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Advances in Mathematics 219 (2008) 554–567
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Spectral property of the Bernoulli convolutions ☆

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Received 2 March 2007; accepted 6 May 2008

Available online 1 July 2008

Communicated by Kenneth Falconer

Abstract

For $0 < \rho < 1$, let μ_ρ be the Bernoulli convolution associated with ρ. Jorgensen and Pedersen [P. Jorgensen, S. Pedersen, Dense analytic subspaces in fractal L^2-spaces, J. Anal. Math. 75 (1998) 185–228] proved that if $\rho = 1/q$ where q is an even integer, then $L^2(\mu_\rho)$ has an exponential orthonormal basis. We show that for any $0 < \rho < 1$, $L^2(\mu_\rho)$ contains an infinite orthonormal set of exponential functions if and only if ρ is the nth root of a fraction p/q where p is an odd integer and q is an even integer.

MSC: primary 42C05, 42A65; secondary 28A78, 28A80

Keywords: Bernoulli convolution; Minimal polynomial; Orthogonality; Spectral set; Self-similarity

1. Introduction

Let μ be a Borel probability measure in $\mathbb{R}^d$. We say that μ is a *spectral measure* if there exists a discrete set Λ such that $E_\Lambda = \{e^{2\pi i\lambda(\cdot)}\colon \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. In this case we call Λ a *spectrum* of μ, and (μ, Λ) a *spectral pair*. Since (μ, Λ) is a spectral pair if and only if for any fixed $t \in \mathbb{R}^d$, $(\mu, t + \Lambda)$ is also a spectral pair, for simplicity we assume that $0 \in \Lambda$.

Spectral measure was first studied by Jorgensen and Pedersen [2], they showed that (see also [8] for a simplified proof):

☆ The research is supported in part by the HKRGC Grant.
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0001-8708/$ – see front matter
doi:10.1016/j.aim.2008.05.004

Theorem 1.1. *Let $k > 2$ be a positive integer. Then the $1/k$-Cantor measure μ on $\mathbb{R}$ is a spectral measure if and only if k is even. Moreover if k is odd, then any orthonormal set E_Λ in $L^2(\mu)$ has at most two elements.*

Recall that the $1/k$-Cantor measure is a special case of the Bernoulli convolution $\mu = \mu_\rho$ with parameter $0 < \rho < 1$ that satisfies

$$\mu(A) = \frac{1}{2}\mu\big(\rho^{-1}A - 1\big) + \frac{1}{2}\mu\big(\rho^{-1}A + 1\big) \tag{1.1}$$

for any Borel subset $A \subseteq \mathbb{R}$. This class of measures has been studied in detail in literature for the case $0 < \rho < 1/2$ (Cantor-type measure) and for $1/2 < \rho < 1$ (overlapping case in the context of iterated functions system, see e.g., [5–7]). Theorem 1.1 was investigated by Łaba and Wang in more detail [3] and for the general Borel measures [4]. In [9,10] Strichartz considered the "mock" Fourier series and Fourier transforms of such spectral Cantor measures analogous to the classical case.

In this note we consider the Bernoulli convolution in (1.1) with parameter $0 < \rho < 1$. We prove

Theorem 1.2. *Let μ be the ρ-Bernoulli convolution defined by* (1.1). *Then $L^2(\mu)$ contains an infinite orthonormal set E_Λ of exponential functions if and only if ρ is the nth root of a fraction p/q where p is odd and q is even.*

The sufficiency of the theorem follows from Theorems 1.1 and 4.4. The main proof is on the necessity. First we note that the Fourier transform of μ is $\widehat{\mu}(t) = \prod_{j=1}^{\infty} \cos(2\pi\rho^j t)$. Let $\beta = \rho^{-1}$. It is easy to check that $\widehat{\mu}(t) = 0$ if and only if $t = a\beta^j/4$ for some positive integer j and some odd integer a. Let E_Λ be an orthogonal set of exponential functions and let $0 \in \Lambda$. It follows easily that $\widehat{\mu}(\lambda) = 0$, $\lambda \in \Lambda$, and the orthogonal property is reduced to the algebraic equation

$$c_1\beta^k - c_2\beta^j = c_3, \tag{1.2}$$

for some k, j and for some odd integers c_1, c_2, and c_3 (Lemma 2.2). This implies that β must be an algebraic number satisfying some polynomials with *odd* integral coefficients as in (1.2), which enables us to reduce them to polynomials of the form

$$f(x) = x^k + x^j + 1 \quad \text{where } f(x) \in \mathbb{Z}_2[x], \tag{1.3}$$

and $\mathbb{Z}_2$ is the residue class of $\mathbb{Z}$ modulo 2.

We prove the following assertion which yields the necessity of Theorem 1.2:

If either (i) $\rho = (\frac{p}{q})^{1/n}$ *where $n \geqslant 1$, p, q are co-prime and q is odd, or* (ii) *ρ is not the nth root of a fraction, then E_Λ is a finite set.*

The proof of case (i) depends on the expression (1.2) (Theorem 4.3). The proof of case (ii) (Theorem 4.2) is more involved, it concerns when $\beta = \rho^{-1}$ is a solution of the polynomials in (1.3) (Lemma 3.5).

The paper is organized as follows. In Section 2, we introduce some basic properties of orthonormal sets of exponential functions and their relation to the algebraic equation (1.2). In Section 3, we use a linear algebra technique to give some conditions for polynomials satisfying (1.3) (Lemma 3.5.) By using this we prove in Section 4 the above assertion (in italics) of finiteness of the orthonormal E_Λ in Theorems 4.2–4.4, which imply Theorem 1.2. The technical proof of the main lemma (Lemma 3.5) is given in Appendix A.

2. Preliminaries

For $0 < \rho < 1$, let μ be the ρ-Bernoulli convolution defined by (1.1) and let $\widehat{\mu}(t) = \int e^{2\pi i t x}\, d\mu(x)$ be its Fourier transform. Then

$$\widehat{\mu}(t) = \widehat{\mu}(\rho t)\cos(2\pi\rho t).$$

By iterating this expression, we have

$$\widehat{\mu}(t) = \widehat{\mu}\big(\rho^n t\big)\prod_{j=1}^{n}\cos\big(2\pi\rho^j t\big) = \prod_{j=1}^{\infty}\cos\big(2\pi\rho^j t\big). \tag{2.1}$$

We let $\mathcal{Z} = \{t \in \mathbb{R}\colon\ \widehat{\mu}(t) = 0\}$. Then $\mathcal{Z} = -\mathcal{Z}$.

