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Equivalence conditions for on-diagonal upper bounds of heat kernels on self-similar spaces ☆

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Abstract

We obtain the equivalence conditions for an on-diagonal upper bound of heat kernels on self-similar measure energy spaces. In particular, this upper bound of the heat kernel is equivalent to the discreteness of the spectrum of the generator of the Dirichlet form, and to the global Poincaré inequality. The key ingredient of the proof is to obtain the Nash inequality from the global Poincaré inequality. We give two examples of families of spaces where the global Poincaré inequality is easily derived. They are the post-critically finite (p.c.f.) self-similar sets with harmonic structure and the products of self-similar measure energy spaces.

Keywords: Self-similar space; Heat kernel; Dirichlet form; On-diagonal upper bound

1. Introduction

The *heat kernel* plays an important role in studying the dynamical properties of fractals. Significant effort has been made by a number of authors to establish the existence and the bounds

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of the heat kernels on fractals (see [1,2,16], and references therein). Typically, the heat kernel $p(t,x,y)$ on a fractal set K satisfies the following estimate

$$t^{-\alpha/\beta}\Phi_1\big(t^{-1/\beta}d(x,y)\big) \leqslant p(t,x,y) \leqslant t^{-\alpha/\beta}\Phi_2\big(t^{-1/\beta}d(x,y)\big), \tag{1.1}$$

for almost all $x, y \in K$ and all $0 < t < t_0$, where Φ_i $(i = 1, 2)$ are positive decreasing functions on $[0, \infty)$, d is a metric on K, and α, β are positive parameters.

Note that estimate (1.1) holds for the classical Gauss–Weierstrass heat kernel in $\mathbb{R}^n$ with $\alpha = n$, $\beta = 2$, and

$$\Phi_1(s) = \Phi_2(s) = \frac{1}{(4\pi)^{n/2}} \exp\left(-\frac{s^2}{4}\right).$$

For certain classes of fractals, the estimates (1.1) hold with the functions

$$\Phi_i(s) = c_i' \exp\big(-c_i'' s^{\gamma_i}\big), \tag{1.2}$$

where γ_i, c_i' and c_i'' are positive constants. Such estimates were proved by Barlow and Perkins [3] for the Sierpinski gasket, by Fitzsimmons et al. [10] for the affine nested fractals, by Barlow and Bass [2] for the (generalized) Sierpinski carpets, by Hambly and Kumagai [16] and Kumagai and Sturm [21] for p.c.f. fractals with *regular* harmonic structures. On-diagonal upper and lower bounds for p.c.f. fractals were obtained earlier by Kigami [20].

The parameter α in (1.1) is in fact the *Hausdorff dimension* of K, whereas β is the *walk dimension* of the heat kernel $p(t,x,y)$, which can be characterized as the largest index of non-trivial Besov spaces on K (see, for example, [15,19,26]). See also [28] for function spaces on fractals.

The purpose of the present paper is to obtain a number of equivalent conditions for the heat kernel upper bound of the form

$$p(t,x,y) \leqslant Ct^{-\theta}, \tag{1.3}$$

for almost all $x, y \in K$ and all $0 < t < t_0$. For general measure spaces with Dirichlet forms, several equivalent conditions for (1.3) are well known. They are the Sobolev inequality [30], the Nash inequality [6], the log-Sobolev inequality [9], and the Faber–Krahn inequality [7,8,13,14]. In the present paper, we emphasize those equivalent conditions for (1.3), which depend on the *self-similarity* of the underlying space.

In Section 2, we introduce the notion of a *self-similar measure energy space* $(K, \{F_i\}, \mu, \mathcal{E})$, where K is a compact metric space, $\{F_i\}_{i=1}^N$ is an iterated function system on K, μ is a self-similar probability measure on K with weight $\{\rho_i\}$, and $(\mathcal{E}, \mathcal{F})$ is a self-similar Dirichlet form with weight $\{r_i^{-1}\}$. Our main result (Theorem 2.2) gives a number of equivalent conditions for the existence of the heat kernel on this space satisfying (1.3). Surprisingly enough, the heat kernel bound (1.3) is equivalent to the discreteness of the spectrum of the generator H of the Dirichlet form $(\mathcal{E}, \mathcal{F})$. Obviously, self-similarity is important for the validity of this kind of result.

Another equivalent condition for (1.3) is the *global Poincaré inequality*. In Section 3 we provide a convenient sufficient condition for the global Poincaré inequality, which, in particular, can be applied on p.c.f. fractals with harmonic structures.

Finally, in Section 4, we consider two kinds of examples of self-similar measure energy spaces—p.c.f. fractals with harmonic structures and product spaces.

2. Main results

Let K be a compact metric space. Let $N \geqslant 2$ be an integer, set $S = \{1, 2, \ldots, N\}$, and let $\{F_i\}_{i \in S}$ be a family of *contractions* on K such that

$$K = \bigcup_{i \in S} F_i(K). \tag{A0}$$

A couple $(K, \{F_i\})$ is called a *self-similar space*. Typically, self-similar spaces arise as follows. Let G be a complete metric space and let $\{F_i\}_{i \in S}$ be a family of contractions on G. Then there exists a unique non-empty compact subset K of G satisfying (A0) (see [18]). Clearly, restricting the mappings F_i to K, we obtain a self-similar space.

Let K be a self-similar space and μ be a measure on K. We say that μ is *self-similar* if μ is a regular Borel measure with total mass 1, which satisfies the identity

$$\mu(A) = \sum_{i \in S} \rho_i \mu\big(F_i^{-1}(A)\big), \tag{A1}$$

for any Borel set $A \subset K$, where $\{\rho_i\}_{i \in S}$ is a fixed sequence of positive numbers such that

$$\sum_{i \in S} \rho_i = 1.$$

Such a measure μ always exists on K (see [18]). We refer to the sequence $\{\rho_i\}_{i \in S}$ as the *weight* of μ.

For any Borel function f and any $1 \leqslant p < \infty$, set

$$\|f\|_p := \left(\int_K \big|f(x)\big|^p \, d\mu(x) \right)^{1/p}$$

and consider the Lebesgue space $L^p(\mu) := L^p(K, \mu)$.

