Lagrangian torus fibrations and homological mirror symmetry for the conifold

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Abstract

We discuss homological mirror symmetry for the conifold from the point of view of the Strominger-Yau-Zaslow conjecture.

1 Introduction

The behavior of strings and branes near the tip of a cone has been studied extensively in string theory. The case when the cone is a Gorenstein affine toric 3-fold is of particular importance, not only from the point of view of mirror symmetry, but also for applications to geometric engineering of Seiberg-Witten theory [KKV97] and the AdS/CFT correspondence [AGM+00].

Let Z be a Gorenstein affine toric 3-fold and $\varphi : X \to Z$ be a crepant resolution. The convex hull $\triangle$ of primitive generators of one-dimensional cones of the fan describing Z as a toric variety is a lattice polygon, which lies on the plane

$$\overline{N} = \{n = (n_1, n_2, n_3) \in N \mid n_3 = 1\}$$

under a suitable choice of a coordinate $N \cong \mathbb{Z}^3$ on the lattice of one-parameter subgroups of the dense torus.

If $\triangle$ contains an interior lattice point, then X is derived-equivalent to the total space $\mathcal{K}_{\mathfrak{X}}$ of the canonical bundle of a 2-dimensional toric Fano stack $\mathfrak{X}$, and homological mirror symmetry for X is related to homological mirror symmetry for $\mathfrak{X}$ by suspension [Sei10]. The case when $\triangle$ does not contain any interior lattice point is more elusive, and we discuss such a case in this paper.

Let Z be the *conifold*, which is a synonym for a 3-dimensional ordinary double point;

$$Z = \{(u_1, v_1, u_2, v_2) \in \mathbb{C}^4 \mid u_1 v_1 = u_2 v_2\}.$$

The lattice polygon $\triangle$ for Z is the unit lattice square, which does not contain any interior lattice points. The smoothing

$$Y = \{(u_1, v_1, u_2, v_2) \in \mathbb{C}^4 \mid u_1 v_1 = u_2 v_2 - \epsilon\}$$

of the conifold is expected to be mirror to the small resolution $\varphi : X \to Z$ (cf. e.g. [ST01, Gro01]).

In this paper, we discuss homological mirror symmetry for the conifold from the point of view of the Strominger-Yau-Zaslow conjecture [SYZ96]. To do this, it is convenient to consider an open subvariety Y^0 of Y, which is the complete intersection in $\mathbb{C}^\times \times \mathbb{C}^4 = \operatorname{Spec} \mathbb{C}[z, z^{-1}, u_1, u_2, v_1, v_2]$ defined by

$$\begin{cases} u_1 v_1 = z - a, \\ u_2 v_2 = z - b. \end{cases} \tag{1.1}$$

Here a and b are distinct non-zero complex numbers, which we assume to be negative real numbers for simplicity in this section.

We equip Y^0 with the restriction ω of the symplectic form on $\mathbb{C}^\times \times \mathbb{C}^4$ obtained as the sum of the cylindrical Kähler form on $\mathbb{C}^\times$ and the Euclidean Kähler form on $\mathbb{C}^4$. Then the map

$$\begin{array}{cccc} \rho: & Y^0 & \to & \mathbb{R}^3 \\ & \cup & & \cup \\ & (z, u_1, v_1, u_2, v_2) & \mapsto & \left(\log|z|, \frac{1}{2}\left(|u_1|^2 - |v_1|^2\right), \frac{1}{2}\left(|u_2|^2 - |v_2|^2\right)\right) \end{array}$$

is a Lagrangian torus fibration, whose discriminant loci is given by the disjoint union of two skew lines

$$\Gamma = \{(\log|a|, 0, \lambda_2) \in B \mid \lambda_2 \in \mathbb{R}\} \cup \{(\log|b|, \lambda_1, 0) \in B \mid \lambda_1 \in \mathbb{R}\}$$

as shown in Figure 1.1.

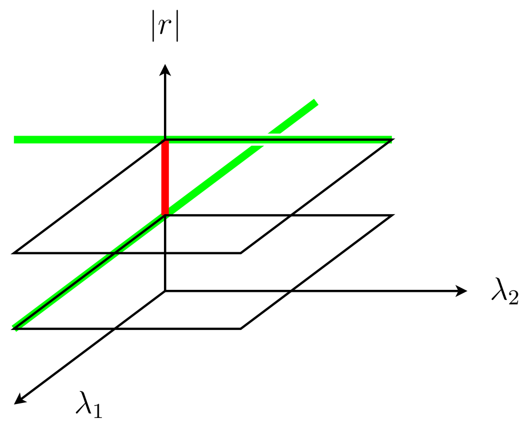

Figure 1.1: The base of the SYZ fibration

The regular fibers of ρ are special with respect to the holomorphic volume form

$$\Omega = d\log z \wedge d\log u_1 \wedge d\log u_2,$$

and we will refer to ρ as the *SYZ fibration*.

The mirror X^0 of Y^0 is identified in [AAK, Theorem 11.1] as the complement of a divisor in the resolved conifold;

$$\begin{aligned} X^0 &= X \setminus D, \\ X &= \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1). \end{aligned}$$

Here, the divisor D is the pull-back of the divisor $\{w_1 w_2 = 0\}$ on the conifold

$$Z = \{(u, v, w_1, w_2) \in \mathbb{C}^4 \mid uv = (1 + w_1)(1 + w_2)\}$$

along the crepant resolution $\varphi : X \to Z$. The natural projection and the inclusion of the zero-section will be denoted by $\pi : X^0 \to \mathbb{P}^1$ and $\iota : \mathbb{P}^1 \hookrightarrow X^0$ respectively. Let $E \subset X^0$ be the image of ι, which is the exceptional locus of the resolution. We write $\mathcal{O}_{X^0}(i) := \pi^* \mathcal{O}_{\mathbb{P}^1}(i)$ and $\mathcal{O}_E(i) := \iota_* \mathcal{O}_{\mathbb{P}^1}(i)$ for short.

To a strongly admissible path γ, the definition of which we defer to Section 3, one can associate an exact non-compact Lagrangian submanifold $L_\gamma \subset Y^0$, which is a section of the SYZ fibration $\rho : Y^0 \to \mathbb{R}^3$. The *SYZ transform* [AP01, LYZ00] of a Lagrangian section of an SYZ fibration is a holomorphic line bundle on the mirror, obtained as a kind of Fourier transform.

Theorem 1.1. *The SYZ transform $\mathcal{L}_\gamma$ of the Lagrangian section L_γ associated with a strongly admissible path $\gamma : \mathbb{R} \to \mathbb{C}^\times \setminus \Delta$ is the line bundle $\mathcal{O}_{X^0}(-w(\gamma))$ on X^0.*

Here $w(\gamma)$ denotes the winding number defined in Section 3. Let γ_0 and γ_1 be admissible paths shown in Figure 1.2. The associated Lagrangian submanifolds of Y^0 will be denoted by $L_0 := L_{\gamma_0}$ and $L_1 := L_{\gamma_1}$, whose winding numbers are 0 and -1 respectively. Let $\mathcal{W}$ be the wrapped Fukaya category of Y^0 consisting of L_0 and L_1.

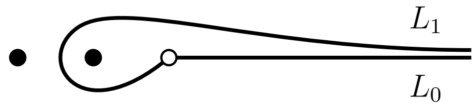

Figure 1.2: Non-compact Lagrangians

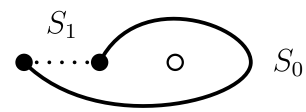

Figure 1.3: Compact Lagrangians

Theorem 1.2. *There is an equivalence*

$$D^b \mathcal{W} \cong D^b \operatorname{coh} X^0 \tag{1.2}$$

of triangulated categories sending L_i to $\mathcal{O}_{X^0}(i)$ for $i = 0, 1$.

There is a natural choice of a pair (S_0, S_1) of Lagrangian 3-spheres in Y^0 which are dual to (L_0, L_1); they are T^2-fibrations over the paths shown in Figure 1.3.

Theorem 1.3. *The SYZ transforms of the Lagrangian 3-spheres S_0 and S_1 are the line bundles $\mathcal{O}_E$ and $\mathcal{O}_E(-1)$ on the exceptional locus E respectively.*

Let $\mathcal{F}_0$ be the Fukaya category of Y^0 consisting of S_0 and S_1, and $\operatorname{coh}_0 X^0$ be the abelian category of coherent sheaves supported on the exceptional locus of the resolution $\varphi : X \to Z$.

Theorem 1.4. *There is an equivalence*

$$D^b\mathcal{F}_0 \cong D^b \operatorname{coh}_0 X^0 \tag{1.3}$$

of triangulated categories sending S_0 and S_1 to $\mathcal{O}_E$ and $\mathcal{O}_E(-1)$ respectively.

This paper is organized as follows: We review the construction of the SYZ mirror for the smoothed conifold from [AAK] in Section 2. In Section 3, we discuss the construction of Lagrangian submanifolds in Y^0 from paths on the z-plane. In Section 4, we recall the definition of the SYZ transform from [AP01, LYZ00] and prove Theorems 1.1 and 1.3. In Section 5, we give an explicit description of the derived category of coherent sheaves on the resolved conifold. In Section 6, we study the wrapped Fukaya category of Y^0 and prove Theorem 1.2. In Section 7, we study A_∞-operations on vanishing cycles in Y^0 and prove Theorem 1.4. In Section 8, we study Floer cohomology of immersed Lagrangian $S^2 \times S^1$. In Section 9, we discuss extension of the main results of this paper to more general small toric Calabi-Yau 3-folds.

Acknowledgment : K. C. is supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CUHK404412). D. P. would like to thank Denis Auroux and Kevin Lin for their very patient explanation of wall-crossing formulas. K. U. is supported by JSPS Grant-in-Aid for Young Scientists No. 24740043.

2 The construction of the SYZ mirror

Recall that Y^0 is given by the complete intersection

$$\begin{aligned} u_1 v_1 &= z - a, \\ u_2 v_2 &= z - b \end{aligned} \tag{2.1}$$

in $\mathbb{C}^\times \times \mathbb{C}^4$, where a and b are distinct negative real numbers. Without loss of generality, we assume that $a < b$. To construct the mirror of Y^0, it is also convenient to regard Y^0 as the complement of the anticanonical divisor

$$H = \{(z, u_1, v_1, u_2, v_2) \in Y \mid z = 0\}$$

in

$$Y = \{(z, u_1, v_1, u_2, v_2) \in \mathbb{C}^5 \mid u_1 v_1 = z - a,\ u_2 v_2 = z - b\}.$$

In the following, we shall briefly review the construction of the mirror for Y^0 (or Y with respect to the divisor H) following the SYZ approach in [Aur07, Aur09]; note that our example is a special case of a much more general construction in [AAK, Section 11] (see also [CLL12, Section 4] and [Cha13, Section 5]).

First of all, there is a Hamiltonian T^2-action on (Y^0, ω):

$$(e^{is}, e^{it}) \cdot (z, u_1, v_1, u_1, v_2) = \left(z, e^{is}u_1, e^{-is}v_1, e^{it}u_2, e^{-it}v_2\right)$$

whose moment map is given by

$$\phi(z, u_1, v_1, u_1, v_2) = \left(\frac{1}{2}\left(|u_1|^2 - |v_1|^2\right), \frac{1}{2}\left(|u_2|^2 - |v_2|^2\right)\right).$$

This action extends to Y and preserves the anticanonical divisor H. The SYZ fibration is given by

$$
\begin{array}{cccc}
\rho : & Y^0 & \to & B := \mathbb{R}_{>0} \times \mathbb{R}^2 \\
& \cup & & \cup \\
& (z, u_1, v_1, u_2, v_2) & \mapsto & (|z|, \phi(z, u_1, v_1, u_2, v_2)) .
\end{array}
\tag{2.2}
$$

Note that we use $|z|$ here instead of $\log |z|$ and the base is $\mathbb{R}_{>0} \times \mathbb{R}^2$ instead of $\mathbb{R}^3$. This harmless change is more convenient for us because we would like to extend this map to $\rho : Y \to \overline{B} := \mathbb{R}_{\geq 0} \times \mathbb{R}^2$ so that the preimage of the boundary $\{0\} \times \mathbb{R}^2$ is precisely given by the hypersurface H.

Let $\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $r \in \mathbb{R}_{>0}$. We denote by

$$
L_{r,\vec{\lambda}} = \{(z, u_1, v_1, u_2, v_2) \in Y \mid |z| = r,\ \phi(z, u_1, v_1, u_2, v_2) = \vec{\lambda}\}
$$

the fiber of ρ over $(r, \vec{\lambda}) \in B = \mathbb{R}_{>0} \times \mathbb{R}^2$. Consider the double conic fibration $f : Y \to \mathbb{C}$ given by projection to the z-coordinate. Then $L_{r,\vec{\lambda}}$ can be viewed as a fibration, via f, over the circle $C_r = \{z \in \mathbb{C}^\times \mid |z| = r\}$ with generic fiber T^2. The fiber $L_{r,\vec{\lambda}}$ is singular precisely when

(i) $r = |a|$ and $\vec{\lambda} = (0, \lambda_2)$; or

(ii) $r = |b|$ and $\vec{\lambda} = (\lambda_1, 0)$;

so the discriminant loci of ρ is the disjoint union of two skew lines

$$
\Gamma = \{(|a|, 0, \lambda_2) \in B \mid \lambda_2 \in \mathbb{R}\} \cup \{(|b|, \lambda_1, 0) \in B \mid \lambda_1 \in \mathbb{R}\} . \tag{2.3}
$$

We denote by $B^{\mathrm{sm}} := B \setminus \Gamma$ the smooth loci of the base of the SYZ fibration. When $L_{r,\vec{\lambda}}$ is smooth, it is a *special* Lagrangian torus in Y^0 with respect to the symplectic form ω and the holomorphic volume form

$$
\Omega = d \log z \wedge d \log u_1 \wedge d \log u_2.
$$

Let

$$
L'_{r,\lambda_1} = \{(u_1, v_1) \in \mathbb{C}^2 \mid |u_1 v_1 + a| = r,\ |u_1|^2 - |v_1|^2 = 2\lambda_1\},
$$

and

$$
L''_{r,\lambda_2} = \{(u_2, v_2) \in \mathbb{C}^2 \mid |u_2 v_2 + b| = r,\ |u_2|^2 - |v_2|^2 = 2\lambda_2\}.
$$

Via the map $f' : \mathbb{C}^2 \to \mathbb{C}$ given by $(u_1, v_1) \mapsto u_1 v_1 + a$, we can think of L'_{r,λ_1} as an S^1-bundle over the circle C_r. Similarly, via the map $f'' : \mathbb{C}^2 \to \mathbb{C}$ given by $(u_2, v_2) \mapsto u_2 v_2 + b$, L''_{r,λ_2} can be thought of as an S^1-bundle over the same circle. Then $L_{r,\vec{\lambda}}$ is nothing but the fibred product

$$
\begin{array}{ccc}
L_{r,\vec{\lambda}} = L'_{r,\lambda_1} \times_{\mathbb{C}} L''_{r,\lambda_2} & \longrightarrow & L''_{r,\lambda_2} \\
\big\downarrow & & \big\downarrow {\scriptstyle f''} \\
L'_{r,\lambda_1} & \xrightarrow{f'} & \mathbb{C}.
\end{array}
$$

To construct the SYZ mirror, we compute the *superpotential* [CO06, FOOO09, FOOO10, AAK] which counts Maslov index two holomorphic discs in Y (caution: not Y^0!) with boundary on the Lagrangian torus fibers of ρ. Our arguments are along the same lines as those in [Aur07, Aur09].

By composing a disc with the holomorphic map $f : Y \to \mathbb{C}$ and applying the maximal principle, we see that Maslov index two discs in $(Y, L_{r,\vec{\lambda}})$ are sections of f over the disc bounded by C_r. Omitting subscripts for convenience, for r large, the Lagrangian $L = L' \times_{\mathbb{C}} L''$ is Hamiltonian isotopic to a Lagrangian of the form

$$(S^1(r_1) \times S^1(r_2)) \times_{\mathbb{C}} (S^1(r_3) \times S^1(r_4)).$$

The $S^1(r_1) \times S^1(r_2)$ component bounds two families of Maslov index two discs, which we will denote by β'_1 and β'_2, and the $S^1(r_3) \times S^1(r_4)$ component also bounds two families of Maslov index two discs, which we will denote by β''_1 and β''_2. Therefore $L' \times_{\mathbb{C}} L''$ bounds four families of Maslov index two discs which we will denote by (β'_i, β''_j) for $i, j = 1, 2$.

Let z_1, z_2, z_3, z_4 be the weights corresponding respectively to

$$(\beta'_1, \beta''_1), (\beta'_1, \beta''_2), (\beta'_2, \beta''_1), (\beta'_2, \beta''_2).$$

Since

$$(\beta'_1, \beta''_2) + (\beta'_2, \beta''_1) - (\beta'_1, \beta''_1) = (\beta'_2, \beta''_2),$$

we have the relation

$$z_2 z_3 / z_1 = z_4.$$

It follows that the superpotential for large r is given by

$$W = z_1 + z_2 + z_3 + \frac{z_2 z_3}{z_1}.$$

Remark 2.1. Note that this is exactly the Hori-Vafa superpotential corresponding to the singular toric variety

$$Z = \{(u_1, v_1, u_2, v_2) \in \mathbb{C}^4 \mid u_1 v_1 - u_2 v_2 = 0\}.$$

In a sense, we can think of r large as corresponding to some 'toric limit'.

Using the description of $L = L_{r,\vec{\lambda}}$ as a fibred product, it is easy to see that

Proposition 2.2. *A Lagrangian torus fiber $L_{r,\vec{\lambda}}$ bounds a nontrivial Maslov index zero holomorphic disc in Y if and only if $r = |a|$ or $r = |b|$. In other words, there are exactly two walls.*

Recall that we have $a < b < 0$ so that $|b| < |a|$. Let α' denote the Maslov index zero disc bounded by the L' factor and α'' the one bounded by the L'' factor. Also let w_1 and w_2 be the corresponding weights. When r is small, the Lagrangian torus L is a fibred product of Chekanov tori $L' \times_{\mathbb{C}} L''$, with each factor bounding one family of discs β'_0 and β''_0 respectively. So L bounds one family of discs with relative homotopy class

(β_0', β_0''). Let u be the weight corresponding to (β_0', β_0''). We conclude that, for small r, the superpotential is simply given by

$$W = u.$$

To analyze the wall-crossing for counting of Maslov index two discs, we first assume that $\lambda_1 > 0$. As r increases and passes through the first wall $r = |b|$, the class (β_0', β_0'') deforms naturally to

$$(\beta_1', \beta_0''),$$

but it may also pick up the Maslov index zero disc α' and deform into

$$(\beta_1' + \alpha', \beta_0'') = (\beta_2', \beta_0'').$$

Similarly, assuming $\lambda_2 > 0$, as r passes through the second wall $r = |a|$, (β_i', β_0'') naturally deforms to (β_i', β_1'') but it may also deform to $(\beta_i', \beta_1'' + \alpha'') = (\beta_i', \beta_2'')$.