We will use the following notation throughout the paper: let $\beta = \rho^{-1}$; $\mathbb{O}$ denotes the set of odd integers; Λ is a discrete set in $\mathbb{R}$ that contains 0, and $E_\Lambda = \{e^{2\pi i\lambda(\cdot)}\colon\ \lambda \in \Lambda\}$.

Lemma 2.1. $\mathcal{Z} = \{t \in \mathbb{R}\colon\ t = a\beta^j/4$ *where* $j \in \mathbb{N}$ *and* $a \in \mathbb{O}\}$.

Proof. If $t = a\beta^j/4$, where $j > 0$ and $a \in \mathbb{O}$, then $\cos(2\pi\rho^j t) = 0$, so $\widehat{\mu}(t) = 0$. Conversely suppose that $\widehat{\mu}(t) = 0$. Since $\widehat{\mu}(0) = 1$, we have $\widehat{\mu}(\rho^n t) \neq 0$ for large n. By (2.1), $\prod_{j=1}^{n}\cos(2\pi\rho^j t) = 0$, which implies $\cos(2\pi\rho^j t) = 0$ for some j. Hence $t = a\beta^j/4$ for some $j > 0$, $a \in \mathbb{O}$. □

Lemma 2.2. *Let Λ be any discrete set containing* 0. *Then E_Λ is an orthonormal set of $L^2(\mu)$ if and only if $(\Lambda - \Lambda)\setminus\{0\} \subseteq \mathcal{Z}$. Furthermore, $\Lambda\setminus\{0\} = \{a_i\beta^{k_i}/4\colon\ k_i \in \mathbb{N},\ a_i \in \mathbb{O},\ i = 1, 2, \ldots, n\}$ where $n \geqslant 2$ is a finite integer or infinity. In particular β satisfies equations of the form*

$$a_i\beta^{k_i} - a_j\beta^{k_j} = b_{ij}\beta^{k_{ij}}, \tag{2.2}$$

where $1 \leqslant i, j \leqslant n$, $a_i, a_j, b_{ij} \in \mathbb{O}$, and $k_i, k_j, k_{ij} \in \mathbb{N}$ are not all equal.

Proof. It is clear that E_Λ is an orthonormal set of $L^2(\mu)$ if and only if $\widehat{\mu}(\lambda - \lambda') = 0$ for any $\lambda, \lambda' \in \Lambda$ and $\lambda \neq \lambda'$, which is equivalent to $(\Lambda - \Lambda)\setminus\{0\} \subseteq \mathcal{Z}$. The second assertion is obvious in view of the fact that $\Lambda\setminus\{0\} \subseteq \mathcal{Z}$ (as $0 \in \Lambda$) and Lemma 2.1 applies.

For the last part let $\lambda_i = a_i\beta^{k_i}/4$ and $\lambda_j = a_j\beta^{k_j}/4$ be any two nonzero elements in Λ, then $\lambda_i - \lambda_j = b_{ij}\beta^{k_{ij}}/4$ (by Lemma 2.1) and (2.2) follows. By dividing the lowest power of β in (2.2), we see that the three powers of β cannot be the same, otherwise, the left-hand side will be an even integer and the right-hand side will be an odd integer. □

From Lemma 2.2, we conclude that there are two alternatives regarding the three powers of β in (2.2):

(A1) *the three powers k_i, k_j, k_{ij} are distinct,* or
(A2) *exactly two of the three powers k_i, k_j, k_{ij} are equal.*

In regard to these two alternatives we have the following observation:

Lemma 2.3. *Suppose that $\beta > 0$ satisfies $a_1\beta^i + a_2\beta^j = a_3\beta^r$ where $a_1, a_2, a_3 \in \mathbb{O}$ and $i, j, r \in \mathbb{N}$. Then either*

(i) *i, j and r are distinct, and in this case β is not the nth root of a fraction; or*
(ii) *exactly two of the i, j or r are equal, and in this case β is the nth root of a fraction p/q, where p, q are co-prime, not both being odd.*

Proof. (i) We can assume that $i > j > r = 0$. Then

$$a_1\beta^i + a_2\beta^j = a_3, \quad a_1, a_2, a_3 \in \mathbb{O}. \tag{2.3}$$

Suppose to the contrary $\beta = (\frac{q}{p})^{1/k}$ where p, q are co-prime. First we claim that both i and j are multiples of k. Let us write $i = i_1k + s$ and $j = j_1k + t$ with $0 \leqslant s, t < k$, then $i_1 \geqslant j_1$. Multiplying (2.3) by p^{i_1}, we obtain

$$n_1\left(\frac{q}{p}\right)^{s/k} + n_2\left(\frac{q}{p}\right)^{t/k} + n_3 = 0.$$

Therefore $\beta = (\frac{q}{p})^{1/k}$ satisfies the equation

$$n_1x^s + n_2x^t + n_3 = 0.$$

Since $x^k - \frac{q}{p}$ is the minimal polynomial of β, in view of $0 \leqslant s, t < k$, a necessary condition for the equation to have solution is when $s = 0$ and $t = 0$. This proves the claim.

Now substitute $\beta = (\frac{q}{p})^{1/k}$ back in (2.3) and multiply throughout by p^{i_1} to obtain

$$q^{j_1}\left(a_1q^{i_1-j_1} + a_2p^{i_1-j_1}\right) = a_3p^{i_1} \tag{2.4}$$

(where $i_1 > j_1$). Since p, q are co-prime, we have $q|a_3$ which implies q is odd. Similarly, write (2.4) as

$$a_1q^{i_1} = a_3p^{i_1} - a_2p^{i_1-j_1}q^{j_1} = \left(a_3p^{j_1} - a_2q^{j_1}\right)p^{i_1-j_1}.$$

We obtain $p|a_1$, so that p is odd. We therefore conclude that the left-hand side of (2.4) is even and the right-hand side is odd, which is impossible. Therefore $\beta \neq (\frac{q}{p})^{1/k}$ and (i) follows.

(ii) Assume that exactly two of the i, j and r are equal. If $i = j > r$, then it is easy to see from (2.3) that β is the nth root of a fraction with odd numerator and even denominator. Similarly if $i = j < r$, then β is the nth root of a fraction with even numerator and odd denominator. $\square$

3. Some lemmas on polynomials

Let $\mathbb{Q}[x]$ and $\mathbb{Z}[x]$ be the polynomials with coefficients in $\mathbb{Q}$ (the rationals) and $\mathbb{Z}$ (the integers) respectively. A nonzero $f(x) \in \mathbb{Z}[x]$ is called a *primitive polynomial* if its coefficients are relatively prime.