Set $K_i := F_i(K)$ for $i \in S$. We further assume that the sets $\{K_i\}_{i \in S}$ do not overlap in the sense that

$$\mu(K_i \cap K_j) = 0 \quad \text{for all distinct } i, j \in S. \tag{A2}$$

Note that (A2) is satisfied if the *open set condition* holds, see, for example, [22]. For any $m \geqslant 1$, any *word* $\omega := i_1 \ldots i_m \in S^m$, and any function $f : K \to \mathbb{R}$, define

$$F_\omega = F_{i_1} \circ \cdots \circ F_{i_m}, \qquad K_\omega = F_\omega(K),$$
$$\rho_\omega = \rho_{i_1} \ldots \rho_{i_m}, \qquad f_\omega = f \circ F_\omega.$$

For the empty word ω, set $F_\omega = \mathrm{id}$.

It follows from (A1) and (A2) that

$$\mu(K_\omega) = \rho_\omega. \tag{2.1}$$

Moreover, for any $f \in L^1(\mu)$ and any $m \geqslant 1$, we have

$$\int_K f(x)\,d\mu(x) = \sum_{\omega\in S^m} \rho_\omega \int_K f_\omega(x)\,d\mu(x) \tag{2.2}$$

(see [1, Theorem 5.28, p. 73]).

Fix $0 < \lambda < 1$ and $\mathbf{q} := (q_1, q_2, \ldots, q_N)$ with $0 < q_i < 1$, and we define a *partition* Λ associated with the data $(\lambda, \mathbf{q})$ as follows

$$\Lambda := \{\omega = i_1 \ldots i_m\colon q_{i_1} \ldots q_{i_{m-1}} \geqslant \lambda > q_{i_1} \ldots q_{i_m}\}.$$

Then it is easy to see that

$$K = \bigcup_{\omega\in\Lambda} K_\omega \quad \text{and} \quad \mu(K_\omega \cap K_\tau) = 0 \quad \text{if } \omega \neq \tau \in \Lambda,$$

which implies the following extension of identity (2.2):

$$\int_K f(x)\,d\mu(x) = \sum_{\omega\in\Lambda} \rho_\omega \int_K f_\omega(x)\,d\mu(x), \tag{2.3}$$

for any $f \in L^1(\mu)$.

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form[1] on $L^2(\mu)$. We say that $(\mathcal{E}, \mathcal{F})$ is *self-similar* if, for any $f \in \mathcal{F}$, the functions $f \circ F_i$ are also in $\mathcal{F}$ for each $i \in S$, and

$$\text{(A3)} \qquad \mathcal{E}(f) = \sum_{i\in S} r_i^{-1} \mathcal{E}(f \circ F_i),$$

where $\{r_i\}_{i\in S}$ is a fixed sequence of positive numbers. The sequence $\{r_i^{-1}\}_{i\in S}$ is referred to as the *weight* of $\mathcal{E}$.

By induction it follows from (A3) that, for any partition Λ,

$$\mathcal{E}(f) = \sum_{\omega\in\Lambda} (r_\omega)^{-1} \mathcal{E}(f_\omega). \tag{2.4}$$

Definition 2.1. Any quadruple $(K, \{F_i\}, \mu, \mathcal{E})$ satisfying conditions (A0)–(A3) is called a self-similar measure energy space.

By the closedness of $(\mathcal{E}, \mathcal{F})$, the space $\mathcal{F}$ is a Hilbert space with the inner product

$$\mathcal{E}_1(u, v) := (u, v) + \mathcal{E}(u, v).$$

[1] We refer the reader to [12] for the definition and properties of the Dirichlet form and related topics.

Any Dirichlet form $(\mathcal{E}, \mathcal{F})$ has a *generator* H, which is a non-negative definite self-adjoint operator in $L^2(\mu)$. Denote by $\lambda_{\mathrm{ess}}(H)$ the bottom (or infimum) of the essential spectrum[2] of H. The operator H gives rise to the *heat semigroup*

$$P_t = \exp(-tH),$$

where $t \geqslant 0$. If the operator P_t has an integral kernel for any $t > 0$, then the latter is called the *heat kernel* of $(\mathcal{E}, \mathcal{F})$, and is denoted by $p(t, x, y)$. Recall that a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *irreducible*, if $1 \in \mathcal{F}$ and $\mathcal{E}(u) = 0$ if and only if u is constant.

Out main result is the following theorem.

Theorem 2.2. *Let $(K, \{F_i\}, \mu, \mathcal{E})$ be a self-similar measure energy space, and let the Dirichlet form $(\mathcal{E}, \mathcal{F})$ be irreducible. Assume that*

$$\eta := \max_{i \in S}\{\rho_i r_i\} < 1. \tag{2.5}$$

Then the following conditions are equivalent:

(1) (Global Poincaré inequality) *There exists a constant $c > 0$ such that, for all $f \in \mathcal{F}$,*

$$\|f\|_2^2 \leqslant c\mathcal{E}(f) + \left(\int_K f \, d\mu\right)^2. \tag{2.6}$$

(2) (Nash inequality) *There exist constants $c, \theta > 0$ such that, for all $f \in \mathcal{F}$,*

$$\|f\|_2^{2(1+1/\theta)} \leqslant c\big(\mathcal{E}(f) + \|f\|_2^2\big)\|f\|_1^{2/\theta}. \tag{2.7}$$

(3) (Diagonal upper bound) *The heat kernel $p(t, x, y)$ of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ exists, and satisfies the estimate*

$$p(t, x, y) \leqslant c \max\big(t^{-\theta}, 1\big), \tag{2.8}$$

for all $t > 0$ and almost all $x, y \in K$, and for some $c, \theta > 0$.

(4) (Trace of heat semigroup) *The trace* $\mathrm{Trace}(P_t)$ *of the heat semigroup admits the estimate*

$$\mathrm{Trace}(P_t) \leqslant c \max\big(t^{-\theta}, 1\big), \tag{2.9}$$

for all $t > 0$ and some $c, \theta > 0$.

(5) (Eigenvalue estimates) *The spectrum of the generator H is discrete and consists of a countable sequence $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$ of eigenvalues counted with multiplicity. Furthermore,*

$$\lambda_k \geqslant c k^{1/\theta}, \tag{2.10}$$

for all $k \geqslant 0$ and for some $c, \theta > 0$.