Hence, the wall-crossing formula for the first wall reads

$$u \mapsto \hat{z}_1(1 + w_1), \tag{2.4}$$

where $w_1 = \hat{z}_3/\hat{z}_1$, and the wall-crossing formulas for the second wall are given by

$$\hat{z}_1 \mapsto z_1(1 + w_2), \;\; \hat{z}_3 \mapsto z_3(1 + w_2), \tag{2.5}$$

where $w_2 = z_2/z_1$. Composing these formulas gives

$$u \mapsto z_1 + z_3 + z_2 + \frac{z_2 z_3}{z_1},$$

so the wall-crossing formulas do make the superpotential for r small agree with that for r large.

Remark 2.3. We comment on transversality and orientation for the above moduli spaces of Maslov index two discs.

For transversality, we may apply the argument in [Aur15, Lemma 7]. We briefly explain how this argument carries over to our situation. In our situation, explicit calculation shows that the Maslov index two discs avoid the fixed point locus of the natural T^2 action on the total space. Therefore, for any map u, we have a short exact sequence

$$0 \to u^*\mathcal{L} \to u^*TX \to u^*TX/\mathcal{L} \to 0,$$

where $\mathcal{L}$ is a trivial rank two bundle with real boundary conditions. Let $\bar{u}$ denote the corresponding map to $\mathbb{C}$, which as remarked above is a section over a disc. It follows that surjectivity of the $\bar{\partial}$ operator on sections of u^*TX with boundary conditions $u^*_{|S^1}(TL)$ reduces to that of the $\bar{\partial}$ operator on the quotient bundle $u^*TX/u^*\mathcal{L} \cong \bar{u}^*T\mathbb{C}$ with the corresponding boundary conditions. The surjectivity of the latter operator is well-known.

Similarly, the argument in the proof of [Aur15, Corollary 8] (in particular its 4th paragraph, which in turn rely on constructions in [FOOO09, Chapter 8] or [Cho04, Proposition 5.2]) adapts directly to determine the signs.

Although we have assumed that $\lambda_1 > 0$ and $\lambda_2 > 0$, the above calculations also work for other cases when $\lambda_1 < 0$ or $\lambda_2 < 0$. So letting $z = \hat{z}_1$ and $v = z_1^{-1}$, the *uncompleted* SYZ mirror $\check{Y}_0$ of the complement $Y^0 = Y \setminus H$ is given by the union of three charts U_1, U_2 and U_3, all algebraically equivalent to $(\mathbb{C}^\times)^3$ and equipped with coordinates (u, w_1, w_2), (z, w_1, w_2) and (v, w_1, w_2) respectively. The wall-crossing formulas then tell us that these charts are glued by

$$u \to z(1 + w_1),\ w_1 \to w_1,\ w_2 \to w_2$$

from U_1 to U_2, and by

$$z \to v^{-1}(1 + w_2),\ w_1 \to w_1,\ w_2 \to w_2$$

from U_2 to U_3.

To have a more concrete description of $\check{Y}_0$, let us consider the singular variety

$$Z = \{(u, v, w_1, w_2) \in \mathbb{C}^4 \mid uv = (1 + w_1)(1 + w_2)\}$$

and its crepant resolution

$$X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Let $X^0 = X \setminus D$, where D is the pull-back of the divisor $\{w_1 w_2 = 0\}$ on Z along the crepant resolution $\varphi : X \to Z$. Then we can write

$$X^0 = \{(u, v, w_1, w_2, [x_1 : x_2]) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^2 \times \mathbb{P}^1 \mid ux_2 = (1 + w_1)x_1,\ (1 + w_2)x_2 = vx_1\}.$$

Observe that U_1 can be embedded into X^0 as the chart where $u \neq 0$ and with coordinates (u, w_1, w_2). Similarly, U_2 is the chart of X^0 where $x_1/x_2 \neq 0$ and with coordinates $(z := x_1/x_2, w_1, w_2)$, while U_3 is the chart of X^0 where $v \neq 0$ and with coordinates (v, w_1, w_2). It is clear that these charts satisfy the above gluing relations. Now we claim that the union of these charts is precisely given by the complement $X^0 \setminus (C_1 \cup C_2)$ where

$$C_1 = \{(u, v, w_1, w_2, [x_1 : x_2]) \in X^0 \mid u = v = 0, w_1 = -1, [x_1 : x_2] = [1 : 0]\},$$
$$C_2 = \{(u, v, w_1, w_2, [x_1 : x_2]) \in X^0 \mid u = v = 0, w_2 = -1, [x_1 : x_2] = [0 : 1]\}.$$

To see this, just notice that any point with $u \neq 0$ or $v \neq 0$ is covered by U_1 and U_3 respectively, and any point whose $\mathbb{P}^1$ coordinate is not equal to $[1 : 0]$ or $[0 : 1]$ is covered by U_2.

Hence we conclude that the uncompleted SYZ mirror of Y with respect to the anti-canonical divisor H is the *Landau-Ginzburg model* $(\check{Y}_0, W)$ with total space[1]

$$\check{Y}_0 = X^0 \setminus (C_1 \cup C_2)$$

and superpotential

$$W = u.$$

Note that $\check{Y}_0$ is an open subvariety of X^0, and the superpotential W naturally extends to X^0, so the above argument also gives a natural completion of this Landau-Ginzburg model:

[1]While the three charts of the uncompleted mirror $\check{Y}_0$ and their gluing are correctly described in the published version of this paper, the explicit formula for $\check{Y}_0$ there is incorrect and should be modified as described here. We thank Luis Diogo for discussions which lead us to the discovery of this error.

Proposition 2.4 ([AAK, Section 11]). *The Landau-Ginzburg model (X^0, W) is the completed, corrected SYZ mirror to Y with respect to the anticanonical divisor H. In particular, X^0 is the completed, corrected SYZ mirror to $Y^0 = Y \setminus H$.*

Remark 2.5. It is natural to speculate that the 'missing points' $X^0 \setminus \check{Y}_0$ correspond to singular fibers $L_u := \rho^{-1}(u)$ of the SYZ fibration $\rho : Y^0 \to \mathbb{R}^3$, where $u \in \Gamma$ is a point in the discriminant locus. In Section 8, we will try to justify this speculation by some Floer-theoretic computations.

3 Lagrangian submanifolds fibred over paths

We introduce a class of Lagrangian submanifolds in Y^0 which are fibred over paths in the z-plane. Let us start with non-compact Lagrangian submanifolds.

Definition 3.1. A smooth path $\gamma : \mathbb{R} \to \mathbb{C}^\times$ on the z-plane such that $\lim_{t\to-\infty} |\gamma(t)| = 0$ and $\lim_{t\to\infty} |\gamma(t)| = \infty$ is said to be *admissible* if it intersects the interval $\epsilon := [a, b]$ transversally and does not intersect the discriminant $\Delta = \{a, b\}$ of the double conic fibration $f : Y^0 \to \mathbb{C}^\times$. The *winding number* $w(\gamma)$ of an admissible path γ is defined as its intersection number with ϵ. We choose the orientation so that a path intersecting ϵ transversally once and in the counterclockwise direction contributes $+1$ to the intersection number.

Let $\gamma : \mathbb{R} \to \mathbb{C}^\times \setminus \Delta$ be an admissible path. The symplectic fibration $f : Y^0 \to \mathbb{C}^\times$ induces a natural horizontal distribution given by symplectic orthogonal to the fiber. Parallel transport with respect to this horizontal distribution gives symplectomorphisms between the smooth fibers of f. A 3-dimensional submanifold $L \subset f^{-1}(\gamma)$ is Lagrangian if and only if it is swept by the parallel transport of a Lagrangian cycle in a fiber along γ (cf. [Aur07, Section 5.1]). Therefore, by fixing $t_0 \in \mathbb{R}$ and choosing a Lagrangian cycle A_0 in the double conic fiber

$$f^{-1}(\gamma(t_0)) = (f')^{-1}(\gamma(t_0)) \times (f'')^{-1}(\gamma(t_0)),$$

one can construct a Lagrangian submanifold $L_{\gamma,A_0} \subset Y$ as the submanifold in $f^{-1}(\gamma)$ swept out by the parallel transport of A_0 along γ.

Notice that the winding number $w(\gamma)$ and the Hamiltonian isotopy class of the Lagrangian submanifold L_γ are invariant when we deform γ in a fixed isotopy class relative to the boundary conditions. In particular, we can always deform γ so that $\gamma(t)$ lies on the positive real axis for $t < -T$ for some fixed $T > 0$. Then we consider the direct product

$$A_t := \{(\gamma(t), u_1, v_1, u_2, v_2) \in f^{-1}(\gamma(t)) \mid u_1, v_1, u_2, v_2 \in \mathbb{R}\},$$

of the real loci (see Figure 6.2) in the factors $(f')^{-1}(\gamma(t))$ and $(f'')^{-1}(\gamma(t))$ of the double conic fiber $f^{-1}(\gamma(t))$ for each $t < -T$. The Lagrangian cycle A_t is invariant under symplectic parallel transport for $t < -T$. We then set

$$L_\gamma := L_{\gamma,A_t},$$

i.e. the submanifold in Y^0 swept out by parallel transport of A_{t_0} (for some fixed $t_0 < -T$). This defines a Lagrangian submanifold in (Y^0, ω) homeomorphic to $\mathbb{R}^3$.

Definition 3.2. An admissible path $\gamma : \mathbb{R} \to \mathbb{C}^\times \setminus \Delta$ is said to be *strongly admissible* if

- $|\gamma| : \mathbb{R} \to \mathbb{R}_{>0}$ is a strictly increasing function.
- The path agrees with a straight line near $z = 0$ and outside of some compact set.

Remark 3.3. The second condition above is necessary for the purposes of defining wrapped Floer cohomology and in particular for the maximum principle of the appendix to hold.

Proposition 3.4. *Let $\gamma : \mathbb{R} \to \mathbb{C}^\times \setminus \Delta$ be a strongly admissible path. Then the Lagrangian submanifold L_γ we define above is a section of the SYZ fibration $\rho : Y^0 \to B$.*

Proof. The proof is essentially the same as that of [CU13, Proposition 3.4]. The restriction of the moment map ϕ to A_t (for t sufficiently small), which is just the direct product of the real loci (Figure 6.2), is injective. Since T^2 acts fiberwise and it acts by symplectomorphisms on Y^0, the symplectic parallel transport induces T^2-equivariant symplectomorphisms between fibers of f. So the restriction of ϕ to a parallel transport of A_t remains injective. Together with the condition that $|\gamma(t)|$ is strictly increasing, we see that L_γ is intersecting each fiber of the SYZ fibration $\rho : Y^0 \to B$ at one point. □

Remark 3.5. Given a strongly admissible path $\gamma : \mathbb{R} \to \mathbb{C}^\times \setminus \Delta$, we can as well choose any Lagrangian cycle $A_0 \subset f^{-1}(\gamma(t_0))$ such that $\phi|_{A_0}$ is an injective map, then the resulting Lagrangian submanifold L_{γ, A_0} is also a section of the SYZ fibration.

An example is given by the path

$$\gamma_0 : \mathbb{R} \to \mathbb{C}^\times, \ t \mapsto e^t,$$

which runs through the whole positive real axis, which is obviously strongly admissible. The corresponding Lagrangian submanifold $L_0 := L_{\gamma_0}$ is simply the real locus in Y which we choose as the *zero-section* of the SYZ fibration.

To construct compact Lagrangian submanifolds in (Y^0, ω), we consider *bounded paths*, which are smooth paths $\sigma : [0, 1] \to \mathbb{C}^\times$ starting from the critical value a of one conic fibration and ending at the critical value b of the other conic fibration. The fiber product of the Lefschetz thimbles of each conic fibrations along a bounded path σ gives a Lagrangian submanifold L_σ of Y^0, which is a T^2 fibration over the bounded path. One S^1-factor collapses to a point on one end and the other S^1-factor collapses to a point on the other end, so that the total space L_σ is homeomorphic to S^3.

Definition 3.6. We call a bounded path $\sigma : [0, 1] \to \mathbb{C}^\times$ going from b to a *strongly admissible* if $|\sigma| : [0, 1] \to \mathbb{R}_{>0}$ is a strictly increasing function and σ intersects the interval $\epsilon^- := [-b, -a]$ transversally.

As in [Cha13], in order to define the SYZ transform, we need to choose a reference path σ_0, relative to which we measure the winding numbers. Since we have chosen the Lagrangian L_0 associated to the positive real axis γ_0 as the zero-section of the SYZ fibration, and we would like the Floer cohomology between the Lagrangians fibered over the two reference paths γ_0 and σ_0 to have the correct dimension, we shall impose the condition that the reference paths γ_0 and σ_0 intersect transversally at one point (with the correct orientation). For this reason, we choose σ_0 to be the path corresponding to the Lagrangian 3-sphere S_0 as shown in Figure 1.3.

Definition 3.7. The *winding number* $w(\sigma)$ of a strongly admissible bounded path $\sigma : [0,1] \to \mathbb{C}^\times$ going from b to a is defined to be the winding number of the concatenation of paths $\overline{\sigma}_0 \circ \sigma$ with respect to the counterclockwise isomorphism $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$, where $\overline{\sigma}_0$ denotes the path σ_0 with reversed orientation.

With this definition, the bounded path σ_1, which corresponds to the Lagrangian 3-sphere S_1 in Figure 1.3, has winding number 1.

It is easy to see that the Lagrangian 3-sphere L_σ associated with a strongly admissible bounded path $\sigma : [0,1] \to \mathbb{C}^\times$ is fibred by T^2 over the line segment (the red line in Figure 1.1)

$$\ell_0 := (|b|, |a|) \times \{\vec{0}\}$$

in the base B of the SYZ fibration, and the T^2 fiber degenerates to an S^1 at both ends $(|a|, \vec{0})$ and $(|b|, \vec{0})$.

4 SYZ transforms

Let $x_1 = -\lambda_1$, $x_2 = -\lambda_2$ and x_3 be affine coordinates (action coordinates) on the smooth locus B^{sm} of the SYZ fibration; note that $x_1 = -\lambda_1$ and $x_2 = -\lambda_2$ are globally defined coordinates. We denote by $\Lambda^\vee \subset T^* B^{\mathrm{sm}}$ the family of lattices locally generated by dx_1, dx_2, dx_3, and let

$$\omega_0 := dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2 + dx_3 \wedge d\xi_3$$

be the standard symplectic structure on the quotient $T^* B^{\mathrm{sm}} / \Lambda^\vee$ of the cotangent bundle $T^* B^{\mathrm{sm}}$ by $\Lambda^\vee$, where (ξ_1, ξ_2, ξ_3) denote the fiber coordinates on $T^* B^{\mathrm{sm}}$. Since we have a global Lagrangian section L_0 (the zero-section) of the SYZ fibration $\rho : Y \to B$, there exists a fiber-preserving symplectomorphism [Dui80]

$$\Theta : (T^* B^{\mathrm{sm}} / \Lambda^\vee, \omega_0) \xrightarrow{\cong} (\rho^{-1}(B^{\mathrm{sm}}), \omega)$$

so that L_0 is mapped to the zero section of $T^* B^{\mathrm{sm}} / \Lambda^\vee$.

We take an open cover $\{U_i\}$ of B^{sm} such that each U_i is contractible. As we have seen in Section 2, the SYZ mirror X^0 is obtained by gluing the open pieces $TU_i / TU_i \cap \Lambda$ together according to the wall-crossing formulas (2.4), (2.5) (and then extending by analytic continuation). Let y_1, y_2, y_3 be the coordinates on TB^{sm} which are dual to the angle coordinates ξ_1, ξ_2, ξ_3 on $T^* B^{\mathrm{sm}} / \Lambda^\vee$. The local complex coordinates on X^0 are then given by $w_1 = \exp 2\pi(x_1 + \sqrt{-1} y_1)$, $w_2 = \exp 2\pi(x_2 + \sqrt{-1} y_2)$ and $\exp 2\pi(x_3 + \sqrt{-1} y_3)$.

Let $L \subset Y^0$ be a Lagrangian cycle, given as the quotient of a translate of the conormal bundle $N^* S$ of an integral affine linear subspace $S \subset B$ by the lattice $N^* S \cap \Lambda^\vee$, and equipped with a flat $U(1)$-connection ∇. The SYZ transform of (L, Y^0) is given by a pair $(C, \check{\nabla})$ consisting of the complex submanifold C, which is given by gluing the open pieces

$$T(S \cap U_i) / T(S \cap U_i) \cap \Lambda$$

according to the wall-crossing formulas (2.4), (2.5), and a $U(1)$-connection $\check{\nabla}$, the $(0,2)$-part of the curvature two form of which is trivial and hence defines a holomorphic line bundle $\check{\mathcal{L}}$ over $C \subset X^0$.

Definition 4.1. We define the *SYZ transform* of the Lagrangian submanifold L equipped with the flat $U(1)$-connection ∇ to be the holomorphic line bundle $\check{\mathcal{L}}$ over the complex submanifold $C \subset X^0$.

We refer the reader to the original papers [LYZ00, AP01] for more details and the precise formulas; see also [Cha13].

By Proposition 3.4, the non-compact Lagrangian submanifold L_γ associated with a strongly admissible path $\gamma : \mathbb{R} \to \mathbb{C}^\times$ is a section of the SYZ fibration $\rho : Y^0 \to B$, so its SYZ transform should produce a holomorphic line bundle over X^0. Via the symplectomorphism Θ, we can write L_γ as a section of $T^*B^{\mathrm{sm}}/\Lambda^\vee$

$$L_\gamma = \{(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) \in T^*B^{\mathrm{sm}}/\Lambda^\vee \mid \xi_j = \xi_j(x_1, x_2, x_3) \text{ for } j = 1, 2, 3\},$$

where $\xi_j = \xi_j(x_1, x_2, x_3)$ $(j = 1, 2, 3)$ are smooth functions on B^{sm}. The condition that L_γ being Lagrangian is then equivalent to saying that the functions ξ_1, ξ_2, ξ_3 satisfy the relations

$$\frac{\partial \xi_j}{\partial x_l} = \frac{\partial \xi_l}{\partial x_j}$$

for $j, l = 1, 2, 3$.

