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ScienceDirect
Journal of Functional Analysis 239 (2006) 542–565
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Laplace operators related to self-similar measures on $\mathbb{R}^d$

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Received 26 October 2005; accepted 7 July 2006

Available online 22 August 2006

Communicated by L. Gross

Abstract

Given a bounded open subset Ω of $\mathbb{R}^d$ $(d \geqslant 1)$ and a positive finite Borel measure μ supported on $\overline{\Omega}$ with $\mu(\Omega) > 0$, we study a *Laplace-type operator* Δ_μ that extends the classical Laplacian. We show that the properties of this operator depend on the *multifractal structure* of the measure, especially on its *lower* L^∞*-dimension* $\underline{\dim}_\infty(\mu)$. We give a sufficient condition for which the Sobolev space $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega, \mu)$, which leads to the existence of an orthonormal basis of $L^2(\Omega, \mu)$ consisting of eigenfunctions of Δ_μ. We also give a sufficient condition under which the Green's operator associated with μ exists, and is the inverse of $-\Delta_\mu$. In both cases, the condition $\underline{\dim}_\infty(\mu) > d-2$ plays a crucial rôle. By making use of the multifractal L^q*-spectrum* of the measure, we investigate the condition $\underline{\dim}_\infty(\mu) > d-2$ for *self-similar measures* defined by iterated function systems satisfying or not satisfying the *open set condition*.

Keywords: Laplacian; Self-similar measure; Eigenvalue; Eigenfunction; L^q-spectrum; L^∞-dimension; Upper regularity of a measure

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[1] Research supported in part by NSFC, Grant No. 10371062 and an HKRGC grant.

[2] Research supported in part by an HKRGC grant.

[3] Research supported in part by an HKRGC grant, a Faculty Research Stipend from Georgia Southern University, and an Academic Excellence Award from the College of Science and Technology of Georgia Southern University.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2006.07.005

1. Introduction

Let $\Omega \subseteq \mathbb{R}^d$ $(d \geqslant 1)$ be a bounded open set, and let dx be the Lebesgue measure on $\mathbb{R}^d$. Denote by $L^2(\Omega) := L^2(\Omega, dx)$. Let $H^1(\Omega)$ be the Sobolev space

$$\langle u, v\rangle_{H^1(\Omega)} := \int_\Omega uv\,dx + \int_\Omega \nabla u \cdot \nabla v\,dx.$$

Let $C_c^\infty(\Omega)$ denote the space of all $C^\infty(\Omega)$ functions with compact support in Ω. Let $H_0^1(\Omega)$ denote the completion of $C_c^\infty(\Omega)$ in the $H^1(\Omega)$ norm. In view of the Poincaré inequality; that is, there exists a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leqslant C\|\nabla u\|_{L^2(\Omega)} \quad \text{for all } u \in H_0^1(\Omega),$$

the space $H_0^1(\Omega)$ admits the equivalent inner product defined by

$$\langle u, v\rangle_{H_0^1(\Omega)} := \int_\Omega \nabla u \cdot \nabla v\,dx.$$

Note that both $H^1(\Omega)$ and $H_0^1(\Omega)$ are Hilbert spaces.

Let μ be a positive finite Borel measure on $\mathbb{R}^d$ with $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$. Since the case $\mu(\Omega) = 0$ is not of interest to our discussions, we will assume throughout this paper that $\mu(\Omega) > 0$. In order to define a *Laplace-type operator* on $L^2(\Omega, \mu)$, we need the following important condition (see [28]):

(C1) There exists a constant $C > 0$ such that, for all $u \in C_c^\infty(\Omega)$,

$$\int_\Omega |u|^2\,d\mu \leqslant C \int_\Omega |\nabla u|^2\,dx.$$

This condition implies that each equivalence class $u \in H_0^1(\Omega)$ contains a unique (in $L^2(\Omega, \mu)$ sense) member $\bar{u}$ that belongs to $L^2(\Omega, \mu)$ and satisfies both conditions below:

(1) There exists a sequence $\{u_n\}$ in $C_c^\infty(\Omega)$ such that $u_n \to \bar{u}$ in $H_0^1(\Omega)$ and $u_n \to \bar{u}$ in $L^2(\Omega, \mu)$;
(2) $\bar{u}$ satisfies the inequality in (C1).

We call $\bar{u}$ the $L^2(\Omega, \mu)$*-representative* of u. Assume condition (C1) holds and define a mapping $I : H_0^1(\Omega) \to L^2(\Omega, \mu)$ by

$$I(u) = \bar{u}.$$

It is straightforward to verify that I is a bounded linear operator. I is not necessarily injective, because it is possible for a non-zero function $u \in H_0^1(\Omega)$ to have an $L^2(\Omega, \mu)$-representative that has zero $L^2(\Omega,\mu)$-norm. To deal with this situation, we consider a subspace $\mathcal{N}$ of $H_0^1(\Omega)$ defined as

$$\mathcal{N} := \{u \in H_0^1(\Omega):\ \|I(u)\|_{L^2(\Omega,\mu)} = 0\}.$$

Then the continuity of I implies that $\mathcal{N}$ is a closed subspace of $H_0^1(\Omega)$. Now let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $H_0^1(\Omega)$. It is clear that $I : \mathcal{N}^\perp \to L^2(\Omega,\mu)$ is injective.

$\mathcal{N}^\perp$ is the very space we will work on in this paper. If no confusion is possible, we will denote $\bar{u}$ simply by u. By condition (C1), we see that $\|u\|_{L^2(\Omega,\mu)} \leqslant C^{1/2}\|u\|_{H_0^1(\Omega)}$ for all $u \in \mathcal{N}^\perp$; that is, $\mathcal{N}^\perp$ is embedded in $L^2(\Omega,\mu)$. If $d \geqslant 2$ and if μ has a point mass in Ω, then condition (C1) fails, since $H_0^1(\Omega)$ contains unbounded functions. We will study condition (C1) in detail in Section 3 for general measures and in Section 5 for self-similar measures.

We remark that condition (C1) is similar to a condition in [34, Chapter 1], which is defined under a different setting, e.g., $\operatorname{supp}(\mu)$ there is assumed to have zero Lebesgue measure and is contained in a C^∞ domain Ω.

Consider a nonnegative bilinear form $\mathcal{E}(\cdot,\cdot)$ on $L^2(\Omega,\mu)$ given by

$$\mathcal{E}(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx \tag{1.1}$$

with *domain* $\operatorname{Dom}(\mathcal{E}) = \mathcal{N}^\perp$. Condition (C1) implies that $(\mathcal{E}, \operatorname{Dom}(\mathcal{E}))$ is a closed quadratic form on $L^2(\Omega,\mu)$ (see Proposition 2.1). Hence, there exists a nonnegative self-adjoint operator H on $L^2(\Omega,\mu)$ such that $\operatorname{Dom}(\mathcal{E}) = \operatorname{Dom}(H^{1/2})$ and

$$\mathcal{E}(u,v) = \langle H^{1/2}u, H^{1/2}v\rangle_{L^2(\Omega,\mu)} \quad \text{for all } u, v \in \operatorname{Dom}(\mathcal{E}).$$

(See, for example, [8].) We write $\Delta_\mu = -H$, and call it a (*Dirichlet*) *Laplacian with respect to* μ. We will show that $u \in \operatorname{Dom}(\Delta_\mu)$ and $-\Delta_\mu u = f$ if and only if $-\Delta u = f\,d\mu$ (or, more precisely, $-\Delta u\,dx = f\,d\mu$) in the sense of distribution (Proposition 2.2).

In this paper, we are interested in the following questions, especially in the case $d > 1$:

(1) What kinds of measures satisfy condition (C1)?
(2) Under what conditions does there exist an orthonormal basis of $L^2(\Omega,\mu)$ consisting of (Dirichlet) eigenfunctions of $-\Delta_\mu$ with discrete spectrum?
(3) Under what conditions is the Green's operator defined with respect to μ the inverse of $-\Delta_\mu$?

It turns out that these problems intertwine one another, and are intimately related to the *lower* L^∞*-dimension* $\underline{\dim}_\infty(\mu)$ and *upper regularity* of the measure μ.

For the one-dimensional case, the answers to the above problems are easier (see, for example, [4]). A class of more general Laplace-type operators on $\mathbb{R}$ was studied by Freiberg [13], and Freiberg and Zähle [14]. For the one- or higher-dimensional case, the first two problems above were investigated by Naimark and M. Solomyak, and M. Solomyak and Verbitsky. They obtained the compactness of the embedding $\operatorname{Dom}(\mathcal{E}) \hookrightarrow L^2(\Omega,\mu)$ [28,29] and the asymptotics of the eigenvalues [32] for *self-similar measures* satisfying the *open set condition*. Recently, Zähle

[37] introduced a pseudo-differential operator $\Delta_\mu := -(D^1_\mu)^2$ on a metric space (X, ρ) equipped with a finite Borel measure μ which is *upper s-regular* (see (3.1)) with s given by

$$s = \sup\left\{t:\ \int_X \rho(x,y)^{-t}\, d\mu(y) < \infty\right\} \quad \text{for } \mu\text{-a.e. } x \in \operatorname{supp}(\mu). \tag{1.2}$$

This pseudo-differential operator Δ_μ is termed a Laplacian on X if it is *local* [37, Section 2].

Recall that the *lower* and *upper* L^∞*-dimensions* of μ are defined by

$$\underline{\dim}_\infty(\mu) = \liminf_{\delta\to 0^+} \frac{\ln(\sup_x \mu(B_\delta(x)))}{\ln\delta},$$
$$\overline{\dim}_\infty(\mu) = \limsup_{\delta\to 0^+} \frac{\ln(\sup_x \mu(B_\delta(x)))}{\ln\delta}, \tag{1.3}$$

where, in each case, the supremum is taken over all $x \in \operatorname{supp}(\mu)$ (see [33]).