[2] The essential spectrum is the part of the spectrum of H which is complement of the discrete spectrum of H.

(6) (Discrete spectrum) *The spectrum of the generator H is discrete, that is $\lambda_{\mathrm{ess}}(H) = +\infty$.*
(7) (Positivity of the essential spectrum) $\lambda_{\mathrm{ess}}(H) > 0$.

Remark 1. The hypothesis of self-similarity (including (2.5)) is used only for the implication (1) $\Rightarrow$ (2). Without this hypothesis, the following equivalences hold in the general setting:

$$(2) \quad \Leftrightarrow \quad (3), \qquad (4) \quad \Leftrightarrow \quad (5), \qquad (1) \quad \Leftrightarrow \quad (7).$$

Indeed, (1) $\Rightarrow$ (7) is obvious from the spectral theory and (2) $\Leftrightarrow$ (3) was proved in [6]. With a certain amount of effort, we can prove that (5) $\Rightarrow$ (4) is true (we omit the detail).

Remark 2. Note that the equivalence (2) $\Leftrightarrow$ (3) holds in general with the same value of θ (see [6]). The equivalence (4) $\Leftrightarrow$ (5) also holds with the same value of θ. However, there are examples of p.c.f. fractals where the best value of θ in (3) is different from the best value of θ in (5) (see [17, Theorem 3.4] and [20, p. 179]).

Remark 3. A. Bendikov and L. Saloff-Coste (private communication) constructed an example of a Dirichlet form on an infinite-dimensional torus $\mathbb{T}^\infty$ with $0 < \lambda_{\mathrm{ess}}(H) < \infty$. Hence, in general the implication (7) $\Rightarrow$ (6) fails. Another example of a Dirichlet form on $\mathbb{T}^\infty$ gives a discrete spectrum with eigenvalues λ_k growing logarithmically in k (see [4]). Hence, the implication (6) $\Rightarrow$ (5) also fails.

Since the discreteness of the spectrum of H is known to be equivalent to the compactness of the embedding $\mathcal{F} \hookrightarrow L^2(\mu)$, we obtain the following corollary.

Corollary 2.3 *(Compact Embedding Theorem). Under the hypotheses of Theorem* 2.2*, each of conditions* (1)–(7) *is equivalent to the fact that the identical embedding $\mathcal{F} \hookrightarrow L^2(\mu)$ is a compact operator.*

The fact that the heat kernel bound (2.8) implies the compactness of the embedding $\mathcal{F} \hookrightarrow L^2(\mu)$ was also proved in [15, Theorem 4.12].

Proof. The proof of Theorem 2.2 will follow the diagram:

$$(1) \quad \Rightarrow \quad (2) \quad \Rightarrow \quad (3) \quad \Rightarrow \quad (4) \quad \Rightarrow \quad (5) \quad \Rightarrow \quad (6) \quad \Rightarrow \quad (7) \quad \Rightarrow \quad (1).$$

The implications (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are trivial. The fact that (2) $\Rightarrow$ (3) was proved in [6, Theorem (2.1), p. 251] (see also [14]). In the sequel we denote by c a positive constant, whose value is unimportant and may change at different occurrences.

(1) $\Rightarrow$ (2). The proof given here is motivated by [1, p. 107], see also [20, p. 173]. Let $q_i = \rho_i r_i$ where $1 \leqslant i \leqslant N$. By hypothesis (2.5), we have $0 < q_i < 1$. For any fixed $0 < \lambda < 1$, consider the partition Λ associated with $(\lambda, \mathbf{q})$. It follows from (2.3) and (2.6) that, for any $f \in \mathcal{F}$,

$$\|f\|_2^2 = \sum_{\tau \in \Lambda} \rho_\tau \int_K f_\tau(x)^2 \, d\mu(x)$$

$$\leqslant \sum_{\tau \in \Lambda} \rho_\tau \left(c\mathcal{E}(f_\tau) + \left(\int_K f_\tau \, d\mu \right)^2 \right)$$

$$\leqslant c \sum_{\tau \in \Lambda} (\rho_\tau r_\tau) r_\tau^{-1} \mathcal{E}(f_\tau) + \sum_{\tau \in \Lambda} \rho_\tau^{-1} \left(\rho_\tau \int_K |f_\tau| \, d\mu \right)^2$$

$$\leqslant c \max_{\tau \in \Lambda} \{\rho_\tau r_\tau\} \sum_{\tau \in \Lambda} r_\tau^{-1} \mathcal{E}(f_\tau) + \left(\min_{\tau \in \Lambda} \{\rho_\tau\} \right)^{-1} \left(\sum_{\tau \in \Lambda} \rho_\tau \int_K |f_\tau| \, d\mu \right)^2$$

$$\leqslant c\big(\lambda \mathcal{E}(f) + \lambda^{-\theta} \|f\|_1^2\big), \tag{2.11}$$

where

$$\theta = \max_{i \in S} \left(\frac{\log \rho_i}{\log \rho_i r_i} \right). \tag{2.12}$$

Here we have used the fact that, for any $\tau \in \Lambda$,

$$\rho_\tau r_\tau < \lambda \leqslant a_0^{-1} \rho_\tau r_\tau, \quad a_0 = \min_{i \in S} \{\rho_i r_i\},$$

and

$$\rho_\tau = (\rho_\tau r_\tau)^{\frac{\log \rho_\tau}{\log(\rho_\tau r_\tau)}} \geqslant (\rho_\tau r_\tau)^{\max_{i \in S} \frac{\log \rho_i}{\log(\rho_i r_i)}} \geqslant (a_0 \lambda)^\theta$$

by noting that $\rho_\tau r_\tau < 1$ and

$$\frac{\log \rho_\tau}{\log(\rho_\tau r_\tau)} \leqslant \max_{i \in S} \frac{\log \rho_i}{\log(\rho_i r_i)}.$$

Clearly (2.11) implies

$$\|f\|_2^2 \leqslant c\big(\lambda\big(\mathcal{E}(f) + \|f\|_2^2\big) + \lambda^{-\theta} \|f\|_1^2\big) \quad (0 < \lambda < 1). \tag{2.13}$$

Note that (2.13) also holds for any $\lambda \geqslant 1$, so it is true for all $\lambda > 0$. Choosing an optimal value of λ, for example,

$$\lambda = \left(\frac{\|f\|_1^2}{\mathcal{E}(f) + \|f\|_2^2} \right)^{\frac{1}{\theta+1}},$$

we arrive at (2.7).