The restriction of the Lagrangian section L_γ to an open set $U_i \subset B^{\mathrm{sm}}$ is transformed to a family of connections $\{\check{\nabla}_{\xi(x)} \mid x \in U_i\}$ which patch together to give a $U(1)$-connection over U_i that can locally be written as

$$\check{\nabla}_{U_i} = d + 2\pi\sqrt{-1}(\xi_1 dy_1 + \xi_2 dy_2 + \xi_3 dy_3)$$

over the open piece $TU_i/TU_i \cap \Lambda \subset X^0$. Since the $(0, 2)$-part of the curvature two form for each connection vanishes and the wall-crossing formulas are holomorphic, these connections glue together to give globally a holomorphic line bundle $\check{\mathcal{L}}_\gamma$ over X^0.

Notice that the isomorphism class of $\check{\mathcal{L}}_\gamma$ is unchanged when we deform L_γ in a fixed Hamiltonian isotopy class (or deforming γ in a fixed homotopy class relative to the boundary conditions $\lim_{t\to-\infty} |\gamma(t)| = 0$ and $\lim_{t\to\infty} |\gamma(t)| = \infty$). Therefore, we will regard this as defining the SYZ transform of the Hamiltonian isotopy class of the Lagrangian submanifold L_γ as an isomorphism class of holomorphic line bundle over X^0.

As an immediate example, the SYZ transformation of the zero section L_0 gives the structure sheaf $\mathcal{O}_{X^0}$ over X^0.

To compute (the isomorphism class of) the line bundle $\check{\mathcal{L}}_\gamma$, note that the degree of its restriction to the exceptional curve $E \cong \mathbb{P}^1$ in X^0 is given by

$$\deg \check{\mathcal{L}}_\gamma|_E = \int_E \frac{\sqrt{-1}}{2\pi} F_{\check{\nabla}} = -\int_E d\xi_3 \wedge dy_3 = -(\xi_3(|b|, \vec{0}) - \xi_3(|a|, \vec{0})).$$

We have the second equality because y_1, y_2 are constant (and $x_i = \lambda_i = 0$ for $i = 1, 2$) on E. Hence the isomorphism class of the line bundle $\check{\mathcal{L}}_\gamma$ is completely determined by the increment of the angle coordinate ξ_3 on the Lagrangian section L_γ from $(0, 0, |b|)$ to $(0, 0, |a|)$ (which is measured with reference to the path γ_0).

Proof of Theorem 1.1. Arguing as in the proof of [CU13, Theorem 1.1], we first deform γ so that $\gamma(\log |b|) = -b$ and $\gamma(\log |a|) = -a$ and $\gamma(t) \in \mathbb{R}_{>0}$ for $t \notin (\log |b|, \log |a|)$

(up to a re-parametrization if necessary). We then further deform $\gamma|_{(\log|b|,\log|a|)}$ to the concatenation of $\gamma_0|_{(\log|b|,\log|a|)}$ (the positive real axis) with a loop winding around the circle $C_{|a|} = \{z \in \mathbb{C}^\times \mid |z| = |a|\}$ for $w(\gamma)$ times. Along γ_0, the angle coordinate ξ_3 is constantly zero, and ξ_3 increases by one when we wind around $C_{|a|}$ once in the counterclockwise direction. Hence, the increment $\xi_3(|b|,\vec{0}) - \xi_3(|a|,\vec{0})$ is precisely given by the winding number $w(\gamma)$. This completes the proof of Theorem 1.1. $\square$

Let γ_0 and γ_1 be admissible paths shown in Figure 1.2 which have winding numbers 0 and -1 respectively. Their associated Lagrangian submanifolds are denoted by $L_0 := L_{\gamma_0}$ and $L_1 := L_{\gamma_1}$ respectively. By Theorem 1.1, the SYZ transform of L_i is precisely given by the line bundle $\mathcal{O}_{X^0}(i)$ for $i = 0, 1$.

Next we consider a strongly admissible bounded path $\sigma : [0,1] \to \mathbb{C}^\times$ going from b to a. Recall that the corresponding compact Lagrangian 3-sphere L_σ is a T^2-fibration over the line segment $\ell_0 = (|b|,|a|) \times \{\vec{0}\}$ in the base $B = \mathbb{R}_{>0} \times \mathbb{R}^2$ of the SYZ fibration $\rho : Y^0 \to B$ such that the T^2-fiber degenerates to an S^1 over the endpoints $(|a|,\vec{0})$ and $(|b|,\vec{0})$ of ℓ_0.

Let $L_\sigma^\circ = L_\sigma \cap \rho^{-1}(\ell_0)$, i.e. L_σ with the two S^1's over the end points of ℓ_0 removed. Then L_σ° is (the quotient by a lattice of) a translate of the conormal bundle of ℓ_0. Recall that the coordinates w_1, w_2 on X^0 are given by $w_1 = \exp 2\pi(x_1 + \sqrt{-1}y_1)$ and $w_2 = \exp 2\pi(x_2 + \sqrt{-1}y_2)$. We equip L_σ with the flat $U(1)$-connection

$$\nabla_0 = d - \pi\sqrt{-1}(d\xi_1 + d\xi_2).$$

Then the SYZ transform of (L_σ, ∇_0) produces the complex submanifold in X^0 defined by $x_1 = x_2 = 0$, $y_1 = y_2 = 1/2$ or simply $w_1 = w_2 = -1$, which is precisely the exceptional locus $E \cong \mathbb{P}^1 \subset X^0$ (cf. [Cha13, Section 2]).

We also get the $U(1)$-connection

$$\check{\nabla} = d + 2\pi\sqrt{-1}\xi_3(x_3,\vec{0})dy_3$$

on E which defines a holomorphic line bundle over E whose degree can be computed as

$$\int_E \frac{\sqrt{-1}}{2\pi} F_{\check{\nabla}} = -\int_E d\xi_3 \wedge dy_3 = -(\xi_3(|a|,\vec{0}) - \xi_3(|b|,\vec{0})) = -w(\sigma).$$

We have the last equality because the increment of the angle coordinate ξ_3 is measured relative to the reference path σ_0 which is computed by the winding number of the loop $\overline{\sigma}_0 \circ \sigma$ and this is by definition $w(\sigma)$. This proves the following:

Theorem 4.2. *The SYZ transform of the compact Lagrangian 3-sphere L_σ associated to a strongly admissible bounded path $\sigma : [0,1] \to \mathbb{C}^\times$ is given by the line bundle $\mathcal{O}_E(-w(\sigma))$ over the exceptional locus $E \subset X^0$.*

Theorem 1.3 is an immediate consequence of Theorem 4.2 since the bounded paths defining S_0 and S_1 have winding numbers 0 and 1 respectively.

5 Coherent sheaves on the resolved conifold

Let $\mathbb{C}^\times$ acts on $\mathbb{C}^4 = \operatorname{Spec}\mathbb{C}[x, y, t_1, t_2]$ in such a way that $\alpha \in \mathbb{C}^\times$ maps (x, y, t_1, t_2) to $(\alpha x, \alpha y, \alpha^{-1} t_1, \alpha^{-1} t_2)$. It is convenient to realize the resolved conifold as the quotient

$$X = (\mathbb{C}^4 \setminus \Sigma)/\mathbb{C}^\times \tag{5.1}$$

where $\Sigma := \{(x, y, t_1, t_2) \in \mathbb{C}^4 \mid x = y = 0\}$. In these coordinates, the morphism

$$\varphi : X \to Z = \{(u, v, w_1, w_2) \in \mathbb{C}^4 \mid uv = (1 + w_1)(1 + w_2)\}$$

to the conifold is given by

$$u = xt_1,\ v = yt_2,\ w_1 = xt_2 - 1, w_2 = yt_1 - 1.$$

Definition 5.1. An object $\mathcal{E}$ in a triangulated category $\mathcal{T}$ is a *tilting object* if

- $\mathcal{E}$ is *acyclic* in the sense that $\operatorname{Ext}^k(\mathcal{E}, \mathcal{E}) = 0$ for any $k \neq 0$, and
- $\mathcal{E}$ is a *classical generator*, in the sense that the smallest, thick, triangulated subcategory generated by $\mathcal{E}$ is all of $\mathcal{T}$.

Note that any classical generator $\mathcal{E}$ *generates* $\mathcal{T}$ in the sense that $\operatorname{Hom}^k(\mathcal{E}, A) = 0$ for some $A \in \mathcal{T}$ and all $k \in \mathbb{Z}$ implies $A \cong 0$ (cf. e.g. [BvdB03, Section 2.1]). The proof of the following theorem can be found in [TU10, Lemma 3.3], and goes back at least to [Ric89, Bon89]:

Theorem 5.2. *Let $\mathcal{E}$ be a tilting object in the derived category $D^b \operatorname{coh} X$ of coherent sheaves on a smooth quasi-projective variety X. Then $D^b \operatorname{coh} X$ is equivalent to the bounded derived category of finitely-generated right modules over* $\operatorname{Hom}(\mathcal{E}, \mathcal{E})$.

The following is well-known (cf. e.g. [VdB04]):

Theorem 5.3. *The direct sum $\mathcal{O}_X \oplus \mathcal{O}_X(1)$ is a tilting object in $D^b \operatorname{coh} X$, whose endomorphism algebra is described by the quiver*

$$\begin{tikzcd}
\mathcal{O} \arrow[r, bend left=40, "x"] \arrow[r, bend left=20, "y"'] & \mathcal{O}(1) \arrow[l, bend left=20, "t_1"'] \arrow[l, bend left=40, "t_2"]
\end{tikzcd}$$

with relations

$$\mathcal{I} = (xt_1y - yt_1x, xt_2y - yt_2x, t_1xt_2 - t_2xt_1, t_1yt_2 - t_2yt_1). \tag{5.2}$$

Let $\{P_{a,i_1,i_2}\}_{(a,i_1,i_2)\in\mathbb{Z}\times\mathbb{N}^2}$ be the basis of $\operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X) \cong \operatorname{Hom}(\mathcal{O}_X(1), \mathcal{O}_X(1)) \cong \Gamma(\mathcal{O}_X)$ defined by

$$P_{a,i_1,i_2} = \begin{cases} u^{-a} w_1^{i_1} w_2^{i_2} & a < 0, \\ v^a w_1^{i_1} w_2^{i_2} & a \geq 0. \end{cases}$$

Similarly, we define the bases $\{Q_{a,i_1,i_2}\}_{(a,i_1,i_2)\in(\mathbb{Z}+\frac{1}{2})\times\mathbb{N}^2}$ and $\{R_{a,i_1,i_2}\}_{(a,i_1,i_2)\in(\mathbb{Z}+\frac{1}{2})\times\mathbb{N}^2}$ of $\mathrm{Hom}(\mathcal{O}_X,\mathcal{O}_X(1))$ and $\mathrm{Hom}(\mathcal{O}_X(1),\mathcal{O}_X)$ as

$$Q_{a,i_1,i_2}=\begin{cases} xu^{-a-1/2}w_1^{i_1}w_2^{i_2} & a<0,\\ yv^{a-1/2}w_1^{i_1}w_2^{i_2} & a\geq 0.\end{cases}$$

and

$$R_{a,i_1,i_2}=\begin{cases} t_1u^{-a-1/2}w_1^{i_1}w_2^{i_2} & a<0,\\ t_2v^{a-1/2}w_1^{i_1}w_2^{i_2} & a\geq 0.\end{cases}$$

respectively.

We have the following elementary algebra calculation.

Proposition 5.4 (cf. [Pas14, Proposition 4.5])**.** *The composition of P_{a,i_1,i_2} is given by*

$$P_{b,j_1,j_2}\cdot P_{a,i_1,i_2}=\sum_{s_1,s_2=0}^{k}\binom{k}{s_1}\binom{k}{s_2}P_{a+b,i_1+j_1+s_1,i_2+j_2+s_2} \tag{5.3}$$

where

$$k=\begin{cases}\min\{|a|,|b|\} & a \text{ and } b \text{ have different signs},\\ 0 & \text{otherwise}.\end{cases}$$

The composition of P_{a,i_1,i_2} and Q_{b,j_1,j_2} is given by

$$Q_{b,j_1,j_2}\cdot P_{a,i_1,i_2}=\sum_{s_1,s_2=0}^{k}\binom{k}{s_1}\binom{k}{s_2}Q_{a+b,i_1+j_1+s_1,i_2+j_2+s_2} \tag{5.4}$$

where

$$k=\begin{cases}\min\{|a|,|b|-1/2\} & a \text{ and } b \text{ have different signs},\\ 0 & \text{otherwise},\end{cases}$$

and similarly for the composition of P_{a,i_1,i_2} and R_{b,j_1,j_2}. The composition of Q_{a,i_1,i_2} and R_{b,j_1,j_2} is given by

$$R_{b,j_1,j_2}\cdot Q_{a,i_1,i_2}=\sum_{s_1}^{k_1}\sum_{s_2=0}^{k_2}\binom{k_1}{s_1}\binom{k_2}{s_2}P_{a+b,i_1+j_1+s_1,i_2+j_2+s_2} \tag{5.5}$$

where

$$k_1=\begin{cases}\min\{|a|-1/2,|b|-1/2\}+1 & a<0 \text{ and } b>0,\\ \min\{|a|-1/2,|b|-1/2\} & a>0 \text{ and } b<0,\\ 0 & \text{otherwise},\end{cases}$$

$$k_2=\begin{cases}\min\{|a|-1/2,|b|-1/2\} & a<0 \text{ and } b>0,\\ \min\{|a|-1/2,|b|-1/2\}+1 & a>0 \text{ and } b<0,\\ 0 & \text{otherwise}.\end{cases}$$

The mirror X^0 is the complement $X \setminus D$ of the divisor $D = \{w_1 w_2 = 0\}$ on X.

Corollary 5.5. *The direct sum $\mathcal{O}_{X^0} \oplus \mathcal{O}_{X^0}(1)$ is a tilting object in $D^b \operatorname{coh} X^0$.*

Proof. The fact that $\mathcal{O}_{X^0} \oplus \mathcal{O}_{X^0}(1)$ is a classical generator follows immediately from the fact that $\mathcal{O}_X \oplus \mathcal{O}_X(1)$ is a classical generator and the equivalence

$$D^b \operatorname{coh} X / D^b \operatorname{coh}_D X \xrightarrow{\sim} D^b \operatorname{coh} X^0$$

of triangulated categories [Orl11, Lemma 2.2]. The acyclicity of $\mathcal{O}_{X^0} \oplus \mathcal{O}_{X^0}(1)$ follows from the acyclicity of $\mathcal{O}_X \oplus \mathcal{O}_X(1)$ and the description

$$H^k(\mathcal{O}_{X^0}(i)) = \varinjlim \left(H^k(\mathcal{O}_X(i)) \xrightarrow{w_1 w_2} H^k(\mathcal{O}_X(i)) \xrightarrow{w_1 w_2} H^k(\mathcal{O}_X(i)) \xrightarrow{w_1 w_2} \cdots \right)$$

of the cohomology as a direct limit [Sei08a, (1.13)]. □

The derived category $D^b \operatorname{coh}_0 X$ of coherent sheaves on X supported on the exceptional locus E of the resolution $\varphi : X \to Z$ is generated by $\mathcal{O}_E$ and $\mathcal{O}_E(-1)[1]$, which are Koszul dual to $\mathcal{O}_X$ and $\mathcal{O}_X(1)$ in the sense that

$$\begin{aligned} \operatorname{Hom}^0(\mathcal{O}_X, \mathcal{O}_E) &= \mathbb{C}, & \operatorname{Hom}^0(\mathcal{O}_X, \mathcal{O}_E(-1)[-1]) &= 0, \\ \operatorname{Hom}^0(\mathcal{O}_X(1), \mathcal{O}_E) &= 0, & \operatorname{Hom}^0(\mathcal{O}_X(1), \mathcal{O}_E(-1)[1]) &= \mathbb{C}. \end{aligned}$$

The endomorphism A_∞-algebra of $\mathcal{O}_E \oplus \mathcal{O}_E(-1)$ is Koszul dual to the endomorphism algebra of $\mathcal{O}_X \oplus \mathcal{O}_X(1)$. A convenient way to describe it is given by the dimer model shown in Figure 5.1.

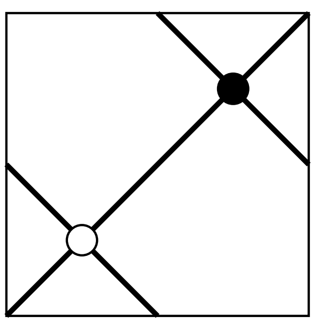

Figure 5.1: The dimer model

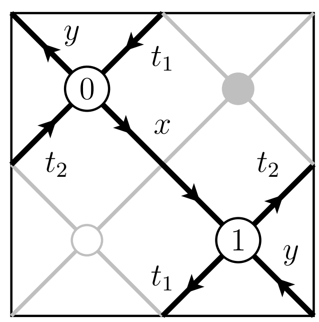

Figure 5.2: The corresponding quiver

It is a graph G drawn on the real 2-torus consisting of two nodes and four edges. One node is painted in black, and the other is painted in white. The dual graph of G is combinatorially identical to G, and we turn each edge of the dual graph into an arrow by giving the orientation such that the white node is on the right of the arrow. This makes the dual graph of G into the quiver $Q = (V, A)$ shown in Figure 5.2 with two vertices $V = \{0, 1\}$ and four arrows $A = \{x, y, t_1, t_2\}$. For each arrow a in the quiver, there are two paths $p_+(a)$ and $p_-(a)$ from the target of a to the source of a; the former goes around the white node, and the latter goes around the black node. Then we can equip the quiver with the relation such that $p_+(a)$ is equivalent to $p_-(a)$ for all arrows; $\mathcal{I} = (p_+(a) - p_-(a))_{a \in A}$. One can easily see that this relation is identical to the one in (5.2).