Theorem 1.1. *Let $d \geqslant 1$ and let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set. Let μ be a finite positive Borel measure on $\mathbb{R}^d$ with* $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$ *and* $\mu(\Omega) > 0$. *Assume that* $\underline{\dim}_\infty(\mu) > d-2$. *Then condition* (C1) *holds. Moreover, the embedding* $\operatorname{Dom}(\mathcal{E}) \hookrightarrow L^2(\Omega,\mu)$ *is compact.*

In Theorem 1.1, we do not assume that μ is a self-similar measure. However, we will prove that for a self-similar measure μ determined by an *iterated function system* (IFS) satisfying the open set condition, the condition $\underline{\dim}_\infty(\mu) > d-2$ is both necessary and sufficient for the compactness of the embedding $\operatorname{Dom}(\mathcal{E}) \hookrightarrow L^2(\Omega,\mu)$ (see Theorem 1.4).

In view of the second question, we have

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set and let μ be a positive finite Borel measure on $\mathbb{R}^d$ with* $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$ *and* $\mu(\Omega) > 0$. *Assume* $\underline{\dim}_\infty(\mu) > d-2$. *Then there exists an orthonormal basis* $\{u_n\}_{n=1}^\infty$ *of* $L^2(\Omega,\mu)$ *consisting of* (*Dirichlet*) *eigenfunctions of* $-\Delta_\mu$. *The eigenvalues* $\{\lambda_n\}_{n=1}^\infty$ *satisfy* $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ *with* $\lim_{n\to\infty}\lambda_n = \infty$. *Moreover, the eigenspace associated with each eigenvalue is finite-dimensional.*

For a bounded domain (i.e., an open connected set) Ω in $\mathbb{R}^d$, assume that a classical Green's function $g(\cdot,\cdot)$ exists on Ω. For $1 \leqslant p \leqslant \infty$, define the *Green's operator* G_μ on $L^p(\Omega,\mu)$ by

$$(G_\mu f)(x) := \int_\Omega g(x,y) f(y)\, d\mu(y).$$

In Section 4, we show that if $\underline{\dim}_\infty(\mu) > d-2$, then $G_\mu(L^2(\Omega,\mu)) \subseteq \operatorname{Dom}(-\Delta_\mu)$, and G_μ is the inverse of $-\Delta_\mu$.

Theorem 1.3. *Let Ω be a bounded domain of $\mathbb{R}^d$ for which the classical Green's function exists. Let μ be a positive finite Borel measure on $\mathbb{R}^d$ with* $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$ *and* $\mu(\Omega) > 0$. *Assume* $\underline{\dim}_\infty(\mu) > d-2$. *Then for any* $f \in L^2(\Omega,\mu)$, $G_\mu f \in \operatorname{Dom}(-\Delta_\mu)$ *and*

$$-\Delta_\mu(G_\mu f) = f.$$

Consequently, $G_\mu(L^2(\Omega,\mu)) \subseteq \operatorname{Dom}(-\Delta_\mu)$ *and* $G_\mu = -\Delta_\mu^{-1}$.

Theorem 1.3 says that the Green's function of Δ_μ with respect to μ is the same as the classical Green's function, provided that $\underline{\dim}_\infty(\mu) > d-2$. This result is natural. In fact, observe that if μ is the Lebesgue measure, then $\Delta_\mu = \Delta$, and so their Green's functions are the same. Clearly the Lebesgue measure satisfies the condition $\underline{\dim}_\infty(\mu) = d > d-2$.

In Section 5, we investigate in detail the condition $\underline{\dim}_\infty(\mu) > d-2$ for invariant measures determined by iterated function systems. Let $\{S_i\}_{i=1}^N$ be an IFS of contractions on $\mathbb{R}^d$; that is, for each i, there exists r_i with $0 < r_i < 1$ such that

$$\left|S_i(x) - S_i(y)\right| \leqslant r_i|x-y| \quad \text{for all } x, y \in \mathbb{R}^d.$$

It is well known (see [11,17]) that there exists a unique non-empty compact set K, called the *attractor* (or *invariant set*) satisfying

$$K = \bigcup_{i=1}^N S_i(K).$$

Moreover, for any set of *probability weights* $\{p_i\}_{i=1}^N$ (that is, $0 < p_i < 1$ and $\sum_{i=1}^N p_i = 1$), there corresponds a unique regular Borel *probability* measure μ, with $\operatorname{supp}(\mu) = K$, satisfying the identity

$$\mu = \sum_{i=1}^N p_i \mu \circ S_i^{-1}. \tag{1.4}$$

We call μ the *invariant measure* associated to the probability weights $\{p_i\}_{i=1}^N$. It follows from our definition that μ must be continuous.

More can be said if the IFS $\{S_i\}_{i=1}^N$ consists of *contractive similitudes*; that is

$$S_i(x) = r_i R_i x + b_i, \quad i = 1, \dots, N, \tag{1.5}$$

where for each i, $0 < r_i < 1$, R_i is a $d \times d$ orthogonal matrix, and $b_i \in \mathbb{R}^d$. In this case, the attractor K is called a (*strictly*) *self-similar set*, an invariant measure μ is called a (*strictly*) *self-similar measure*, and identity (1.4) is called a *self-similar identity*. It follows from a result of Peres and B. Solomyak [30] that for such a μ (and, in fact, for the more general class of self-conformal measures) $\underline{\dim}_\infty(\mu) = \overline{\dim}_\infty(\mu) =: \dim_\infty(\mu)$ (see Remark 5.3).

Recall that an IFS $\{S_i\}_{i=1}^N$ satisfies the *open set condition* (OSC) if there exists a non-empty bounded open set U, called a *basic open set*, such that $\bigcup_{i=1}^N S_i(U) \subseteq U$ and $S_i(U) \cap S_j(U) = \emptyset$ for any $i \neq j$. In this case, for any associated self-similar measure μ, we have that $\operatorname{supp}(\mu) = K \subseteq \overline{U}$.

For a self-similar measure associated with an IFS of contractive similitudes satisfying the OSC, we have

Theorem 1.4. *Let $\{S_i\}_{i=1}^N$ be an IFS of contractive similitudes on $\mathbb{R}^d$ $(d \geqslant 1)$ satisfying the OSC, and let μ be an associated self-similar measure. Assume that Ω is a bounded open subset of $\mathbb{R}^d$ with* $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$ *and $\mu(\Omega) > 0$. Then the following conditions are equivalent*:

(a) *Condition* (C1) *holds, and the embedding* $\operatorname{Dom}(\mathcal{E}) \hookrightarrow L^2(\Omega, \mu)$ *is compact*;

(b) $\bar{A} := \max_{1 \leqslant i \leqslant N}\{p_i r_i^{-(d-2)}\} < 1$;
(c) $\underline{\dim}_\infty(\mu) > d - 2$;
(d) *μ is upper s-regular for some $s > d - 2$.*

In particular, all the conditions hold on $\mathbb{R}^2$.

From this theorem, we see that for the above class of measures, the condition $\underline{\dim}_\infty(\mu) > d - 2$ in Theorem 1.1 is sharp. The equivalence of (a) and (b) has already been established by Naimark and M. Solomyak (see [28,29]). Our main proof is on (b) implying (c), for which we make use of the (*lower*) *L^q-spectrum $\tau(q)$* of a measure. Recall that

$$\tau(q) := \liminf_{\delta \to 0^+} \frac{\ln(\sup \sum_i \mu(B_\delta(x_i))^q)}{\ln \delta}, \quad q \in \mathbb{R}, \tag{1.6}$$

where $\{B_\delta(x_i)\}_i$ is a countable family of disjoint closed δ-balls centered at $x_i \in \operatorname{supp}(\mu)$, and the supremum is taken over all such families (see [5,21]).

For iterated function systems with overlaps (that is, the open set condition fails), it is in general not easy to verify the condition $\underline{\dim}_\infty(\mu) > d - 2$. Nevertheless, we show that this condition holds on $\mathbb{R}^2$ for invariant measures defined by iterated function systems of *bi-Lipschitz contractions* (Lemma 5.1), provided that the attractor K is not a singleton.

For $d \geqslant 3$, we will show that the condition $\underline{\dim}_\infty(\mu) > d - 2$ can be verified, provided the (lower) L^q-spectrum $\tau(q)$ can be computed. The computation of $\tau(q)$ in the absence of the open set condition is an interesting and challenging problem. It has been studied extensively for iterated function systems satisfying certain weak separation condition (see [12,18,19,22,23,25]). Using the fact that $\underline{\dim}_\infty(\mu) > d - 2$ if and only if there exists some $q > 0$ such that $\tau(q)/q > d - 2$ (Lemma 5.7), we can verify the condition $\underline{\dim}_\infty(\mu) > d - 2$ by computing $\tau(q)$. We show that if the IFS satisfies the *weak separation condition** (WSC*) in [25], then $\tau(q)$, $q \in \mathbb{N}$, can be computed (Theorem 5.9).

This paper is organized as follows. In Section 2, we define the Laplacian Δ_μ and study some of its properties. In Section 3, we make use of $\underline{\dim}_\infty(\mu)$ to study the compactness of the embedding $\operatorname{Dom}(\mathcal{E}) \hookrightarrow L^2(\Omega, \mu)$, and prove Theorems 1.1 and 1.2. In Section 4 we study the Green's operator and prove Theorem 1.3. In Section 5, we prove Theorem 1.4, and investigate the condition $\underline{\dim}_\infty(\mu) > d - 2$, especially for invariant measures defined by various classes of iterated function systems.

2. Fractal Laplace operators

Throughout this section, we let $\Omega \subseteq \mathbb{R}^d$ $(d \geqslant 1)$ be a bounded open set, and μ be a positive finite Borel measure on $\mathbb{R}^d$ with $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. We assume that condition (C1) holds. Under this condition, we will introduce the fractal Laplacian Δ_μ, and study its basic properties.