Lemma 3.1. *Suppose that $f(x) \in \mathbb{Z}[x]$ and $\beta > 1$ is a real root of $f(x)$. Let $g(x) \in \mathbb{Q}[x]$ be a minimal polynomial of β. Then there is $h(x) \in \mathbb{Z}[x]$ and an integer N such that $Ng(x) \in \mathbb{Z}[x]$ and $f(x) = Ng(x)h(x)$.*

The proof is simple, we write $f(x) = g(x)r(x)$ for some $r(x) \in \mathbb{Q}[x]$. By factoring the scalar factors, we can write $f(x) = \frac{M}{L} g'(x)r'(x)$ where M, L are co-prime, and $g'(x), r'(x) \in \mathbb{Z}[x]$ are primitive. Using $f(x) \in \mathbb{Z}[x]$ it is not hard to show that $L = 1$ and the lemma follows.

Corollary 3.2. *Suppose that $\{f_i(x)\}_{i=1}^{\infty} \subset \mathbb{Z}[x]$ is a sequence of primitive polynomials and $\beta > 1$ is a real root of $f_i(x)$ for all i. Let $g(x) \in \mathbb{Q}[x]$ be a minimal polynomial of β. Then there is a sequence of primitive polynomials $h_i(x) \in \mathbb{Z}[x]$ and an integer N with $Ng(x) \in \mathbb{Z}[x]$ such that $Ng(x)$ is primitive and $f_i(x) = Ng(x)h_i(x)$ for every i.*

Proof. By Lemma 3.1, we have $f_i(x) = N_i g(x)h_i(x)$. Since $f_i(x) \in \mathbb{Z}[x]$ is primitive, the same is true for $N_i g(x)$ and $h_i(x)$, and hence each N_i must be the least common multiple of the denominators of the coefficients in $g(x)$. Therefore $N_i = N$ for every i. □

Condition (2.2) gives rise to the above sequence of polynomials $\{f_i(x)\}_{i=1}^{\infty}$ which have the form $c_1x^m + c_2x^n + c_3$ where the c_i's are odd integers. Let $g_\beta(x)$ be the minimal polynomial of β. If β is a solution of $f_i(x)$, then $g_\beta(x) \mid f_i(x)$, i.e., $f_i(x) = g_\beta(x)h_i(x)$ for some $h_i(x)$. We use $\widetilde{g}_\beta(x)$ to denote the primitive polynomial $Ng(x)$ associated with β in the corollary and call it the *integral minimal polynomial* of β. We now study some necessary conditions for $\widetilde{g}_\beta(x) \mid f_i(x)$, which count the number of equations of (2.2) and hence the cardinality of E_Λ in case (A1) in Section 2 (Theorem 4.2).

The polynomials with odd integral coefficients mentioned above are more conveniently handled as follows. Let $\mathbb{Z}_2$ be the residue class of $\mathbb{Z}$ modulo 2, identified with $\{0, 1\}$ and endowed with the regular binary algebraic operations. Obviously, we can reduce the polynomials in $\mathbb{Z}[x]$ to $\mathbb{Z}_2[x]$, and the relation $f(x) = g(x)h(x)$ in $\mathbb{Z}[x]$ can be reduced to $f(x) = g(x)h(x)$ in $\mathbb{Z}_2[x]$ (using similar notation).

Let $f(x), g(x), h(x) \in \mathbb{Z}_2[x]$ with

$$f(x) = \sum_{k=0}^{m+n} c_k x^k, \quad g(x) = \sum_{i=0}^{n} a_i x^i \quad \text{and} \quad h(x) = \sum_{j=0}^{m} b_j x^j.$$

Here $a_0 = a_n = b_0 = b_m = 1$ and the other coefficients are either 0 or 1. Obviously $f(x) = g(x)h(x)$ if and only if

$$c_k = \sum_{i+j=k} a_i b_j \quad \text{for } k = 0, 1, \ldots, m+n. \tag{3.1}$$

This relation can be written as a matrices expression: $\mathbf{c} = A\mathbf{b}$ where A is an $(m+n+1)\times(m+1)$ matrix defined by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_{n-1} & \vdots & a_1 & 1 & 0 & \cdots & 0 \\ 1 & a_{n-1} & \vdots & a_1 & 1 & 0 & \vdots \\ 0 & 1 & a_{n-1} & \vdots & a_1 & \ddots & 0 \\ 0 & 0 & 1 & a_{n-1} & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 1 & a_{n-1} & \ddots & a_1 \\ 0 & 0 & 0 & 0 & 1 & a_{n-1} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & a_{n-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2}$$

$\mathbf{b} = (1, b_1, \ldots, b_{m-1}, 1)^t$, and $\mathbf{c} = (1, c_1, \ldots, c_{m+n-1}, 1)^t$. Note that all the entries are either 0 or 1 and the operations are binary.

Remark 3.3. Given $f(x)$ and $g(x)$, there is an $h(x)$ such that $f(x) = g(x)h(x)$ if and only if the linear system $A\mathbf{x} = \mathbf{c}$ has a solution $\mathbf{x} = \mathbf{b}$. From the standard linear equations theory, this is equivalent to the rank of the $(m+n+1)\times(m+2)$ augmented matrix $(A \mid \mathbf{c})$ equaling $m+1$, i.e., the bottom n rows in the reduced row echelon form of $(A \mid \mathbf{c})$ are zero.

Remark 3.4. To obtain the echelon form, we multiply on the left of $(A \mid \mathbf{c})$ by a sequence of $(m+n+1)\times(m+n+1)$ square binary invertible matrices E_i, $i = 1, 2, \ldots, m+1$:

$$E_i = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & \mathbf{0} & \\ & & 1 & & & & \\ & & & E & & & \\ & & & & 1 & & \\ & \mathbf{0} & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

where

$$E = \begin{pmatrix} 1 & & & & \\ a_1 & 1 & & & \\ \vdots & & \ddots & & \\ a_{n-1} & & & 1 & \\ 1 & & & & 1 \end{pmatrix}$$

is an $(n+1)\times(n+1)$ invertible matrix, its position in E_i is starting from the (i,i)-entry of E_i. We call E the *row operation matrix* associated with $g(x)$.