Now the endomorphism A_∞-algebra of $\mathcal{O}_E \oplus \mathcal{O}_E(-1)[1]$ is described as follows [FU10, Definition 2.1 and Proposition 2.2]:

- The vertices 0 and 1 of Q correspond to objects $\mathcal{O}_E$ and $\mathcal{O}_E(-1)[1]$ respectively.
- For a pair v and w of vertices, the space of morphisms is given by

$$\operatorname{Hom}^i(v,w) = \begin{cases} \mathbb{C} \cdot \mathrm{id}_v & i = 0 \text{ and } v = w, \\ \operatorname{span}\{a \mid a : w \to v\} & i = 1, \\ \operatorname{span}\{a^\vee \mid a : v \to w\} & i = 2, \\ \mathbb{C} \cdot \mathrm{id}_v^\vee & i = 3 \text{ and } v = w, \\ 0 & \text{otherwise.} \end{cases}$$

- Non-zero A_∞-operations are

$$\mathfrak{m}_2(x, \mathrm{id}_v) = \mathfrak{m}_2(\mathrm{id}_w, x) = x$$

  for any $x \in \operatorname{Hom}(v,w)$,

$$\mathfrak{m}_2(a, a^\vee) = \mathrm{id}_v^\vee$$

  and

$$\mathfrak{m}_2(a^\vee, a) = \mathrm{id}_w^\vee$$

  for any arrow a from v to w,

$$\mathfrak{m}_k(a_1, \dots, a_k) = a_0.$$

  for any cycle $(a_0, \dots, a_k)$ of the quiver going around a white node, and

$$\mathfrak{m}_k(a_1, \dots, a_k) = -a_0.$$

  for any cycle $(a_0, \dots, a_k)$ of the quiver going around a black node.
- The pairing

$$\langle \bullet, \bullet \rangle : \operatorname{Hom}(w,v) \otimes \operatorname{Hom}(v,w) \to \mathbb{C}[3]$$

  defined by

$$\langle a^\vee, a \rangle = \langle \mathrm{id}_v^\vee, \mathrm{id}_v \rangle = 1$$

  and zero otherwise makes the endomorphism A_∞-algebra into a cyclic A_∞-algebra of dimension three.

To be more explicit, one has

$$\mathrm{Hom}^i(\mathcal{O}_E,\mathcal{O}_E)=\begin{cases}\mathbb{C}\cdot \mathrm{id}_{\mathcal{O}_E} & i=0,\\ \mathbb{C}\cdot \mathrm{id}_{\mathcal{O}_E}^\vee & i=3,\\ 0 & \text{otherwise},\end{cases}$$

$$\mathrm{Hom}^i(\mathcal{O}_E(-1)[1],\mathcal{O}_E(-1)[1])=\begin{cases}\mathbb{C}\cdot \mathrm{id}_{\mathcal{O}_E(-1)[1]} & i=0,\\ \mathbb{C}\cdot \mathrm{id}_{\mathcal{O}_E(-1)[1]}^\vee & i=3,\\ 0 & \text{otherwise},\end{cases}$$

$$\mathrm{Hom}^i(\mathcal{O}_E(-1)[1],\mathcal{O}_E)=\begin{cases}\mathbb{C}\cdot x\oplus\mathbb{C}\cdot y & i=1,\\ \mathbb{C}\cdot t_1^\vee\oplus\mathbb{C}\cdot t_2^\vee & i=2,\\ 0 & \text{otherwise},\end{cases}$$

$$\mathrm{Hom}^i(\mathcal{O}_E,\mathcal{O}_E(-1)[1])=\begin{cases}\mathbb{C}\cdot t_1\oplus\mathbb{C}\cdot t_2 & i=1,\\ \mathbb{C}\cdot x^\vee\oplus\mathbb{C}\cdot y^\vee & i=2,\\ 0 & \text{otherwise},\end{cases}$$

with A_∞-operations

$$\begin{aligned}&\mathfrak{m}_3(y,t_1,x)=-t_2^\vee, && \mathfrak{m}_3(t_2,y,t)=-x^\vee, && \mathfrak{m}_3(x,t_2,y)=-t_1^\vee, && \mathfrak{m}_3(t_1,x,t_2)=-y^\vee,\\ &\mathfrak{m}_3(y,t_2,x)=t_1^\vee, && \mathfrak{m}_3(t_1,y,t_2)=x^\vee, && \mathfrak{m}_3(x,t_1,y)=t_2^\vee, && \mathfrak{m}_3(t_2,x,t_1)=y^\vee,\end{aligned}$$

and

$$\begin{aligned}&\mathfrak{m}_2(x,x^\vee)=\mathrm{id}_{\mathcal{O}_E}^\vee, && \mathfrak{m}_2(y,y^\vee)=\mathrm{id}_{\mathcal{O}_E}^\vee, && \mathfrak{m}_2(s^\vee,s)=\mathrm{id}_{\mathcal{O}_E}^\vee, && \mathfrak{m}_2(t_1^\vee,t_1)=\mathrm{id}_{\mathcal{O}_E}^\vee,\\ &\mathfrak{m}_2(t_2,t_2^\vee)=\mathrm{id}_{\mathcal{O}_E(-1)[1]}^\vee, && \mathfrak{m}_2(t_1,t_1^\vee)=\mathrm{id}_{\mathcal{O}_E(-1)[1]}^\vee, && \mathfrak{m}_2(x^\vee,x)=\mathrm{id}_{\mathcal{O}_E(-1)[1]}^\vee, && \mathfrak{m}_2(y^\vee,y)=\mathrm{id}_{\mathcal{O}_E(-1)[1]}^\vee.\end{aligned}$$

All the other non-zero A_∞-operations just say that $\mathrm{id}_{\mathcal{O}_E}$ and $\mathrm{id}_{\mathcal{O}_E(-1)[1]}$ are the identity elements for $\mathfrak{m}_2$.

6 Wrapped Fukaya category

We prove Theorem 1.2 in this section. For technical reasons, Floer theory on a non-compact fibration such as the one we are considering requires a modification of the symplectic form so that

- the symplectic monodromy is trivial along the horizontal boundary of the fibration.
- the flow of the Hamiltonians H_i below fiber over the flow in the base for Hamiltonian vector-field of H_b in the base.

We refer the reader to the appendix for a more detailed discussion of the geometric setup for Floer cohomology of fibrations. We set $a=\sqrt{-1}$ and $b=-\sqrt{-1}$ as in Figure 6.1 for convenience in this section. We take wrapping Hamiltonians of the forms

$$H_i=H_b+H_{f_1,i}+H_{f_2,i}, \tag{6.1}$$

where the Hamiltonian H_b is an admissible Hamiltonian in the base (see the appendix for this definition) and wraps the z-plane as shown in Figure 6.3, and the fiber Hamiltonians $H_{f_i,i}$ are admissible Hamiltonians in the fiber which wrap the fiber either as in Figure 6.4 or Figure 6.5. We assume that each H_i is *Lefschetz admissible* in the sense of Section A.2 or McLean [McL09]. Let $\phi_t : Y^0 \to Y^0$ be the time t flow by the wrapping Hamiltonian H_i. The *wrapped Floer cohomology* is defined as

$$\mathrm{Hom}_{\mathcal{W}_i}(L_j, L_k) = \lim_{t\to\infty} \mathrm{Hom}_{\mathcal{F}}(\phi_t L_j, L_k)$$

where $\mathrm{Hom}_{\mathcal{F}}(\phi_t L_j, L_k)$ is the ordinary Floer cohomology.

Remark 6.1. Our choice of Hamiltonian is slightly different from that in [AS10], but is very suitable for analyzing fibrations. In the appendix, we provide some details concerning wrapped Floer cohomology as well as the relationship between the two approaches. It is also important to note that, while we don't construct A_∞-operations on our wrapped Floer cohomology, all of our wrapped Floer groups are concentrated in degree zero and thus any such enhancement would actually be quasi-isomorphic to its cohomology algebra.

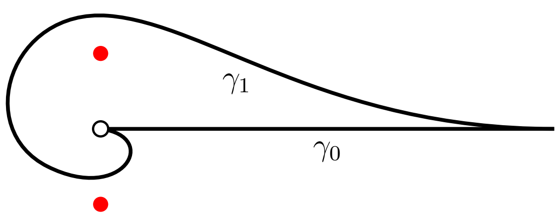

Figure 6.1: The paths on the base

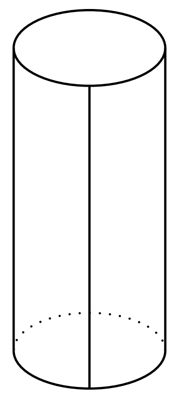

Figure 6.2: The Lagrangian on the fiber

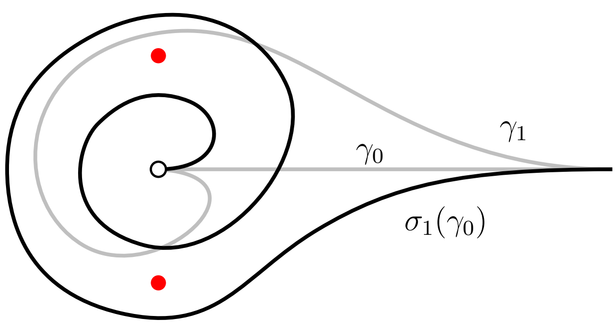

Figure 6.3: Wrapping the base

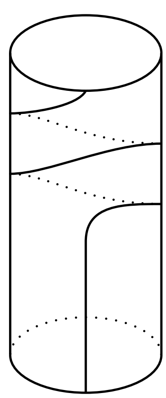

Figure 6.4: Wrapping the fiber by $H_{f,1}$

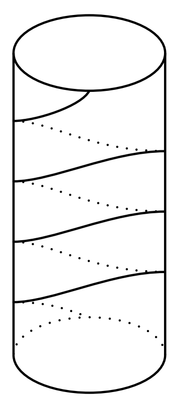

Figure 6.5: Wrapping the fiber by $H_{f,2}$

Proposition 6.2. *There is a ring isomorphism*

$$\bigoplus_{i,j=0}^{1} \mathrm{Hom}_{\mathcal{W}_1}(L_i, L_j) \xrightarrow{\sim} \bigoplus_{i,j=0}^{1} \mathrm{Hom}\left(\mathcal{O}_X(i), \mathcal{O}_X(j)\right). \tag{6.2}$$

Proof. Let us first consider the composition

$$\mathrm{Hom}(\phi_n(L_0), L_0) \otimes \mathrm{Hom}(\phi_{m+n}(L_0), \phi_n(L_0)) \to \mathrm{Hom}(\phi_{m+n}(L_0), L_0). \tag{6.3}$$

In the appendix, the product in wrapped Floer cohomology is defined using solutions to a perturbed holomorphic curve equation. The argument of [Pas14, Proposition 7.2] allows us to show that, in this situation, this is equivalent to the usual product in Lagrangian Floer theory which counts J-holomorphic triangles with boundary on L_0, $\phi_n(L_0)$, and $\phi_{m+n}(L_0)$.

The intersection points in $\phi_n(L_0) \cap L_0$ can be labeled as p_{a,i_1,i_2} as in Figure 6.6. We view the z-plane as a cylinder, which is obtained by identifying the horizontal edges of the rectangle in Figure 6.6. We choose a coordinate on the rectangle in such a way that the top right and the bottom left corners have coordinates $(1,1)$ and $(-1,-1)$ respectively.

Intersections between the Lagrangians $\phi_n(L_0)$ and L_0 are parameterized by triplets of integers (a, i_1, i_2). The integer $a \in [-n+1, n-1]$ parametrizes the intersection point of the z-projections $\sigma_n(\gamma_0)$ and γ_0 of the Lagrangians $\phi_n(L_0)$ and L_0. The integers i_1 and i_2 in $[0, \lfloor (n-|a|)/2 \rfloor]$ parametrize the intersection points on the fiber just as in [Pas14, Section 3.3.4].

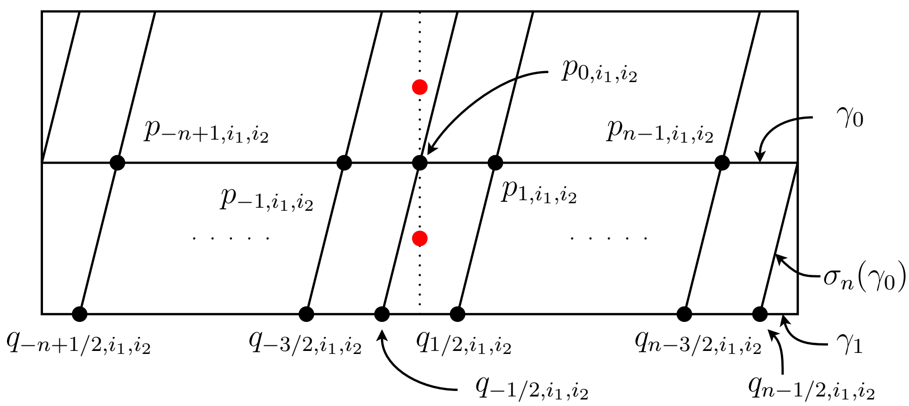

Figure 6.6: Intersections of Lagrangians

Our arguments will be based upon the following adaptation of Pascaleff's theorem [Pas14, Proposition 4.4] to this setting. Its proof follows *mutatis-mutandis* from Pascaleff's paper.

Lemma 6.3. *Let L, L', and L'' be Lagrangian submanifolds of Y^0 fibered over paths γ, γ' and γ'' in $\mathbb{C}^\times$. Assume that a holomorphic triangle $u : D^2 \to \mathbb{C}^\times$ bounded by γ, γ' and γ'' with vertices $o \in \gamma \cap \gamma'$, $o' \in \gamma' \cap \gamma''$ and $o'' \in \gamma \cap \gamma''$ intersects the discriminants a and b in $\mathbb{C}^\times$ exactly d_1 and d_2 times respectively. Then holomorphic sections over u contributes to the triangle product* $\mathrm{Hom}(L', L'') \otimes \mathrm{Hom}(L, L') \to \mathrm{Hom}(L, L'')$ *as*

$$\mathfrak{m}_2(o'_{j_1,j_2}, o_{i_1,i_2}) = \sum_{s_1=0}^{d_1} \sum_{s_2=0}^{d_2} \binom{d_1}{s_1} \binom{d_2}{s_2} o''_{i_1+j_1+s_1, i_2+j_2+s_2},$$

where $o_{i_1,i_2} \in L \cap L'$ is the intersection point above $o \in \gamma \cap \gamma'$, which is the i_1-th one from the bottom in the u_1v_1-direction and the i_2-th one from the bottom in the u_2v_2-direction.

The universal cover of the cylinder in Figure 6.6 is an infinite strip $\{(s,t) \in \mathbb{R}^2 \mid -1 \leq s \leq 1\}$. A lift of the z-projection $\gamma_{0,n}$ of the wrapped Lagrangian $\phi_n(L_0)$ to the universal cover is given by a line with slope n, passing through $(0,k)$ with $k \in \mathbb{Z}$. The discriminants of the conic fibrations are given by $(0,1/4)$ and $(0,-1/4)$ respectively. The projection of the intersection point $p_{b,j_1,j_2} \in \operatorname{Hom}(\phi_n(L_0), L_0)$ has the s-coordinate b/n, and we choose the lift to the universal cover to be $(b/n, 0)$.

Consider the lift of $\gamma_{0,n}$ passing through $(b/n, 0)$. The induced lift of the intersection point corresponding to $p_{a,i_1,i_2} \in \operatorname{Hom}(\phi_{m+n}(L_0), \phi_n(L_0))$ will have coordinate $(a/m, na/m - b)$. If we then take the lift of $\gamma_{0,m+n}$ passing though this point, it intersects with the lift of γ_0 at $((a+b)/(m+n), 0)$ as shown in Figure 6.7 or Figure 6.8 depending on the order of a and b.

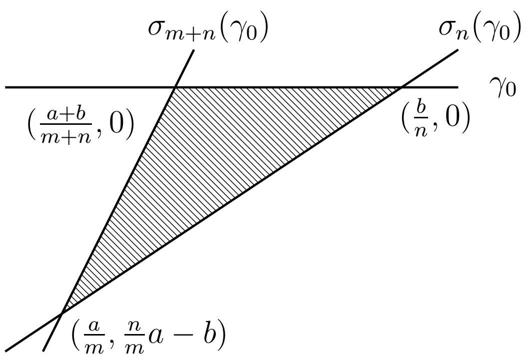

Figure 6.7: The case $a < b$

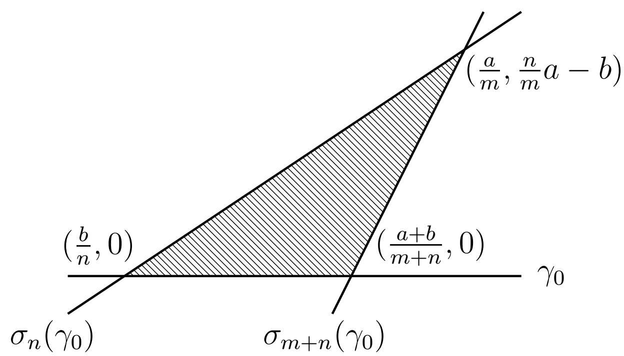

Figure 6.8: The case $b < a$

In either case, one can see from Figure 6.9 or Figure 6.10 that the triangle hits both of the discriminants $(0, -1/4 + \mathbb{Z})$ and $(0, 1/4 + \mathbb{Z})$ k times, where k is $\min\{|a|, |b|\}$ if a and b has different signs and 0 otherwise. Then one has

$$\mathfrak{m}_2(p_{b,j_1,j_2}, p_{a,i_1,i_2}) = \sum_{s_1,s_2=0}^{k} \binom{k}{s_1}\binom{k}{s_2} p_{a+b,i_1+j_1+s_1,i_2+j_2+s_2} \tag{6.4}$$

by Pascaleff's formula, in agreement with (5.3).