(3) $\Rightarrow$ (4). By definition, we have

$$\operatorname{Trace}(P_t) = \sum_k (P_t v_k, v_k), \tag{2.14}$$

where $\{v_k\}$ is an orthonormal basis in $L^2(\mu)$, and this definition does not depend on the choice of the basis. Let us show that

$$\operatorname{Trace}(P_{2t}) = \int_K \int_K p(t,x,y)^2 \, d\mu(y) \, d\mu(x). \tag{2.15}$$

Noticing that

$$P_t v(x) = \int_K p(t,x,y) v(y) \, d\mu(y) = \big(p(t,x,\cdot), v\big),$$

we obtain from (2.14) and $P_{2t} = P_t^2$ that

$$\begin{aligned} \operatorname{Trace}(P_{2t}) &= \sum_k \big(P_t^2 v_k, v_k\big) = \sum_k (P_t v_k, P_t v_k) \\ &= \sum_k \int_K \big(p(t,x,\cdot), v_k\big)^2 \, d\mu(x). \end{aligned} \tag{2.16}$$

Expanding $p(t,x,\cdot)$ in the basis $\{v_k\}$ we obtain

$$p(t,x,\cdot) = \sum_k \big(p(t,x,\cdot), v_k\big) v_k \tag{2.17}$$

whence by the Parseval identity

$$\sum_k \big(p(t,x,\cdot), v_k\big)^2 = \big\| p(t,x,\cdot) \big\|_2^2. \tag{2.18}$$

Hence, it follows from (2.16) and (2.18) that

$$\operatorname{Trace}(P_{2t}) = \int_K \big\| p(t,x,\cdot) \big\|_2^2 \, d\mu(x), \tag{2.19}$$

giving (2.15). Since $\mathcal{E}$ is irreducible, we have that

$$\int_K p(t,x,y) \, d\mu(y) = 1 \quad (t > 0, x \in K), \tag{2.20}$$

see, for example, [1, Lemma 4.10]. Finally, using (2.8) and (2.20), we obtain from (2.15) that

$$\operatorname{Trace}(P_{2t}) \leqslant \sup_{x,y \in K} p(t,x,y) \int_K \int_K p(t,x,y) \, d\mu(y) \, d\mu(x) \leqslant c \max\big(t^{-\theta}, 1\big),$$

proving (2.9).

(4) $\Rightarrow$ (5). It is a general fact that if a positive-definite self-adjoint operator in a Hilbert space has a finite trace, then its spectrum is discrete away from 0, see, for example, [5]. Hence, the spectrum of the operator $P_t = e^{-tH}$ consists of a discrete part and possibly 0. By the spectral mapping theorem, if $\lambda \in \operatorname{Spec}(H)$, then $e^{-t\lambda} \in \operatorname{Spec}(P_t)$. Since $e^{-t\lambda}$ is positive, it belongs to the discrete spectrum of P_t, whence we conclude that λ belongs to the discrete spectrum of H. This proves that all the spectrum of H is discrete.

Now let $\{\lambda_k\}_{k=0}^{\infty}$ be the eigenvalues with eigenfunctions φ_k which forms an orthonormal basis of $L^2(\mu)$. Using (2.14) and the fact that

$$p(t,x,y) = \sum_k e^{-\lambda_k t}\varphi_k(x)\varphi_k(y)$$

for all $t > 0$ and μ-almost all $x, y \in K$, we have

$$\operatorname{Trace}(P_t) = \sum_k (P_t\varphi_k, \varphi_k) = \sum_k \bigl(e^{-\lambda_k t}\varphi_k, \varphi_k\bigr) = \sum_k e^{-\lambda_k t}.$$

Therefore, by hypothesis (2.9) we see that, for all $t > 0$,

$$\sum_{k=0}^{\infty} e^{-\lambda_k t} \leqslant c \max\bigl(t^{-\theta}, 1\bigr) =: h(t).$$

Assuming that the sequence $\{\lambda_k\}$ is enumerated in an increasing order, we obtain that, for any $k \geqslant 1$,

$$k e^{-\lambda_k t} \leqslant h(t),$$

which yields that

$$\lambda_k \geqslant \frac{1}{t} \log \frac{k}{h(t)}, \quad t > 0.$$

For k large enough, choose t so that $k = ect^{-\theta}$ and $t < 1$. For such a t, we have $h(t) = ct^{-\theta} = k/e$, and so

$$\lambda_k \geqslant c' k^{1/\theta}, \tag{2.21}$$

where $c' > 0$. Now (2.21) is true for large enough k, but by adjusting the value of c', we see that this inequality holds for all $k \geqslant 0$.

(7) $\Rightarrow$ (1). The irreducibility of $(\mathcal{E}, \mathcal{F})$ implies that 0 is a *simple* eigenvalue of H with eigenfunction 1. Therefore, the rest of the spectrum of H coincides with the spectrum of H restricted to the subspace $\mathcal{Q}$ of $L^2(\mu)$, which is the orthogonal complement of 1, that is

$$\mathcal{Q} = \left\{ f \in L^2(\mu) \colon \int_K f(x)\, d\mu(x) = 0 \right\}.$$

Hence, we see that

$$\inf\{\lambda\colon \lambda \in \operatorname{Spec}(H) \setminus \{0\}\} = \inf_{f \in \mathcal{F} \cap \mathcal{Q} \setminus \{0\}} \frac{\mathcal{E}(f)}{\|f\|_2^2}. \tag{2.22}$$

The Poincaré inequality means exactly that the right-hand side of (2.22) is positive. Thus it is enough to show that the left-hand side of (2.22) is positive. Indeed, since $\lambda_{\text{ess}} = \lambda_{\text{ess}}(H) > 0$, it suffices to show that

$$\inf\left\{\lambda\colon \lambda \in \operatorname{Spec}(H) \cap \left(0, \frac{1}{2}\lambda_{\text{ess}}\right)\right\} > 0. \tag{2.23}$$