With the binary operation, it is clear that $E = E^{-1}$ and $E(1, a_1, \ldots, a_{n-1}, 1)^t = (1, 0, \ldots, 0)^t$. It follows that $E_1 A$ reduces the first column of A to $(1, 0, \ldots, 0)^t$ and the other columns remain unchanged; E_2E_1A reduces the second column of E_1A to $(0, 1, 0, \ldots, 0)^t$ and the other columns remain unchanged. Finally $E_{m+1}\cdots E_1A$ is the matrix with 1 on the diagonal and 0 elsewhere, which is the desired echelon form. In order for $A\mathbf{x} = \mathbf{c}$ to have a solution, it is necessary that $E_{m+1}\cdots E_1\mathbf{c}$ has zeros on the last n entries. This idea leads to the following main lemma.

We define a sequence of $(n+1)$-dimensional vectors $\{\mathbf{v}_i\}_{i=1}^{\infty}$ generated by E: Let $\mathbf{v}_0 = (1, 0, 0, \ldots, 0, 0)^t$. Define $\mathbf{v}_1 = E\mathbf{v}_0 = (1, a_1, \ldots, a_{n-1}, 1)^t$, and inductively

$$\mathbf{v}_i = E(\sigma\mathbf{v}_{i-1})$$

where $\sigma\mathbf{v} = (v_1, \ldots, v_n, 0)$ for $\mathbf{v} = (v_0, \ldots, v_n)$.

We note that the entries of each $\mathbf{v}_i$ are either 0 or 1, hence for i large, some of the $\mathbf{v}_i$'s must be repeated, say, $\mathbf{v}_i = \mathbf{v}_j$. It is easy to prove that (Lemma A.1(iii)) we can reduce this to $\mathbf{v}_1 = \mathbf{v}_j$ (for a different j). Let $\ell \geqslant 1$ be the smallest integer such that $\mathbf{v}_{\ell+1} = \mathbf{v}_1$, we say that ℓ is the *period* of $\{\mathbf{v}_i\}_{i=1}^{\infty}$. From the definition of ℓ, we have $\mathbf{v}_i = \mathbf{v}_{j\ell+i}$ for any $i, j \geqslant 1$.

The following main lemma gives some restrictions on β being a solution of certain specific polynomials $f(x) \in \mathbb{Z}_2[x]$, its technical proof is in Appendix A.

Lemma 3.5. *Let $\widetilde{g}_\beta(x) \in \mathbb{Z}_2[x]$ be the reduced (integral) minimal polynomial of β of degree n and let ℓ be defined as above. Suppose that $f(x) \in \mathbb{Z}_2[x]$ is of the form $f(x) = x^{m+n} + x^k + 1$, and that $\widetilde{g}_\beta(x) \mid f(x)$, then $k \not\equiv 0 \pmod{\ell}$ and $m + n \not\equiv 0 \pmod{\ell}$.*

4. The theorems

Lemma 4.1. *Let E_Λ be an orthonormal set of $L^2(\mu)$ and let $\Lambda_i = \{t \in \Lambda\colon\ t = a\beta^i/4$ for some $a \in \mathbb{O}\}$, $i \geqslant 1$. If $\rho^n \neq \frac{p}{q}$ for any positive integer n, where p, q are co-prime, then $\#\Lambda_i \leqslant 1$ for all i. If in addition we assume that both p and q are odd, then $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$.*

Proof. We assume to the contrary $\#\Lambda_i \geqslant 2$ for some $i \geqslant 1$. Let $\lambda, \lambda' \in \Lambda_i$ with $\lambda = a\beta^i/4$ and $\lambda' = a'\beta^i/4$, where $a, a' \in \mathbb{O}$, then there exists $j \in \mathbb{N}$ and $b \in \mathbb{O}$ such that $\lambda - \lambda' = b\beta^j/4$. It follows that $\beta^i(a - a') = b\beta^j$. Obviously, $i \neq j$ (otherwise, $a - a' = b$). Thus β^{i-j} is a fraction which contradicts the assumption. The first assertion follows.

Note that for $i > j$, $\Lambda_i \cap \Lambda_j \neq \emptyset$ is equivalent to $a\beta^i/4 = b\beta^j/4$, for some $a, b \in \mathbb{O}$, i.e., β^{i-j} is a quotient of two odd integers. So if $\rho^n \neq \frac{p}{q}$, where both p and q are odd, then $\Lambda_i \cap \Lambda_j = \emptyset$. □

Theorem 4.2. *Suppose that ρ is not the nth root of a fraction, and that E_Λ is an orthonormal set of $L^2(\mu_\rho)$. Then Λ is finite.*

Proof. Suppose that Λ is infinite. Write $\Lambda \setminus \{0\} = \{a_i\beta^{n_i}/4\colon\ n_i \in \mathbb{N},\ a_i \in \mathbb{O},\ i = 1, 2, \ldots\}$ (Lemma 2.2). Since $\beta^n \neq \frac{p}{q}$ for any positive integer n, by Lemma 4.1, we can actually assume that $n_1 < n_2 < \cdots < n_i < \cdots$ is a strictly increasing sequence. Let ℓ be the period of $\{\mathbf{v}_j\}_{j=1}^{\infty}$ determined by the minimal polynomial of β. Since $n_1 < n_2 < \cdots < n_i < \cdots$, there exists $0 \leqslant$

$r \leqslant \ell - 1$ such that $n_i \equiv r \pmod{\ell}$ for infinitely many i. Without loss of generality, we assume that $n_i \equiv r \pmod{\ell}$ for all n_i, hence $n_i - n_j \equiv 0 \pmod{\ell}$ for all such n_i, n_j.

Let $\lambda_i = a_i \beta^{n_i}/4$ for each i, we have $\widehat{\mu}(\lambda_1 - \lambda_i) = 0$ for all $i > 1$, i.e., there are $n_i, j_i \in \mathbb{N}$, $a_i, b_i \in \mathbb{O}$, such that for $i > 1$,

$$a_1 \beta^{n_1} - a_i \beta^{n_i} = b_i \beta^{j_i}. \tag{4.1}$$

Obviously, $j_i \neq n_1$ for all $i > 1$, otherwise, $\beta^{n_i - n_1} = \frac{p}{q}$ for some integers p and q. Suppose that there is an index $j_i > n_1$. By dividing both sides of (4.1) by β^{n_1}, the equation becomes $a_1 - a_i \beta^{n_i - n_1} = b_i \beta^{j_i - n_1}$ and the corresponding polynomial in $\mathbb{Z}_2[x]$ is

$$f(x) = 1 + x^{n_i - n_1} + x^{j_i - n_1}.$$

Let $\widetilde{g}_\beta(x)$ be the reduced (integral) minimal polynomial in $\mathbb{Z}_2[x]$. Then $f(x) = \widetilde{g}_\beta(x) h(x)$ for some $h(x) \in \mathbb{Z}_2[x]$. Lemma 3.5 implies that $n_i - n_1 \not\equiv 0 \pmod{\ell}$, a contradiction.