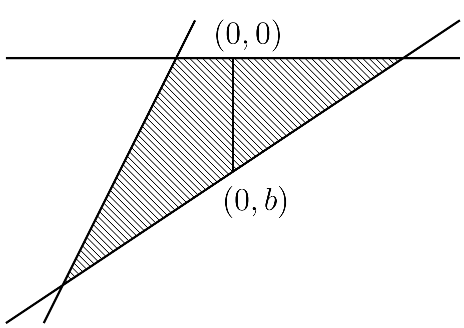

Figure 6.9: The case $0 < b < -a$

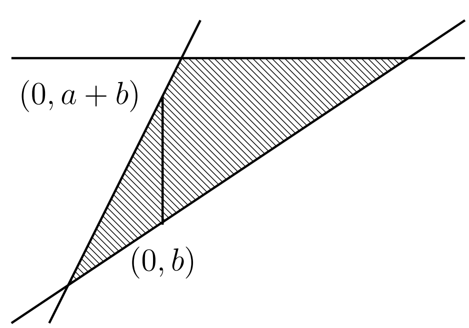

Figure 6.10: The case $0 < -a < b$

Next we consider the composition

$$\mathrm{Hom}(\phi_n(L_1), L_0) \otimes \mathrm{Hom}(\phi_{m+n}(L_0), \phi_n(L_1)) \to \mathrm{Hom}(\phi_{m+n}(L_0), L_0).$$

A lift of the z-projection $\gamma_{1,n}$ of the wrapped Lagrangian $\phi_n(L_1)$ to the universal cover is given by a line with slope n passing through $(0, k+1/2)$ with $k \in \mathbb{Z}$. The intersections of the curves on the z-planes are as in Figure 6.7 or Figure 6.8 again, with $\gamma_{0,n}$ replaced with $\gamma_{1,n}$ and a and b being half-integers. One can see from Figure 6.9 and Figure 6.10 that the triangle hits the discriminants at $(0, -1/4+\mathbb{Z})$ and $(0, 1/4+\mathbb{Z})$ for k_1 times and k_2 times respectively, where $k_1 = \min\{|a|-1/2, |b|-1/2\}+1$ and $k_2 = \min\{|a|-1/2, |b|-1/2\}$ if a and b have different signs, and $k_1 = k_2 = 0$ otherwise. Then one has

$$\mathfrak{m}_2(r_{b,j_1,j_2}, q_{a,i_1,i_2}) = \sum_{s_1=0}^{k_1} \sum_{s_2=0}^{k_2} \binom{k}{s_1} \binom{k}{s_2} p_{a+b-1, i_1+j_1+s_1, i_2+j_2+s_2}. \tag{6.5}$$

This is in complete agreement with (5.5). Other compositions can be calculated similarly, and Proposition 6.2 is proved. □

The choice of partial wrapping function $H_{f_i,1}$ corresponds to the fact that the mirror of the resolved conifold X is in fact the Landau-Ginzburg model (Y^0, u_1+u_2). See [AAK] for more discussion. Since wrapping by $H_{f_i,2}$ corresponds to multiplication by w_i, one obtains the following:

Corollary 6.4. *There is a ring isomorphism*

$$\bigoplus_{i,j=0}^{1} \mathrm{Hom}_{\mathcal{W}_2}(L_i, L_j) \xrightarrow{\sim} \bigoplus_{i,j=0}^{1} \mathrm{Hom}\left(\mathcal{O}_{X^0}(i), \mathcal{O}_{X^0}(j)\right). \tag{6.6}$$

Theorem 1.2 is an immediate consequence of Corollary 5.5, Corollary 6.4, and Theorem A.3.

7 Mirror symmetry for vanishing cycles

For a path $\gamma : [0,1] \to \mathbb{C}^\times$ on the z-plane, the union

$$S_\gamma := \bigcup_{t \in [0,1]} \left\{ (\gamma(t), u_1, v_1, u_2, v_2) \in Y^0 \;\middle|\; |u_1| = |v_1|,\ |u_2| = |v_2| \right\} \tag{7.1}$$

gives a compact Lagrangian submanifold of Y^0, which has boundaries in general. Let S_0 and S_1 be Lagrangian 3-spheres in Y^0, which are obtained in this way from the paths shown in Figure 1.3. We prove Theorem 7.1 below in this section. Theorem 1.4 follows immediately since $D^b \operatorname{coh}_0 X^0$ is generated by $\mathcal{O}_E$ and $\mathcal{O}_E(-1)$ as a triangulated category.

Theorem 7.1. *Let $\mathcal{F}_0$ be the Fukaya category of Y^0 consisting of S_0 and S_1. Then $\mathcal{F}_0$ is quasi-equivalent to the full subcategory of (the dg enhancement of) $D^b \operatorname{coh} X^0$ consisting of $\mathcal{O}_E$ and $\mathcal{O}_E(-1)$.*

Proof. First note that the union $S_0 \cup S_1$ is *exact* in the sense that the symplectic form ω vanishes on $\pi_2(Y^0, S_0 \cup S_1)$. To see this, one can look at the exact sequence

$$\cdots \to \pi_2(Y^0) \to \pi_2(Y^0, S_0 \cup S_1) \to \pi_1(S_0 \cup S_1) \to \pi_1(Y^0) \to \cdots$$

of homotopy groups; the symplectic form ω vanishes on the image of $\pi_2(Y^0)$ since Y^0 is an exact symplectic manifold, whereas the group $\pi_1(S_0 \cup S_1) \cong \mathbb{Z}$ injects to $\pi_1(Y^0)$.

The exactness of $S_0 \cup S_1$ allows us to use the chain model for the Fukaya category of the plumbing by Abouzaid [Abo11, Appendix A]. Let Q_1 and Q_2 be a pair of graded exact Lagrangian submanifolds in an exact symplectic manifold equipped with a trivialization of the canonical bundle. Assume that Q_1 and Q_2 intersect cleanly along a submanifold B, which consists of r connected components $B^1, \dots, B^k$;

$$B = Q_1 \cap Q_2, \qquad B = B^1 \sqcup \cdots \sqcup B^r.$$

Since Q_1 and Q_2 are Lagrangian submanifolds intersecting cleanly along B, the normal bundles $N_{Q_1}B$ and $N_{Q_2}B$ are isomorphic as real vector bundles. Choose closed tubular neighborhoods N_i of Q_i and triangulations $\mathcal{Q}_i$ of Q_i such that $\mathcal{Q}_i$ induce triangulations $\mathcal{N}_i$ of N_i and the isomorphism $N_{Q_1}B \cong N_{Q_2}B$ induces a cellular homeomorphism $\mathcal{N}_1 \cong \mathcal{N}_2$. Let $\mathcal{N} = \mathcal{N}^1 \sqcup \cdots \sqcup \mathcal{N}^r$ be the decomposition of the abstract simplicial complex $\mathcal{N} = \mathcal{N}_1 \cong \mathcal{N}_2$ into connected components. Then the chain model for the Fukaya category consisting of Q_i is given by

$$\begin{aligned}
\operatorname{Hom}(Q_i, Q_i) &= C^*(\mathcal{Q}_i), \\
\operatorname{Hom}(Q_1, Q_2) &= \bigoplus_{k=1}^{r} C^*(\mathcal{N}^k)[m_k], \\
\operatorname{Hom}(Q_2, Q_1) &= \bigoplus_{k=1}^{r} C^*(\mathcal{N}^k, \partial \mathcal{N}^k)[-m_k].
\end{aligned}$$

Here integers m_k comes from the gradings of the Lagrangian submanifolds.

Now we apply this construction to the case when $Q_1 = S_0 \cong \mathbb{S}^3$, $Q_2 = S_1 \cong \mathbb{S}^3$, $B = B^1 \sqcup B^2 = \mathbb{S}^1 \sqcup \mathbb{S}^1$ and $\mathcal{N}^k \cong \mathbb{D}^2 \times \mathbb{S}^1$;

$$\begin{aligned}
\operatorname{Hom}(S_i, S_i) &\cong C^*(\mathbb{S}^3), \\
\operatorname{Hom}(S_1, S_0) &\cong C^*(\mathbb{D}^2 \times \mathbb{S}^1)[-1] \oplus C^*(\mathbb{D}^2 \times \mathbb{S}^1)[-1], \\
\operatorname{Hom}(S_0, S_1) &\cong C^*(\mathbb{D}^2 \times \mathbb{S}^1, \partial \mathbb{D}^2 \times \mathbb{S}^1)[1] \oplus C^*(\mathbb{D}^2 \times \mathbb{S}^1, \partial \mathbb{D}^2 \times \mathbb{S}^1)[1].
\end{aligned}$$

In this formula, cochains denote simplicial cochains with respect to a suitable triangulation. We have chosen the gradings on S_0 and S_1 in such a way that $m_1 = m_2 = -1$. We view each copy of $\mathbb{S}^3$ via its Hopf decomposition

$$\mathbb{S}^3 = \mathbb{D}^2 \times \mathbb{S}^1 \cup_{\mathbb{T}^2} \mathbb{D}^2 \times \mathbb{S}^1.$$

In our example, we can work with the smaller cellular model described below, which can be easily seen to be that we get a quasi-isomorphic dg-category if we choose to view each $\mathbb{D}^2$ as a two simplex Δ_2 and $\mathbb{S}^1$ as the union of three one simplices Δ_1 in the usual way.

We have the cochain models

$$C^*(\mathcal{N}^1) = \mathrm{span}_{\mathbb{C}} \left\{ \begin{tikzcd}[row sep=tiny, column sep=small] & x \arrow[ddr] & xy \arrow[dr] & \\ 1 & & & yz \\ & y & z & \end{tikzcd} \right\},$$

$$C^*(\mathcal{N}^2) = \mathrm{span}_{\mathbb{C}} \left\{ \begin{tikzcd}[row sep=tiny, column sep=small] & x \arrow[ddr] & xy \arrow[dr] & \\ 1 & & & yw \\ & y & w & \end{tikzcd} \right\},$$

for $C^*(\mathcal{N}^k) = C^*(\mathbb{D}^2 \times \mathbb{S}^1)$. Arrows show the differential in such a way that $\boldsymbol{d}(x) = z$ and similarly for other arrows. The cohomological degrees are given by $\deg(x) = \deg(y) = 1$ and $\deg(z) = \deg(w) = 2$. The elements w and z are the cellular cochains which are dual to the disc $\mathbb{D}^2$ as shown in Figure 7.1. We use the same symbols x and y for those generators which will be identified under the Hopf gluing. In the first copy of $\mathbb{D}^2 \times \mathbb{S}^1$, x is the cochain dual to the boundary of $\mathbb{D}^2$ and y is the cellular cochain dual to the $\mathbb{S}^1$ factor. The roles of these cochains are reversed in the second copy of $\mathbb{D}^2 \times \mathbb{S}^1$. The chain

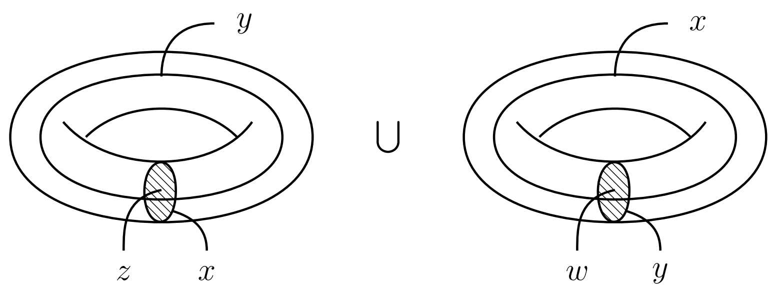

Figure 7.1: Hopf decomposition of $\mathbb{S}^3$

model for one copy of $C^*(\partial\mathbb{D}^2 \times \mathbb{S}^1)$ is given by

$$C^*(\partial\mathbb{D}^2 \times \mathbb{S}^1) = \mathrm{span}_{\mathbb{C}} \left\{ \begin{array}{ccc} & x & \\ 1 & & xy \\ & y & \end{array} \right\},$$

and similarly for the other copy. Accordingly, we have the chain model

$$C^*(\mathbb{S}^3) = \mathrm{span}_{\mathbb{C}} \left\{ \begin{tikzcd}[row sep=tiny, column sep=small] & & xy \arrow[dr] \arrow[ddr] & \\ & x \arrow[dr] & & yz \\ 1 & & z & \\ & y \arrow[dr] & & xw \\ & & w & \end{tikzcd} \right\},$$

for $C^*(\mathbb{S}^3)$ where $\boldsymbol{d}(xy) = yz + xw$. Using these basic models, we construct the chain level model for the category as follows: For $C^*(\mathbb{S}^3)$, we take the above cochain algebra. For the other groups, we preserve the letters corresponding to the above models to make clear the geometric origins of the generators and use $\overrightarrow{m}$ to denote a morphism in $\mathrm{Hom}(S_1, S_0)$ and $\overleftarrow{m}$ to denote a morphism in $\mathrm{Hom}(S_0, S_1)$. For $\mathrm{Hom}(S_0, S_1)$ we take as a basis:

$$\overleftarrow{z} \quad \overleftarrow{yz} \quad \overleftarrow{w} \quad \overleftarrow{xw}, \quad \boldsymbol{d} = 0.$$

We have that $\mathrm{Hom}(\mathrm{S}_1, \mathrm{S}_0)$ is the sum of two complexes,

$$\overrightarrow{u_1} \quad \overrightarrow{x_1} \quad \overrightarrow{y_1} \quad \overrightarrow{z_1} \quad \overrightarrow{xy_1} \quad \overrightarrow{yz_1},$$

$$\overrightarrow{u_2} \quad \overrightarrow{x_2} \quad \overrightarrow{y_2} \quad \overrightarrow{w_2} \quad \overrightarrow{xy_2} \quad \overrightarrow{xw_2}.$$

The differential is as in the model for $C^*(\mathbb{D}^2 \times \mathbb{S}^1)$. Compositions are the natural ones described in [Abo11, Section 2.1].

Lemma 7.2. *We have*

$$\begin{aligned}
\mathfrak{m}_3(\overrightarrow{u_1}, \overleftarrow{z}, \overrightarrow{u_2}) &= \overrightarrow{x_2}, \\
\mathfrak{m}_3(\overleftarrow{w}, \overrightarrow{u_1}, \overleftarrow{z}) &= \overleftarrow{yz}, \\
\mathfrak{m}_3(\overrightarrow{u_2}, \overleftarrow{w}, \overrightarrow{u_1}) &= -\overrightarrow{y_1}, \\
\mathfrak{m}_3(\overleftarrow{z}, \overrightarrow{u_2}, \overleftarrow{w}) &= -\overleftarrow{xw}.
\end{aligned}$$

The other m_3's are determined by the natural cyclic Calabi-Yau structure.

Proof. Given a dga $(V, \boldsymbol{d})$, we choose

- an injection $\mathrm{i} : \mathbb{H}^*(V) \to V$,
- a projection operator $\mathrm{P} : V \to \mathrm{i}(\mathbb{H}^*(V))$ such that $\mathrm{P}|_{\mathrm{i}(\mathbb{H}^*(V))} = \mathrm{id}_{\mathrm{i}(\mathbb{H}^*(V))}$, and
- a chain homotopy Q such that $\mathrm{id} - [\boldsymbol{d}, \mathrm{Q}] = \mathrm{P}$.

Then we define a series of linear maps

$$\lambda_n : V^{\otimes n} \to V$$

by setting

$$\lambda_2(v_1, v_2) = v_1 \cdot v_2$$

and inductively define

$$\begin{aligned}
\lambda_n(v_1, \cdots, v_n) := (-1)^{n-1}[\mathrm{Q}\lambda_{n-1}(v_1, \cdots, v_{n-1})]v_n - (-1)^{n \deg(v_1)} v_1[\mathrm{Q}\lambda_{n-1}(v_2, \cdots, v_n)] \\
- \sum_{k,l \geq 2} (-1)^{k+(l-1)(\deg(v_1)+\cdots+\deg(v_k))}[\mathrm{Q}\lambda_k(v_1, \cdots, v_k)][\mathrm{Q}\lambda_l(v_{k+1}, \cdots, v_n)].
\end{aligned}$$

for $n \geq 3$. Now the operators

$$\mathfrak{m}_n : \mathbb{H}^*(V)^{\otimes n} \to \mathbb{H}^*(V)$$

are defined by $\mathfrak{m}_n = \mathrm{P} \circ \lambda_n$.

Theorem 7.3 ([Mer99])**.** *The operators $\mathfrak{m}_n$ define the structure of an A_∞-algebra on $\mathbb{H}^*(V)$ quasi-isomorphic to $(V, \boldsymbol{d})$.*

We now compute Q in our setting. Since the differential vanishes on our model for $\mathrm{Hom}(S_0, S_1)$, the operator Q also vanishes on $\mathrm{Hom}(S_0, S_1)$. On $\mathrm{Hom}(S_0, S_0)$ and $\mathrm{Hom}(S_1, S_1)$, we can set

$$\mathrm{Q}(z) = x,\ \mathrm{Q}(w) = y,\ \mathrm{Q}(yz) = \frac{xy}{2},\ \mathrm{Q}(xw) = \frac{xy}{2}$$

and everything else to be zero. In the first summand of $\mathrm{Hom}(S_1, S_0)$, the operator Q is given by

$$\mathrm{Q}(\overrightarrow{z_1}) = \overrightarrow{x_1},\ \mathrm{Q}(\overrightarrow{yz_1}) = \overrightarrow{xy_1}.$$

A similar formula holds in the second summand.

To compute $\mathfrak{m}_3(\overrightarrow{u_1}, \overleftarrow{z}, \overrightarrow{u_2})$, we notice that $\overleftarrow{z} \cdot \overrightarrow{u_2} = 0$, so that

$$\begin{aligned}\mathfrak{m}_3(\overrightarrow{u_1}, \overleftarrow{z}, \overrightarrow{u_2}) &= (1 - [d, \mathrm{Q}])(\mathrm{Q}(\overrightarrow{u_1} \cdot \overleftarrow{z}) \cdot \overrightarrow{u_2}) \\ &= (1 - [d, \mathrm{Q}])(\mathrm{Q}(z) \cdot \overrightarrow{u_2}) \\ &= (1 - [d, \mathrm{Q}])(x \cdot \overrightarrow{u_2}) \\ &= (1 - [d, \mathrm{Q}])(\overrightarrow{x_2}) \\ &= \overrightarrow{x_2}.\end{aligned}$$

The other formulas can be calculated similarly, and Lemma 7.2 is proved. ◻

We also have the following result:

Lemma 7.4. *All A_∞-operations $\mathfrak{m}_n$ for $n \geq 4$ vanish.*

Proof. We argue using Merkulov's formula by showing that a higher product cannot have a non-trivial component in any of the cohomology classes. First we notice that no cohomology class can be written as a product of two cochains which are in the image of Q. To avoid repeated arguments, we will demonstrate why we cannot have

$$\mathfrak{m}_n(x_1, \cdots, x_n) = \overleftarrow{yz}.$$

All other cases can be addressed using the same type of arguments.