However, by the definition of λ_{ess}, we see that the spectrum of H inside the interval $[0, \lambda_{\text{ess}})$ is discrete. Hence, the spectrum inside $(0, \frac{1}{2}\lambda_{\text{ess}})$ consists of a finite number of eigenvalues with finite multiplicity. Therefore, we see that 0 is not a limit point of the spectrum, proving (2.23). □

3. Global Poincaré inequality

Let $(K, \{F_i\}, \mu, \mathcal{E})$ be a self-similar measure energy space with the weights $\{\rho_i\}_{i \in S}$ and $\{r_i^{-1}\}_{i \in S}$, as defined in Section 2. Fix a point $q_0 \in K$, set

$$Q := \{F_1(q_0), F_2(q_0), \ldots, F_N(q_0)\}$$

and consider the following condition: there exists $c_0 > 0$ such that

(A4) $$|f(q) - f(q_0)|^2 \leqslant c_0 \mathcal{E}(f) \quad \text{for all } f \in \mathcal{F} \text{ and } q \in Q.$$

Lemma 3.1. *Assume that* (A4) *holds. Then, for any sequence of positive numbers* $\{a_l\}_{l=0}^{\infty}$ *satisfying*

$$\sum_{l=0}^{\infty} a_l^{-1} < \infty, \tag{3.1}$$

there exists a constant c such that, for any $k \geqslant 1$ *and all* $f \in \mathcal{F}$,

$$\sum_{\tau \in S^k} \mu(K_\tau)\big(f_\tau(q_0) - f(q_0)\big)^2 \leqslant c \sum_{l=0}^{k-1} a_l \eta^l \mathcal{E}(f), \tag{3.2}$$

where η *is defined in* (2.5).

Proof. Fix $f \in \mathcal{F}$, $k \geqslant 1$ and consider $\tau := i_1 i_2 \ldots i_k \in S^k$. Set

$$x_0 = q_0 \quad \text{and} \quad x_l = F_{i_1 \ldots i_l}(q_0) \quad \text{for } 1 \leqslant l \leqslant k.$$

Observing that $F_{i_{l+1}}(q_0) \in Q$, we apply (A4) with the function $f \circ F_{i_1 \ldots i_l}$ to obtain that, for any $0 \leqslant l \leqslant k-1$,

$$\begin{aligned}\big(f(x_{l+1})-f(x_l)\big)^2 &= \big(f\circ F_{i_1\dots i_{l+1}}(q_0)-f\circ F_{i_1\dots i_l}(q_0)\big)^2\\ &= \big(f\circ F_{i_1\dots i_l}\big(F_{i_{l+1}}(q_0)\big)-f\circ F_{i_1\dots i_l}(q_0)\big)^2\\ &\leqslant c_0\mathcal{E}(f\circ F_{i_1\dots i_l}).\end{aligned}\tag{3.3}$$

Set $c_1 := \sum_{l=0}^{\infty} a_l^{-1} < \infty$. It follows from (3.3) that

$$\begin{aligned}\big(f_\tau(q_0)-f(q_0)\big)^2 &= \big(f(x_k)-f(x_0)\big)^2\\ &= \left(\sum_{l=0}^{k-1} a_l^{-1/2}a_l^{1/2}\big(f(x_l)-f(x_{l+1})\big)\right)^2\\ &\leqslant \left(\sum_{l=0}^{\infty} a_l^{-1}\right)\sum_{l=0}^{k-1} a_l\big(f(x_l)-f(x_{l+1})\big)^2\\ &\leqslant c_1c_0\sum_{l=0}^{k-1} a_l\mathcal{E}(f\circ F_{i_1\dots i_l}).\end{aligned}\tag{3.4}$$

Summing up in $\tau\in S^k$, we obtain that

$$\sum_{\tau\in S^k}\mu(K_\tau)\big(f_\tau(q_0)-f(q_0)\big)^2\leqslant c_1c_0I_k(f),\tag{3.5}$$

where

$$I_k(f):=\sum_{\tau=i_1\dots i_k\in S^k}\mu(K_\tau)\sum_{l=0}^{k-1}a_l\mathcal{E}(f\circ F_{i_1\dots i_l}).$$

Noting that $\mu(K_{i_1\dots i_k})=\rho_{i_1}\dots\rho_{i_k}$ and $\sum_{i\in S}\rho_i=1$, we have

$$\begin{aligned}I_k(f) &= \sum_{i_1,\dots,i_k\in S}\mu(K_{i_1\dots i_k})\sum_{l=0}^{k-1}a_l\mathcal{E}(f\circ F_{i_1\dots i_l})\\ &= \sum_{l=0}^{k-1}\left(a_l\sum_{i_1,\dots,i_l\in S}\rho_{i_1}\dots\rho_{i_l}\mathcal{E}(f\circ F_{i_1\dots i_l})\right)\\ &\leqslant \sum_{l=0}^{k-1}a_l\eta^l\mathcal{E}(f),\end{aligned}\tag{3.6}$$

where the last inequality follows from

$$\sum_{i_1,\dots,i_l\in S}\rho_{i_1}\dots\rho_{i_l}\mathcal{E}(f\circ F_{i_1\dots i_l})=\sum_{\tau\in S^l}(\rho_\tau r_\tau)(r_\tau)^{-1}\mathcal{E}(f\circ F_\tau)$$

$$\leqslant \eta^l \sum_{\tau \in S^l} (r_\tau)^{-1} \mathcal{E}(f \circ F_\tau) = \eta^l \mathcal{E}(f).$$

Finally, (3.5) and (3.6) yield (3.2). □

Theorem 3.2. *Let* $(K, \{F_i\}, \mu, \mathcal{E})$ *be a self-similar measure energy space. Assume further that the Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *is regular and satisfies* (A4). *If* (2.5) *holds, then the global Poincaré inequality* (2.6) *holds with a constant* $c = c(c_0, \eta)$.

Consequently, under the hypotheses of Theorem 3.2, all conditions (1)–(7) of Theorem 2.2 hold as well.

