigation of the SYZ Conjecture. Section 3 discusses the mirror symmetry for toric Fano manifolds, where quantum corrections arise due to the anticanonical toric divisor. Following [7], we demonstrate how to construct and apply SYZ mirror transformations in this case. The final section contains a brief discussion of possible generalizations.

Acknowledgments. The authors are grateful to the organizers of the conference "New developments in Algebraic Geometry, Integrable Systems and Mirror Symmetry" held in Kyoto University in January 2008 for giving them an opportunity to participate in such a stimulating and fruitful event. Thanks are also due to Hiroshi Iritani and Cheol-Hyun Cho for many useful discussions. Finally, we thank the referee for several helpful comments. K.-W. C. was partially supported by Harvard University and the Croucher Foundation Fellowship. N.-C. L. was partially supported by RGC grants from the Hong Kong Government.

2. SYZ MIRROR TRANSFORMATIONS WITHOUT CORRECTIONS

In this section, we review the construction of SYZ mirror transformations for semi-flat Calabi-Yau manifolds and see how they were applied in the study of semi-flat mirror symmetry.

2.1. **Semi-flat SYZ mirror transformations.** Denote by $N \cong \mathbb{Z}^n$ a rank-n lattice and $M = \mathrm{Hom}(N, \mathbb{Z})$ the dual lattice. Let $D \subset M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ be a convex domain. [4] Then the tangent bundle $TD = D \times \sqrt{-1}M_{\mathbb{R}}$ is naturally a complex manifold with complex coordinates $x_j + \sqrt{-1}y_j$, $j = 1, \ldots, n$, where $x_1, \ldots, x_n \in \mathbb{R}$ and $y_1, \ldots, y_n \in \mathbb{R}$ are respectively the base coordinates on D and fiber coordinates on $M_{\mathbb{R}}$. We have the standard holomorphic volume form $\Omega_{TD} = d(x_1 + \sqrt{-1}y_1) \wedge \ldots \wedge d(x_n + \sqrt{-1}y_n)$ on TD. By taking fiberwise quotient by the lattice $M \subset M_{\mathbb{R}}$, we can compactify the fiber directions to give the complex manifold

$$Y = TD/M = D \times \sqrt{-1}T_M,$$

where T_M denotes the torus $M_{\mathbb{R}}/M$. The complex coordinates on Y are naturally given by $z_j = \exp(x_j + \sqrt{-1}y_j)$, $j = 1, \ldots, n$, where $y_1, \ldots, y_n \in \mathbb{R}/2\pi\mathbb{Z}$ are now coordinates on T_M. Note that Y is biholomorphic to an open part of $(\mathbb{C}^*)^n =$

[4] More generally, instead of a convex domain, one may consider a smooth affine manifold.

$TM_{\mathbb{R}}/M$. The projection to D is a torus bundle, which we denote by $\nu_Y : Y \to D$. The holomorphic n-form Ω_{TD} descends to give the holomorphic volume form

$$\Omega_Y = \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}$$

on Y. As mentioned in the introduction, Ω_Y in turn determines the complex structure on Y: a 1-form α is of $(1,0)$-type if and only if $\alpha \lrcorner \Omega_Y = 0$. Further, if ϕ is an elliptic solution of the *real Monge-Ampère equation*

$$\det\Big(\frac{\partial^2 \phi}{\partial x_j \partial x_k}\Big) = const,$$

then the Kähler form

$$\omega_Y := \sqrt{-1}\partial\bar{\partial}\phi = \sum_{j,k} \phi_{jk} dx_j \wedge dy_k,$$

with ϕ_{jk} denoting $\frac{\partial^2 \phi}{\partial x_j \partial x_k}$, gives a Calabi-Yau metric on Y, and

$$\nu_Y : Y \to D$$

becomes a special Lagrangian torus bundle (*SYZ fibration*). In summary, we have the following structures on the complex n-dimensional semi-flat Calabi-Yau manifold Y:

Riemannian metric	$g_Y = \sum_{j,k} \phi_{jk}(dx_j \otimes dx_k + dy_j \otimes dy_k)$
Holomorphic volume form	$\Omega_Y = \bigwedge_{j=1}^n (dx_j + \sqrt{-1} dy_j)$
Symplectic form	$\omega_Y = \sum_{j,k} \phi_{jk} dx_j \wedge dy_k$
SYZ fibration	$\nu_Y : Y \to D$

As suggested in the monumental work Strominger-Yau-Zaslow [40], the mirror of Y, which is another Calabi-Yau manifold we denote by X, should be given by the moduli space of pairs (L, ∇), where L is a special Lagrangian torus fiber in Y, and ∇ is a flat $U(1)$-connection on the trivial complex line bundle $L \times \mathbb{C} \to L$. This is nothing but the total space of the torus fibration $\mu_X : X = D \times \sqrt{-1}T_N \to D$, where $T_N = N_{\mathbb{R}}/N = (T_M)^\vee$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, which is dual to $\nu_Y : Y \to D$. This is called *T-duality* in physics. Furthermore, X can naturally be viewed as the fiberwise quotient of the cotangent bundle $T^*D = D \times \sqrt{-1}N_{\mathbb{R}}$ by the lattice $N \subset N_{\mathbb{R}}$. In particular, the standard symplectic form $\omega_{T^*D} = \sum_{j=1}^n dx_j \wedge du_j$ descends to give a symplectic form

$$\omega_X = \sum_{j=1}^n dx_j \wedge du_j$$

on $X = T^*D/N$, where $u_1, \ldots, u_n \in \mathbb{R}/2\pi\mathbb{Z}$ are coordinates on T_N. Through the metric

$$g_X = \sum_{j,k} (\phi_{jk} dx_j \otimes dx_k + \phi^{jk} du_j \otimes du_k),$$

where (ϕ^{jk}) is the inverse matrix of (ϕ_{jk}), we obtain a complex structure on X with complex coordinates given by $d\log(w_j) = \sum_{k=1}^n \phi_{jk} dx_k + \sqrt{-1} du_j$. There is a corresponding holomorphic volume form which can be written as

$$\Omega_X = \frac{dw_1}{w_1} \wedge \ldots \wedge \frac{dw_n}{w_n} = \bigwedge_{j=1}^{n} (\sum_{k=1}^{n} \phi_{jk} dx_k + \sqrt{-1} du_j).$$

The projection map

$$\mu_X : X \to D$$

now naturally becomes a special Lagrangian torus fibration. In summary, we have the following structures on X:

Riemannian metric	$g_X = \sum_{j,k} (\phi_{jk} dx_j \otimes dx_k + \phi^{jk} du_j \otimes du_k)$
Holomorphic volume form	$\Omega_X = \bigwedge_{j=1}^{n} (\sum_{k=1}^{n} \phi_{jk} dx_k + \sqrt{-1} du_j)$
Symplectic form	$\omega_X = \sum_{j=1}^{n} dx_j \wedge du_j$
SYZ fibration	$\mu_X : X \to D$

We remark that both Y and X admit natural Hamiltonian T^n-actions, but while $\mu : X \to D$ is a moment map for the T_N-action on X, $\nu : Y \to D$ is not a moment map for the T_M-action on Y. In fact, a moment map $\mu_Y : Y \to N_{\mathbb{R}}$ for the T_M-action on Y is given by

$$\mu_Y = L_\phi \circ \nu_Y,$$

where $L_\phi : D \to N_{\mathbb{R}}$ is the *Legendre transform* of ϕ defined by

$$L_\phi(x_1, \ldots, x_n) = d\phi_x = \Big(\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n} \Big).$$

Since ϕ is convex, the image $D^* = L_\phi(D)$ is an open convex subset of $(M_{\mathbb{R}})^* = N_{\mathbb{R}}$. (For this and other properties of the Legendre transform, see the book of Guillemin [22], Appendix 1.) In the action coordinates $x^1, \ldots, x^n$ of D^*, which are given by $\frac{\partial x^j}{\partial x_k} = \phi_{jk}$, the various structures on Y can be rewritten as:

Riemannian metric	$g_Y = \sum_{j,k} (\phi^{jk} dx^j \otimes dx^k + \phi_{jk} dy_j \otimes dy_k)$
Holomorphic volume form	$\Omega_Y = \bigwedge_{j=1}^{n} (\sum_{k=1}^{n} \phi^{jk} dx^k + \sqrt{-1} dy_j)$
Symplectic form	$\omega_Y = \sum_{j=1}^{n} dx^j \wedge dy_j$
SYZ fibration	$\mu_Y : Y \to D^*$

We call X the *SYZ mirror manifold* of Y (and vice versa) since the symplectic (resp. complex) geometry of X and the complex (resp. symplectic) geometry of Y are interchanged under the *semi-flat SYZ mirror transformation*, which is described as follows.