The only way to write $\overleftarrow{yz}$ as a non-trivial product is as $y \cdot \overleftarrow{z}$. Using Merkulov's formula and the above observation, we can assume without loss of generality that $\mathrm{Q}\lambda_{n-1}(x_1, \cdots, x_{n-1})$ has non-trivial coefficient in the basis vector y and that x_n has a non-trivial component in the basis vector $\overleftarrow{z}$. This would in turn imply that $\lambda_{n-1}(x_1, \cdots, x_{n-1})$ has non-trivial coefficient in w, which is not possible unless $n = 3$ because w cannot be written as the product of cochains, $s_1 s_2$, where either s_1 or s_2 is in the image of Q. ◻

Lemma 7.2 and Lemma 7.4 show that the A_∞-operations on $\mathcal{F}_0$ is identical to those for the endomorphism algebra of $\mathcal{O}_E \oplus \mathcal{O}_E(-1)$ described in Section 5, and Theorem 7.1 is proved. ◻

For the remainder of this section, we offer an alternative approach to Theorem 7.1, which stands on a conjecture that we were not able to verify, but hope is not outside the reach of current technology. Let $\mathcal{CO} : SH^0(Y^0) \to WF(L, L')$ be the open closed string map considered by Abouzaid [Abo10]. This makes the derived category of the wrapped Fukaya category of Y^0 into a triangulated category over $SH^0(Y^0)$, and allows one to talk about the *Serre functor* over $SH^0(Y^0)$ in the sense of [BK04, Definition 2.5]. A triangulated category over $SH^0(Y^0)$ is *Calabi-Yau* if the Serre functor over $SH^0(Y^0)$ is isomorphic to the shift functor $\bullet[n]$ for some $n \in \mathbb{Z}$.

Conjecture 7.5. *The morphism $\mathcal{CO}$ makes the derived category of the wrapped Fukaya category of Y^0 into a Calabi-Yau category over $SH^0(Y^0)$.*

Let $\mathcal{D}$ be a triangulated category and $\mathcal{N} \subset \mathcal{D}$ be a full triangulated subcategory. We will always assume that triangulated categories have enhancements in terms of dg categories [BK90] or A_∞-categories (cf. e.g. [Kel01]). The *right orthogonal* to $\mathcal{N}$ is the full subcategory $\mathcal{N}^\perp \subset \mathcal{D}$ consisting of objects M satisfying $\operatorname{Hom}(N, M) = 0$ for any $N \in \mathcal{N}$. The *left orthogonal* ${}^\perp\mathcal{N}$ is defined similarly. The subcategory $\mathcal{N}$ is said to be *right admissible* if the embedding $I : \mathcal{N} \hookrightarrow \mathcal{D}$ has a right adjoint functor $Q : \mathcal{D} \to \mathcal{N}$. Left admissibility is defined similarly as the existence of a left adjoint functor, and $\mathcal{N}$ is said to be *admissible* if it is both right and left admissible.

A subcategory $\mathcal{N}$ is right admissible if and only if for any $X \in \mathcal{D}$, there exists a distinguished triangle $N \to X \to M \to N[1]$ with $N \in \mathcal{N}$ and $M \in \mathcal{N}^\perp$. Such a triangle is unique up to isomorphism, and one has $Q(X) = N$ in this case. If $\mathcal{N}$ is right admissible, then the quotient category $\mathcal{D}/\mathcal{N}$ is equivalent to $\mathcal{N}^\perp$. Analogous statements also hold for left admissible subcategories. A sequence $(\mathcal{N}_1, \dots, \mathcal{N}_n)$ of admissible subcategories in a triangulated category $\mathcal{D}$ is called a *semiorthogonal decomposition* $\mathcal{N}_j \subset \mathcal{N}_i^\perp$ for any $1 \le j < i \le n$, and $\mathcal{N}_1, \dots, \mathcal{N}_n$ generates $\mathcal{D}$ as a triangulated category. A semiorthogonal decomposition will be denoted by

$$\mathcal{D} = \langle \mathcal{N}_1, \dots, \mathcal{N}_n \rangle .$$

An object E of $\mathcal{D}$ is *almost exceptional* if $\operatorname{Ext}^i(E, E) = 0$ for $i \neq 0$ and the algebra $A := \operatorname{Hom}(E, E)$ has finite homological dimension [BK04, Definition 2.1]. Let $\mathcal{E}$ be the smallest full subcategory of $\mathcal{D}$ containing E and closed under cones and direct summands. Then one has a semiorthogonal decomposition

$$\mathcal{D} = \left\langle \mathcal{E},\ \mathcal{E}^\perp \right\rangle$$

as in [Bon89, Theorem 3.2]; the object $N \in \mathcal{E}$ in the decomposition $N \to X \to M \to N[1]$ of an object $X \in \mathcal{D}$ is given by

$$N = \hom^\bullet(E, X) \overset{\mathbb{L}}{\otimes}_A E,$$

and $M \in \mathcal{E}^\perp$ is the mapping cone

$$M = \operatorname{Cone}\left(\hom^\bullet(E, X) \overset{\mathbb{L}}{\otimes}_A E \xrightarrow{\mathrm{ev}} X\right)$$

of the evaluation morphism.

An alternative proof of Theorem 7.1 assuming Conjecture 7.5. Corollary 6.4 and Corollary 5.5 show that $L_0 \oplus L_1$ is an almost exceptional object in $D^b\widetilde{\mathcal{W}}$, so that one has a semiorthogonal decomposition

$$D^b\widetilde{\mathcal{W}} = \left\langle D^b\mathcal{W}^\perp,\ D^b\mathcal{W} \right\rangle . \tag{7.2}$$

Conjecture 7.5 implies that (7.2) is an orthogonal decomposition;

$$D^b\widetilde{\mathcal{W}} = D^b\mathcal{W}^\perp \oplus D^b\mathcal{W}. \tag{7.3}$$

Since $\operatorname{End}(S_0) \cong H^0(S^3) \cong \mathbb{C}$, the objects S_0 is indecomposable and belongs to either $D^b\mathcal{W}$ or $D^b\mathcal{W}^\perp$. The latter is impossible since $\operatorname{Hom}(L_0, S_0) = \mathbb{C}$. This implies that $S_0 \in D^b\mathcal{W}$, and similarly for S_1. The fact

$$\operatorname{Hom}^i(L_j, S_k) = \begin{cases} \mathbb{C} & i = 0 \text{ and } j = k, \\ 0 & \text{otherwise} \end{cases}$$

shows that S_i goes to $\mathcal{O}_E$ and $\mathcal{O}_E(-1)$ under the derived equivalence

$$\mathcal{W} \cong D^b \operatorname{coh} X^0,$$

and Theorem 7.1 is proved. □

8 Immersed Lagrangian $S^2 \times S^1$

In the construction of the SYZ mirror in Section 2, we first considered the fibers away from the discriminant locus to obtain $\check{Y}_0$, and then added fibers above the discriminant locus 'by hand' to obtain a partial compactification X^0. It is reasonable to speculate that points on $X^0 \setminus \check{Y}_0$ can be identified with singular fibers $L_u := \rho^{-1}(u)$ of the original SYZ fibration $\rho : Y^0 \to \mathbb{R}^3$, where $u \in \Gamma$ is a point on the discriminant. In this section, we give Floer cohomology computations in favor of this speculation.

Set $a = -1$ and $b = -1/2$ for simplicity, and consider a point $(1, 0, \lambda) \in \Gamma$ on the discriminant (2.3) for the SYZ fibration (2.2). The fiber $L_{1,0,\lambda} := \rho^{-1}(1, 0, \lambda)$ is given by

$$L_{1,0,\lambda} = \left\{ (z, u_1, v_1, u_2, v_2) \in Y^0 \;\middle|\; |z| = 1,\ |u_1| = |v_1|,\ |u_2|^2 = |v_2|^2 + 2\lambda \right\}.$$

The Lagrangian $L_{1,0,\lambda}$ is an $S^1 \times S^1$-fibration over the unit circle on the z-plane shown in Figure 8.1 such that the first S^1-component degenerates to a point above $z = -1$. It follows that $L_{1,0,\lambda}$ is an immersed $S^2 \times S^1$, where S^2 is immersed in the (z, u_1, v_1)-direction in such a way that both the north pole and the south pole are mapped to $(z, u_1, v_1) = (-1, 0, 0)$, and S^1 is embedded in the (u_2, v_2)-direction. We equip $L_{1,0,\lambda}$ with the trivial spin structure and the grading such that the unique intersection point of $L_{1,0,\lambda}$ and L_0 has Maslov index zero. We consider the pair $(L_{1,0,\lambda}, \nabla_\alpha)$ of $L_{1,0,\lambda}$ and a flat $U(1)$ connection ∇_α on the trivial complex line bundle $L_{1,0,\lambda} \times \mathbb{C} \to L_{1,0,\lambda}$, where $\alpha \in U(1)$ is the holonomy along the generator of $\pi_1(S^2 \times S^1) \cong \mathbb{Z}$.

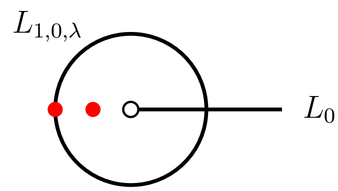

Figure 8.1: The immersed Lagrangian $L_{1,0,\lambda}$

Immersed Lagrangian Floer theory [Aka05, AJ10] gives a structure of an A_∞-algebra on $H^*((L_{1,0,\lambda}, \nabla_\alpha); \mathbb{C})$. The following lemma is a corollary of a result of Abouzaid, which can be found in [Sei, Lemma 11.6]:

Lemma 8.1. *The A_∞-algebra on $H^*((L_{1,0,\lambda}, \nabla_\alpha); \mathbb{C})$ is quasi-isomorphic to the exterior algebra $\Lambda^*(\mathbb{C}^3)$ equipped with the trivial differential.*

Proof. The immersed Lagrangian $L_{1,0,\lambda}$ is exact in the sense that the symplectic form ω vanishes on $\pi_2(Y^0, L_{1,0,\lambda})$, since it is homotopic to $S_0 \cup S_1$ appearing in Theorem 7.1. It follows that the A_∞-structure on $H^*((L_{1,0,\lambda}, \nabla_\alpha); \mathbb{C})$ does not depend on ∇_α and can be computed by the chain model of Abouzaid [Abo11].

The Abouzaid model for $L_{1,0,\lambda}$ is the tensor product of the Abouzaid model for an immersed Lagrangian S^2 and the cochain complex $C^*(S^1)$ for a circle. Since the Abouzaid model for the immersed S^2 is quasi-isomorphic to $\Lambda^*(\mathbb{C}^2)$ by [Sei, Lemma 11.6] and the cochain complex $C^*(S^1)$ is quasi-isomorphic to $\Lambda(\mathbb{C})$, their tensor product is quasi-isomorphic to $\Lambda(\mathbb{C}^2) \otimes \Lambda(\mathbb{C}) \cong \Lambda^*(\mathbb{C}^3)$. $\square$

Since the immersed Lagrangian $L_{1,0,\lambda}$ is not exact in the sense that the pull-back of the Liouville 1-form (i.e., the 1-form θ on Y^0 such that $\omega = d\theta$) represents a non-trivial cohomology class on $H^1(S^2 \times S^1)$, one has to work over the Novikov field

$$\Lambda_{\mathbb{C}} = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \,\middle|\, a_i \in \mathbb{C},\ \lim_{i\to\infty} \lambda_i = \infty \right\}$$

to define the Floer cohomology $\operatorname{Hom}(L_i, (L_{1,0,\lambda}, \nabla_\alpha))$ for $i = 0, 1$. We replace the mirror manifold X^0 with the variety $X^0_\Lambda := X^0 \otimes_{\mathbb{C}} \Lambda$ over Λ accordingly. Let $p_{\lambda,\alpha}$ be the point on X^0_Λ given by $(u, v, w_1, w_2, [x : y]) = (0, 0, -1, \alpha T^\lambda, [0 : 1])$.

Lemma 8.2. *The Floer cohomology* $\operatorname{Hom}(L_0, (L_{1,0,\lambda}, \nabla_\alpha)) \oplus \operatorname{Hom}(L_1, (L_{1,0,\lambda}, \nabla_\alpha))$ *as a module over* $\oplus_{i,j=0}^1 \operatorname{Hom}(L_i, L_j)$ *is isomorphic to the module* $\operatorname{Hom}(\mathcal{O}_{X^0_\Lambda} \oplus \mathcal{O}_{X^0_\Lambda}(1), \mathcal{O}_{p_{\lambda,\alpha}})$ *over* $\operatorname{End}(\mathcal{O}_{X^0_\Lambda} \oplus \mathcal{O}_{X^0_\Lambda}(1))$.

Proof. The intersection $L_0 \cap L_{1,0,\lambda}$ consists of a single point $q_\lambda = (1, \sqrt{2}, \sqrt{2}, u_2, v_2) \in Y^0$, where $(u_2, v_2) \in (\mathbb{R}^{>0})^2$ is defined by $u_2 v_2 = 3/2$ and $u_2^2 - v_2^2 = 2\lambda$. The A_∞-operation $\mathfrak{m}_2(q_\lambda, p_{a,i,j})$ in immersed Lagrangian Floer theory is given by the virtual count

$$\mathfrak{m}_2(q_\lambda, p_{a,i,j}) = \sum_{\phi_n(q) \in L_{1,0,\lambda} \cap \phi_n(L_0)} \sum_{\varphi \in [\overline{\mathcal{M}}(q)]^{\mathrm{virt}}} \operatorname{sgn}(\varphi) \operatorname{hol}(\nabla_\alpha, \varphi(\partial D^2)) T^{\int_D^2 \varphi^* \omega} \cdot q$$

over the moduli space $\overline{\mathcal{M}}(q)$ of holomorphic maps $\varphi : (D^2, (z_0, z_1, z_2)) \to Y_0$ from a disk with three marked points on the boundary satisfying

- $\varphi(z_0) = \phi_n(q)$, $\varphi(z_1) = p_{a,i_1,i_2}$, and $\varphi(z_2) = q_\lambda$,
- $\varphi(\partial_0 D^2) \subset \phi_n(L_0)$, $\varphi(\partial_1 D^2) \subset L_0$, and $\varphi(\partial_2 D^2) \subset L_{1,0,\lambda}$.

Here $\partial_i D^2 \subset \partial D^2$ is the interval between z_i and z_{i+1}. The sign $\operatorname{sgn}(\varphi) = \pm 1$ comes from the orientation on the moduli space (cf. [Sei08b, Section 11] and [FOOO09, Chapter 8]).

In immersed Lagrangian Floer theory, one counts only maps φ such that $\varphi|_{\partial_2 D^2} : \partial_2 D^2 \to L_{1,0,\lambda}$ lifts to a map $\partial_2 D^2 \to S^2 \times S^1$. One can see from Figure 8.2 that this is possible only for $a = 0$, so that

$$\mathfrak{m}_2(q_\lambda, p_{a,i,j}) = 0 \tag{8.1}$$

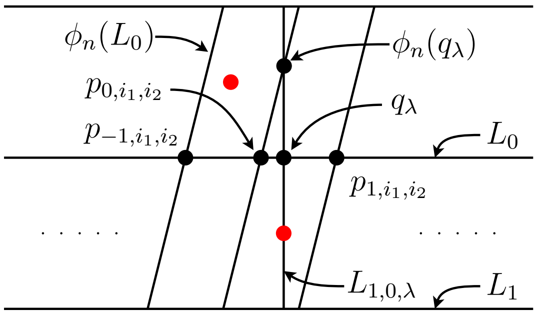

Figure 8.2: Intersections on the base

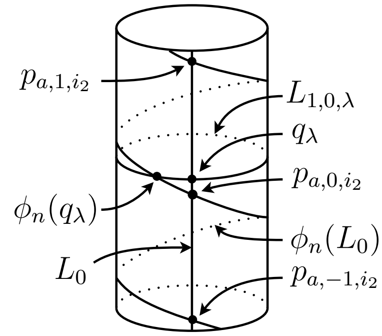

Figure 8.3: Intersections on the fiber

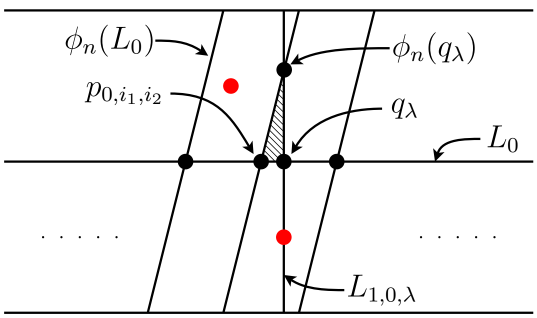

Figure 8.4: A triangle on the base

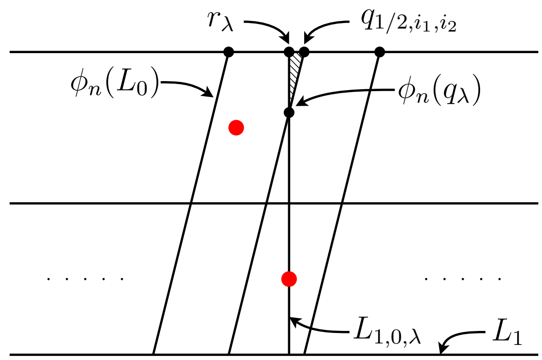

Figure 8.5: Another triangle on the base

for $a \neq 0$ and any $i, j \in \mathbb{Z}$. One can also see from Figures 8.2 and 8.3 that for each $(i,j) \in \mathbb{Z}^2$, there is a unique holomorphic triangle bounded by $\phi_n(L_0)$, L_0 and $L_{1,0,\lambda}$ two of whose vertices are $p_{0,i,j}$ and q_λ. The projection of this unique triangle to the z-plane is shown in Figure 8.4. The third vertex of this triangle is $\phi_n(q_\lambda)$, which is the unique intersection point of $\phi_n(L_0)$ and $L_{1,0,\lambda}$. Since L_0 and hence $\phi_n(L_0)$ carry the trivial spin structure and the trivial flat $U(1)$-bundle, the factor $\operatorname{sgn}(\varphi)\operatorname{hol}(\nabla_\alpha, \varphi(\partial D^2))$ comes entirely from the holonomy and the spin structure along $\partial_2 D^2$. Since $\varphi(\partial_2 D^2)$ wraps j times along the S^1-factor of $S^2 \times S^1$, one has $\operatorname{hol}(\nabla_\alpha, \varphi(\partial D^2)) = \alpha^j$. The trivial spin structure on $S^2 \times S^1$ induces the trivial spin structure on the S^1-factor, and the non-trivial spin structure on the equator of the S^2-factor (cf. e.g. [Mil63, Page 201]). This non-trivial spin structure contributes to the sign as $\operatorname{sgn}(\varphi) = (-1)^i$, cf. [Sei11, Section 9e]. This sign can also be determined by the ring isomorphism $\operatorname{Hom}_{\mathcal{W}_1}(L_0, L_0) \cong H^0(\mathcal{O}_{X^0})$ and the associativity. The area of $\varphi(D^2)$ is $j\lambda$ up to an additive overall constant which can be absorbed in the definition of the generator of the Floer cohomology, and one obtains

$$\mathfrak{m}_2(q_\lambda, p_{0,i,j}) = (-1)^i (\alpha_2 T^\lambda)^j q_\lambda$$

for any $i, j \in \mathbb{Z}$.