Proof. Fix a function $f \in \mathcal{F} \cap C(K)$ and $k \geqslant 1$. By hypothesis (A2), when τ varies in S^k, the cells K_τ form a partition of K up to a set of μ-measure 0. Therefore, for μ-almost all $x \in K$, there exists exactly one $\tau \in S^k$ such that $x \in K_\tau$. For such an x, set $f_k(x) := f_\tau(q_0)$. Obviously, the function $f_k(x)$ is defined for μ-almost all $x \in K$, and is constant on any cell K_τ.

Set $a_l = \eta^{-l/2}$ and observe that the sequence $\{a_l\}_{l=0}^\infty$ satisfies (3.1) since $\eta < 1$. Thus, we obtain from (3.2) that

$$\begin{aligned}\int_K \big(f_k(x) - f(q_0)\big)^2 \, d\mu(x) &= \sum_{\tau \in S^k} \int_{K_\tau} \big(f_\tau(q_0) - f(q_0)\big)^2 \, d\mu(x) \\ &= \sum_{\tau \in S^k} \mu(K_\tau) \big(f_\tau(q_0) - f(q_0)\big)^2 \\ &\leqslant c \sum_{l=0}^{k-1} a_l \eta^l \mathcal{E}(f) \leqslant c\mathcal{E}(f). \end{aligned} \tag{3.7}$$

Since f is continuous, K is compact, and F_i's are contractive, it is easy to see that $f_k(x) \to f(x)$ for μ-almost all $x \in K$ as $k \to \infty$. Hence, letting $k \to \infty$ in (3.7), we see that

$$\int_K \big(f(x) - f(q_0)\big)^2 \, d\mu(x) \leqslant c\mathcal{E}(f)$$

for all $f \in \mathcal{F} \cap C(K)$. Thus, upon setting

$$\bar{f} = \int_K f(x) \, d\mu(x),$$

we obtain that

$$\int_K f^2 \, d\mu - (\bar{f})^2 = \int_K (f - \bar{f})^2 \, d\mu = \inf_{\xi \in \mathbb{R}} \int_K (f - \xi)^2 \, d\mu \leqslant c\mathcal{E}(f),$$

whence (2.6) follows. Finally, by the regularity of $(\mathcal{E}, \mathcal{F})$, the set $\mathcal{F} \cap C(K)$ is dense in $\mathcal{F}$, which allows us to extend (2.6) to all $f \in \mathcal{F}$. □

Given a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on K, define an *effective resistance* $R(x, y)$ for any two points $x, y \in K$ by

$$R(x, y) = \sup_{f \in \mathcal{F}, \mathcal{E}(f) \neq 0} \frac{|f(x) - f(y)|^2}{\mathcal{E}(f)}. \tag{3.8}$$

Hypothesis (A4) means that

$$R(q, q_0) \leqslant c_0 \quad \text{for any } q \in Q,$$

that is, R is assumed to be bounded on a *finite* set of points in K. It was shown in [20, Lemma 3.3.7, p. 86] that, for a p.c.f. self-similar fractal K with a harmonic structure, the estimate

$$\sup_{x, y \in K} R(x, y) < \infty \tag{3.9}$$

holds if and only if the harmonic structure is *regular* (see the next section for details). If (3.9) holds, then the Poincaré inequality (2.6) trivially follows from

$$\left| f(y) - f(x) \right|^2 \leqslant \sup_{x, y \in K} R(x, y) \mathcal{E}(f),$$

see [16, Lemma 3.1, pp. 438–439]. However, for non-regular harmonic structures, the Poincaré inequality cannot be obtained in this way, whereas Theorem 3.2 still can be applied.

4. Examples

In this section, we consider two ways of constructing self-similar measure energy spaces. The first example is p.c.f. fractals with harmonic structure and the second is products of self-similar measure energy spaces.

4.1. Post-critically finite self-similar fractals

Let G be a complete metric space and $\{F_i\}_{i \in S}$ be a family of contractions in G, where $S = \{1, \ldots, N\}$ and $N \geqslant 2$. Fix a finite set $V_0 = \{p_1, \ldots, p_D\} \subset G$ which consists of $D \geqslant 2$ distinct points. For any $m \geqslant 1$, define the sets $V_m \subset G$ by induction as follows:

$$V_m = \bigcup_{i \in S} F_i(V_{m-1}).$$

Assume that

(B1) $$V_0 \subset V_1,$$

which implies that the sequence $\{V_m\}_{m \geqslant 1}$ is increasing.

Consider the set V_1 as a graph: two points $x, y \in V_1$ are neighbors in V_1 if there exists $i \in S$ such that $x, y \in F_i(V_0)$. We say that V_1 is *connected* if, for any pair x, y in V_1, there exists a finite sequence $x = x_0, x_1, \dots, x_k = y$ such that x_{l-1} and x_l are neighbors for any $l = 1, \dots, k$. In the sequel, assume that

(B2) $$V_1 \text{ is connected.}$$

Let us introduce a quadratic form $\mathcal{E}_0$ on V_0 as follows. Fix a symmetric $D \times D$ matrix (c_{ij}) of non-negative reals and, for any function $f : V_0 \to \mathbb{R}$, set

$$\mathcal{E}_0(f) = \sum_{i,j=1}^{D} c_{ij}\big(f(p_i) - f(p_j)\big)^2. \tag{4.1}$$

The numbers c_{ij} are termed the *conductances* of the *graph* V_0. Assume that $\mathcal{E}_0$ is *irreducible*, that is

(B3) $$\mathcal{E}_0(f) = 0 \quad \text{implies} \quad f \equiv \text{const} \quad \text{on } V_0.$$

Given $\mathcal{E}_0$, we inductively define a quadratic form $\mathcal{E}_m$ on V_m by

$$\mathcal{E}_m(f) = \sum_{i \in S} r_i^{-1} \mathcal{E}_{m-1}(f \circ F_i), \tag{4.2}$$

for every function f on V_m, where r_i are positive constants. By (4.2), we have

$$\mathcal{E}_m(f) = \sum_{\omega \in S^m} (r_\omega)^{-1} \mathcal{E}_0(f_\omega) \tag{4.3}$$

for all $m \geqslant 1$, where $r_\omega = r_{i_1} \dots r_{i_m}$ for $\omega = i_1 \dots i_m$.