Let Q be a quadratic form with domain $\operatorname{Dom}(Q)$ on a Hilbert space $\mathcal{H}$, with inner product $\langle\cdot,\cdot\rangle$. Define Q_* by $Q_*(u, v) = Q(u, v) + \langle u, v\rangle$. Recall that the form $(Q, \operatorname{Dom}(Q))$ is *closed* if the space $\mathrm{Dom}(Q)$ is complete under the norm $Q_*(u,u)^{1/2}$. Define another nonnegative bilinear form $\mathcal{E}_*(\cdot,\cdot)$ on $L^2(\Omega,\mu)$ by

$$\mathcal{E}_*(u,v) := \mathcal{E}(u,v) + \langle u,v\rangle_{L^2(\Omega,\mu)} = \int_\Omega \nabla u\cdot\nabla v\,dx + \int_\Omega uv\,d\mu. \tag{2.1}$$

It is clear that $\mathcal{E}_*(\cdot,\cdot)$ is an inner product on $\mathrm{Dom}(\mathcal{E})$.

Proposition 2.1. *Let $\Omega \subseteq \mathbb{R}^d$ $(d \geqslant 1)$ be a bounded open set, and let μ be a positive finite Borel measure on $\mathbb{R}^d$ with $\mathrm{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. Let $\mathcal{E}$ and $\mathcal{E}_*$ be the quadratic forms defined as in* (1.1) *and* (2.1)*, respectively. Assume* (C1) *holds. Then we have*

(a) $\mathrm{Dom}(\mathcal{E})$ *is dense in* $L^2(\Omega,\mu)$.
(b) $(\mathcal{E}_*, \mathrm{Dom}(\mathcal{E}))$ *is a Hilbert space.*

Proof. (a) Note that $C_c(\Omega)$, the space of continuous functions with compact support in Ω, is dense in $L^2(\Omega,\mu)$. Next, $C_c^\infty(\Omega)$ is dense in $C_c(\Omega)$ in the supremum norm, and by using $\mu(\Omega) < \infty$, we see that $C_c^\infty(\Omega)$ is also dense in $C_c(\Omega)$ in the $L^2(\Omega,\mu)$-norm. Now let $u \in L^2(\Omega,\mu)$ and let $\{u_n\}$ be a sequence in $C_c^\infty(\Omega)$ converging to u in the $L^2(\Omega,\mu)$-norm. Write $u_n = u_n^0 + u_n^\perp$, where $u_n^0 \in \mathcal{N}$ and $u_n^\perp \in \mathcal{N}^\perp = \mathrm{Dom}(\mathcal{E})$. It is clear that $u_n^\perp \to u$ in $L^2(\Omega,\mu)$. This proves (a).

(b) Under assumption (C1), the norm induced by $\mathcal{E}_*$ is equivalent to the norm $\|\cdot\|_{H_0^1(\Omega)}$. Hence $(\mathcal{E}_*, \mathrm{Dom}(\mathcal{E}))$ is complete. □

It follows from Proposition 2.1 that under condition (C1), the quadratic form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ is closed on $L^2(\Omega,\mu)$. Hence, there exists a nonnegative self-adjoint operator H on $L^2(\Omega,\mu)$ such that $\mathrm{Dom}(H) \subseteq \mathrm{Dom}(H^{1/2}) = \mathrm{Dom}(\mathcal{E})$ and

$$\mathcal{E}(u,v) = \langle H^{1/2}u, H^{1/2}v\rangle_{L^2(\Omega,\mu)} \quad \text{for all } u,v \in \mathrm{Dom}(\mathcal{E}).$$

Moreover, $u \in \mathrm{Dom}(H)$ if and only if $u \in \mathrm{Dom}(\mathcal{E})$ and there exists $f \in L^2(\Omega,\mu)$ such that $\mathcal{E}(u,v) = \langle f,v\rangle_{L^2(\Omega,\mu)}$ for all $v \in \mathrm{Dom}(\mathcal{E})$. Note that for all $u \in \mathrm{Dom}(H)$ and $v \in \mathrm{Dom}(\mathcal{E})$,

$$\int_\Omega \nabla u\cdot\nabla v\,dx = \mathcal{E}(u,v) = \langle Hu, v\rangle_{L^2(\Omega,\mu)}. \tag{2.2}$$

Let $\mathcal{D}(\Omega)$ denote the space of *test functions* consisting of $C_c^\infty(\Omega)$ equipped with the following topology: a sequence $\{u_n\}$ converges to a function u in $\mathcal{D}(\Omega)$ if there exists a compact $K \subseteq \Omega$ such that $\mathrm{supp}(u_n) \subseteq K$ for all n, and for any partial derivative D^s of order s, the sequence $\{D^s u_n\}$ converges to $D^s u$ uniformly on K (see [36, p. 29]). Denote by $\mathcal{D}'(\Omega)$ the space of distributions, the dual space of $\mathcal{D}(\Omega)$.

Proposition 2.2. *Assume that condition* (C1) *holds. For $u \in \mathrm{Dom}(\mathcal{E})$ and $f \in L^2(\Omega,\mu)$, the following conditions are equivalent*:

(a) $u \in \mathrm{Dom}(H)$ *and* $Hu = f$;

(b) $-\Delta u = f\,d\mu$ *in the sense of distribution*; *that is, for any* $v \in \mathcal{D}(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega vf\,d\mu. \tag{2.3}$$

Proof. Assume that (a) holds. We have, for any $v \in \mathcal{D}(\Omega)$, and for any $s \geqslant 0$,

$$\left|\int_\Omega D^s vf\,d\mu\right| \leqslant \|f\|_{L^1(\Omega,\mu)} \left\|D^s v\right\|_\infty \leqslant \big(\mu(\Omega)\big)^{1/2} \|f\|_{L^2(\Omega,\mu)} \left\|D^s v\right\|_\infty.$$

Thus $f\,d\mu$ defines a continuous linear functional on $\mathcal{D}(\Omega)$, and so it is a distribution.

Moreover, we see from (2.2) that

$$\int_\Omega vf\,d\mu = \langle Hu, v\rangle_{L^2(\Omega,\mu)} = \mathcal{E}(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx,$$

for any $v \in \mathcal{D}(\Omega)$. Hence (b) holds.

Conversely, assume that (b) holds. Since $\mathcal{D}(\Omega)$ is dense in $\mathrm{Dom}(\mathcal{E})$, one can show, by using condition (C1), that (2.3) also holds for all $v \in \mathrm{Dom}(\mathcal{E})$. Hence, we see that $\mathcal{E}(u,v) = \langle f, v\rangle_{L^2(\Omega,\mu)}$ for all $v \in \mathrm{Dom}(\mathcal{E})$. This implies that $u \in \mathrm{Dom}(H)$ and $Hu = f$. Therefore, (a) follows. □

In the sequel, we denote $-H$ by Δ_μ and call Δ_μ a *Laplacian with respect to* μ. Proposition 2.2 says that for any $u \in \mathrm{Dom}(\Delta_\mu)$, $\Delta u = \Delta_\mu u\,d\mu$ in the sense of distribution. We rewrite (2.2) as

$$\int_\Omega \nabla u \cdot \nabla v\,dx = \mathcal{E}(u,v) = \langle -\Delta_\mu u, v\rangle_{L^2(\Omega,\mu)} \tag{2.4}$$

for $u \in \mathrm{Dom}(\Delta_\mu)$ and $v \in \mathrm{Dom}(\mathcal{E})$.

The following theorem shows that for any $f \in L^2(\Omega,\mu)$, the equation

$$\Delta_\mu u = f, \quad u|_{\partial\Omega} = 0,$$

has a unique solution in $L^2(\Omega,\mu)$.

Theorem 2.3. *Assume that condition* (C1) *holds. Then, for any* $f \in L^2(\Omega,\mu)$, *there exists a unique* $u \in \mathrm{Dom}(\Delta_\mu)$ *such that* $\Delta_\mu u = f$. *The operator*

$$\Delta_\mu^{-1} : L^2(\Omega,\mu) \to \mathrm{Dom}(\Delta_\mu), \quad f \mapsto u,$$

is bounded and has norm at most C, *the constant in condition* (C1).

Proof. Let $f \in L^2(\Omega,\mu)$. Define a linear functional T_f on $\mathrm{Dom}(\mathcal{E})$ by

$$T_f(v) = -\int_\Omega fv\,d\mu, \quad v \in \mathrm{Dom}(\mathcal{E}).$$

Then, by condition (C1),

$$\left|T_f(v)\right| \leqslant \|f\|_{L^2(\Omega,\mu)}\|v\|_{L^2(\Omega,\mu)} \leqslant C\|f\|_{L^2(\Omega,\mu)}\mathcal{E}(v,v)^{1/2}.$$

Hence T_f is continuous. By the Riesz representation theorem, there exists a unique $u \in \mathrm{Dom}(\mathcal{E})$ such that

$$\|u\|_{H_0^1(\Omega)} = \|T_f\| \leqslant C\|f\|_{L^2(\Omega,\mu)}, \tag{2.5}$$

and for all $v \in \mathrm{Dom}(\mathcal{E})$,

$$-\int_\Omega f v\, d\mu = T_f(v) = \mathcal{E}(u,v).$$

Therefore $\Delta u = f\, d\mu$ in the sense of distribution. By Proposition 2.2, we have that $u \in \mathrm{Dom}(\Delta_\mu)$ and $\Delta_\mu u = f$. The last assertion follows from (2.5). □

3. The L^∞-dimension and compactness of Δ_μ^{-1}

Let μ be a finite positive Borel measure on $\mathbb{R}^d$ with bounded support. In this section we are concerned with the condition for which assumption (C1) holds. As a result, we will obtain a sufficient condition for the operator $(\Delta_\mu)^{-1}$ to be compact. The case of self-similar measures will be discussed in Section 5.