First recall that the dual torus $T_M = (T_N)^\vee$ can be interpreted as the moduli space of flat $U(1)$-connections on the trivial complex line bundle over T_N. More precisely, given $y = (y_1, \ldots, y_n) \in M_{\mathbb{R}} \cong \mathbb{R}^n$, we have a flat $U(1)$-connection

$$\nabla_y = d + \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} y_j du_j$$

on $T_N \times \mathbb{C} \to \mathbb{C}$. The *holonomy* of ∇_y is given by the map

$$\mathrm{hol}_{\nabla_y} : N \to U(1),\ v \mapsto e^{-\sqrt{-1}\langle y, v \rangle}.$$

Hence, ∇_y is gauge equivalent to the trivial connection if and only if $y \in M \cong (2\pi\mathbb{Z})^n$. Moreover this construction gives all flat $U(1)$-connections on the trivial complex line bundle over T_N up to unitary gauge transformations. The universal $U(1)$-bundle, i.e. the Poincaré line bundle $\mathcal{P}$, is given by the trivial complex line bundle $(T_N \times T_M) \times \mathbb{C} \to T_N \times T_M$ equipped with the connection $d + \frac{\sqrt{-1}}{2}\sum_{j=1}^n (y_j du_j - u_j dy_j)$. The curvature of this connection is the two form

$$F = \sqrt{-1}\sum_{j=1}^n dy_j \wedge du_j.$$

Now consider the relative version of this picture. Let $X \times_D Y = D \times \sqrt{-1}(T_N \times T_M)$ be the fiber product of the dual torus bundles $\mu : X \to D$ and $\nu : Y \to D$. By abuse of notations, we still use $\mathcal{P}$ and $F = \sqrt{-1}\sum_{j=1}^n dy_j \wedge du_j \in \Omega^2(X \times_D Y)$ to denote the fiberwise universal line bundle and curvature two form respectively.

Definition 2.1. *The semi-flat SYZ mirror transformation*

$$\mathcal{F}^{sf} : \Omega^*(X) \to \Omega^*(Y)$$

is defined by

$$\begin{aligned}\mathcal{F}^{sf}(\alpha) &= \frac{1}{(2\pi\sqrt{-1})^n}\pi_{Y,*}(\pi_X^*(\alpha) \wedge e^{\sqrt{-1}F}) \\ &= \frac{1}{(2\pi\sqrt{-1})^n}\int_{T_N} \pi_X^*(\alpha) \wedge e^{\sqrt{-1}F},\end{aligned}$$

where $\pi_X : X \times_D Y \to X$ and $\pi_Y : X \times_D Y \to Y$ are the two projections.

What is crucial is that this Fourier-Mukai-type transformation transforms the symplectic structure on X to the complex structure on Y in the sense of the following two propositions. These already appeared in [7], Proposition 3.2. We include their proofs, which are somewhat interesting, here for completeness.

Proposition 2.1.

$$\mathcal{F}^{sf}(e^{\sqrt{-1}\omega_X}) = \Omega_Y.$$

Proof.

$$\begin{aligned}\mathcal{F}^{\mathrm{sf}}(e^{\sqrt{-1}\omega_X}) &= \frac{1}{(2\pi\sqrt{-1})^n}\int_{T_N} \pi_X^*(e^{\sqrt{-1}\omega_X}) \wedge e^{\sqrt{-1}F} \\ &= \frac{1}{(2\pi\sqrt{-1})^n}\int_{T_N} e^{\sqrt{-1}\sum_{j=1}^n (dx_j + \sqrt{-1}dy_j)\wedge du_j} \\ &= \frac{1}{(2\pi\sqrt{-1})^n}\int_{T_N} \bigwedge_{j=1}^n \left(1 + \sqrt{-1}(dx_j + \sqrt{-1}dy_j) \wedge du_j\right) \\ &= \frac{1}{(2\pi)^n}\int_{T_N} \left(\bigwedge_{j=1}^n (dx_j + \sqrt{-1}dy_j)\right) \wedge du_1 \wedge \ldots \wedge du_n \\ &= \Omega_Y,\end{aligned}$$

where we have $\int_{T_N} du_1 \wedge \ldots \wedge du_n = (2\pi)^n$ in the final step. $\square$

As a mirror transformation, $\mathcal{F}^{\mathrm{sf}}$ should have the *inversion property*. This is the following proposition.

Proposition 2.2. *If we define the inverse transform* $(\mathcal{F}^{sf})^{-1} : \Omega^*(Y) \to \Omega^*(X)$ *by*

$$\begin{aligned}(\mathcal{F}^{sf})^{-1}(\alpha) &= \frac{1}{(2\pi\sqrt{-1})^n}\pi_{X,*}(\pi_Y^*(\alpha)\wedge e^{-\sqrt{-1}F}) \\ &= \frac{1}{(2\pi\sqrt{-1})^n}\int_{T_M}\pi_Y^*(\alpha)\wedge e^{-\sqrt{-1}F},\end{aligned}$$

then we have

$$(\mathcal{F}^{sf})^{-1}(\Omega_Y) = e^{\sqrt{-1}\omega_X}.$$

Proof.

$$\begin{aligned}(\mathcal{F}^{\mathrm{sf}})^{-1}(\Omega_Y) &= \frac{1}{(2\pi\sqrt{-1})^n}\int_{T_M}\pi_Y^*(\Omega_Y)\wedge e^{-\sqrt{-1}F} \\ &= \frac{1}{(2\pi\sqrt{-1})^n}\int_{T_M}\left(\bigwedge_{j=1}^n(dx_j+\sqrt{-1}dy_j)\right)\wedge e^{\sum_{j=1}^n dy_j\wedge du_j} \\ &= \frac{1}{(2\pi\sqrt{-1})^n}\int_{T_M}\bigwedge_{j=1}^n\left((dx_j+\sqrt{-1}dy_j)\wedge e^{dy_j\wedge du_j}\right) \\ &= \frac{1}{(2\pi\sqrt{-1})^n}\int_{T_M}\bigwedge_{j=1}^n\left(dx_j+\sqrt{-1}dy_j+dx_j\wedge dy_j\wedge du_j\right) \\ &= \frac{1}{(2\pi)^n}\int_{T_M}\bigwedge_{j=1}^n\left((1+\sqrt{-1}dx_j\wedge du_j)\wedge dy_j\right) \\ &= \frac{1}{(2\pi)^n}\int_{T_M}\bigwedge_{j=1}^n\left(e^{\sqrt{-1}dx_j\wedge du_j}\wedge dy_j\right) \\ &= \frac{1}{(2\pi)^n}\int_{T_M}e^{\sqrt{-1}\sum_{j=1}^n dx_j\wedge du_j}\wedge dy_1\wedge\ldots\wedge dy_n \\ &= e^{\sqrt{-1}\omega_X}.\end{aligned}$$

□

By exactly the same arguments, one can also show that

$$\mathcal{F}^{\mathrm{sf}}(\Omega_X) = e^{\sqrt{-1}\omega_Y},\ (\mathcal{F}^{\mathrm{sf}})^{-1}(e^{\sqrt{-1}\omega_Y}) = \Omega_X.$$

If we take into account the *B-fields*, then the semi-flat SYZ transformation will give an identification between the moduli space of complexified Kähler structures on X with the moduli space of complex structures on Y, and vice versa. For this and transformations of other geometric structures, we refer the reader to Leung [31].