It remains to check the case when $j_i < n_1$ for all $i > 1$. By the pigeonhole principle, there are infinitely many identical j_i's. Suppose $j_2 = j_3$; using (4.1) we have $a_1 \beta^{n_1} - a_2 \beta^{n_2} = b_2 \beta^{j_2}$ and $a_1 \beta^{n_1} - a_3 \beta^{n_3} = b_3 \beta^{j_2}$. Subtracting the two identities, noting that $\widehat{\mu}(\lambda_2 - \lambda_3) = 0$ and using (2.2) again, we obtain $(b_2 - b_3)\beta^{j_2} = a_3 \beta^{n_3} - a_2 \beta^{n_2} = a\beta^m$ for some odd integer a. This implies that $\beta^n = p/q$ for some integers p and q and some positive integer n, a contradiction.

Therefore Λ is finite. □

The above theorem resolves the first alternative (A1) in Section 2 (see also Lemma 2.3). We now discuss the situation concerning the second alternative (A2).

Theorem 4.3. *Let $\rho = (\frac{p}{q})^{1/n}$ where $n \geqslant 1$, p, q are co-primes, and let E_Λ be an orthonormal set of $L^2(\mu_\rho)$. If q is odd, then Λ is finite.*

Proof. We see from Lemma 2.3(ii) that p, q cannot be both odd, so we assume that p is even and q is odd.

For each i, let $\Lambda_i = \{a\beta^i/4\colon\ a \in \mathbb{O}\}$ be a discrete set such that E_{Λ_i} is an orthonormal set of $L^2(\mu_\rho)$. We first prove the following claims.

Claim 1. Let $\lambda, \lambda' \in \Lambda_j$ and $\lambda - \lambda' = a_r \beta^r/4 \in \mathcal{Z}$, then $r < j$: The assumption implies that there are three odd integers a, b and c such that $a\beta^j - b\beta^j - c\beta^r = 0$ (Lemma 2.2) and it is clear that $r \neq j$. If $r > j$, then we get $a - b - c\beta^{r-j} = 0$. This implies that β is an $(r-j)$th root of a fraction with an even numerator and an odd denominator, a contradiction. Hence $r < j$.

Claim 2. Let $\lambda = a\beta^j/4$, $\lambda' = b\beta^i/4$, $\lambda - \lambda' \in \mathcal{Z}$ and $i < j$, then $\lambda - \lambda' = c\beta^j/4$, where $a, b, c \in \mathbb{O}$: The assumption implies that $\lambda - \lambda' = c\beta^r/4$ for some $c \in \mathbb{O}$. We will show that $r = j$. Since β is the nth root of a fraction, by Lemma 2.3, either $r = i$ or $r = j$. If $r = i$, then $a\beta^{j-i} - b - c = 0$ implying that β is the $(j-i)$th root of a fraction with an even numerator and an odd denominator, a contradiction. Therefore $r = j$.

Claim 3. $\#\Lambda_i \leqslant 2^i$ for every $i \geqslant 1$: We prove this by induction. Let $\lambda \in \Lambda_1$, then $\lambda = a\beta/4$ for some $a \in \mathbb{O}$. If there were two distinct members λ and λ' in Λ_1, using Claim 1 we see that

$\lambda - \lambda' \notin \mathcal{Z}$, hence $\#\Lambda_1 = 1$. Assume that $\#\Lambda_i \leqslant 2^i$ for all $i < n$. Suppose that $\#\Lambda_n > 2^n$. Let $\lambda_0, \lambda_1, \dots, \lambda_{2^n}$ be the distinct elements in Λ_n. By Claim 1, we have for each $i = 1, \dots, 2^n$,

$$\lambda_0 - \lambda_i = a_i \beta^{j_i}/4 \quad \text{for some } j_i < n \quad \text{and} \quad a_i \in \mathbb{O}.$$

Since $1 + 2 + \cdots + 2^{n-1} < 2^n$, by the pigeonhole principle there exists $1 \leqslant j < n$ and $\lambda_{n_i} \in \Lambda_n$, for $i = 1, 2, \dots, 2^j + 1$, such that

$$\alpha_i := \lambda_0 - \lambda_{n_i} = a_{n_i} \beta^j/4. \tag{4.2}$$

Write $A = \{\alpha_i \colon i = 1, 2, \dots, 2^j + 1\}$ and let $\alpha_i, \alpha_l \in A$, then

$$\alpha_i - \alpha_l = (\lambda_0 - \lambda_{n_i}) - (\lambda_0 - \lambda_{n_l}) = (\lambda_{n_l} - \lambda_{n_i}) \in \mathcal{Z}.$$

Hence $A \subseteq \Lambda_j$ for some orthonormal set E_{Λ_j} of $L^2(\mu_\rho)$. It follows that $\#\Lambda_j \geqslant 2^j + 1$, contradicting the inductive assumption. This proves Claim 3.