We state the relation between the upper (or lower) regularity and lower (or upper) L^∞-dimension of μ. We say that μ is *upper s-regular* for $s > 0$, if there exists some $c > 0$ such that, for all $x \in \mathrm{supp}(\mu)$ and all $0 \leqslant r \leqslant \mathrm{diam}(\mathrm{supp}(\mu))$,

$$\mu\big(B_r(x)\big) \leqslant c r^s. \tag{3.1}$$

The *lower s-regularity* is defined by reversing the inequality.

Lemma 3.1. *Assume that μ is a finite positive Borel measure on $\mathbb{R}^d$ with bounded support.*

(a) *If μ is upper* (*respectively lower*) *s-regular for some $s > 0$, then $\underline{\dim}_\infty(\mu) \geqslant s$* (*respectively $\overline{\dim}_\infty(\mu) \leqslant s$*).
(b) *Conversely, if $\underline{\dim}_\infty(\mu) \geqslant s$* (*respectively $\overline{\dim}_\infty(\mu) \leqslant s$*) *for some $s > 0$, then μ is upper* (*respectively lower*) *α-regular for any $0 < \alpha < s$* (*respectively $\alpha > s$*).

Proof. The conclusion (a) directly follows from the definitions in (1.3) and (3.1). To show (b), let $\underline{\dim}_\infty(\mu) \geqslant s$ and $0 < \alpha < s$. By the definition in (1.3), there exist $r_0, \varepsilon > 0$ such that, for any $0 < r < r_0$,

$$\frac{\ln(\sup_x \mu(B_r(x)))}{\ln r} \geqslant s - \varepsilon \geqslant \alpha + \varepsilon,$$

which implies that

$$\mu\big(B_r(x)\big) \leqslant r^{\alpha+\varepsilon} \leqslant cr^\alpha \tag{3.2}$$

for all $x \in \operatorname{supp}(\mu)$. Note that (3.2) also holds for $r \geqslant r_0$ by adjusting the value of c, since μ is finite and has compact support. Thus μ is upper α-regular. Similarly, one can show that if $\overline{\dim}_\infty(\mu) \leqslant s$ and $\alpha > s$, then μ satisfies (3.1) with α replacing s. $\square$

Let Ω be a bounded open subset of $\mathbb{R}^d$. Note that if the *unit ball*

$$B_0 := \left\{u \in C_c^\infty(\Omega)\colon \|u\|_{H_0^1(\Omega)} \leqslant 1\right\}$$

is *relatively compact* in $L^2(\Omega, \mu)$, then condition (C1) holds and the embedding $\mathrm{Dom}(\mathcal{E}) \hookrightarrow L^2(\Omega, \mu)$ is compact. The following theorem, based on a result in [27], is crucial in establishing the relative compactness of B_0 in $L^2(\Omega, \mu)$.

Theorem 3.2. *Let $d \geqslant 2$ and $2 < q < \infty$, and let μ be a finite positive Borel measure on $\mathbb{R}^d$ with bounded support. Let $B = \{u \in C_c^\infty(\mathbb{R}^d)\colon \|u\|_{H_0^1(\mathbb{R}^d)} \leqslant 1\}$.*

(a) *If $\underline{\dim}_\infty(\mu) > q(d-2)/2$, then B is relatively compact in $L^q(\mathbb{R}^d, \mu)$.*
(b) *If $\underline{\dim}_\infty(\mu) < q(d-2)/2$, then B is not relatively compact in $L^q(\mathbb{R}^d, \mu)$.*

Proof. We will use the following result. For $q > 2$, the ball $B = \{u \in C_c^\infty(\mathbb{R}^d)\colon \|u\|_{H_0^1(\mathbb{R}^d)} \leqslant 1\}$ is relatively compact in $L^q(\mathbb{R}^d, \mu)$ if and only if

$$\lim_{\delta \to 0^+} \sup_{x \in \mathbb{R}^d;\, r \in (0,\delta)} r^{1-d/2} \mu\big(B_r(x)\big)^{1/q} = 0 \quad \text{for } d > 2, \quad \text{and} \tag{3.3}$$

$$\lim_{\delta \to 0^+} \sup_{x \in \mathbb{R}^d;\, r \in (0,\delta)} |\ln r|^{1/2} \mu\big(B_r(x)\big)^{1/q} = 0 \quad \text{for } d = 2 \tag{3.4}$$

(see Maz'ja [27, p. 386]).

(a) Since $\underline{\dim}_\infty(\mu) > q(d-2)/2$, by Lemma 3.1(b), there is $\alpha > q(d-2)/2$ such that μ is upper α-regular; that is, for all $r > 0$ and all $x \in \operatorname{supp}(\mu)$, $\mu(B_r(x)) < cr^\alpha$.

If $d > 2$, we obtain

$$\sup_{x \in \mathbb{R}^d;\, r \in (0,\delta)} r^{1-d/2} \mu\big(B_r(x)\big)^{1/q} < c^{1/q} \delta^{(\alpha - q(d-2)/2)/q},$$

which implies (3.3). If $d = 2$, we have

$$\sup_{x \in \mathbb{R}^d;\, r \in (0,\delta)} |\ln r|^{1/2} \mu\big(B_r(x)\big)^{1/q} < c^{1/q} |\ln \delta|^{1/2} \delta^{\alpha/q},$$

and so (3.4) holds.

(b) For $d > 2$, it is straightforward to show that

$$\lim_{\delta \to 0^+} \sup_{x \in \mathbb{R}^d;\, r \in (0,\delta)} r^{1-d/2} \mu\big(B_r(x)\big)^{1/q} = 0 \quad \Rightarrow \quad \underline{\dim}_\infty(\mu) \geqslant \frac{q(d-2)}{2}.$$

Since the inequality $\underline{\dim}_\infty(\mu) \geqslant 0$ always holds, the case $d = 2$ is trivial. Hence, if $\underline{\dim}_\infty(\mu) < q(d-2)/2$, then B is not relatively compact in $L^q(\mathbb{R}^d, \mu)$. $\square$

We are now in a position to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. For the case $d = 1$, the conclusion of the theorem follows from the fact that $H_0^1(\Omega)$ is compactly embedded in $C(\overline{\Omega})$, the space of continuous functions on $\overline{\Omega}$ (cf. [1]). It remains to consider the case $d \geqslant 2$.

Let $s := \underline{\dim}_\infty(\mu) > d - 2$. Choose q so that $2 < q < 2s/(d-2)$. Since $\underline{\dim}_\infty(\mu) = s > q(d-2)/2$, we see from the above theorem that the unit ball B is relatively compact in $L^q(\mathbb{R}^d, \mu)$. Since μ is a finite measure, the space $L^q(\mathbb{R}^d, \mu)$ is embedded in $L^2(\mathbb{R}^d, \mu)$. Consequently, the unit ball B is relatively compact in $L^2(\mathbb{R}^d, \mu)$. Noting that $B_0 \subset B$, we obtain that B_0 is relatively compact in $L^2(\Omega, \mu)$. Thus, condition (C1) holds, and the embedding $\mathrm{Dom}(\mathcal{E}) \hookrightarrow L^2(\Omega, \mu)$ is compact. □

Proof of Theorem 1.2. This theorem is a direct consequence of Theorem 1.1. In fact, since $\underline{\dim}_\infty(\mu) > d - 2$, the embedding $\mathrm{Dom}(\mathcal{E}) \hookrightarrow L^2(\Omega, \mu)$ is compact by Theorem 1.1. A standard argument implies that the operator $-\Delta_\mu$ possesses a sequence of eigenfunctions $\{u_n\}_{n=1}^\infty$ that forms a complete orthonormal basis of $L^2(\Omega, \mu)$, with corresponding positive eigenvalues λ_n converging to ∞ as $n \to \infty$. Moreover, each eigenvalue is of finite multiplicity (see, for example, [8]). □

The domain and spectrum $\mathrm{Spec}(-\Delta_\mu)$ of $-\Delta_\mu$ can be characterized by the eigenfunctions $\{u_n\}$ and eigenvalues $\{\lambda_n\}$ of $-\Delta_\mu$ as follows:

(a) $\mathrm{Dom}(-\Delta_\mu) = (-\Delta_\mu)^{-1}(L^2(\Omega, \mu)) = \{\sum_{n=1}^\infty a_n u_n \colon \sum_{n=1}^\infty a_n^2 \lambda_n^2 < \infty\}$;
(b) $\mathrm{Spec}(-\Delta_\mu) = \overline{\{\lambda_n\}}$.

The proofs of these are standard; we omit the details.

4. Green's operator

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain (i.e., open and connected). Let μ be a positive finite Borel measure with $\mathrm{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$ as before. Throughout this section, we assume that the Green's function $g(x, y)$ for the classical Laplacian Δ exists on Ω. We will prove that this Green's function $g(x, y)$ is also the Green's function for Δ_μ, if condition (C2) holds. We show that (C2) is true if $\underline{\dim}_\infty(\mu) > d - 2$ (see Proposition 4.1). Finally, we prove Theorem 1.3.

Note that if $u \in C^2(\Omega)$, we have

$$u(x) = \int_\Omega g(x, y)(-\Delta u)(y)\, dy \quad (x \in \Omega). \tag{4.1}$$

For $f \in C^1(\Omega)$, the equation

$$-\Delta u = f \quad \text{with } u|_{\partial\Omega} = 0 \tag{4.2}$$

possesses a unique solution in $C^2(\Omega)$ given by

$$u(x) = \int_\Omega g(x, y) f(y)\, dy. \tag{4.3}$$

Note that for $d=1$ and $\Omega=(a,b)$,

$$g(x,y)=\begin{cases}(x-a)(b-y) & \text{if } x\leqslant y,\\ (y-a)(b-x) & \text{if } x>y.\end{cases}$$

For $d\geqslant 2$,

$$g(x,y)=\begin{cases}-\frac{1}{2\pi}\ln|x-y|+h(x,y) & \text{if } d=2,\\ -\frac{1}{|x-y|^{d-2}}+h(x,y) & \text{if } d\geqslant 3,\end{cases} \tag{4.4}$$

where $h(x,\cdot)$ is harmonic in $x\in\Omega$ for any fixed $y\in\Omega$, and $h(x,y)=h(y,x)$ is continuous on $\overline{\Omega}\times\overline{\Omega}$. The function g is equal to 0 for $x\in\Omega$ and $y\in\partial\Omega$ or for $y\in\Omega$ and $x\in\partial\Omega$ (see [10]).