2.2. **Transformations of branes.** Lying at the heart of the SYZ Conjecture is the basic but important observation that a point $z = \exp(x+\sqrt{-1}y) \in Y$ defines a flat $U(1)$-connection ∇_y on the trivial complex line bundle over the special Lagrangian torus fiber $L_x = \mu_X^{-1}(x)$. Now, the point $z \in Y$ together with its structure sheaf $\mathcal{O}_z$ can be considered as a *B-brane* on Y; while the pair $(L_x, \mathbb{L}_y)$, where $\mathbb{L}_y$ denotes the flat $U(1)$-bundle $(L_x \times \mathbb{C}, \nabla_y)$, gives an *A-brane* on X. This implements the simplest case of correspondence between branes on mirror manifolds via SYZ transformations:

$$(L_x, \mathbb{L}_y) \longleftrightarrow (z, \mathcal{O}_z).$$

The space of infinitestimal deformations of the A-brane $(L_x, \mathbb{L}_y)$, which is given by $H^1(L_x, \mathbb{R}) \times H^1(L_x, \sqrt{-1}\mathbb{R}) = H^1(L_x, \mathbb{C})$, is canonically identified with the tangent space $T_z Y$, the space of infinitestimal deformations of the sheaf $\mathcal{O}_z$.

On the other hand, consider a section $L = \{(x, u(x)) \in X : x \in D\}$ of $\mu_X : X \to D$. The submanifold L is Lagrangian if and only if (locally) there exists a function f such that $u_j = \frac{\partial f}{\partial x_j}$. By the above observation (now used in the opposite way), a point $(x, u(x)) \in L$ determines a flat $U(1)$-connection $\nabla_{u(x)}$ on the trivial complex line bundle over the fiber $(L_x)^\vee = \nu_Y^{-1}(x)$. The family of points $\{(x, u(x)) : x \in D\}$ thus patch together to give the $U(1)$-connection

$$\nabla_L = d_Y - \frac{\sqrt{-1}}{2} \sum_{j=1}^n u_j(x) dy_j$$

on a certain complex line bundle over Y; its curvature two form is given by

$$F_L = d_Y\Big(- \frac{\sqrt{-1}}{2} \sum_{j=1}^n u_j(x) dy_j\Big) = -\frac{\sqrt{-1}}{2} \sum_{j,k} \frac{\partial u_j}{\partial x_k} dx_k \wedge dy_j,$$

and, in particular,

$$F_L^{2,0} = \frac{1}{8} \sum_{j<k} \Big(\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j}\Big) \frac{dz_j}{z_j} \wedge \frac{dz_k}{z_k}.$$

We conclude that ∇_L is integrable, i.e. $F_L^{2,0} = 0$, if and only if L is Lagrangian. More generally, we can equip L with a flat $U(1)$-bundle $\mathbb{L} = (L \times \mathbb{C}, d_L + \alpha)$, where $\alpha \in \Omega^1(L, \mathbb{R})$ is a closed (and hence exact) one-form. The A-brane $(L, \mathbb{L})$ is then transformed to the $U(1)$-connection

$$\nabla_{L,\mathbb{L}} = \nabla_L + \alpha,$$

which again is integrable if and only if L is Lagrangian. Furthermore, one can prove that $\nabla_{L,\mathbb{L}}$ satisfies the *deformed Hermitian-Yang-Mills equations* if and only if L is special Lagrangian (see Leung-Yau-Zaslow [32] and Leung [31] for the detailed proofs). $\nabla_{L,\mathbb{L}}$ is a connection on the holomorphic line bundle over Y given by the semi-flat SYZ transformation of $\mathbb{L}$:

$$\mathcal{L}_{L,\mathbb{L}} = \pi_{Y,*}(\pi_X^*(\iota_* \mathbb{L}) \otimes \mathcal{P}),$$

where $\iota : L \hookrightarrow X$ is the inclusion map. In conclusion, the A-brane $(L, \mathbb{L})$ is transformed to the B-brane $(Y, \mathcal{L}_{L,\mathbb{L}})$ through semi-flat SYZ transformations:

$$(L, \mathbb{L}) \longleftrightarrow (Y, \mathcal{L}_{L,\mathbb{L}}).$$

3. SYZ mirror transformations with corrections

In the previous section, we see that T-duality and SYZ mirror transformations can be applied successfully to give a geometric understanding of the mirror symmetry for semi-flat Calabi-Yau manifolds. However, no quantum corrections were involved in this case due to the absence of holomorphic curves and discs. The existence of quantum corrections is also closely related to the singularities of the Lagrangian torus fibrations, which again are not present in the semi-flat case. In this section, following [7], we are going to discuss how SYZ mirror transformations can be applied to a case where quantum corrections *do* exist, namely, the mirror symmetry for toric Fano manifolds.

3.1. **Mirror symmetry for toric Fano manifolds.** We begin with a more detailed description of the mirror picture for toric Fano manifolds [17], [29], [27]. Let $\bar{P} \subset M_{\mathbb{R}}$ be a smooth reflexive polytope given by the inequalities

$$\langle x, v_i \rangle \geq \lambda_i, \quad i = 1, \ldots, d,$$

where $v_1, \ldots, v_d \in N$ are primitive vectors and $\langle \cdot, \cdot \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ is the dual pairing. This determines a toric Fano manifold $\bar{X}$, together with a Kähler structure $\omega_{\bar{X}}$. Unlike the case of Calabi-Yau manifolds, the mirror of $\bar{X}$ is not another compact Kähler manifold, but a Landau-Ginzburg model: a pair (Y, W) consisting of a noncompact Kähler manifold Y, which (as a complex manifold) is biholomorphic to (a bounded domain of) $(\mathbb{C}^*)^n$, and the Laurent polynomial

$$W = e^{\lambda_1} z^{v_1} + \ldots + e^{\lambda_d} z^{v_d} : Y \to \mathbb{C},$$

which is called the superpotential. Here z^{v_i} denotes the monomial $z_1^{v_{i1}} \ldots z_n^{v_{in}}$ in the coordinates $z_1, \ldots, z_n$ of Y. For example, if $P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq t\}$, then $\bar{X} = \mathbb{C}P^2$ and the mirror Landau-Ginzburg model is given by the Laurent polynomial $W(z_1, z_2) = z_1 + z_2 + \frac{e^{-t}}{z_1 z_2}$ on $Y = (\mathbb{C}^*)^2$.

Among the many mirror symmetry predictions are the following conjectures:

Conjecture 3.1.

1. *The small quantum cohomology ring $QH^*(\bar{X})$ of $\bar{X}$ is isomorphic to the Jacobian ring $Jac(W)$ of W, where*

$$Jac(W) = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] / \langle \partial_1 W, \ldots, \partial_n W \rangle,$$

and ∂_j denotes $z_j \frac{\partial}{\partial z_j}$.

2. *(Homological mirror symmetry, see* [29], [39], [37]*) There are equivalences of triangulated categories*

$$\begin{aligned} D^b Coh(\bar{X}) &\cong D^{\pi} Fuk(Y, W) \\ D^{\pi} Fuk(\bar{X}) &\cong D_{Sing}(Y, W) \end{aligned}$$

where $D^{\pi} Fuk(Y, W)$ is (a suitably defined version of) the derived Fukaya category of the Landau-Ginzurg model (Y, W) and $D_{Sing}(Y, W)$ is the category of singularities of (Y, W).

Substantial evidences [19], [25], [39], [41], [5], [6], [1], [2], [9], [8] have been found for these conjectures, while evidence in the Calabi-Yau and other non-toric cases is much rarer. This is partly due to the fact that geometric structures on toric varieties are highly computable and explicit, making them an exceptionally fertile testing ground for techniques and conjectures.