The intersection $L_1 \cap L_{1,0,\lambda}$ also consists of a single point r_λ, and one can show $\mathfrak{m}_2(r_\lambda, p'_{0,i,j}) = (-1)^i (\alpha_2 T^\lambda)^j r_\lambda$ for any $i, j \in \mathbb{Z}$ by exactly the same argument as above, where $p'_{a,i,j} \in \phi_n(L_1) \cap L_1$ are defined similarly as $p_{a,i,j} \in \phi_n(L_0) \cap L_0$. One can also see from Figure 8.5 and the same argument as above that the composition $\operatorname{Hom}(L_1, L_{1,0,\lambda}) \otimes$

$\mathrm{Hom}(L_0, L_1) \to \mathrm{Hom}(L_0, L_{1,0,\lambda})$ is given by

$$\mathfrak{m}_2(r_\lambda, q_{a,i_1,i_2}) = \begin{cases} (-1)^i (\alpha_2 T^\lambda)^j q_\lambda, & a = 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

The composition $\mathrm{Hom}(L_0, L_{1,0,\lambda}) \otimes \mathrm{Hom}(L_1, L_0) \to \mathrm{Hom}(L_1, L_{1,0,\lambda})$ can be computed similarly, and Lemma 8.2 is proved. □

9 Small toric Calabi-Yau 3-folds

Let Y^0 be the complete intersection in $\mathbb{C}^\times \times \mathbb{C}^4 = \operatorname{Spec} \mathbb{C}[z, z^{-1}, u_1, u_2, v_1, v_2]$ defined by

$$\begin{cases} u_1 v_1 = (z - a_1) \cdots (z - a_k), \\ u_2 v_2 = (z - b_1) \cdots (z - b_l). \end{cases} \tag{9.1}$$

The SYZ mirror for Y^0 is the complement

$$X^0 = X \setminus D$$

of a divisor D in a crepant resolution X of the toric variety whose fan polytope is shown in Figure 9.1.

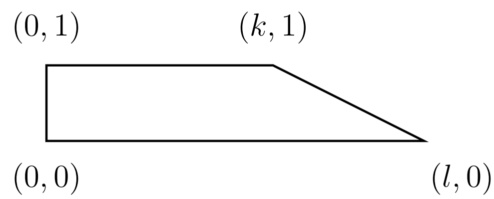

Figure 9.1: The fan polytope

Here, the fan polytope of a toric variety is the convex hull of the primitive generators of one-dimensional cones of the fan. The fan structure induces a polyhedral decomposition of the fan polytope, and the fan polytope equipped with this polyhedral decomposition is called a *toric diagram*.

The construction of the SYZ mirror of a complete intersection in [AAK, Section 11] shows that primitive generators of one-dimensional cones of the fan for X are given by $(0,1,0), (1,1,0), \dots, (k,1,0)$, and $(0,0,1), (1,0,1), \dots, (l,0,1)$.

One can map these points by the automorphism of $N \cong \mathbb{Z}^3$ sending (n_1, n_2, n_3) to (n_1, n_2, n_2+n_3), so that the fan polytope is the quadrangle on the hyperplane $\{(n_1, n_2, n_3) \in N_\mathbb{R} \mid n_3 = 1\}$ shown in Figure 9.1. The toric Calabi-Yau 3-fold X is *small* in the sense that the resolution $X \to Z = \operatorname{Spec} \mathbb{C}[X]$ does not have 2-dimensional fibers (in other words, the toric variety X has no compact toric divisors).

It is sometimes convenient to consider a stacky resolution $\mathcal{X}$ of Z, whose toric diagram is obtained by the triangulation of the fan polytope. Let consider the case when the fan for $\mathcal{X}$ has two 3-dimensional cones, one of which is generated by

$$v_1 = (0,0,1), \; v_3 = (0,1,0), \; \text{and } v_4 = (l,0,1),$$

and the other is generated by

$$v_2 = (k, 1, 0),\ v_3,\ \text{and } v_4.$$

Let $\varphi : \mathbb{Z}^4 \to N \cong \mathbb{Z}^3$ be the homomorphism sending the i-th standard basis $e_i \in \mathbb{Z}^4$ to $v_i \in N$ for $i = 1, \dots, 4$. Then the toric stack $\mathcal{X}$ is the quotient

$$\mathcal{X} := [(\mathbb{C}^4 \setminus \Sigma)/K],$$

where $\Sigma := \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 \mid x_1 = x_2 = 0\}$ is the Stanley-Reisner locus and

$$\begin{aligned} K &= \operatorname{Ker}\left(\varphi \otimes \mathbb{C}^\times : (\mathbb{C}^\times)^4 \to N_{\mathbb{C}^\times} \cong (\mathbb{C}^\times)^3\right) \\ &= \left\{(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in (\mathbb{C}^\times)^4 \;\middle|\; \alpha_2^k \alpha_4^l = \alpha_2\alpha_3 = \alpha_1\alpha_4 = 1\right\} \\ &\cong K_1 \times K_2. \end{aligned}$$

Here K_1 and K_2 are the subgroups of K given by

$$K_1 := \left\{(\alpha^k, \alpha^l, \alpha^{-l}, \alpha^{-k}) \in (\mathbb{C}^\times)^4 \;\middle|\; \alpha \in \mathbb{C}^\times\right\} \cong \mathbb{C}^\times, \tag{9.2}$$

$$K_2 := \left\{(\alpha, 1, 1, \alpha^{-1}) \in (\mathbb{C}^\times)^4 \;\middle|\; \alpha \in \mathbb{C}^\times,\ \alpha^g = 1\right\} \cong \mathbb{Z}/g\mathbb{Z} \tag{9.3}$$

where $g = \gcd(k, l)$. This shows that the toric stack $\mathcal{X}$ is the total space of the direct sum of two line bundles on the quotient stack $\mathbb{X} := \mathbb{X}_{k,l} := [\mathbb{P}(k', l')/(\mathbb{Z}/g\mathbb{Z})]$ of the weighted projective line $\mathbb{P}(k', l')$ for $k = gk'$ and $l = gl'$.

The toric stack $\mathbb{X}$ has the following description due to Geigle and Lenzing [GL87]: Let $S = \mathbb{C}[x_1, x_2]$ be the polynomial ring in two variables, graded by the abelian group $L = \mathbb{Z} \cdot \vec{x}_1 \oplus \mathbb{Z} \cdot \vec{x}_2/(k\vec{x}_1 - l\vec{x}_2)$ of rank one as $\deg(x_i) = \vec{x}_i$ for $i = 1, 2$. Then L can naturally be identified with the group $\operatorname{Hom}(K, \mathbb{C}^\times)$ of characters of K and one has

$$\mathbb{X} \cong [(\operatorname{Spec} S \setminus \mathbf{0})/K].$$

The Picard group of $\mathbb{X}$ can be identified with L, and the line bundle on $\mathbb{X}$ associated with an element $\vec{x} \in L$ will be denoted by $\mathcal{O}_{\mathbb{X}}(\vec{x})$. The canonical bundle of $\mathbb{X}$ is $\mathcal{O}_{\mathbb{X}}(\vec{\omega})$ for $\vec{\omega} = -\vec{x}_1 - \vec{x}_2$, and the stack $\mathcal{X}$ is the total space of the direct sum

$$\mathcal{X} \cong \mathcal{O}_{\mathbb{X}}(-\vec{x}_1) \oplus \mathcal{O}_{\mathbb{X}}(-\vec{x}_2)$$

of line bundles.

Choose a_i and b_j in such a way that all of them are on the unit circle and mutually distinct. Let $(\gamma_i)_{i=0}^{k+l-1}$ be a collection of strongly admissible paths, such that for any connected component of $S^1 \setminus \Delta$ for $S^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\}$ and $\Delta = \{a_1, \dots, a_k, b_1, \dots, b_l\}$, there is a unique i such that γ_i intersects it. Let $\mathcal{W}$ be the wrapped Fukaya category of Y consisting of $L_i := L_{\gamma_i}$ for $i = 0, \dots, k + l - 1$. Define the collection $(\mathcal{L}_i)_{i=0}^{k+l-1}$ of line bundles on $\mathcal{X}$ inductively by $\mathcal{L}_0 = \mathcal{O}_{\mathcal{X}}$ and

$$\mathcal{L}_i = \begin{cases} \mathcal{L}_i \otimes \pi^*(\mathcal{O}(\vec{x}_1)) & a_j \text{ for some } j \text{ lies between } \gamma_{i-1} \text{ and } \gamma_i, \\ \mathcal{L}_i \otimes \pi^*(\mathcal{O}(\vec{x}_2)) & b_j \text{ for some } j \text{ lies between } \gamma_{i-1} \text{ and } \gamma_i, \end{cases}$$

where $\pi : \mathcal{X} \to \mathbb{X}$ is the natural projection.

Then the proof of Theorem 1.2 can be easily adapted to prove the following:

Theorem 9.1. *There is an equivalence*

$$D^b\mathcal{W} \cong D^b \operatorname{coh} \mathcal{X}^0 \tag{9.4}$$

of triangulated categories sending L_i to $\mathcal{L}_i$ for $i = 0, \dots, k+l-1$.

The manifold Y^0 comes in a family $\mathcal{Y}^0 \to S$ over the configuration space

$$S = \left\{ (a_1, \dots, b_l) \in (\mathbb{C}^\times)^{k+l} \;\middle|\; \text{all the points } a_1, \dots, b_l \text{ are distinct} \right\} \big/ \mathfrak{S}_k \times \mathfrak{S}_l,$$

in such a way that $\mathcal{Y}^0$ is the submanifold of $\mathbb{C}^\times \times \mathbb{C}^4 \times S$ defined by the same equations (9.1) as Y^0. This family is locally trivial as a family of symplectic manifolds by Moser's theorem. The fundamental group $\mathscr{A}_{k,l} := \pi_1(S)$ is called the *mixed annular braid group*, and the symplectic monodromy gives a homomorphism

$$\phi : \mathscr{A}_{k,l} \to \pi_0(\operatorname{Symp}(Y^0, \omega))$$

to π_0 of the symplectomorphism group of Y^0.

Choose the point

$$(a_1, a_2, \dots, b_l) = (\zeta_{k+l}, \zeta_{k+l}^2, \dots, \zeta_{k+l}^{k+l}) \in S, \quad \zeta_{k+l} = \exp(2\pi\sqrt{-1}/(k+l))$$

as a base point, and let γ_i be the line segment from ζ_{k+l}^i to ζ_{k+l}^{i+1}. The half-twist T_i along γ_i interchanges ζ_{k+l}^i and ζ_{k+l}^{i+1}, and one can see that T_i for $i \neq k, k+l$ and $(T_i)^2$ for $i = k, k+l$ belong to $\mathscr{A}_{k,l}$. Let S_i be the compact Lagrangian submanifold of Y^0 defined by the path γ_i as in (7.1). The Lagrangian S_i is homeomorphic to S^3 if $i = k, k+l$ and $S^2 \times S^1$ otherwise.

- For $i \neq k, k+l$, we can identify a neighborhood of S_i with $T^*S^1 \times T^*S^2$, and the symplectic monodromy along T_i is given by the symplectic Dehn twist in the T^*S^2 factor.
- For $i = k, k+l$, the symplectic monodromy along $(T_i)^2$ is the Dehn twist along the Lagrangian S_i.

It is an interesting problem to explore the relation between this action and the mixed braid group action on the derived category of coherent sheaves on X^0 studied by Donovan and Segal [DS15].

A Lefschetz wrapped Floer cohomology

A.1 Liouville domains and wrapped Floer cohomology

An exact symplectic manifold with contact type boundary, or a *Liouville domain* for short, is a pair (M, θ) of a compact manifold M with boundary and a one-form θ on M called the *Liouville one-form* such that

- the two-form $\omega := d\theta$ is a symplectic form on M, and

- the *Liouville vector field* Z defined by $\iota_Z\omega = \theta$ points strictly outward along the boundary ∂M.

The restriction $\alpha := \theta|_{\partial M}$ of the Liouville one-form is a contact one-form on ∂M. The *Reeb vector field* R on ∂M is defined by $R \in \operatorname{Ker} \alpha$ and $\alpha(R) = 1$. The *symplectic completion* $\hat{M}$ of M is obtained by gluing the positive part

$$(\partial M \times [1, \infty), d(r\alpha))$$

of the symplectization of ∂M onto M;

$$\hat{M} := M \cup_{\partial M} (\partial M \times [1, \infty)) .$$

Let L be a compact Lagrangian submanifold L of M such that

- $\theta|_L$ is exact; $\theta|_L = df$,
- L intersects ∂M transversally, and
- $\theta|_L$ vanishes to infinite order along the boundary $\partial L := L \cap \partial M$.

In this setting, the completion

$$\hat{L} := L \cup_{\partial L} (\partial L \times [1, \infty)) ,$$

of L is a Lagrangian submanifold of $\hat{M}$.

A Hamiltonian function $H \in C^\infty(\hat{M})$ is *admissible* if there are constants $K > 0$, $a > 0$, and b such that

$$H(x, r) = ar + b, \qquad \forall (x, r) \in \partial M \times [K, \infty) \subset \hat{M}. \tag{A.1}$$

The constant a is called the *slope* of H. An almost complex structure J on $\hat{M}$ is *admissible* if it is of contact type outside a compact set;

$$dr = \theta \circ J, \qquad \forall (x, r) \in \partial M \times [K, \infty) \subset \hat{M}. \tag{A.2}$$

A *Reeb chord* of length w is a trajectory $x : [0, w] \to \partial M$ of the flow along R such that $x(0) \in L$ and $x(w) \in L$. An *integer Reeb chord* is a Reeb chord of integer length. A *Hamiltonian chord* is defined similarly as a trajectory of the Hamiltonian vector field starting and ending at L. If we write the time t Hamiltonian flow as $\varphi_t : \hat{M} \to \hat{M}$, then a Hamiltonian chord of length w corresponds to an intersection point $p \in L \cap \varphi_w(L)$. A Hamiltonian chord is *non-degenerate* if the corresponding intersection is transversal.

Fix an admissible Hamiltonian H of unit slope. If $\dim M \geq 4$, then by perturbing L by an exact symplectic isotopy if necessary, we may assume [AS10, Lemmas 8.1 and 8.2] that

- there are no integer Reeb chords,
- all integer Hamiltonian chords are non-degenerate, and

- no point of L is both a starting point of an integer Hamiltonian chord and an endpoint of an integer Hamiltonian chord, which may or may not be the same chord.

For an integer w, the set of Hamiltonian chords of length w will be denoted by $\mathcal{X}_w$. The set $\mathcal{X}_w$ is finite since all the integer Hamiltonian chords are non-degenerate. The *action* of $x \in \mathcal{X}_w$ is defined by

$$A_{wH}(x) = \int_0^1 \left(x^*\theta - wH(x(t))dt \right) + H(x(1)) - H(x(0)).$$

The Floer complex is defined as the direct sum

$$CF^*(\hat{L}; wH) := \bigoplus_{x \in \mathcal{X}_w} \mathbb{C}[x],$$

equipped with the grading coming from the Maslov index. The differential δ on $CF(\hat{L}; wH)$ is given by counting solutions to Floer's equation

$$\begin{cases} u : \mathbb{R} \times [0,1] \to \hat{M}, \\ u(\mathbb{R} \times \{0,1\}) \subset \hat{L}, \\ \lim_{s \to \pm\infty} u(s, \cdot) = x_\pm(\cdot), \\ \partial_s u + J_t(\partial_t u - wX_H) = 0 \end{cases}$$

up to the $\mathbb{R}$-action by translation in the s-direction. Here, one has to choose a t-dependent almost complex structure to achieve transversality in Floer's equation. The conditions (A.1) and (A.2) gives the maximum principle for u, which ensures the compactness of the moduli space. The continuation map

$$\varphi : CF^*(\hat{L}; wH) \to CF^*(\hat{L}; (w+1)H)$$

is defined as the sum

$$\varphi(x_+) = \sum_{u \in \mathcal{M}(x_-, x_+)} \pm x_-$$

over the set $\mathcal{M}(x_-, x_+)$ of solutions to the continuation equation

$$\begin{cases} u : \mathbb{R} \times [0,1] \to \hat{M}, \\ u(\mathbb{R} \times \{0,1\}) \subset \hat{L}, \\ \lim_{s \to \pm\infty} u(s, \cdot) = x_\pm(\cdot), \\ \partial_s u + J_{s,t}(\partial_t u - X_s) = 0 \end{cases}$$

where $J_{s,t}$ is an (s,t)-dependent almost complex structure and X_s is the Hamiltonian vector field of an s-dependent Hamiltonian H_s which coincides with $(w+1)H$ and wH when $s \ll 0$ and $s \gg 0$ respectively. A standard argument in Floer theory shows that the continuation map is a chain map, which is independent of the choice of almost complex structures up to chain homotopy. The *wrapped Floer cohomology* is defined by

$$HW(\hat{L}) := \varinjlim_w HF(\hat{L}; wH).$$

The continuation map is defined more generally for a family H_s of admissible Hamiltonians with monotonically decreasing slope, and satisfies the transitive law; if one divides a family $\{H_s\}_s$ from $H_{-\infty}$ to H_∞ smoothly into two, then the diagram

$$\begin{tikzcd} HF(\hat{L}; H_\infty) \arrow[rr, "\varphi"] \arrow[dr, "\varphi_+"'] & & HF(\hat{L}; H_{-\infty}) \\ & HF(\hat{L}; H_0) \arrow[ur, "\varphi_-"'] & \end{tikzcd}$$

consisting of continuation maps commutes. We say that a family $\{H_p\}_p$ of admissible Hamiltonians is *cofinal* if the slope of H_p goes to infinity as p goes to infinity. The wrapped Floer cohomology can also be defined as the limit of Floer cohomologies with respect to any cofinal family of non-degenerate admissible Hamiltonians.