It is known that the Green's function exists for any domain $\Omega\subseteq\mathbb{R}^2$ which can be conformally mapped onto the closed unit disk (see [6, p. 377]). In $\mathbb{R}^3$, the Green's function exists for any domain Ω each of its boundary points is a vertex of a tetrahedron lying outside of Ω (see [7, pp. 290–292]). In [26], it was proved that the Green's function for $-\Delta$ exists for any regular domain $\Omega\subseteq\mathbb{R}^d$ $(d\geqslant 2)$. See also [2].

Recall that $g(x,y)\geqslant 0$ for all $x,y\in\Omega$. We introduce the following condition:

(C2) $$\sup_{x\in\Omega}\int_{\Omega} g(x,y)\,d\mu(y)\leqslant C<\infty \quad \text{for some constant } C>0.$$

Note that this condition automatically holds for the case $d=1$.

Proposition 4.1. *Let Ω be a bounded domain in $\mathbb{R}^d$ for which the Green's function $g(\cdot,\cdot)$ exists, and let μ be a positive finite Borel measure with* $\operatorname{supp}(\mu)\subseteq\overline{\Omega}$. *Assume $\underline{\dim}_\infty(\mu)>d-2$. Then condition* (C2) *holds.*

Proof. Assume that $\underline{\dim}_\infty(\mu)>d-2$. By Lemma 3.1(b), we see that μ is upper α-regular for some $\alpha>d-2$; that is, there exists a constant $c>0$ such that for all $x\in\operatorname{supp}(\mu)$ and all $r>0$,

$$\mu\big(B_r(x)\big)<cr^{\alpha}. \tag{4.5}$$

In order to prove (C2), we divide the proof into the following two cases: $d=2$ and $d>2$. (The case $d=1$ is clear.)

Case 1. $d=2$. By (4.4), it suffices to prove that there exists some constant $C>0$ such that

$$\int_{\Omega}\big|\ln|x-y|\big|\,d\mu(y)\leqslant C \tag{4.6}$$

for all $x\in\Omega$. Indeed, letting $r_0:=\operatorname{diam}(\Omega)$, we have that

$$\int_{\Omega}\big|\ln|x-y|\big|\,d\mu(y)=\int_{|y-x|<1}\big|\ln|x-y|\big|\,d\mu(y)+\int_{1\leqslant|y-x|\leqslant r_0}\big|\ln|x-y|\big|\,d\mu(y). \tag{4.7}$$

The second integral on the right-hand side of (4.7) is bounded for all $x \in \Omega$, since Ω is bounded and $\mu(\Omega) < \infty$. The first integral is also uniformly bounded in x, by noting that, using (4.5),

$$\begin{aligned}\int_{|y-x|<1} \big|\ln|x-y|\big|\, d\mu(y) &= \sum_{k=1}^{\infty} \int_{2^{-k}\leqslant|y-x|<2^{-(k-1)}} \big|\ln|x-y|\big|\, d\mu(y) \\ &\leqslant \sum_{k=1}^{\infty} \big(\ln 2^k\big)\mu\big(B_{2^{-(k-1)}}(x)\big) \\ &\leqslant c(\ln 2)\sum_{k=1}^{\infty} k 2^{-\alpha(k-1)} < \infty.\end{aligned}$$

This proves (C2) for the case $d = 2$.

Case 2. $d > 2$. The proof is similar to that of the case $d = 2$. By (4.4), it is sufficient to prove that there exists a constant $C > 0$ such that

$$\int_{\Omega} |x-y|^{-(d-2)}\, d\mu(y) \leqslant C \tag{4.8}$$

for all $x \in \Omega$. This is true, since

$$\int_{\Omega} |x-y|^{-(d-2)}\, d\mu(y) = \int_{|x-y|<1} |x-y|^{-(d-2)}\, d\mu(y) + \int_{1\leqslant|x-y|\leqslant r_0} |x-y|^{-(d-2)}\, d\mu(y).$$

The second integral on the right-hand side is clearly bounded. The first one is estimated as follows, using (4.5) again:

$$\begin{aligned}\int_{|x-y|<1} |x-y|^{-(d-2)}\, d\mu(y) &= \sum_{k=1}^{\infty} \int_{2^{-k}\leqslant|y-x|<2^{-(k-1)}} |x-y|^{-(d-2)}\, d\mu(y) \\ &\leqslant \sum_{k=1}^{\infty} 2^{k(d-2)}\mu\big(B_{2^{-(k-1)}}(x)\big) \\ &\leqslant c2^{\alpha}\sum_{k=1}^{\infty} 2^{-k(\alpha-(d-2))} < \infty.\end{aligned}$$

This proves (C2) for $d > 2$. $\square$

For $1 \leqslant p \leqslant \infty$, we define the *Green's operator* G_μ on $L^p(\Omega, \mu)$ by

$$(G_\mu f)(x) := \int_{\Omega} g(x,y) f(y)\, d\mu(y) \quad (x \in \Omega).$$

We remark that this operator and its generalizations have been studied by many authors (see, e.g., [3,15,16,34,35]). Clearly, G_μ is self-adjoint by using the fact that $g(\cdot,\cdot)$ is symmetric. Moreover, by (C2), we obtain that G_μ is bounded on $L^p(\Omega,\mu)$ for any $1 \leqslant p \leqslant \infty$; that is, for all $f \in L^p(\Omega,\mu)$,

$$\|G_\mu f\|_{L^p(\Omega,\mu)} \leqslant C\|f\|_{L^p(\Omega,\mu)}, \tag{4.9}$$

where C is the same constant as in (C2). Indeed, it is easy to see from (C2) that (4.9) holds for $p = 1$ or $p = \infty$. For $1 < p < \infty$, by using Hölder's inequality and (C2), we obtain

$$\begin{aligned}
\|G_\mu f\|^p_{L^p(\Omega,\mu)} &= \int_\Omega \left| \int_\Omega g(x,y) f(y)\, d\mu(y) \right|^p d\mu(x) \\
&\leqslant \int_\Omega \left\{ \int_\Omega g(x,y) \left|f(y)\right|^p d\mu(y) \right\} \left\{ \int_\Omega g(x,y)\, d\mu(y) \right\}^{p-1} d\mu(x) \\
&\leqslant C^p \|f\|^p_{L^p(\Omega,\mu)}.
\end{aligned} \tag{4.10}$$

Thus (4.9) also holds for $1 < p < \infty$. We will show below that the operator G_μ is also bounded from $L^p(\Omega,\mu)$ to $L^p(\Omega,dx)$ for any $1 \leqslant p \leqslant \infty$, if condition (C2) holds.

Proposition 4.2. *Let $\Omega \subseteq \mathbb{R}^d$ $(d \geqslant 1)$ be a bounded domain for which the classical Green's function $g(\cdot,\cdot)$ exists. Assume that condition* (C2) *holds. Then there exists some $C > 0$ such that, for all $f \in L^p(\Omega,\mu)$ with $1 \leqslant p \leqslant \infty$,*

$$\|G_\mu f\|_{L^p(\Omega,dx)} \leqslant C\|f\|_{L^p(\Omega,\mu)}. \tag{4.11}$$

Proof. Note that the Lebesgue measure $\mathcal{L}$ satisfies $\underline{\dim}_\infty(\mathcal{L}) = d > d-2$, and so (C2) holds for $\mathcal{L}$ by using Proposition 4.1; that is,

$$\sup_{x\in\Omega} \int_\Omega g(x,y)\, dy \leqslant C < \infty.$$

Let $1 < p < \infty$. Similar to (4.10), we have that

$$\begin{aligned}
\int_\Omega \left|G_\mu f(x)\right|^p dx &= \int_\Omega \left| \int_\Omega g(x,y) f(y)\, d\mu(y) \right|^p dx \\
&\leqslant \int_\Omega \left\{ \int_\Omega g(x,y) \left|f(y)\right|^p d\mu(y) \right\} \left\{ \int_\Omega g(x,y)\, d\mu(y) \right\}^{p-1} dx \\
&\leqslant C^{p-1} \int_\Omega \left\{ \int_\Omega g(x,y)\, dx \right\} \left|f(y)\right|^p d\mu(y) \\
&\leqslant C^p \|f\|^p_{L^p(\Omega,\mu)},
\end{aligned}$$

showing that (4.11) holds for $1 < p < \infty$. The cases $p = 1$ and $p = \infty$ are clear. $\square$

Proof of Theorem 1.3. We first claim that $G_\mu f \in H_0^1(\Omega)$ for all $f \in L^2(\Omega,\mu)$. The proof given here is motivated by [3, Proposition 3.1].

For $f \in L^2(\Omega,\mu)$, let $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$ be the positive and negative parts of f, respectively. Then $f^+,\ f^- \in L^2(\Omega,\mu)$, and

$$G_\mu f = G_\mu f^+ - G_\mu f^-.$$

We show that $G_\mu f^+ \in H_0^1(\Omega)$. To do this, it suffices to prove (see [15, Theorem 10] or [16, Theorem 9]) that

$$\int_\Omega \big(G_\mu f^+\big)(x) f^+(x)\, d\mu(x) < \infty. \tag{4.12}$$

But this easily follows by noting that $\|G_\mu f^+\|_{L^2(\Omega,\mu)} \leqslant C\|f^+\|_{L^2(\Omega,\mu)}$ (see (4.10)) and

$$\int_\Omega \big(G_\mu f^+\big)(x) f^+(x)\, d\mu(x) \leqslant \big\|G_\mu f^+\big\|_{L^2(\Omega,\mu)} \big\|f^+\big\|_{L^2(\Omega,\mu)} \leqslant C\big\|f^+\big\|^2_{L^2(\Omega,\mu)}.$$

Thus, $G_\mu f^+ \in H_0^1(\Omega)$. Similarly, $G_\mu f^- \in H_0^1(\Omega)$, and hence the claim follows.