One of these explicit structures: the Lagrangian torus fibration on $\bar{X}$ given by the moment map $\mu_{\bar{X}} : \bar{X} \to \bar{P}$ of the Hamiltonian T_N-action on $(\bar{X}, \omega_{\bar{X}})$, is particularly important in the SYZ spproach and in the constructions of SYZ mirror transformations. Let

$$\mu_X : X \to P$$

be the restriction of the moment map to the open dense T_N-orbit $X = \bar{X} \setminus D_\infty$, where $D_\infty = \bigcup_{i=1}^d D_i$ is the anticanonical toric divisor, and P is the interior of $\bar{P}$. In the symplectic (or action-angle) coordinates,

$$X = T^*P/N = P \times \sqrt{-1}T_N$$

and the restriction of $\omega_{\bar{X}}$ to X is nothing but the standard symplectic structure

$$\omega_X = \sum_{j=1}^n dx_j \wedge du_j,$$

where $x_1, \ldots, x_n \in \mathbb{R}$ and $u_1, \ldots, u_n \in \mathbb{R}/2\pi\mathbb{Z}$ are respectively the base coordinates on P and fiber coordinates on T_N (see Abreu [3]). Now we are in exactly the same situation as in the previous section and it is tempting to assert that the mirror manifold Y predicted by Physicists is given by the SYZ mirror manifold of X, which is $TP/M = P \times \sqrt{-1}T_M$. This is indeed nearly the case.

Proposition 3.1 (Proposition 3.1 in [7]). *The mirror manifold $Y = (\mathbb{C}^*)^n$ predicted by Physicists contains the SYZ mirror manifold $TP/M = P \times \sqrt{-1}T_M$ of $X = \bar{X} \setminus D_\infty$ as a bounded domain*

$$\{(z_1, \ldots, z_n) \in Y : |e^{\lambda_i} z^{v_i}| < 1 \text{ for } i = 1, \ldots, d\}.$$

Equivalently, the SYZ mirror manifold is given by the preimage of $P \subset M_\mathbb{R} = \mathbb{R}^n$ under the Log map

$$Log : (\mathbb{C}^*)^n \to \mathbb{R}^n,\ (z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|).$$

The same result also appeared in Auroux's paper [4] (Proposition 4.2). Also included in his paper was a discussion of the issue that the SYZ mirror manifold (a bounded domain in $(\mathbb{C}^*)^n$) is "smaller" than Hori-Vafa's mirror manifold (the whole $(\mathbb{C}^*)^n$). There is evidence (say, in Abouzaid's works [1], [2]) showing that one should work with the SYZ mirror manifold, instead of the whole $(\mathbb{C}^*)^n$, in studying mirror symmetry. In any case, we will use and work with the SYZ mirror manifold, i.e. the bounded domain in $(\mathbb{C}^*)^n$, and denote it by Y henceforth.

In terms of the coordinates $z_1 = \exp(-x_1 - \sqrt{-1}y_1), \ldots, z_n = \exp(-x_n - \sqrt{-1}y_n) \in \mathbb{C}^*$ of $Y \subset (\mathbb{C}^*)^n$, the holomorphic volume form is given by the standard one on $(\mathbb{C}^*)^n$:

$$\Omega_Y = \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}$$

and the torus fibration $\nu_Y : Y \to P$ is the restriction of the Log map. We remark that metrically we are *not* considering $X = \bar{X} \setminus D_\infty$ as a Calabi-Yau manifold; instead of the semi-flat Calabi-Yau metric, we use the T_N-invariant Kähler metric on $\bar{X}$ (and the corresponding dual metric on Y). These are defined (cf. Guillemin [22] and Abreu [3]) using the strictly convex function $\phi_P : P \to \mathbb{R}$ given by

$$\phi_P(x) = \frac{1}{2} \sum_{i=1}^d l_i(x) \log l_i(x),$$

where $l_i(x) = \langle x, v_i \rangle - \lambda_i$ for $i = 1, \ldots, d$, instead of a solution of the real Monge-Ampère equation. For example, this gives the standard Fubini-Study metric on $\bar{X} = \mathbb{C}P^n$. Using these metrics and the corresponding holomorphic volume forms, X and Y are almost Calabi-Yau manifolds and the torus fibers of μ_X and ν_Y are special Lagrangian submanifolds (also see Section 2 in Auroux [4]).

3.2. **SYZ transformations for toric Fano manifolds.** By applying the semi-flat SYZ mirror transformation or T-duality, we can obtain the mirror manifold Y. But where comes the superpotential $W : Y \to \mathbb{C}$? Recall that, in applying T-duality, we have completely ignored the compactification of X, which is given by adding the anticanonical toric divisor $D_\infty = \bigcup_{i=1}^{d} D_i$. As suggested in the foundational work of Fukaya-Oh-Ohta-Ono [14], this has tremendous effect on the Floer theory of the Lagrangian torus fibers of $\mu_X : X \to P$, and this is indeed where quantum corrections by holomorphic discs come into play.

As have been discussed in the introduction, motivated by the idea of using Morse theory on the free loop space $\mathcal{L}\bar{X}$ to construct the quantum cohomology $QH^*(\bar{X})$, we introduce the subspace $LX \subset \mathcal{L}\bar{X}$ consisting of those loops which are geodesic in the Lagrangian torus fibers of the moment map $\mu_X : X \to P$, i.e.

$$LX = \{\gamma \in \mathcal{L}\bar{X} : \gamma \text{ is a geodesic in } L_x = \mu_X^{-1}(x) \text{ for some } x \in P\}.$$

Concretely, we have

$$LX = X \times N = P \times \sqrt{-1}T_N \times N,$$

and we consider it as a (trivial) $\mathbb{Z}^n$-cover of X, $\pi : LX \to X$. Notice that, for each Lagrangian torus fiber L_x, $x \in P$, we have a canonical identification $\pi_1(L_x) \cong N$.

We are going to define a function Ψ on LX in terms of the counting of holomorphic discs in $\bar{X}$ of minimal Maslov index. This will recapture the information of the compactification of X by D_∞, which we have ignored previously, and Ψ serves as the object in the A-model of $\bar{X}$ mirror to the superpotential W. To do this, let's first recall the fundamental results of Cho-Oh [9] on the classification of holomorphic discs in $\bar{X}$ with boundary in Lagrangian torus fibers of $\mu_X : X \to P$.

Let $L_x = \mu_X^{-1}(x)$ be the Lagrangian torus fiber in X over a point $x \in P$. Then the relative homotopy group $\pi_2(\bar{X}, L_x)$ is generated by the Maslox index two classes $\beta_1, \ldots, \beta_d$, which are represented by holomorphic discs in $(\bar{X}, L_x)$. Note that we have, $\partial\beta_i = v_i$, for $i = 1, \ldots, d$, where $\partial : \pi_2(\bar{X}, L_x) \to \pi_1(L_x) \cong N$ is the natural boundary map. In [9], Cho and Oh proved that, for $i = 1, \ldots, d$ and for each point $p \in L_x$, there is a unique (up to automorphism of the domain) Maslov index two J-holomorphic disc $\varphi_i : (D^2, \partial D^2) \to (\bar{X}, L_x)$ in the class β_i which passes through p and intersects the toric divisor D_i at an interior point.[5] Here J is the complex structure on $\bar{X}$ determined by the fan Σ dual to $\bar{P}$.