Now to complete the proof of the theorem, we assume that $\#\Lambda = \infty$ and decompose Λ into $\bigcup_{i=1}^\infty \Lambda_i$ as in Lemma 4.1. Let r be the smallest index for which $\Lambda_r \neq \emptyset$ and let $\lambda = a(\frac{q}{p})^r/4 \in \Lambda_r$. Since Λ is infinite, $\Lambda = \bigcup_{i=r}^\infty \Lambda_i$ is a disjoint union (Lemma 4.1) with $\#\Lambda_i \leqslant 2^i$ for each i. Note that $0 < \rho = (\frac{p}{q})^{1/n} < 1$ and p is even, so $q \geqslant 3$. Choose $i > r$ such that $q^{i-r} > a$ and $\Lambda_i \neq \emptyset$. Let $\lambda' = b(\frac{q}{p})^i/4 \in \Lambda_i$. By Claim 2, $\lambda - \lambda' = c(\frac{q}{p})^i/4$, for some odd integer c. This implies that there is an even integer l such that

$$a\left(\frac{q}{p}\right)^r = l\left(\frac{q}{p}\right)^i.$$

Multiplying both sides by $p^i q^{-r}$ we have

$$ap^{i-r} = lq^{i-r}.$$

Since p is even and q is odd and they are co-prime, q^{i-r} must be a factor of a. This is impossible by the choice of i. We conclude that Λ is finite. □

Theorem 1.1 shows that $\mu = \mu_{1/q}$, where q is even, is a spectral measure with a spectrum Λ, where Λ, according to the construction in [2], can be chosen to be

$$\Lambda = \left\{ \sum_{i=1}^n \frac{q^i \varepsilon_i}{4} \colon \varepsilon_i = 0 \text{ or } 1,\ n \in \mathbb{N} \right\}.$$

Next we show that this E_Λ is also an infinite orthonormal set of $L^2(\mu_\rho)$ for $\rho = (\frac{p}{q})^{1/n}$ where $n \geqslant 1$, p is odd and q is even.

Theorem 4.4. *Let $\rho = (\frac{p}{q})^{1/n}$ where $n \geqslant 1$, p is odd and q is even. Let E_Λ be any orthonormal set of $L^2(\mu_{1/q})$. Then E_Λ is orthonormal in $L^2(\mu_\rho)$, but it is not complete if $n > 1$.*

Proof. Let $\mathcal{Z}'$ be the set of zeros of $\widehat{\mu_\rho}$. Let $\lambda, \lambda' \in \Lambda$ with $\lambda \neq \lambda'$, by Lemmas 2.1–2.2, there exist $i > 0$, $b \in \mathbb{O}$ such that

$$\lambda - \lambda' = bq^i/4 = bp^i\big((q/p)^{1/n}\big)^{ni}/4 \in \mathcal{Z}'.$$

This shows that E_Λ is an orthonormal set of $L^2(\mu_\rho)$.

To prove the final statement we recall that the completeness will imply that $\sum_{\lambda\in\Lambda} |\widehat{\mu}(\lambda - x)|^2 \equiv 1$, $x \in \mathbb{R}$ [2] which we show cannot happen.

Let $\widehat{\mu_\rho} = \prod_{i=1}^{\infty} \cos(2\pi\rho^i t)$ be the Fourier transform of μ_ρ. Also let ν be the $(\frac{p}{q})$-Bernoulli convolution. Then $\widehat{\nu}(t) = \prod_{i=1}^{\infty} \cos(2\pi\rho^{ni} t)$ and $\sum_{\lambda\in\Lambda} |\widehat{\nu}(\lambda - x)|^2 \leqslant 1$ for all $x \in \mathbb{R}$ by the Bessel's inequality. Moreover $\widehat{\mu_\rho}(t) = \prod_{j=0}^{n-1} \widehat{\nu}(\rho^{-j} t)$.

Now choose a point $x \in \mathbb{R}$ such that $0 < \prod_{j=1}^{n-1} |\widehat{\nu}(\rho^{-j}(-x))|^2 < 1$. Then

$$\begin{aligned}
\sum_{\lambda\in\Lambda} \big|\widehat{\mu_\rho}(\lambda - x)\big|^2 &= \sum_{\lambda\in\Lambda} \prod_{j=0}^{n-1} \big|\widehat{\nu}\big(\rho^{-j}(\lambda - x)\big)\big|^2 \\
&= \sum_{\lambda\in\Lambda\setminus\{0\}} \prod_{j=0}^{n-1} \big|\widehat{\nu}\big(\rho^{-j}(\lambda - x)\big)\big|^2 + \prod_{j=0}^{n-1} \big|\widehat{\nu}\big(\rho^{-j}(-x)\big)\big|^2 \\
&< \sum_{\lambda\in\Lambda\setminus\{0\}} \big|\widehat{\nu}(\lambda - x)\big|^2 + \big|\widehat{\nu}(-x)\big|^2 \\
&= \sum_{\lambda\in\Lambda} \big|\widehat{\nu}(\lambda - x)\big|^2 \leqslant 1.
\end{aligned}$$

Hence E_Λ cannot be complete in $L^2(\mu_\rho)$. □

Proof of Theorem 1.2. The sufficiency follows from Theorem 4.4, and the necessity follows from Theorems 4.2 and 4.3. □

We have not been able to prove Theorem 4.4 for $n = 1$, nor to show the existence of exponential orthonormal basis for $L(\mu_\rho)$ when $\rho = p/q$, $p > 1$ is odd and q is even.

Bernoulli convolutions can be put into a more general framework of *self-similar measures* in $\mathbb{R}^d$ [1,6]:

$$\mu = \sum_{j=1}^{m} w_j \mu \circ \varphi_j^{-1}$$

where $\sum_{j=1}^{m} w_j = 1$, $\varphi_j(x) = A^{-1}(x + b_j)$, $j = 1, \ldots, m$, with $b_j \in \mathbb{R}^d$, A is an $d \times d$ matrix in $\mathbb{R}^d$ and is expanding (all eigenvalues has moduli > 1). However it is not clear to what extent we can conclude that μ is a spectral measure (see [3,8]). Indeed it was conjectured in [3] that for $\varphi_j(x) = \rho(x + a_j)$, $1 \leqslant j \leqslant m$, in order that the self-similar measure to be a spectral measure, it is necessary that $\rho = 1/q$ for some integer $q \geqslant 2$ with the w_j all equal. We have the following conclusion for the Bernoulli convolution in (1.1) if we consider different weights w and $1 - w$:

Proposition 4.5. *Let E_Λ be any orthonormal set of $L^2(\mu_{\rho,w})$, where $w \neq 1/2$, then E_Λ contains only the constant function.*

Proof. Let $\mu = \mu_{\rho,w}$. A direct calculation gives

$$\widehat{\mu}(t) = \prod_{j=1}^{\infty}\bigl(we^{2\pi i\rho^j t} + (1-w)e^{-2\pi i\rho^j t}\bigr) = \prod_{j=1}^{\infty}\bigl(\cos 2\pi\rho^j t + i(2w-1)\sin 2\pi\rho^j t\bigr).$$

Hence

$$\bigl|\widehat{\mu}(t)\bigr|^2 = \prod_{j=1}^{\infty}\bigl(\cos^2\bigl(2\pi\rho^j t\bigr) + (2w-1)^2\sin^2\bigl(2\pi\rho^j t\bigr)\bigr).$$