The *triangle product* on wrapped Floer cohomology is defined by counting solutions of the inhomogeneous Cauchy-Riemann equation

$$\begin{cases} u : S \to \hat{M}, \\ u(\partial S) \subset \hat{L}, \\ \lim_{s \to \pm\infty} u(\epsilon^k(s, \cdot)) = x^k(\cdot), \qquad k = 0, 1, 2, \\ (du_z - X_{u(z)} \otimes \gamma_z) \circ j + J_{z,u(z)} \circ (du_z - X_{u(z)} \otimes \gamma_z) = 0, \end{cases}$$

where

- $w^k \in \mathbb{N}$ and $x^k \in \mathcal{X}_{w^k}$ for $k = 0, 1, 2$,
- $S = D^2 \setminus \{\zeta^0, \zeta^1, \zeta^2\}$ is a disk with three points on the boundary removed,
- $\epsilon^0 : (-\infty, 0] \times [0, 1] \to S$ and $\epsilon^{1,2} : [0, \infty) \times [0, 1] \to S$ are *strip-like ends*,
- j is the complex structure on S,
- $\{J_z\}_{z \in S}$ is a family of almost complex structures on $\hat{M}$,
- γ is a one-form on S satisfying
  - $\gamma|_{\partial S} = 0$,
  - $d\gamma < 0$ on S,
  - $d\gamma = 0$ in a neighborhood of ∂S,
  - $(\epsilon^k)^* \gamma = w^k dt$ on the strip-like ends, and
- $X \otimes \gamma \in \operatorname{Hom}(TS, u^* T\hat{M})$ is obtained by composing $\gamma \in C^\infty(T^*S)$ with $u^* X \in C^\infty(u^* TM)$.

A more careful discussion on the wrapped Floer cohomology can be found in [Rit13]. To define higher A_∞-operations, one takes the homotopy colimit of the Floer cochain complex instead of the colimit of the cohomology, and use moduli spaces of *stable popsicle maps* [AS10].

A.2 Lefschetz fibrations and wrapped Floer cohomology

This section follows McLean [McL09] closely. An *exact Lefschetz fibration* is a proper map $\pi : E \to S$ from a compact manifold E with corners to a compact surface S with boundary satisfying the following:

- ∂E consists of the *vertical boundary* $\partial_v E := \pi^{-1}(\partial S)$ and the *horizontal boundary* $\partial_h E := \partial E \setminus \partial_v E$ meeting in a codimension 2 corner.

- π is a C^∞-map with finitely many critical points $E^{\mathrm{crit}} \subset E$ with critical values $S^{\mathrm{crit}} \subset S$. Every critical point is non-degenerate in the sense that the Hessian is non-degenerate. Different critical points have distinct critical values.

- E is equipped with a one-form Θ such that $\Omega = d\Theta$ is a symplectic form on $E_s \setminus E^{\mathrm{crit}}$ for every $s \in S$, where $E_s := \pi^{-1}(s)$ is the fiber of π.

- There is a neighborhood N of $\partial_h E$ such that $\pi|_N : N \to S$ is a product fibration $S \times U$, where U is a neighborhood of ∂F in F. We require that $\Theta|_N$ is a pull-back from the second factor.

- There are integrable complex structures J_0 (resp. j_0) defined on a neighborhood of E^{crit} (resp. S^{crit}) such that π is (J_0, j_0)-holomorphic near E^{crit}.

- Ω is a Kähler form for J_0 near E^{crit}.

There is a natural connection for π given by the horizontal distribution defined as the Ω-orthogonal to the tangent space to the fiber. Parallel transport with respect to this connection gives exact symplectomorphisms between smooth fibers of π. We write a smooth fiber of π considered as an abstract exact symplectic manifold as F.

We say that E is a *compact convex Lefschetz fibration* if $(F, \Theta|_F)$ is a Liouville domain. Choose a Liouville one-form θ_S on the base S. Then there is a constant $K > 0$ such that for all $k \geq K$, one has

- $\omega := \Omega + k\pi^*\omega_S$ is a symplectic form

- the ω-dual λ of $\theta := \Theta + k\pi^*\theta_S$ is transverse to ∂E and pointing outward

by [McL09, Theorem 2.15]. One can complete a compact convex Lefschetz fibration to a *complete convex Lefschetz fibration* $\pi : \hat{E} \to \hat{S}$ in a natural way, whose base is the completion $\hat{S}$ of the base S and whose fiber is a completion $\hat{F}$ of the fiber F. The completion $\hat{E}$ can be partitioned into

- $E \subset \hat{E}$,

- $A := F_e \times \hat{S}$ where F_e is the cylindrical end of $\hat{F}$, and

- $B := \hat{E} \setminus (A \cup E)$

as in [McL09, Figure 1].

The completion $\hat{E}$ is isomorphic to the completion $\hat{M}$ of a Liouville domain M, obtained by smoothing out the corner of E. We write the radial coordinates for cylindrical ends of $\hat{E}$, $\hat{S}$ and $\hat{F}$ as r, r_S and r_F. There exists a positive constant ϖ such that $r_S \leq \varpi r$ and $r_F \leq \varpi r$ by [McL09, Lemma 5.7].

A map $H : \hat{E} \to \mathbb{R}$ is a *Lefschetz admissible Hamiltonian* if $H|_A = \pi^* H_S + \pi_1^* H_F$ outside some large compact set [McL09, Definition 2.21]. Here H_S and H_F are admissible Hamiltonians on $\hat{S}$ and $\hat{F}$ such that $H_S = 0$ on $S \subset \hat{S}$ and $H_F = 0$ on $F \subset \hat{F}$ respectively [McL09, Page 1905], and $\pi_1 : A = F_e \times \hat{S} \to F_e$ is the first projection.

Let $\gamma : [0,1] \to S$ be a path on the base such that $\gamma((0,1)) \subset S \setminus S^{\mathrm{crit}}$. Recall that a *Lagrangian submanifold fibered over* γ is a Lagrangian submanifold L of E obtained as the trajectory of the parallel transport along γ of a Lagrangian submanifold L_s in a fiber $E_s = \pi^{-1}(s)$. We assume that L is exact, L intersects ∂E transversally, and $\theta|_L$ vanishes to infinite order along ∂L. If an endpoint of γ is in the interior of S, then it must be a critical value of π. If exactly one endpoint of γ is in the interior of S, then L is a Lefschetz thimble. If both endpoints of γ are in the interior of S, then L is a Lagrangian sphere. The Lagrangian $L \subset M$ can be completed to a Lagrangian $\hat{L} \subset \hat{M}$ by first taking the completion $\hat{L}_s := L_s \cup_{\partial L_s} ([1,\infty) \times \partial L_s) \subset \hat{E}_s$ in the fiber direction and then taking its parallel transport along $\hat{\gamma} = \gamma \cup_{\partial\gamma} ([1,\infty) \times \partial\gamma) \subset \hat{S}$. Since $\hat{L} \cap A$ is the product $(\hat{L}_s \setminus L_s) \times \hat{\gamma}$ and $\hat{L} \cap B$ is the product $L_s \times (\hat{\gamma} \setminus \gamma)$, one has a maximum principle which applies to Lefschetz admissible H:

Lemma A.1. *For any Floer trajectory $u : D \to \hat{E}$, the functions $r_S \circ u$ and $r_F \circ u$ do not admit local maxima for large r_S and r_F.*

This allows one to define the Floer differential and the continuation map, which gives the *Lefschetz wrapped Floer cohomology*

$$HW_l^*(\hat{L}) := \varinjlim_w HF^*(\hat{L}; wH).$$

The Lefschetz wrapped Floer cohomology $HW_l^*(\hat{L})$ does not depend on the choice of a Lefschetz admissible Hamiltonian just as in the case of the ordinary wrapped Floer cohomology.

Theorem A.2. *One has an isomorphism*

$$HW^*(\hat{L}) \cong HW_l^*(\hat{L}) \tag{A.3}$$

of graded rings.

The isomorphism (A.3) is obtained by

$$\begin{aligned} HW^*(\hat{L}) &\cong \varinjlim_p HF^*(\hat{L}; \varrho_p) && \text{(A.4)}\\ &\cong \varinjlim_p HF^*_{[-\epsilon,\infty)}(\hat{L}; \varrho_p) && \text{(A.5)}\\ &\cong \varinjlim_p HF^*_{[-\epsilon,\infty)}(\hat{L}; K_p) && \text{(A.6)}\\ &\cong \varinjlim_p HF^*_{[-\epsilon,\infty)}(\hat{L}; G_p) && \text{(A.7)}\\ &\cong \varinjlim_p HF^*(\hat{L}; G_p) && \text{(A.8)}\\ &\cong HW_l^*(\hat{L}), && \text{(A.9)} \end{aligned}$$

which is an adaptation of the proof of [McL09, Theorem 2.22]. Here ϱ_p is a Hamiltonian function on $\hat{M}$ satisfying

(i) $\varrho_p < 0$ on M,

(ii) ϱ_p goes to zero in the C^2 norm on M as p goes to infinity,

(iii) ϱ_p depends only on the radial coordinate on the cylindrical end;

$$\varrho_p(x, r) = h_p(r), \qquad \forall (x, r) \in \partial M \times [1, \infty) \subset \hat{M}.$$

(iv) $h_p'(r) \geq 0$ and $h_p''(r) \geq 0$ for all $r \in [1, \infty)$,

(v) $h_p'(r) = p$ for $r \in [2, \infty)$, and

(vi) for any $K > 0$ and any $r \in (1, \infty)$, there is an integer N such that

$$r h_p'(r) - h_p(r) > K, \qquad \forall p > N.$$

The sequence $\{\varrho_p\}_p$ is a cofinal family of admissible Hamiltonians, so that the isomorphism (A.4) comes from the definition of the wrapped Floer cohomology.

The condition (ii) implies that for any $\epsilon > 0$, the action of any Hamiltonian chord of ϱ_p in M is greater than $-\epsilon$ for sufficiently large p. The condition (iii) implies that the Hamiltonian vector field of ϱ_p in the cylindrical end is $h_p'(r)$ times the Reeb vector field on ∂M. It follows that Hamiltonian chords of length one are in one-to-one correspondence with Reeb chords of length $h_p'(r)$, and the action of a Hamiltonian chord $(x, r) : [0, 1] \to M \times [1, \infty)$ is given by

$$\begin{aligned} A_{\varrho_p}(x, r) &= \int_0^1 \big(x^*\theta - \varrho(x, r))dt\big) + f(x(1)) - f(x(0)), \\ &= r h_p'(r) - h_p(r) + f(x(1)) - f(x(0)). \end{aligned}$$

The condition $\theta|_{\partial L} = 0$ implies $\theta|_{\hat{L} \setminus L} = 0$, so that the primitive function $f(x) = \int^x \theta$ is constant on each connected component of $\hat{L} \setminus L$ and hence bounded. Then the condition (vi) shows that the actions of Hamiltonian chords on the cylindrical end are positive for sufficiently large p. As a result, one obtains the isomorphism (A.5), where $HF^*_{[-\epsilon,\infty)}(\hat{L}; \varrho_p)$ is the subgroup of $HF^*(\hat{L}; \varrho_p)$ generated by chords of action greater than or equal to ϵ.

The construction of K_p from ϱ_p proceeds in two steps [Her00, McL09]: First one modifies ϱ_p to a Hamiltonian ς_p which is constant outside a large compact set κ while only adding chords of action less than $-\epsilon$. Then one adds to ς_p a Lefschetz admissible Hamiltonian L_p, which is zero in the region κ but has action bounded above, so that Hamiltonian chords of $K_p := L_p + \varsigma_p$ outside κ have action less than $-\epsilon$. For a suitable choice of a family of admissible almost complex structures,

- there is a bijection between Hamiltonian chords of K_p of action greater than $-\epsilon$ and Hamiltonian chords of ϱ_p, and

- this bijection inducing an isomorphism of the moduli spaces of Floer trajectories.

This gives the isomorphism (A.6). The sequence $\{G_p\}_p$ is a cofinal family of Lefschetz admissible Hamiltonians such that

- there are sequences p_i and q_i of positive integers such that

$$K_{p_i} \le G_{q_i} \le K_{p_{i+1}}$$

  for all i, and

- all Hamiltonian chords of G_p have action greater than $-\epsilon$.

This induces the isomorphisms (A.7) and (A.8). The isomorphism (A.9) comes from the cofinality of $\{G_p\}_p$ and Theorem A.2 is proved.

A.3 Fiber products of Lefschetz fibrations

Let $\pi_1 : E_1 \to S$ and $\pi_2 : E_2 \to S$ be exact Lefschetz fibrations and consider the fiber product

$$\begin{tikzcd} & E := E_1 \times_S E_2 \arrow[dl, "\tau_1"'] \arrow[dd, "\pi"] \arrow[dr, "\tau_2"] & \\ E_1 \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & S & \end{tikzcd} .$$

By smoothing corners, we obtain a Liouville domain M with a Liouville one-form $\theta = \tau_1^*\Theta_1 + \tau_2^*\Theta_2 + k\pi^*\theta_S$ for sufficiently large k whose completion $\hat{M}$ is symplectomorphic to $\hat{E} := \hat{E}_1 \times_{\hat{S}} \hat{E}_2$. We have fiberwise cylindrical coordinates r_{F_i}, $i = 1, 2$ and a cylindrical coordinate r_S on the base. We say that a Hamiltonian $H : E \to \mathbb{R}$ is *fibered admissible* if

$$H = \pi^* H_S + \tau_1^* H_{F_1} + \tau_2^* H_{F_2}$$

where

- H_S is an admissible Hamiltonian on $\hat{S}$, and
- H_{F_i} is a Hamiltonian on $\hat{E}_i$ which is
  - zero on $E_i \cup B_i$, and
  - a pull-back of an admissible Hamiltonian of $(F_i)_e$ on $A_i := (F_i)_e \times \hat{S}$.

Lagrangian submanifolds L_i of E_i fibered over a common path $\gamma : [0, 1] \to S$ gives a Lagrangian submanifold $L := L_1 \times_\gamma L_2$ of E, which can be completed to a Lagrangian submanifold $\hat{L}$ of $\hat{E}$. Although $\pi : E \to S$ is not a Lefschetz fibration but a Bott-Morse analog of a Lefschetz fibration, the proof of Theorem A.2 can be adapted in a straightforward way to prove the following:

Theorem A.3. *One has an isomorphism*

$$HW^*(\hat{L}) \cong \varinjlim_p HF^*(\hat{L}; pH) \tag{A.10}$$

of rings.

The right hand side does not depend on the choice of a fibered admissible Hamiltonian H, or a cofinal family $\{H_p\}_p$ of fibered admissible Hamiltonians in general. One starts with a cofinal family $\{\varrho_p\}_p$ of admissible Hamiltonians, truncate outside a large compact set κ to obtain ς_p, then adds a fibered admissible Hamiltonian L_p supported outside of κ to obtain $K_p = \varsigma_p + L_p$. This process can be performed without changing chords with actions greater than $-\epsilon$, and one obtains the isomorphism (A.10).

A.4 Symplectic cohomology and the bulk-boundary map

In view of McLean's work, it is also natural to discuss the implication of the calculations in this paper for symplectic cohomology. Our treatment here is less detailed because the discussion which follows is complementary to our main topic.

Theorem A.4. *Let L be an admissible Lagrangian which is also a section of the SYZ fibration for the conifold. Then we have an isomorphism of rings $SH^0(\hat{M}) \to WF(L)$.*

Proof. Consider a fibered admissible Hamiltonian H as in the main part of this paper and assume that the discriminant points are generically positioned away from L inside of M. For an appropriate choice of H as above, Hamiltonian orbits are precisely:

- T^3 submanifolds on $\hat{M}$, which fiber over circles in the base
- one-dimensional tori of orbits living in the fibers over the discriminant locus

Standard Morse-Bott theory allows one to find a perturbation which has exactly 2^m orbits corresponding to generators of $H^*(T^m)$ for each submanifold of Hamiltonian orbits. For our purposes, it will be sufficient to consider the T^3 submanifolds which fiber over circles in the base because the orbits of the second type have grading at least 2. We will be interested in the $SH^0(\hat{M})$, which is generated by the cochains arising from $H^0(T^3)$.

A priori there could be a differential

$$\partial : SH^0(\hat{M}) \to SH^1(\hat{M})$$

However, curves contributing to this differential would necessarily preserve the free homotopy class of the projection of the chord to $\mathbb{C}^*$. An energy estimate similar to that in [Sei, Theorem 18.10] shows that curves connecting Morse-Bott submanifolds which live in the same fiber must live entirely within the fiber and hence there are no such differentials.

The essential geometric idea which underlies our theorem is that generators of wrapped Floer homology between L and $\phi_H(L)$ consist of a single Hamiltonian chord in each of the T^3 submanifolds, which we may think of as being geometrically the same as the corresponding generator in $SH^0(\hat{M})$. To turn this observation into a precise statement, we note that Abouzaid has defined a closed-open unital ring homomorphism

$$\mathcal{CO} : SH^*(\hat{M}) \to WF(L).$$

This map is defined by counting solutions to a perturbed J-holomorphic curve equation with one interior puncture which is required to be asymptotic to our Hamiltonian orbit.

In our setting this map can be completely calculated. More precisely the map sends the class in $H^0(T^3)$ to the unique Hamiltonian chord of L in each submanifold worth of orbits. The non-trivial component of our map corresponds to "local" curves, e.g. curves which do not escape some fixed neighborhood of the orbits. Using elementary Morse-Bott analysis, one can show that these correspond to the classical intersection $T^3 \cap L$. As before, there can be no non-trivial curves connecting different Morse-Bott submanifolds. In particular, this map induces an isomorphism

$$SH^0(\hat{M}) \to WF(L).$$

□

Our computations in this paper therefore allow us to compute $SH^0(\hat{M})$ as well:

Corollary A.5. $SH^0(\hat{M}) \cong \mathbb{C}[u, v, w_1, w_1^{-1}, w_2, w_2^{-1}]/(uv = (1 + w_1)(1 + w_2))$

Remark A.6. While finishing this paper, we noticed that Pascaleff [Pas] has very recently proven a similar theorem in his study of log Calabi-Yau surfaces. While our notion of Lagrangian section comes from an SYZ fibration, Pascaleff considers Lagrangian sections of an SYZ fibration defined only in a neighborhood of the compactifying divisor for log Calabi-Yau surfaces. It would be interesting to study the relationship between these approaches in more detail.

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