Definition 3.1. *For $i = 1, \ldots, d$, define $\Psi_i : LX \to \mathbb{R}$ by*

$$\Psi_i(p, v) = \begin{cases} n_i(p) \exp(-\frac{1}{2\pi} \int_{\beta_i} \omega_{\bar{X}}) & \text{if } v = v_i \\ 0 & \text{if } v \neq v_i, \end{cases}$$

for $(p, v) \in LX = X \times N$, where $n_i(p)$ is the algebraic number of Maslov index two J-holomorphic discs in $(\bar{X}, L_{\mu_X(p)})$ in the class β_i which pass through p. Then set

$$\Psi = \Psi_1 + \ldots + \Psi_d : LX \to \mathbb{R}.$$

[5] Another way to state this result is the following. Let $\mathcal{M}_1(\beta_i)$ be the moduli space of J-holomorphic discs $\varphi : (D^2, \partial D^2) \to (\bar{X}, L_x)$ in the class β_i with 1 boundary marked point. Let $ev : \mathcal{M}_1(\beta_i) \to L_x$ be the evaluation map at the boundary marked point. Then the result of Cho and Oh says that $ev_*[\mathcal{M}_1(\beta_i)] = [L_x]$ as n-cycles in L_x. See also Sections 3.1 and 4 in Auroux [4].

By their definitons, the T_N-invariant functions $\Psi_1, \ldots, \Psi_d$ carry enumerative meaning, although by Cho and Oh's result, we always have $n_i(p) = 1$, for all i and any p. One may think of the T_N-invariant function Ψ as recording which cycle $v \in N = \pi_1(L_x)$ collapses to a point as one goes towards the anticanonical toric divisor D_∞, or equivalently, which geodesic loop $\gamma \in LX$ bounds a holomorphic disc of Maslov index two.

Remark 3.1. *Before showing how to transform Ψ to get the superpotential W, we remark that the T_N-invariant function $\Phi : LX \to \mathbb{R}$ introduced in* [7]*, Definition 2.1, is nothing but the "exponential" of Ψ, i.e.*

$$\Phi = Exp\ \Psi,$$

where $Exp\ \Psi$ is defined as $\sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\Psi \star \ldots \star \Psi}_{k \text{ times}}$ in which $\star$ denotes the convolution product of a certain class of functions on LX with respect to the lattice N. Now each point $q = (q_1, \ldots, q_l)$ $(l = d - n)$ in the Kähler cone $\mathcal{K}(\bar{X}) \subset H^2(\bar{X}, \mathbb{R})$ determines a symplectic structure $\omega_{\bar{X}}$ on $\bar{X}$ and we can choose the polytope $\bar{P} = \{x \in M_{\mathbb{R}} : \langle x, v_i \rangle \geq \lambda_i, \quad i = 1, \ldots, d\}$ such that $v_1 = e_1, \ldots, v_n = e_n$ is the standard basis of $N = \mathbb{Z}^n$, $\lambda_1 = \ldots = \lambda_n = 0$ and $\lambda_{n+a} = \log q_a$ for $a = 1, \ldots, l$. We thus get two families of functions $\{\Psi_q\}_{q \in \mathcal{K}}$ and $\{\Phi_q\}_{q \in \mathcal{K}}$. By the symplectic area formula of Cho-Oh ([9]*, Theorem 8.1), we have*

$$\int_{D^2} \varphi_i^* \omega_{\bar{X}} = \int_{\beta_i} \omega_{\bar{X}} = 2\pi(\langle x, v_i \rangle - \lambda_i),$$

for $i = 1, \ldots, d$. Hence, for any $(p, v) \in LX$,

$$\Psi_i(p, v) = \begin{cases} e^{-\langle x, v_i \rangle} & \text{if } v = v_i \\ 0 & \text{if } v \neq v_i, \end{cases}$$

for $i = 1, \ldots, n$, and

$$\Psi_{n+a}(p, v) = \begin{cases} q_a e^{-\langle x, v_{n+a} \rangle} & \text{if } v = v_{n+1} \\ 0 & \text{if } v \neq v_{n+a}, \end{cases}$$

for $a = 1, \ldots, l$, where $x = \mu_X(p)$. It follows that

$$q_a \frac{\partial \Phi_q}{\partial q_a} = \Phi_q \star \Psi_{n+a}$$

for $a = 1, \ldots, l$, which is the first part of Proposition 1.1 in [7].

On the other hand, the functions $\Psi_1, \ldots, \Psi_d$ are intimately related to the small quantum cohomology $QH^*(\bar{X})$ of $\bar{X}$, as was shown in the following

Proposition 3.2 (Second part of Proposition 1.1 in [7])**.** *Assume that $\bar{X}$ is a product of projective spaces. Then we have a natural isomorphism of $\mathbb{C}$-algebras*

$$QH^*(\bar{X}) \cong \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]/\mathfrak{L}$$

where $\mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]$ is the polynomial algebra generated by $\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}$ with respect to the convolution product $\star$, and $\mathfrak{L}$ is the ideal generated by linear relations: $\sum_{i=1}^{d} a_i \Psi_i \sim \sum_{i=1}^{d} b_i \Psi_i$ if and only if the corresponding divisors $\sum_{i=1}^{d} a_i D_i$ and $\sum_{i=1}^{d} b_i D_i$ are linearly equivalent.

Remark 3.2. *By employing Givental's mirror theorem* [19]*, one can in fact show that the proposition holds for all toric Fano manifolds. See Remark 2.3 in* [7] *for details.*

We need the assumption that $\bar{X}$ is a product of projective spaces as we are intended for a geometric understanding of the isomorphism in Proposition 3.2 by using *tropical geometry*. This is briefly described as follows (see Subsection 2.2 in [7] for details). One first defines a tropical version $QH^*_{trop}(\bar{X})$ of the small quantum cohomology ring of $\bar{X}$. Since $\bar{X}$ is a product of projective spaces, we have a one-to-one correspondences between the J-holomorphic curves in $\bar{X}$ which have contribution to the quantum product in $QH^*(\bar{X})$ and those tropical curves in $N_{\mathbb{R}}$ which have contribution to the tropical quantum product in $QH^*_{trop}(\bar{X})$, by the *correspondence theorem* of Mikhalkin [33] and Nishinou-Siebert [36]. From this follows the canonical isomorphism

$$QH^*(\bar{X}) \cong QH^*_{trop}(\bar{X}).$$

Then comes a simple but important observation: *Each tropical curve which has contribution to the tropical quantum product in $QH^*_{trop}(\bar{X})$ is obtained by gluing tropical discs in $N_{\mathbb{R}}$.*[6] On the other hand, these tropical discs are exactly corresponding to the families of Maslov index two J-holomorphic discs in $\bar{X}$ with boundary in Lagrangian torus fibers, which were used to define the functions $\Psi_1, \ldots, \Psi_d$. Hence, we naturally have another canonical isomorphism

$$QH^*_{trop}(\bar{X}) \cong \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]/\mathfrak{L}.$$

For example, let us take a look at the case of $\bar{X} = \mathbb{C}P^2$. See Figure 3.1 below.

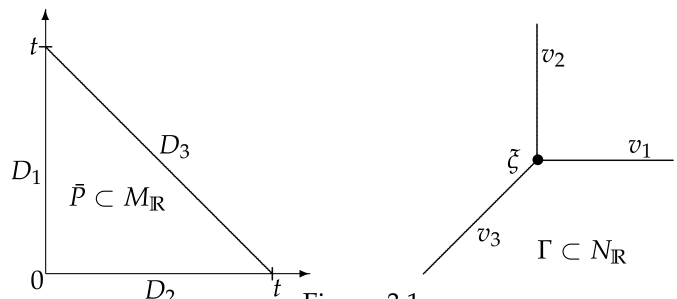

Figure 3.1

Denote by $\{e_1, e_2\}$ the standard basis of $N = \mathbb{Z}^2$. We have $v_1 = (1,0), v_2 = (0,1), v_3 = (-1,-1)$, and the polytope $\bar{P} \subset M_{\mathbb{R}} \cong \mathbb{R}^2$ is defined by the inequalities

$$x_1 \geq 0, \ x_2 \geq 0, \ x_1 + x_2 \leq t,$$

where $t > 0$. There are three toric divisors D_1, D_2, D_3 corresponding to three functions $\Psi_1, \Psi_2, \Psi_3 \in C^\infty(LX)$ defined by

$$\begin{aligned}
\Psi_1(p,v) &= \begin{cases} e^{-x_1} & \text{if } v = (1,0) \\ 0 & \text{otherwise,} \end{cases} \\
\Psi_2(p,v) &= \begin{cases} e^{-x_2} & \text{if } v = (0,1) \\ 0 & \text{otherwise,} \end{cases} \\
\Psi_3(p,v) &= \begin{cases} e^{-(t-x_1-x_2)} & \text{if } v = (-1,-1) \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