Let $\eta_w = (2w-1)^2$, then $0 \leqslant \eta_w < 1$. Observe that the function

$$f(x) = \cos^2 x + \eta_w \sin^2 x, \quad x \in \mathbb{R},$$

attains the maximum value 1 when $\sin x = 0$, and attains the minimum value η_w when $\cos x = 0$. Since $\eta_w = 0$ if and only if $w = 1/2$. It follows that if $w \neq 1/2$, then for all $t \in \mathbb{R}$, $\cos^2 t + \eta_w \sin^2 t > 0$. This implies $\widehat{\mu}(t) \neq 0$ and the result follows. □

Acknowledgments

The authors would like to express their gratitude to Professor R. Strichartz for communications during the preparation of the manuscript, and to Professors D.J. Feng and X.G. He for some valuable discussions. They thank the referee for constructive remarks and grammatical corrections making the paper more readable.

Appendix A

In this section we will present a proof of Lemma 3.5. Recall from Section 3 that $\widetilde{g}_\beta(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{Z}_2[x]$ is the reduced (integral) minimal polynomial of β, $\mathbf{v}_0 = (1, 0, 0, \ldots, 0, 0)^t$, $\mathbf{v}_1 = E\mathbf{v}_0 = (1, a_1, \ldots, a_{n-1}, 1)^t$, and inductively

$$\mathbf{v}_i = E(\sigma\mathbf{v}_{i-1})$$

where $\sigma\mathbf{v} = (v_1, \ldots, v_n, 0)$ for $\mathbf{v} = (v_0, \ldots, v_n)$.

Lemma A.1. *For the sequence $\{\mathbf{v}_i\}_{i=1}^{\infty}$, writing $\mathbf{v}_i = (v_{i0}, \ldots, v_{in})^t$, we have for each $i \geqslant 1$,*

(i) $v_{i0} = v_{in}$;
(ii) $\mathbf{v}_i$ *contains at most* $(n-1)$ *consecutive zeros, hence* $\mathbf{v}_i \neq (0, 0, \ldots, 0)^t$;
(iii) *for* $i, j > 1$, $\mathbf{v}_i = \mathbf{v}_j$ *implies that* $\mathbf{v}_{i-1} = \mathbf{v}_{j-1}$.

Proof. A direct calculation of $\mathbf{v}_i = E(\sigma\mathbf{v}_{i-1}) = E(v_{(i-1)1}, \ldots, v_{(i-1)n}, 0)^t$ yields $v_{i0} = v_{(i-1)1} = v_{in}$ and (i) follows.

To prove (ii) we first observe that E is invertible, the construction implies all the $\mathbf{v}_i \neq \mathbf{0}$. Now using (i), we see that either $v_{i0} = v_{in} = 1$ or 0. The first case clearly implies the statement in (ii); for the second case, if $\mathbf{v}_i$ contains more than $(n-1)$ consecutive zeros, then $\mathbf{v}_i = \mathbf{0}$, which is impossible.

To prove (iii), we observe that

$$E(v_{(i-1)1}, \ldots, v_{(i-1)n}, 0)^t = \mathbf{v}_i = \mathbf{v}_j = E(v_{(j-1)1}, \ldots, v_{(j-1)n}, 0)^t,$$

hence

$$(v_{(i-1)1}, \ldots, v_{(i-1)n}, 0)^t = E^{-1}\mathbf{v}_i = E^{-1}\mathbf{v}_j = (v_{(j-1)1}, \ldots, v_{(j-1)n}, 0)^t.$$

It follows that $v_{(i-1)k} = v_{(j-1)k}$ for $k = 1, \ldots, n$. By (i), we also have $v_{(i-1)0} = v_{(i-1)n} = v_{(j-1)n} = v_{(j-1)0}$. Therefore $\mathbf{v}_{i-1} = \mathbf{v}_{j-1}$. $\square$

Lemma A.2. *Let ℓ be the period of $\{\mathbf{v}_i\}_{i=1}^{\infty}$. If $k = j\ell + 1 > n$, where n is the degree of $\widetilde{g}_\beta(x)$, then $\mathbf{v}_{k-n} = (1, 0, \ldots, 0, 1)^t$.*

Proof. We first show that $\mathbf{v}_{k-1} = (0, 1, 0, \ldots, 0)^t$. Since $\mathbf{v}_1 = E\mathbf{v}_0$, using the assumption that $\mathbf{v}_k = \mathbf{v}_1$ we obtain

$$(v_{(k-1)1}, \ldots, v_{(k-1)n}, 0)^t = E^{-1}\mathbf{v}_1 = \mathbf{v}_0 = (1, 0, \ldots, 0)^t.$$

Also by Lemma A.1(i), we have $v_{(k-1)0} = v_{(k-1)n} = 0$, hence $\mathbf{v}_{k-1} = (0, 1, 0, \ldots, 0)^t$.

Similarly since $E(\sigma\mathbf{v}_{k-2}) = \mathbf{v}_{k-1} = (0, 1, 0, \ldots, 0)^t$, by $E^{-1} = E$ we have

$$(v_{(k-2)1}, \ldots, v_{(k-2)n}, 0)^t = E^{-1}(0, 1, 0, \ldots, 0)^t = (0, 1, 0, \ldots, 0)^t.$$

Again by Lemma A.1(i), $v_{(k-2)0} = v_{(k-2)n} = 0$, which implies that $\mathbf{v}_{k-2} = (0, 0, 1, 0, \ldots, 0)^t$. Continuing in this fashion we obtain $\mathbf{v}_{k-(n-1)} = (0, 0, \ldots, 0, 1, 0)^t$. It follows that

$$(v_{(k-n)1}, \ldots, v_{(k-n)n}, 0)^t = E^{-1}(0, 0, \ldots, 0, 1, 0)^t = (0, 0, \ldots, 0, 1, 0)^t.$$

This together with $v_{(k-n)0} = v_{(k-n)n} = 1$ yields $\mathbf{v}_{k-n} = (1, 0, \ldots, 0, 1)^t$. $\square$

Proof of Lemma 3.5. Let A be an $(m+n+1) \times (m+1)$ matrix defined by (3.2) and $\mathbf{b} = (1, b_1, \ldots, b_{m-1}, 1)^t$. Let $\mathbf{c}$ be an $(m+n+1)$ vector defined by $\mathbf{c} = (1, 0, \ldots, 0, 1, 0, \ldots, 0, 1)^t$, representing the coefficients of $f(x)$. The assumption implies that $f(x) = \widetilde{g}_\beta(x)h(x)$ for some $h(x) \in \mathbb{Z}_2[x]$. Hence by Remarks 3.3–3.4, the last n entries of $E_{m+1} \cdots E_1\mathbf{c}$ are zero.