Next, we show that $G_\mu f \in \mathrm{Dom}(\mathcal{E})$. Let $u \in \mathcal{N}$ and let $\{u_n\}$ be a sequence in $C_c^\infty(\Omega)$ such that $u_n \to u$ in $H_0^1(\Omega)$ and $u_n \to u$ in $L^2(\Omega,\mu)$. Then

$$\begin{aligned}
\langle G_\mu f, u\rangle_{H_0^1(\Omega)} &= \lim_{n\to\infty} \int_\Omega \bigg(\int_\Omega g(x,y) f(y)\, d\mu(y)\bigg) \Delta u_n(x)\, dx \\
&= \lim_{n\to\infty} \int_\Omega \bigg(\int_\Omega g(x,y) \Delta u_n(x)\, dx\bigg) f(y)\, d\mu(y) \quad \text{(Fubini)} \\
&= \lim_{n\to\infty} \int_\Omega u_n(y) f(y)\, d\mu(y) \quad \big(\text{by (4.1)}\big).
\end{aligned}$$

Thus,

$$\big|\langle G_\mu f, u\rangle_{H_0^1(\Omega)}\big| \leqslant \lim_{n\to\infty} \|u_n\|_{L^2(\Omega,\mu)} \|f\|_{L^2(\Omega,\mu)} = 0$$

and hence $G_\mu f \in \mathcal{N}^\perp = \mathrm{Dom}(\mathcal{E})$.

Lastly, we show that for any f in $L^2(\Omega,\mu)$, $-\Delta_\mu(G_\mu f) = f$. Since $G_\mu f \in \mathrm{Dom}(\mathcal{E})$, it suffices to show, in view of Proposition 2.2, that $-\Delta(G_\mu f) = f\, d\mu$ in the sense of distribution. For any $v \in \mathcal{D}(\Omega)$, it can be derived by using Fubini's theorem and (4.1) as above that

$$\int_\Omega v\Delta(G_\mu f)\, dx = \int_\Omega (\Delta v) G_\mu f\, dx = -\int_\Omega f(y) v(y)\, d\mu(y),$$

proving that $-\Delta(G_\mu f) = f d\mu$ in the sense of distribution. The rest of Theorem 1.3 follows easily from Theorem 2.3. $\square$

5. Self-similar measures

For an invariant measure μ defined by an iterated function system $\{S_i\}_{i=1}^N$ of contractions on $\mathbb{R}^d$, we can strengthen Theorems 1.1 and 1.2 further. For $\omega = (i_1, \dots, i_n)$, we let $S_\omega = S_{i_1} \circ \cdots \circ S_{i_n}$ and for the invariant set K, we let $K_\omega = S_\omega(K)$.

We call $\{S_i\}_{i=1}^N$ an iterated function system of *bi-Lipschitz contractions* if for each $i = 1, \dots, N$, there exist c_i, r_i with $0 < c_i \leqslant r_i < 1$ such that

$$c_i|x-y| \leqslant \big|S_i(x) - S_i(y)\big| \leqslant r_i|x-y| \quad \text{for all } x, y \in \mathbb{R}^d. \tag{5.1}$$

Lemma 5.1. *Let μ be an invariant measure of an IFS $\{S_i\}_{i=1}^N$ of bi-Lipschitz contractions on $\mathbb{R}^d$. Suppose the attractor K is not a singleton. Then μ is upper s-regular for some $s > 0$, and hence $\underline{\dim}_\infty(\mu) > 0$.*

Proof. Let c_i, r_i, $i = 1, \dots, N$, be given as in (5.1) and let $\{p_i\}_{i=1}^N$ be the associated probability weights. Since K is not a singleton, there are indices ω_1, ω_2 of the same length such that $K_{\omega_1} \cap K_{\omega_2} = \emptyset$. Hence, without loss of generality, we assume that $K_1 \cap K_2 = \emptyset$. There exists $r_0 > 0$ such that for any $x \in \mathbb{R}^d$, the ball $B_{r_0}(x)$ intersects at most one of K_1, K_2. Let $p = \min\{p_1, p_2\} < 1$ and let $c = \min_{1 \leqslant i \leqslant N}\{c_i\}$. Set

$$\phi(r) := \sup_{x \in \mathbb{R}^d} \mu\big(B_r(x)\big) \quad (r \geqslant 0).$$

For $x \in \mathbb{R}^d$ and $0 < r \leqslant r_0$, either $B_r(x) \cap K_1 = \emptyset$ or $B_r(x) \cap K_2 = \emptyset$. We only consider the former case (the latter case can be treated in a similar way). By using the fact that $S_i^{-1}(B_r(x)) \subseteq B_{r/c}(S_i^{-1}(x))$, we obtain

$$\begin{aligned}
\mu\big(B_r(x)\big) &= \sum_{i=1}^N p_i \mu\big(S_i^{-1}\big(B_r(x)\big)\big) = \sum_{i \neq 1} p_i \mu\big(S_i^{-1}\big(B_r(x)\big)\big) \\
&= \left(\sum_{i \neq 1} p_i\right) \sup_{x \in \mathbb{R}^d} \mu\big(B_{r/c}\big(S_i^{-1}(x)\big)\big) \leqslant (1-p_1)\phi\left(\frac{r}{c}\right) \\
&\leqslant (1-p)\phi\left(\frac{r}{c}\right).
\end{aligned}$$

It follows that

$$\phi(r) \leqslant (1-p)\,\phi\left(\frac{r}{c}\right) \quad (0 < r \leqslant r_0).$$

Therefore, for any $n \geqslant 0$ and any $0 < r \leqslant r_0$,

$$\phi\big(c^n r\big) \leqslant (1-p)\phi\big(c^{n-1} r\big) \leqslant \cdots \leqslant (1-p)^n \phi(r).$$

This implies that

$$\mu\big(B_{r_0 c^n}(x)\big) \leqslant C\big(r_0 c^n\big)^s,$$

where $s = \ln(1-p)/\ln c$ and $C = \exp(-\ln(1-p)\ln r_0/\ln c)$. Hence μ is upper s-regular. The last assertion follows from Lemma 3.1. □

It follows directly from Lemmas 3.1 and 5.1 that on $\mathbb{R}^2$ the above measure μ satisfies $\underline{\dim}_\infty(\mu) > 0 = d-2$. Hence by Theorem 1.1, we have

Corollary 5.2. *Let $\{S_i\}_{i=1}^N$ be an IFS of bi-Lipschitz contractions on $\mathbb{R}^2$ defined as in* (5.1)*, let μ be an invariant measure, and let Ω be a bounded open subset of $\mathbb{R}^2$ with $\mathrm{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. Then the embedding $\mathrm{Dom}(\mathcal{E}) \hookrightarrow L^2(\Omega,\mu)$ is compact. Consequently, the conclusions of Theorems* 1.2 *and* 1.3 *hold for such a measure μ.*

In order to prove Theorem 1.4, recall that if the IFS $\{S_i\}_{i=1}^N$ of contractive similitudes satisfies the OSC, then for any self-similar measure μ, the corresponding $\tau(q)$ is differentiable and satisfies

$$\sum_{i=1}^N p_i^q r_i^{-\tau(q)} = 1, \quad q \in \mathbb{R}, \tag{5.2}$$

where r_i and p_i are the contraction ratio and probability weight associated to S_i, respectively (see [5,21]). We show in the following remark that the L^∞-dimension of such measures, $\dim_\infty(\mu)$, actually exists.

Remark 5.3. Peres and B. Solomyak [30] proved that for self-conformal measures μ, and thus for all (strictly) self-similar measures, the limit defining $\tau(q)$ in (1.6) actually exists. We will show that this implies

$$\underline{\dim}_\infty(\mu) = \overline{\dim}_\infty(\mu) =: \dim_\infty(\mu).$$

To see this let $q \geqslant 0$ and note that there exists a constant $c > 0$ such that

$$\sup_x \mu\big(B_\delta(x)\big)^q \leqslant \sup \sum_i \mu\big(B_\delta(x_i)\big)^q \leqslant c\delta^{-d} \sup_x \mu\big(B_\delta(x)\big)^q,$$

where the first and third suprema are taken over all $x \in \mathrm{supp}(\mu)$, and the second one is taken over all families of disjoint δ-balls with centers $x_i \in \mathrm{supp}(\mu)$. After taking the logarithm, dividing through by $\ln\delta$ and q, and then taking lim inf and lim sup as $\delta \to 0^+$, we have

$$\frac{-d}{q} + \liminf_{\delta\to 0^+} \frac{\ln\sup_x \mu(B_\delta(x))}{\ln\delta} \leqslant \frac{\tau(q)}{q} \leqslant \liminf_{\delta\to 0^+} \frac{\ln\sup_x \mu(B_\delta(x))}{\ln\delta},$$
$$\frac{-d}{q} + \limsup_{\delta\to 0^+} \frac{\ln\sup_x \mu(B_\delta(x))}{\ln\delta} \leqslant \frac{\tau(q)}{q} \leqslant \limsup_{\delta\to 0^+} \frac{\ln\sup_x \mu(B_\delta(x))}{\ln\delta}.$$

Note that the limit of $\tau(q)/q$ exists since $\tau(q)$ is concave. Now letting $q \to \infty$ yields the assertion.

We will also need the following remark.