[6]This idea was recently generalized by Gross [20] to understand tropically the big quantum cohomology and mirror symmetry of $\mathbb{C}P^2$.

for $(p,v) \in LX$ and $(x_1,x_2) = \mu_X(p) \in P$, respectively. The small quantum cohomology ring is given by

$$\begin{aligned} QH^*(\mathbb{C}P^2) &= \mathbb{C}[D_1,D_2,D_3]/\langle D_1-D_3, D_2-D_3, D_1*D_2*D_3-q\rangle \\ &= \mathbb{C}[H]/\langle H^3-q\rangle, \end{aligned}$$

where we have, by abuse of notations, also use $D_i \in H^2(\mathbb{C}P^2,\mathbb{C})$ to denote the cohomology class Poincaré dual to D_i, $H \in H^2(\mathbb{C}P^2,\mathbb{C})$ is the hyperplane class and $q = e^{-t}$. Fix any point $p \in \mathbb{C}P^2 \setminus D_\infty$, then the quantum corrections, which appear in the relation

$$D_1 * D_2 * D_3 = H^3 = q,$$

is due to the unique holomorphic curve $\varphi : (\mathbb{P}^1; x_1, x_2, x_3, x_4) \to \mathbb{C}P^2$ of degree 1 (i.e. a line) with 4 marked points such that $\varphi(x_4) = p$ and $\varphi(x_i) \in D_i$, for $i = 1,2,3$. Let $x = \mu_X(p) \in P$ and $L_x = \mu_X^{-1}(x)$ be the Lagrangian torus fiber containing p. Using tropical geometry, one sees that there is a tropical curve Γ in $N_\mathbb{R}$ with three unbounded edges in the directions v_1, v_2, v_3 and the vertex mapped to $\xi = \mathrm{Log}(p) \in N_\mathbb{R}$, which is corresponding to this holomorphic curve (see Figure 3.1 above). Here, we identify X with $(\mathbb{C}^*)^2$, and $\mathrm{Log} : X = (\mathbb{C}^*)^2 \to N_\mathbb{R} = \mathbb{R}^2$ is the Log map we defined in Proposition 3.1. It is obvious that Γ can

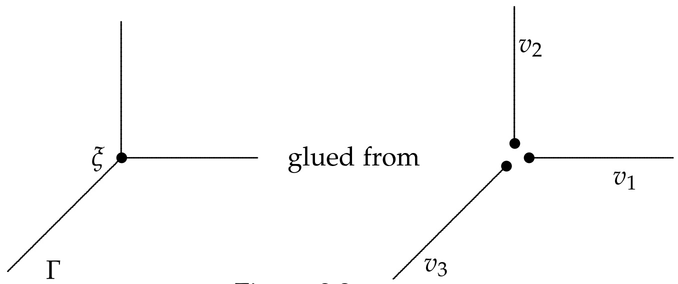

Figure 3.2

be obtained by gluing the three half lines emanating from the point $\xi \in N_\mathbb{R}$ in the directions v_1, v_2, v_3. See Figure 3.2. These half lines are the tropical discs which are corresponding to the three families of Maslov index two holomorphic discs $\varphi_1, \varphi_2, \varphi_3$ respectively. We see that the above quantum relation corresponds exactly to the equation

$$\Psi_1 \star \Psi_2 \star \Psi_3 = q$$

in $\mathbb{C}[\Psi_1^{\pm 1}, \Psi_2^{\pm 1}]$.

Without the assumption that $\bar{X}$ is a product of projective spaces, the tropical interpretation will break down. This is because for general toric Fano manifolds, the holomorphic curves which contribute to the small quantum product may have components mapped into the anticanonical toric divisor D_∞. An example is provided by the exceptional curve in the blowup of $\mathbb{C}P^2$ at one T_N-invariant point (see Example 3 in Section 4 in [7]). Now the problem is that tropical geometry cannot be used to count these holomorphic curves. In other words, there are no tropical curves corresponding to such holomorphic curves (cf. Rau [38]).

Now it's time to return to the main theme of this section, namely, we can construct and apply SYZ mirror transformations to the study of mirror symmetry for toric Fano manifolds. First we equip $LX = X \times N$ with the symplectic structure $\pi^*(\omega_X)$, which we denote again by ω_X. Also let $\mu_{LX} : LX \to P$ be the composition map $\mu_X \circ \pi$. Analog to the semi-flat case, we consider the fiber product $LX \times_P Y = P \times N \times \sqrt{-1}(T_N \times T_M)$ of the fibrations $\mu_{LX} : LX \to P$ and $\nu_Y : Y \to P$. Note that we have a covering map $LX \times_P Y \to X \times_P Y$. Pulling back the universal curvature two-form $F = \sqrt{-1}\sum_{j=1}^n dy_j \wedge du_j \in \Omega^2(X \times_P Y)$, we get a two-form on $LX \times_P Y$, which we again denote by F. We further define the *holonomy function* $\mathrm{hol} : LX \times_P Y \to \mathbb{C}$ by

$$\mathrm{hol}(p, v, z) = \mathrm{hol}_{\nabla_y}(v) = e^{-\sqrt{-1}\langle y,v\rangle}$$

for $(p, v) \in LX, z = \exp(-x - \sqrt{-1}y) \in Y$ such that $\mu_X(p) = \nu_Y(z) = x$. The SYZ mirror transformation for toric Fano manifolds is constructed as *a combination of the semi-flat SYZ transformation $\mathcal{F}^{sf}$ and fiberwise Fourier series*.

Definition 3.2. *The SYZ mirror transformation $\mathcal{F} : \Omega^*(LX) \to \Omega^*(Y)$ for $\bar{X}$ is defined by*

$$\begin{aligned}\mathcal{F}(\alpha) &= (-2\pi\sqrt{-1})^{-n}\pi_{Y,*}(\pi_{LX}^*(\alpha) \wedge e^{\sqrt{-1}F}hol)\\ &= (-2\pi\sqrt{-1})^{-n}\int_{N\times T_N}\pi_{LX}^*(\alpha) \wedge e^{\sqrt{-1}F}hol,\end{aligned}$$

where $\pi_{LX} : LX \times_P Y \to LX$ and $\pi_Y : LX \times_P Y \to Y$ are the two natural projections.

The basic properties of $\mathcal{F}$ are similar to those of other Fourier-type transformations, and in particular, it satisfies the inversion property with the *inverse SYZ mirror transformation* $\mathcal{F}^{-1} : \Omega^*(Y) \to \Omega^*(LX)$ defined by

$$\begin{aligned}\mathcal{F}^{-1}(\alpha) &= (-2\pi\sqrt{-1})^{-n}\pi_{LX,*}(\pi_Y^*(\alpha) \wedge e^{-\sqrt{-1}F}\mathrm{hol}^{-1})\\ &= (-2\pi\sqrt{-1})^{-n}\int_{T_M}\pi_Y^*(\alpha) \wedge e^{-\sqrt{-1}F}\mathrm{hol}^{-1}.\end{aligned}$$

In [7], the SYZ mirror transformation was, for the first time, used to study the appearance of the superpotential W as quantum corrections. More precisely, we showed that

Theorem 3.1 (First part of Theorem 1.1 in [7]). *The SYZ mirror transformation (or fiberwise Fourier series) of the function Ψ, defined in terms of the counting of Maslov index two J-holomorphic discs in the toric Fano manifold $\bar{X}$ with boundary in Lagrangian torus fibers, gives the superpotential $W : Y \to \mathbb{C}$ on the mirror manifold:*

$$\mathcal{F}(\Psi) = W.$$

Furthermore, we can incorporate the symplectic structure ω_X to give the holomorphic volume form of the Landau-Ginzburg model (Y, W) in the sense that

$$\mathcal{F}(e^{\sqrt{-1}\omega_X+\Psi}) = e^W\Omega_Y.$$

Conversely, we have

$$\mathcal{F}^{-1}(W) = \Psi,\ \mathcal{F}^{-1}(e^W\Omega_Y) = e^{\sqrt{-1}\omega_X+\Psi}.$$

Remark 3.3.