Write

$$\mathbf{c}=\begin{pmatrix}1\\0\\\vdots\\0\\0\\0\\\vdots\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\\vdots\\0\\1\\0\\\vdots\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\\vdots\\0\\0\\0\\\vdots\\0\\1\end{pmatrix}=\mathbf{c}_1+\mathbf{c}_2+\mathbf{c}_3.$$

Consider the sequence $\{\mathbf{v}_i\}_{i=1}^{m+1}$. It follows that $\mathbf{v}_0$ is the first $(n+1)$ entries of $\mathbf{c}_1$; $\mathbf{v}_1$ is the first $(n+1)$ entries of $E_1\mathbf{c}_1$ and for $i \leqslant m+1$, $\mathbf{v}_i$ is the consecutive $n+1$ entries of $E_i\cdots E_1\mathbf{c}_1$, starting from the ith entry.

(i) We prove $k \not\equiv 0 \pmod{\ell}$. Suppose otherwise, we claim that

$$E_{m+1}E_m\cdots E_1(\mathbf{c})=(1,t_1,\ldots,t_{k-1},0,0,\ldots,0,1)^t$$

for some $t_1,\ldots,t_{k-1}\in\mathbb{Z}_2$. It follows that the last n entries of $E_{m+1}E_m\cdots E_1(\mathbf{c})$ are not all zero. This is a contradiction, and we conclude that $k \not\equiv 0 \pmod{\ell}$.

To prove the claim, note that $k\equiv 0 \pmod{\ell}$ and $\mathbf{v}_i$'s has period ℓ, we have $\mathbf{v}_{k+1}=\mathbf{v}_1=(1,a_1,\ldots,a_{n-1},1)$, hence

$$E_{k+1}E_k\cdots E_1\mathbf{c}_1=(1,t_1,\ldots,t_{k-1},\underbrace{1,a_1,\ldots,a_{n-1},1}_{\mathbf{v}_{k+1}},0,\ldots,0)^t$$

for some $t_1,\ldots,t_{k-1}\in\mathbb{Z}_2$. Next note that the digit 1 in $\mathbf{c}_2$ is at the $(k+1)$th entry, hence $E_k\cdots E_1\mathbf{c}_2=\mathbf{c}_2$ (see Remark 3.4) and

$$E_{k+1}E_k\cdots E_1\mathbf{c}_2=(0,0,\ldots,0,1,a_1,\ldots,a_{n-1},1,0,\ldots,0)^t,$$

where the first digit 1 in the column is at the $(k+1)$th entry. Finally for $\mathbf{c}_3$, it is trivial to check that $E_i\mathbf{c}_3=\mathbf{c}_3$ for all $1\leqslant i\leqslant m+1$. Using binary addition on $\mathbb{Z}_2$ and the three observations for $\mathbf{c}_1,\mathbf{c}_2,\mathbf{c}_3$, we have

$$E_{m+1}E_m\cdots E_1(\mathbf{c}_1+\mathbf{c}_2+\mathbf{c}_3)=(1,t_1,\ldots,t_{k-1},0,0,\ldots,0,1)^t$$

and the claim is proved.

(ii) To prove $m+n\not\equiv 0 \pmod{\ell}$, we suppose to the contrary that $m+n=\ell j$ for some j, which means $m+1=(\ell j+1)-n$. By Lemma A.2, we have $\mathbf{v}_{m+1}=(1,0,\ldots,0,1)$, which implies, as above,

$$E_{m+1}\cdots E_1\mathbf{c}_1=(1,t_1,\ldots,t_{m-1},\underbrace{1,0,\ldots,0,1}_{\mathbf{v}_{m+1}})^t.$$

Note that $E_{m+1}\cdots E_1\mathbf{c}_3=\mathbf{c}_3$, hence

$$E_{m+1}\cdots E_1(\mathbf{c}_1+\mathbf{c}_3)=(1,t_1,\ldots,t_{m-1},1,0,\ldots,0,0)^t,$$

where the last n entries are zero. Also note that the last $(n+1)$ entries of $E_{m+1}\cdots E_1\mathbf{c}_2$ equal $\mathbf{v}_j$ for some j depending on the value of k. By Lemma A.1(ii), $\mathbf{v}_j$ contains at most $(n-1)$ consecutive zeros, hence the last n entries in $E_{m+1}\cdots E_1(\mathbf{c}_1+\mathbf{c}_2+\mathbf{c}_3)$ contain at least one nonzero entry, a contradiction. □

References

[1] K. Falconer, Fractal Geometry: Mathematical Foundations and Applications, John Wiley & Sons, 1990.
[2] P. Jorgensen, S. Pedersen, Dense analytic subspaces in fractal L^2-spaces, J. Anal. Math. 75 (1998) 185–228.
[3] I. Łaba, Y. Wang, On spectral Cantor measures, J. Funct. Anal. 193 (2002) 409–420.
[4] I. Łaba, Y. Wang, Some properties of spectral measures, Appl. Comput. Harmon. Anal. 20 (2006) 149–157.
[5] K.S. Lau, S.M. Ngai, Multifractal measures and a weak separation condition, Adv. Math. 141 (1999) 45–96.
[6] K.S. Lau, S.M. Ngai, H. Rao, Iterated function systems with overlaps and self-similar measures, J. London Math. Soc. 63 (2001) 249–268.
[7] Y. Peres, W. Schlag, B. Solomyak, Sixty years of Bernoulli convolutions, in: Fractal Geometry and Stochastics II, in: Progr. Probab., vol. 46, Birkhäuser, Basel, 2000, pp. 39–65.
[8] R. Strichartz, Remarks on “Dense analytic subspaces in fractal L^2-spaces” by P.E.T. Jorgensen and S. Pedersen, J. Anal. Math. 75 (1998) 229–231.
[9] R. Strichartz, Mock Fourier series and transforms associated with certain Cantor measures, J. Anal. Math. 81 (2000) 209–238.
[10] R. Strichartz, Convergence of mock Fourier series, J. Anal. Math. 20 (2006) 333–353.