Remark 5.4. If μ is a self-similar measure defined by an IFS of contractive similitudes satisfying the OSC, then $\mu(K_i \cap K_j) = 0$ for any $i \neq j$. Moreover, $\mu(K_\omega) = p_\omega \mu(K) = p_\omega$ for any word ω.

To see this we recall that if μ_0 is the self-similar measure with *natural weights* $p_i = r_i^\alpha$, where α is the similarity (or Hausdorff) dimension of the attractor, then there exists a basic open set U with $\mu_0(U) = 1$ (see [31]). For a self-similar measure μ associated with arbitrary probability weights $p_i > 0$, either $\mu(U) = 1$ or $\mu(U) = 0$ (see [24]). It follows from $\mu_0(U) = 1$ that $\mu(U) = 1$. Now, by observing that $K_i \cap K_j \subseteq \overline{U}_i \cap \overline{U}_j$, we have $\mu(K_i \cap K_j) = 0$.

To see the second assertion in the remark, we notice that $\mu(K_i \cap K_j) = 0$ for $i \neq j$ implies that

$$\sum_i \mu(K_i) = \mu\Bigl(\bigcup_i K_i\Bigr) = \mu(K) = 1. \tag{5.3}$$

On the other hand, the self-similarity of μ implies that

$$\mu(K_i) = p_i \mu(K) + \sum_{j \neq i} p_j \mu\bigl(S_j^{-1}(K_i)\bigr) \geqslant p_i. \tag{5.4}$$

Combining (5.3) and (5.4) we have $\mu(K_i) = p_i$ for each i. Repeating the above procedure, we see that $\mu(K_\omega) = p_\omega$ for any word ω.

Proof of Theorem 1.4. The implication (a) $\Rightarrow$ (b) was proved in [29, Proposition 2], where the technical condition $\mu(\partial\Omega) = 0$ is required. In fact, this condition can be dropped, since we can always find a point $x_i \in \operatorname{supp}(\mu) =: K$ differing from the fixed point of S_i for each i, and then run the same proof as in [29]. (Here we are using the condition $\mu(K_i \cap K_j) = 0$ for any $i \neq j$, so that $\int_\Omega |U_n|^2\, d\mu = \int_\Omega |U_0|^2\, d\mu$; see [29, p. 283] for the definition for U_n.)

The implication (c) $\Rightarrow$ (a) is shown in Theorem 1.1. The equivalence between (c) and (d) is stated in Lemma 3.1. Note that the OSC and the self-similarity of μ are not used in establishing these implications.

It remains to prove the implication (b) $\Rightarrow$ (c), in which we need the OSC. Assume that (b) holds; that is, $\bar{A} < 1$. By (5.2), we have that

$$\tau'(q) = \frac{\sum_{i=1}^N p_i^q r_i^{-\tau(q)} \ln p_i}{\sum_{i=1}^N p_i^q r_i^{-\tau(q)} \ln r_i}.$$

By the definition of $\bar{A}$, we see that $\ln p_i \leqslant (d-2)\ln r_i + \ln \bar{A}$ for all $i = 1, \dots, N$. Consequently, by noting that $\sum_{i=1}^N p_i^q r_i^{-\tau(q)} \ln r_i < 0$ for $0 < r_i < 1$ and using (5.2) again, we obtain that

$$\begin{aligned}
\tau'(q) &\geqslant \frac{\sum_{i=1}^N p_i^q r_i^{-\tau(q)}[(d-2)\ln r_i + \ln \bar{A}]}{\sum_{i=1}^N p_i^q r_i^{-\tau(q)} \ln r_i} \\
&\geqslant d - 2 + \frac{\ln \bar{A}}{\sum_{i=1}^N r_i^{-\tau(q)} p_i^q \ln r_i} \geqslant d - 2 + \frac{\ln \bar{A}}{\ln r},
\end{aligned}$$

where $r = \min_{1\leqslant i\leqslant N} r_i$. On the other hand, it is known (see, e.g., [21]) that

$$\dim_\infty(\mu) = \lim_{q\to\infty} \tau'(q).$$

Consequently, $\dim_\infty(\mu) \geqslant d - 2 + \ln \bar{A}/\ln r > d - 2$, and so (c) holds. $\square$

In view of Theorem 1.4, the following proposition is useful in estimating the lower bound of $\dim_\infty(\mu)$.

Proposition 5.5. *Let $\{S_i\}_{i=1}^N$ be an IFS of contractive similitudes on $\mathbb{R}^d$ satisfying the OSC, and let μ be the associated self-similar measure with probability weights $\{p_i\}_{i=1}^N$. Then*

$$\dim_\infty(\mu) \geqslant \min_{1\leqslant i\leqslant N} \left\{ \frac{\ln p_i}{\ln r_i} \right\}, \tag{5.5}$$

where r_i is the contraction ratio of S_i.

Proof. Under the OSC, we have $\mu(K_i \cap K_j) = 0$ for any $i \neq j$ and $\mu(K_\omega) = p_\omega$ for any word ω (see Remark 5.4). For $0 < r < 1$, let

$$\Lambda(r) = \bigl\{\omega = (i_1, \dots, i_n):\ r_{i_1}\cdots r_{i_n} < r \leqslant r_{i_1}\cdots r_{i_{n-1}}\bigr\}. \tag{5.6}$$

(Intuitively, for each $\omega \in \Lambda(r)$, K_ω has diameter approximately r.) It is easy to see that $K = \bigcup_{\omega\in\Lambda(r)} K_\omega$. Let $s = \min_{1\leqslant i\leqslant N}\{\ln p_i/\ln r_i\}$. Then

$$\mu(K_\omega) = p_{i_1}\cdots p_{i_n} \leqslant (r_{i_1}\cdots r_{i_n})^s < r^s. \tag{5.7}$$

On the other hand, the OSC implies that there exists a constant $C > 0$ such that for each $x_0 \in K$, the ball $B_r(x_0)$ intersects at most C sets of the form K_ω, $\omega \in \Lambda(r)$ (see [11, Section 9.2]). Therefore, it follows from (5.7) that

$$\mu\bigl(B_r(x_0)\bigr) < Cr^s. \tag{5.8}$$

Therefore μ is upper s-regular, and hence $\dim_\infty(\mu) \geqslant s$ by Lemma 3.1. $\square$

While it is in general difficult to estimate the lower bound of $\underline{\dim}_\infty(\mu)$ for an invariant measure, it is straightforward to obtain an upper bound for $\overline{\dim}_\infty(\mu)$.

Proposition 5.6. *Let $\{S_i\}_{i=1}^N$ be an IFS of contractions on $\mathbb{R}^d$ with contraction ratio r_i for each i, i.e.,*

$$\bigl|S_i(x) - S_i(y)\bigr| \leqslant r_i|x - y| \quad \textit{for all } x, y \in \mathbb{R}^d,$$

and let μ be the associated invariant measure with probability weights $\{p_i\}_{i=1}^N$. Then

$$\overline{\dim}_\infty(\mu) \leqslant \max_{1\leqslant i\leqslant N} \left\{ \frac{\ln p_i}{\ln r_i} \right\}. \tag{5.9}$$

Proof. Assume without loss of generality that the attractor K satisfies $\operatorname{diam}(K) \leqslant 1$. Note that $\operatorname{supp}(\mu) = K$. For $0 < r < 1$, let $\Lambda(r)$ be the index set defined as in (5.6). Let $x_0 \in K$. Then there exists $\omega \in \Lambda(r)$ such that $x_0 \in K_\omega$. Observe that such a K_ω is contained in the ball $B_r(x_0)$. To see this, we write $x_0 = S_\omega(z_0)$ for some $z_0 \in K$. For any $x \in K_\omega$, by writing $x = S_\omega(z)$ with $z \in K$, we have

$$|x - x_0| = \big|S_\omega(z) - S_\omega(z_0)\big| \leqslant r_\omega |z - z_0| \leqslant r_\omega \operatorname{diam}(K) < r,$$

showing that $K_\omega \subseteq B_r(x_0)$. Therefore, for $0 < r < 1$ we have

$$\mu\big(B_r(x_0)\big) \geqslant \mu(K_\omega) = \sum_{\tau\in\Lambda(r)} p_\tau \mu\big(S_\tau^{-1}(K_\omega)\big) \geqslant p_\omega \mu\big(S_\omega^{-1}(K_\omega)\big) = p_\omega \geqslant (r_\omega)^s > c_0 r^s,$$

where $s = \max_{1\leqslant i\leqslant N}\{\ln p_i / \ln r_i\}$ and $c_0 = (\min_i\{r_i\})^s$. (Here we have used the fact that $r_\omega \geqslant \min_i\{r_i\} r_{i_1} \cdots r_{i_{n-1}} \geqslant \min_i\{r_i\} r$.) Hence μ is lower s-regular, and $\overline{\dim}_\infty(\mu) \leqslant s$ by using Lemma 3.1. □

In order to study some important IFSs of contractive similitudes that do not satisfy the OSC, Lau and Ngai [21] generalized the OSC by introducing a weaker notion of separation on the IFSs called the *weak separation condition* (WSC). The properties of IFSs satisfying the WSC have been studied extensively in a series of papers [9,19–23,25]. In particular, by making use of the renewal equation, they have given algorithms to calculate the L^q-spectrum $\tau(q)$ for $q = 2$ as well as for integers $q > 2$ for self-similar measures defined by several important classes of IFSs satisfying the WSC [12,25]. For such IFSs, we can make use of the following relationship to obtain a lower bound for $\underline{\dim}_\infty(\mu)$ through $\tau(q)$.