1. *We shall mention that the fact that the superpotential W can be computed in terms of the counting of Maslov index two holomorphic discs in $\bar{X}$ with boundary in Lagrangian torus fibers was originally due to Cho and Oh* [9]. *The key*

point of our result is that there is an explicit Fourier-Mukai-type transformation, namely, the SYZ mirror transformation $\mathcal{F}$, that gives the superpotential W by transforming an object (the function Ψ) in the A-model of $\bar{X}$.

2. *Apparently, the statements written here are slightly different from those in Theorem 1.1 in* [7], *but realizing that $\Phi = Exp\ \Psi$, it is easy to see that they are in fact the same statements.*
3. *The complex oscillatory integrals*

$$\int_\Gamma e^W \Omega_Y$$

of the n-form $e^W \Omega_Y$ over Lefschetz thimbles $\Gamma \subset Y$ (defined by the singularities of $W : Y \to \mathbb{C}$), which satisfy certain Picard-Fuchs equations, play the role of periods for Calabi-Yau manifolds. This is why we call $e^W \Omega_Y$ the holomorphic volume form of the Landau-Ginzburg model (Y, W).

On the other hand, we also showed that the SYZ mirror transformation (which, in this case, is fiberwise Fourier series) $\mathcal{F}(\Psi_i)$ of the function Ψ_i is nothing but the monomial $e^{\lambda_i} z^{v_i}$ on Y, for $i = 1, \ldots, d$. Since the Jacobian ring $Jac(W)$ of the superpotential W is generated by the monomials $e^{\lambda_1} z^{v_1}, \ldots, e^{\lambda_d} z^{v_d}$, by Proposition 3.2, the SYZ mirror transformation realizes a natural isomorphism between the small quantum cohomology $QH^*(\bar{X})$ and the Jacobian ring $Jac(W)$.

Theorem 3.2 (Second part of Theorem 1.1 in [7])**.** *The SYZ mirror transformation $\mathcal{F}$ induces a natural isomorphism of $\mathbb{C}$-algebras*

$$\mathcal{F} : QH^*(\bar{X}) \xrightarrow{\cong} Jac(W),$$

which takes the quantum product, now realized as a convolution product, to the ordinary product of Laurent polynomials, provided that $\bar{X}$ is a product of projective spaces.

In the example of $\bar{X} = \mathbb{C}P^2$, the superpotential is the Laurent polynomial $W(z_1, z_2) = z_1 + z_2 + \frac{q}{z_1 z_2}$ on $Y = (\mathbb{C}^*)^2$, where $q = e^{-t}$. Its logarithmic partial derivatives are given by

$$\partial_1 W = z_1 - \frac{q}{z_1 z_2},\ \partial_2 W = z_2 - \frac{q}{z_1 z_2},$$

so that the Jacobian ring is given by

$$\begin{aligned} Jac(W) &= \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}] / \langle z_1 - \frac{q}{z_1 z_2}, z_2 - \frac{q}{z_1 z_2} \rangle \\ &= \mathbb{C}[Z_1, Z_2, Z_3] / \langle Z_1 - Z_3, Z_2 - Z_3, Z_1 Z_2 Z_3 - q \rangle, \end{aligned}$$

where the monomials $Z_1 = z_1$, $Z_2 = z_2$ and $Z_3 = \frac{q}{z_1 z_2}$ are the SYZ mirror transformations (i.e. fiberwise Fourier series) of the functions Ψ_1, Ψ_2 and Ψ_3 respectively.

Remark 3.4.

1. *In* [10], *Coates, Corti, Iritani and Tseng formulated the mirror symmetry conjecture for toric manifolds (and orbifolds) as an isomorphism of graded $\frac{\infty}{2}$VHS between the A-model $\frac{\infty}{2}$VHS associated to a toric manifold and the B-model $\frac{\infty}{2}$VHS associated to the mirror Landau-Ginzburg model (see also Iritani* [28]*). It is desirable to have this isomorphism, which contains more information than the isomorphism in the above theorem, realized by SYZ mirror transformations.*

2. *In* [15] *(and also* [16]*), Fukaya-Oh-Ohta-Ono applied the machinery developed in* [14] *to the case of toric manifolds. They considered Floer cohomology with coefficients in the Novikov ring, instead of* $\mathbb{C}$ *used here and in Auroux's paper* [4]*. They have results on the superpotential even in the non-Fano toric case. The isomorphism* $QH^*(\bar{X}) \cong Jac(W)$ *(over the Novikov ring) was also discussed and proved in their work (Theorem 1.9 in* [15]*). Their proof is combinatorial, using Batyrev's presentation of the small quantum cohomology ring for toric Fano manifolds, the validity of which in turn relies on Givental's mirror theorem. They claimed that a more conceptual and geometric proof for toric, not necessarily Fano, manifolds will appear in a sequel to their paper.*

3.3. **Transformation of branes.** This subsection is an attempt to understand the correspondence between A-branes of the toric Fano manifold $\bar{X}$ and B-branes of the mirror Landau-Ginzburg model (Y, W) via SYZ mirror transformations.

We will deal with the simplest case of the correspondence. So let $L_x = \mu_{\bar{X}}^{-1}(x)$ be the Lagrangian torus fiber of $\bar{X}$ over a point $x \in P$. We equip L_x with a flat $U(1)$-bundle $\mathbb{L}_y = (L_x \times \mathbb{C}, \nabla_y)$, where ∇_y is the flat $U(1)$-connection corresponding to $y \in (L_x)^\vee$. The mirror of the A-brane $(L_x, \mathbb{L}_y)$ is given, according to SYZ, by the B-brane $(z = \exp(-x - \sqrt{-1}y) \in Y, \mathcal{O}_z)$. In other words, the correspondence on the level of objects is the same as in the semi-flat Calabi-Yau case. Quantum corrections will emerge and make a difference when we consider their endomorphisms.

According to Hori (see [26], Chapter 39), the endomorphism algebra $\mathrm{End}(z, \mathcal{O}_z)$ of the B-brane $(z, \mathcal{O}_z)$, as a $\mathbb{C}$-vector space, is given by the cohomology of the complex

$$(\bigwedge^* T_z Y, \delta = \iota_{\partial W(z)}),$$

where $\iota_{\partial W(z)}$ is contraction with the vector $\partial W(z) = \sum_{j=1}^n \partial_j W(z)(\partial_j)_z$ and here again ∂_j denotes $z_j \frac{\partial}{\partial z_j}$. The following elementary proposition shows that the introduction of the superpotential W "localizes" the category B-branes to the critical points of W.

Proposition 3.3. *The endomorphism* $End(z, \mathcal{O}_z)$ *is nontrivial if and only if* $z \in Y$ *is a critical point of the superpotential* $W : Y \to \mathbb{C}$*, and in which case,* $End(z, \mathcal{O}_z)$ *is isomorphic to* $\bigwedge^* T_z Y$ *as* $\mathbb{C}$*-vector spaces.*

On the other hand, the endomorphism algebra of the A-brane $(L_x, \mathbb{L}_y)$ in the (derived) Fukaya category is given by the *Floer cohomology ring* $HF(L_x, \mathbb{L}_y)$,[7] which in turn, as a $\mathbb{C}$-vector space, is given by the cohomology of the Floer complex

$$(C^*(L_x, \mathbb{C}), \delta = m_1)$$

where $m_1 = m_1(L_x, \mathbb{L}_y)$ denotes the Floer differential. In [9], [8], Cho and Oh explicitly computed the Floer differential m_1. Recall that $H^1(L_x, \mathbb{C})$, viewed as the space of infinitestimal deformations of the pair $(L_x, \mathbb{L}_y)$, is canonically isomorphic to $T_z Y$. Let $C_1, \ldots, C_n$ be the basis of $H^1(L_x, \mathbb{C})$ corresponding to $(\partial_1)_z, \ldots, (\partial_n)_z$.