The irreducibility of $\mathcal{E}_0$ implies that of $\mathcal{E}_1$. Together with the connectivity of V_1, this yields that there exists a constant $c_0 > 0$ that, for any function f on V_1 and for any two points $p, q \in V_1$,

$$\big(f(p) - f(q)\big)^2 \leqslant c_0 \mathcal{E}_1(f). \tag{4.4}$$

Assume further that, for any function f on V_1,

(B4) $$\mathcal{E}_1(f) \geqslant \mathcal{E}_0(f).$$

Set

$$V_* = \bigcup_{m=0}^{\infty} V_m,$$

and observe that

$$V_* = \bigcup_{i \in S} F_i(V_*). \tag{4.5}$$

For any function f on V_*, we have by (4.2)

$$\mathcal{E}_{m+1}(f) - \mathcal{E}_m(f) = \sum_{i \in S} r_i^{-1}\bigl(\mathcal{E}_m(f \circ F_i) - \mathcal{E}_{m-1}(f \circ F_i)\bigr).$$

Therefore, condition (B4) implies that the sequence $\{\mathcal{E}_m(f)\}_{m=1}^{\infty}$ is increasing in m by induction. Thus, for any function $f : V_* \to \mathbb{R}$, we can define

$$\mathcal{E}(f) = \lim_{m \to \infty} \mathcal{E}_m(f) \tag{4.6}$$

(where so far we allow $\mathcal{E}(f) = +\infty$). It follows from (4.4) that, for any function f on V_*,

$$\bigl(f(p) - f(q)\bigr)^2 \leqslant c_0 \mathcal{E}(f) \quad \text{for all } p, q \in V_1. \tag{4.7}$$

Let K be the closure of V_* in G. It is obvious from (4.5) that K satisfies (A0). It is easy to verify that K is compact; hence K is a self-similar space. For any function f on K, define $\mathcal{E}(f) = \mathcal{E}(f|_{V_*})$ and set

$$\mathcal{F} := \bigl\{f \in C(K):\ \mathcal{E}(f) < \infty\bigr\}. \tag{4.8}$$

It follows from (4.1) that $(\mathcal{E}, \mathcal{F})$ satisfies the Markov property: if $f \in \mathcal{F}$ then $g = (0 \vee f) \wedge 1$ is also in $\mathcal{F}$ and $\mathcal{E}(g) \leqslant \mathcal{E}(f)$. Clearly $(\mathcal{E}, \mathcal{F})$ is irreducible. Moreover, (4.2) and (4.7) imply that $(\mathcal{E}, \mathcal{F})$ satisfies conditions (A3) and (A4).

Finally, for any sequence $\{\rho_i\}_{i \in S}$ of positive numbers such that $\sum_{i \in S} \rho_i = 1$, there exists a Borel regular measure μ on K satisfying (A1).

In order to conclude that $(K, \{F_i\}, \mu, \mathcal{E})$ is a self-similar measure energy space, we still need to verify condition (A2), and to ensure that the form $(\mathcal{E}, \mathcal{F})$ is closed with $\mathcal{F}$ dense in $L^2(\mu)$. This can be done under additional conditions as follows.

A particularly interesting case of the above construction is p.c.f. fractals introduced by Kigami, see the details in [20, Chapter 1]. Let $(K, \{F_i\})$ be a connected p.c.f. fractal with the *boundary* $V_0 := \{p_1, p_2, \ldots, p_D\}$ $(D \geqslant 2)$. We may introduce a sequence of quadratic forms $\mathcal{E}_m$ on V_m exactly as above. We say that K possesses a *harmonic structure*, if there exist a $D \times D$ matrix $J := (c_{ij})$ and a vector $\mathbf{r} := (r_1, r_2, \ldots, r_N)$ such that

$$\inf_g \bigl\{\mathcal{E}_1(g, g):\ g = f \text{ on } V_0\bigr\} = \mathcal{E}_0(f) \tag{4.9}$$

for all $f : V_0 \to \mathbb{R}$. The harmonic structure $(J, \mathbf{r})$ is said to be *regular* if $r_i < 1$ for all $i \in S$. It is easy to verify that the harmonic structure is regular if and only if $\eta < 1$ and $\theta < 1$ where η and θ are defined by (2.5) and (2.12), respectively.

It is still an open question whether or not a general p.c.f. fractal possesses a harmonic structure although a positive answer was obtained for certain classes of p.c.f. fractals, see [23,24,27]. Assuming that $(J, \mathbf{r})$ is a harmonic structure on $(K, \{F_i\})$. Then condition (4.9) implies that $\mathcal{E}_1 \geqslant \mathcal{E}_0$ and hence one can obtain the quadratic form $(\mathcal{E}, \mathcal{F})$ on K as above.

Definition 4.1. We say that a collection $(K, \{F_i\}, \mu, \mathcal{E})$ is a *harmonic p.c.f. fractal* if $(K, \{F_i\})$ is a connected p.c.f. fractal with contractions F_i, and μ is a self-similar measure on K, and the quadratic form $\mathcal{E}$ is associated with a harmonic structure as above.

For a harmonic p.c.f. fractal, the assumptions (A0)–(A3) in Section 2 hold; in particular, (A2) follows from

$$F_i(K) \cap F_j(K) = F_i(V_0) \cap F_j(V_0) \quad (i \neq j),$$

see [20, Proposition 1.3.5, p. 19]. In general, $(\mathcal{E}, \mathcal{F})$ is not necessarily a closable form. At this point, a harmonic p.c.f. fractal is not included the *variational fractal* introduced by Mosco [25]. However, Kigami [20, Theorem 3.4.6, p. 92] proved that, a harmonic p.c.f. fractal $(K, \{F_i\}, \mu, \mathcal{E})$ with weights $\{\rho_i\}$ and $\{r_i^{-1}\}$ satisfying (2.5) is a self-similar measure energy space satisfying (A4). Thus, the global Poincaré inequality follows by Theorem 3.2, which implies also conditions (2)–(7) of Theorem 2.2. In particular, we obtain the existence of the heat kernel satisfying (2.8). The latter result was obtained in [20, Theorem 5.3.1, p. 172], where the on-diagonal lower bound of $p(t,x,y)$ was also proved by using a probabilistic approach.