Lemma 5.7. *Let μ be any finite positive Borel measure on $\mathbb{R}^d$ $(d \geqslant 1)$ with compact support. Then $q\,\underline{\dim}_\infty(\mu) \geqslant \tau(q)$ for all $q \in \mathbb{R}$; moreover, $\lim_{q\to\infty} \tau(q)/q = \underline{\dim}_\infty(\mu)$. In particular, $\underline{\dim}_\infty(\mu) > d - 2$ if and only if there exists some $q_0 > 0$ such that $\tau(q_0)/q_0 > d - 2$.*

Proof. The first inequality is hinted in the proof of [21, Proposition 3.4]. Indeed, this inequality easily follows from (1.6) and the fact that

$$\sup_x \mu\big(B_\delta(x)\big)^q \leqslant \sup \sum_i \mu\big(B_\delta(x_i)\big)^q,$$

where $\{B_\delta(x_i)\}$ is a collection of disjoint closed δ-balls with centers $x_i \in \operatorname{supp}(\mu)$. That $\lim_{q\to\infty} \tau(q)/q = \underline{\dim}_\infty(\mu)$ is proved in [21, Proposition 3.4]. □

It follows from Lemma 5.7 that if we can compute $\tau(q)$ for positive integers q, then we may be able to verify the condition $\underline{\dim}_\infty(\mu) > d - 2$. This can be done for an interesting class of IFSs. We note that for $q > 0$, the function $\tau(q)$ has the following equivalent definition (see [20,21]):

$$\tau(q) = \sup\left\{\alpha\colon \limsup_{h\to 0^+} \frac{1}{h^{d+\alpha}} \int_{\mathbb{R}^d} \mu\big(B_h(x)\big)^q\, dx < \infty\right\}.$$

This formula enables us to compute $\tau(q)$ in terms of the spectral radius of some finite transition matrix, if the IFS satisfies a certain weak separation condition to be stated below. We generalize the method in [25] and refer the reader to [22,25] for details. Let $S_i:\mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \dots, N$, be an IFS of contractive similitudes on $\mathbb{R}^d$, with the same contraction ratio, defined by

$$S_i(x) = \rho R_i(x + d_i) = \rho R_i x + b_i, \tag{5.10}$$

where $0 < \rho < 1$, R_i is orthogonal, and $b_i, d_i \in \mathbb{R}^d$. Let μ be the self-similar measure associated with the probability weights $\{p_i\}_{i=1}^N$. In the following we denote a composition $f \circ g$ by fg for simplicity. Fix an integer $q \geqslant 2$, let $\Sigma_n = \{\mathbf{i} = (i_1, \dots, i_n)\colon\ 1 \leqslant i_j \leqslant N\}$ for $n \geqslant 1$, and define

$$\mathcal{S} = \left\{ \mathbf{s} = \left(S_{\mathbf{i}_1}^{-1} S_{\mathbf{j}_1}, \dots, S_{\mathbf{i}_{q-1}}^{-1} S_{\mathbf{j}_{q-1}}\right)\colon\ (\mathbf{i}_k,\ \mathbf{j}_k) \in \bigcup_{n=1}^{\infty} (\Sigma_n \times \Sigma_n) \text{ for } 1 \leqslant k \leqslant q-1 \right\}.$$

Define an infinite Markov matrix T with state space $\mathcal{S}$ by

$$T(\mathbf{s}) = \sum_{\mathbf{s}' \in \mathcal{S}} T_{\mathbf{s},\mathbf{s}'} \mathbf{s}', \quad \mathbf{s} = (\zeta_1, \dots, \zeta_{q-1}) \in \mathcal{S},$$

where

$$T_{\mathbf{s},\mathbf{s}'} = \sum_{i, i_1, \dots, i_{q-1}=1}^{N} \left\{ p_{i_1} \dots p_{i_{q-1}} p_i\colon\ \left(S_{i_1}^{-1} \zeta_1 S_i, \dots, S_{i_{q-1}}^{-1} \zeta_{q-1} S_i\right) = \mathbf{s}' \right\}.$$

For $\alpha \geqslant 0$, $h > 0$, and $\mathbf{s} = (\zeta_1, \dots, \zeta_{q-1}) \in \mathcal{S}$, define

$$\Phi_{\mathbf{s}}(h) := \frac{1}{h^{d+\alpha}} \int_{\mathbb{R}^d} \mu\left(B_h\left(\zeta_1(x)\right)\right) \cdots \mu\left(B_h\left(\zeta_{q-1}(x)\right)\right) \mu\left(B_h(x)\right) dx.$$

We denote the vector $\{\Phi_{\mathbf{s}}(h)\}_{\mathbf{s} \in \mathcal{S}}$ by $\boldsymbol{\Phi}(h)$ and let $\langle \mathcal{S} \rangle$ be the linear space spanned by $\mathcal{S}$. For any $\mathbf{v} = \sum_{\mathbf{s}} v_{\mathbf{s}} \mathbf{s} \in \langle \mathcal{S} \rangle$, let

$$\Phi_{\mathbf{v}}(h) := \sum_{\mathbf{s}} v_{\mathbf{s}} \Phi_{\mathbf{s}}(h).$$

It can be proved by applying the self-similar identity and a change of variables (see [25, Proposition 4.2]) that for $\mathbf{s} \in \mathcal{S}$,

$$\Phi_{\mathbf{s}}(h) = \rho^{-\alpha} \Phi_{T(\mathbf{s})}\left(\frac{h}{\rho}\right).$$

Note that $\operatorname{supp}(\mu)$ is contained in the ball with center 0 and radius $(\max|b_i|)/(1-\rho) = \rho/(1-\rho)\max|d_i|$.

Definition 5.8. Let $\{S_i\}_{i=1}^N$ be defined as in (5.10) and let $q \geqslant 2$ be an integer. Let $C = 2\max|b_i|/(1-\rho)$ and let

$$\widetilde{\mathcal{S}} = \{\mathbf{s} = (\zeta_1, \ldots, \zeta_{q-1}) \in \mathcal{S}\colon\ |\zeta_i(0)| \leqslant C \text{ for } i = 1, \ldots, q-1\}.$$

We say that $\{S_i\}_{i=1}^N$ satisfies the *weak separation condition** (WSC*) if for $q = 2$ (and hence for all integers $q \geqslant 2$), the set $\widetilde{\mathcal{S}}$ is finite.

This definition is equivalent to that in [25], where various classes of IFSs satisfying the WSC* are illustrated. If the WSC* holds, then T can be written as

$$T = \begin{bmatrix} \widetilde{T} & 0 \\ Q & T' \end{bmatrix},$$

where $\widetilde{T}$ is a sub-Markov matrix on the states $\widetilde{S}$. By the WSC*, $\widetilde{T}$ is a finite matrix.

Now, we choose an irreducible component of $\widetilde{T}$ as follows. Denote by $\iota = (I, \ldots, I)$ (($q-1$)-coordinates) the identity map in $\tilde{\mathcal{S}}$. Let $\widetilde{\mathcal{S}}_\iota$ be the $\widetilde{T}$-irreducible component of $\widetilde{\mathcal{S}}$ that contains ι; that is, $\mathbf{s} \in \widetilde{\mathcal{S}}_\iota$ if and only if there exist $m, n \geqslant 1$ such that $T^{(m)}_{\iota,\mathbf{s}}, T^{(n)}_{\mathbf{s},\iota} > 0$ (where $T^{(n)}_{\mathbf{s},\mathbf{s}'}$ denotes the $(\mathbf{s}, \mathbf{s}')$ entry of $\widetilde{T}^n$). Let T_ι be the truncated square matrix of $\widetilde{T}$ on $\widetilde{\mathcal{S}}_\iota$. Then T_ι is irreducible.

The following theorem generalizes [25, Theorem 4.1].

Theorem 5.9. *Suppose the IFS $\{S_i\}_{i=1}^N$ defined as in* (5.10) *satisfies the WSC*. Let $\lambda_{\max}$ be the maximal eigenvalue of T_I. Then*

$$\tau(q) = \frac{\ln \lambda_{\max}}{\ln \rho}.$$

For IFSs satisfying the WSC* but not the OSC, we can use Theorem 5.9 to calculate the values of $\tau(q)$ for integers $q > 0$. If there exists some positive integer q_0 such that $\tau(q_0)/q_0 > d-2$, then by Lemma 5.7, the condition $\dim_\infty(\mu) > d-2$ holds. Plenty of examples of such IFSs on $\mathbb{R}^d$, where $d \geqslant 3$, can be constructed to illustrate this; we briefly mention one below.

Example 5.10. Let $\{S_i\}_{i=1}^9$ be an IFS on $\mathbb{R}^3$ defined by $S_i(x) = x/2 + b_i$, where $b_1 = (0,0,0)$, $b_2 = (1/2, 0, 0)$, $b_3 = (0, 1/2, 0)$, $b_4 = (1/2, 1/2, 0)$, $b_5 = (0, 0, 1/2)$, $b_6 = (1/2, 0, 1/2)$, $b_7 = (0, 1/2, 1/2)$, $b_8 = (1/2, 1/2, 1/2)$, $b_9 = (1/4, 0, 0)$.

It is easy to see that $\{S_i\}_{i=1}^9$ does not satisfy the OSC. However, it satisfies the WSC* (see [25, Example 2.4]). In the case $p_i = 1/9$ for all $i = 1, \ldots, 9$, we have that $9^2\lambda_{\max} = (\sqrt{113} + 11)/2$, the largest root of the polynomial $x^2 - 11x + 2$. Thus, $\tau(2) = \ln\lambda_{\max}/\ln(1/2) \approx 2.9048785171\ldots$. Since $\tau(2)/2 = 1.4524392585\ldots > d-2$, Proposition 5.7 implies that $\dim_\infty(\mu) > d-2$.

Acknowledgments

The authors thank D.-J. Feng for some discussions, in particular for the help on Lemma 5.1. We thank A. Teplyaev and M. Zähle for their advices and their help in correcting the original formulation of condition (C1). We thank H. Triebel for making the manuscript of his new book available. We also thank the referee for some valuable comments leading to the improvement of the paper and for pointing out the references [2,10]. Part of this work was carried out while the first and third authors were visiting the Department of Mathematics of the Chinese University of Hong Kong. They are very grateful to members of the Department for their hospitality and support during their visit.

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