[7]We use $\mathbb{C}$ as the coefficient ring, instead of the Novikov ring.

Then the results of Cho and Oh stated that $m_{1,\beta_i}(C_j) = C_j \cdot \partial\beta_i = v_i^j$ and

$$\begin{aligned} m_1(C_j) &= \sum_{i=1}^{d} m_{1,\beta_i}(C_j) \exp(-\frac{1}{2\pi}\int_{\beta_i} \omega_X) \mathrm{hol}_{\nabla_y}(\partial\beta_i) \\ &= \sum_{i=1}^{d} v_i^j z^{v_i} = \partial_j W(z). \end{aligned}$$

This shows that $m_1 = \iota_{\partial W(z)}$ on $H^1(L_x, \mathbb{C}) = T_z Y$, and $m_1 = 0$ on $H^1(L_x, \mathbb{C})$ if and only if z is a critical point of W. The following result proved by Cho-Oh in [9] is parallel to the above proposition.

Theorem 3.3 (Cho-Oh [9])**.** *The Floer cohomology $HF(L_x, \mathbb{L}_y)$ is nontrivial and isomorphic to $H^*(L_x, \mathbb{C})$ if and only if $m_1 = 0$ on $H^1(L_x, \mathbb{C})$.*

We conclude that

Theorem 3.4. *The Floer cohomology $HF(L_x, \mathbb{L}_y)$ of the A-brane $(L_x, \mathbb{L}_y)$ is isomorphic to the endomorphism algebra $End(z, \mathcal{O}_z)$ of the mirror B-brane $(z, \mathcal{O}_z)$ as $\mathbb{C}$-vector spaces.*

It is intriguing to see whether this isomorphism can be realized by explicit SYZ mirror transformations.

Remark 3.5. *In* [8]*, Cho proved that the Floer cohomology ring $HF(L_x, \mathbb{L}_y)$, equipped with the product structure given by $m_2 = m_2(L_x, \mathbb{L}_y)$, is a Clifford algebra generated by $H^1(L_x, \mathbb{C})$ with the bilinear form given by the Hessian of W: $Q(C_j, C_k) = \partial_j\partial_k W(z)$. This implies that the isomorphism in Theorem 3.4 is in fact an isomorphism of $\mathbb{C}$-algebras. This confirms a prediction by Physicists. See the paper of Cho* [8] *for details.*

4. Further questions

The results described in this article represent the first step in our program which is aimed at exploring mirror symmetry via SYZ mirror transformations. In particular, they showed that these transformations can be applied successfully to explain the mirror symmetry for toric Fano manifolds, a case where quantum corrections do exist. However, we shall emphasize that the quantum corrections in the toric Fano case, which are due to the anticanonical toric divisor, are much simpler than those in the general case (Gross-Siebert [21], Auroux [4]), where quantum corrections may arise due to contributions from the proper singular Lagrangian fibers of the Lagrangian torus fibrations and complicated wall-crossing phenomena start to interfere. In terms of affine geometry, this means that the bases of the Lagranigan torus fibrations in the toric case are affine manifolds with boundary but without singularities, while in the general case, the bases are affine manifolds with *both* boundary and singularities (and in the semi-flat case, the bases are affine manifolds without boundary and singularities). Certainly much more work remains to be done in the future. In this final section, we will comment on several possible future research directions. The discussion is going to be rather speculative.

4.1. **Toric Fano manifolds.** We have seen that the simplest correspondence between A-branes on a toric Fano manifold $\bar{X}$ and B-branes on the mirror Landau-Ginzburg model (Y, W), namely

$$(L_x, \mathbb{L}_y) \longleftrightarrow (z, \mathcal{O}_z),$$

is compatible with the SYZ philosophy. It is desirable to see how other A-branes on X are transformed to the corresponding mirror B-branes on (Y, W). An interesting and important example would be the Lagrangian submanifold $\mathbb{R}P^n \subset \mathbb{C}P^n$ for odd n, which can be viewed as a multi-section of the moment map of $\mathbb{C}P^n$. Employing the SYZ approach, the mirror B-brane is expected to be a trivial rank-2^n holomorphic vector bundle over Y, equipped with some additional information related to W. A possible choice of this additional information would be a *matrix factorization* of W; currently, it is widely believed that the category of B-branes on (Y, W) is given by the category of matrix factorizations of W. This was first proposed by Kontsevich, see Orlov [37] for details. The relation between these matrix factorizations and the computation of Floer cohomology will be the key to a complete understanding of the correspondences of branes.

On the other hand, we have not even touched the correspondence between B-branes on $\bar{X}$ and A-branes on (Y, W). As we mentioned in the introduction, the results of Seidel [39], Ueda [41], Auroux-Katzarkov-Orlov [5], [6] and Abouzaid [1], [2] have provided substantial evidences for this half of the Homological Mirror Symmetry Conjecture. In particular, Abouzaid [2] made use of an idea originated from the SYZ conjecture, namely, the mirror of a Lagrangian section should be a holomorphic line bundle. His results also showed that the correspondence is in line with the SYZ picture. Recently, Fang [11] and Fang-Liu-Treumann-Zaslow [12] proved a version of Homological Mirror Symmetry for toric manifolds by explicitly using T-duality. It is an interesting question whether one can construct an explicit SYZ mirror transformation to realize the correspondence between B-branes on $\bar{X}$ and A-branes on (Y, W).

4.2. **Toric non-Fano or non-toric Fano manifolds.** As in the case of toric Fano manifolds, non-toric Fano manifolds such as Grassmannians and flag manifolds admit natural Lagrangian torus fibrations, provided by Gelfand-Cetlin integrable systems (see, for example, Guillemin-Sternberg [23]), which are convenient for applying SYZ mirror transformations. While mirror symmetry for these manifolds has been studied for some time by Givental [18] and others, new tools and new ideas are needed if we want to apply SYZ mirror transformations to these examples. The recent works of Nishinou-Nohara-Ueda [34], [35] have shed some light on this case. In particular, they obtained a classification the holomorphic discs in flag manifolds with boundary in Lagrangian torus fibers, which should be very useful in the constructions of SYZ mirror transformations.

On the other hand, the mirror symmetry for toric non-Fano manifolds is also not well understood too. As can be seen from the works of Givental [19], the mirror map between the complexified Kähler and complex moduli spaces in this case is a nontrivial *coordinate change*, instead of an identity map as in the toric Fano case. In Auroux [4], nontrivial coordinate changes and wall-crossing phenomena were also observed in constructing the superpotentials for the mirrors of non-toric examples. Hence, the definitions of the SYZ mirror transformations may have to be adjusted to incorporate the nontrivial mirror map and also wall-crossing phenomena. For this, we will have to make the construction of SYZ mirror transformations local. A very preliminary attempt to this is made in Section 5 in [7].

4.3. **Calabi-Yau manifolds.** The ultimate goal of our program is no doubt to apply SYZ mirror transformations to get a better understanding of the mirror symmetry for Calabi-Yau manifolds and the SYZ Conjecture. Works of Fukaya [13], Kontsevich-Soibelman [30] and Gross-Siebert [21] have laid an important foundation for understanding the SYZ framework for both Calabi-Yau and non-Calabi-Yau manifolds. In view of the fact that toric varieties have played an important role in the constructions of Gross and Siebert, it would be nice if we can incorporate our methods with their new techniques to study SYZ mirror transformations for Calabi-Yau manifolds; and hopefully, this would let us reveal geometrically the secret of mirror symmetry.

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