4.2. Products of self-similar spaces

Let $(K, \{F_i\}_{i\in S}, \mu, \mathcal{E})$ be a self-similar measure energy space with the weights $\{\rho_i\}$, $\{r_i^{-1}\}$. We say that the weights of this space are *homogeneous* with coefficient η if

$$\rho_i r_i = \eta \quad \text{for all } i \in S.$$

Let now $(K', \{F_i'\}, \mu', \mathcal{E}')$ and $(K'', \{F_j''\}, \mu'', \mathcal{E}'')$ be two self-similar measure energy spaces, respectively, with the weights $\{\rho_i'\}$, $\{(r_i')^{-1}\}$ and $\{\rho_j''\}$, $\{(r_j'')^{-1}\}$. Consider the product space

$$K := K' \times K''.$$

Clearly K is a self-similar space with the family of contractions

$$\{F_{ij}\} := \{F_i' \otimes F_j''\},$$

because

$$K = \bigcup_{i,j} (F_i' \otimes F_j'')(K).$$

Consider the product measure on K

$$\mu := \mu' \otimes \mu''.$$

It is not hard to see that μ is a self-similar measure on K with the weight $\{\rho_i'\rho_j''\}$, and so conditions (A1)–(A2) hold.

Define an energy form $\mathcal{E}$ on K by

$$\mathcal{E}(f) := \int_{K''} \mathcal{E}'\big(f(\cdot, x'')\big)\, d\mu''(x'') + \int_{K'} \mathcal{E}''\big(f(x', \cdot)\big)\, d\mu'(x') \tag{4.10}$$

for $f \in L^2(\mu)$, and set

$$\mathcal{F} = \{ f \in L^2(\mu) \colon \mathcal{E}(f) < \infty \}.$$

Proposition 4.2. *Let $(K', \{F_i'\}, \mu', \mathcal{E}')$ and $(K'', \{F_j''\}, \mu'', \mathcal{E}'')$ be two self-similar measure energy spaces with the weights $\{\rho_i'\}, \{(r_i')^{-1}\}$ and $\{\rho_j''\}, \{(r_j'')^{-1}\}$, respectively, and let the forms $(\mathcal{E}', \mathcal{F}')$ and $(\mathcal{E}'', \mathcal{F}'')$ be irreducible. Assume that the both pairs of weights are homogeneous with the same coefficient η, that is,*

$$\rho_i' r_i' = \rho_j'' r_j'' = \eta \quad \text{for all } i \text{ and } j. \tag{4.11}$$

Then the energy form $\mathcal{E}$ defined in (4.10) *is self-similar with weight $\{\eta(r_i' r_j'')^{-1}\}$, that is*

$$\mathcal{E}(f,g) = \sum_{i,j} \eta \left(r_i' r_j''\right)^{-1} \mathcal{E}\left(f \circ \left(F_i' \otimes F_j''\right), g \circ \left(F_i' \otimes F_j''\right)\right) \tag{4.12}$$

for $f, g \in \mathcal{F}$. Moreover, $(\mathcal{E}, \mathcal{F})$ is an irreducible Dirichlet form on $L^2(\mu)$, and $(K, \{F_{ij}\}, \mu, \mathcal{E})$ is a self-similar measure energy space, whose weights are homogeneous *with the same coefficient η.*

Note that the homogeneity of the weights of the forms $(\mathcal{E}', \mathcal{F}')$, $(\mathcal{E}'', \mathcal{F}'')$ is essential for the self-similarity of $(\mathcal{E}, \mathcal{F})$.

Proof. The self-similarity of $\mathcal{E}$ was proved in [29, Lemma 2.2]. The Markov property and the irreducibility of $\mathcal{E}$ follow directly from definition (4.10). The closedness of $(\mathcal{E}, \mathcal{F})$ was proved in [11] (see also [29, Corollary 2.7] for the case of discrete spectrum). Hence, $(\mathcal{E}, \mathcal{F})$ is an irreducible Dirichlet form. The weights $\{\rho_i' \rho_j''\}$ and $\{\eta(r_i' r_j'')^{-1}\}$ of the product space are also homogeneous with the same coefficient η because

$$\rho_i' \rho_j'' \left(\eta^{-1} r_i' r_j''\right) = \eta^{-1} \left(\rho_i' r_i'\right)\left(\rho_i'' r_i''\right) = \eta.$$

This completes the proof. $\square$

In the view of Proposition 4.2, the procedure of taking products can be iterated. Namely, if $\{(K^{(n)}, \{F_i^{(n)}\}, \mu^{(n)}, \mathcal{E}^{(n)})\}$ is a finite sequence of a self-similar measure energy spaces with homogeneous weights with the same coefficient η, then the product

$$K := K^{(1)} \times \cdots \times K^{(n)}$$

has also the structure of a self-similar measure energy space defined as above.

Note that the products of fractals are infinitely ramified fractals, and hence they are not p.c.f. fractals. This gives examples of self-similar measure energy spaces which are not p.c.f. fractals. Another family of examples are the Sierpinski carpets.

As an example of applications of the above results, let us prove the following statement.

Corollary 4.3. *Let $\{(K^{(n)}, \{F_i^{(n)}\}, \mu^{(n)}, \mathcal{E}^{(n)})\}$ be a finite sequence of p.c.f. fractals with homogeneous weights with the same coefficient $\eta < 1$. Then their product $(K, \{F_i\}, \mu, \mathcal{E})$ satisfies all conditions* (1)–(7) *of Theorem* 2.2.

Proof. By Theorem 3.2, each space $K^{(n)}$ satisfies the global Poincaré inequality. By Theorem 2.2, the generator of $\mathcal{E}^{(n)}$ has discrete spectrum. Then, it is easy to see that the generator of the form $\mathcal{E}$ on the product space has also discrete spectrum. By Proposition 4.2, the product space $(K, \{F_i\}, \mu, \mathcal{E})$ has homogeneous weights with the same coefficient $\eta < 1$. Hence, Theorem 2.2 applies and yields the claim. □

Corollary 2.3 implies then that, under the above conditions, the compact embedding theorem holds on the product space, too. The latter was also proved by Strichartz for the product of two p.c.f. fractals with regular harmonic structure (see [29, Corollary 2.7]).

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