Characterization of tile digit sets with prime determinants ☆

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Abstract

For an expanding integral $s \times s$ matrix A with $|\det A| = p$, it is well known that if $\mathcal{D} = \{d_0, \ldots, d_{p-1}\} \subset \mathbb{Z}^s$ is a complete set of coset representatives of $\mathbb{Z}^s / A\mathbb{Z}^s$, then $T(A, \mathcal{D})$ is a self-affine tile. In this paper we show that if p is a prime, such $\mathcal{D}$ actually characterizes the tile digit sets provided that $\mathrm{span}(\mathcal{D}) = \mathbb{R}^s$. This result is known for $s = 1$, the one-dimensional case [R. Kenyon, in: Contemp. Math., vol. 135, 1992, pp. 239–264] and the question for $s > 1$ has been considered by Lagarias and Wang [J. London Math. Soc. 53 (1996) 21–49] under some other conditions. The proof here involves a new setup to study the zeros of the mask $m(\xi) = p^{-1} \sum_{j=0}^{p-1} e^{2\pi i \langle \xi, d_j \rangle}$. It can also be generalized to consider the existence of a compactly supported L^1-solution of the refinement equation (scaling function) with positive coefficients.
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1. Introduction

Let $M_s(\mathbb{Z})$ be the set of $s \times s$ matrices with integral entries. Let $A \in M_s(\mathbb{Z})$ be an *expanding* matrix, i.e., all its eigenvalues have moduli > 1. Let $\mathcal{D} = \{d_0, d_1, \ldots, d_{N-1}\} \subset \mathbb{Z}^s$ be a set of N (> 1) distinct integral vectors. We call $\mathcal{D}$ a *digit set* and $(A, \mathcal{D})$ a *self-affine pair*. Without loss of generality, we assume that $d_0 = 0$ in $\mathcal{D}$. Let

$$\psi_j(x) = A^{-1}(x + d_j), \quad j = 0, 1, \ldots, N - 1.$$

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They are contractions with respect to a suitable norm on $\mathbb{R}^s$; the finite family $\{\psi_j\}_{j=0}^{N-1}$ is called an *iterated function system* (IFS). It is well known that there exists a unique nonempty compact set $T := T(A, \mathcal{D})$ satisfying the set-valued functional equation $T = \bigcup_{j=0}^{N-1} \psi_j(T)$. Equivalently,

$$AT = \bigcup_{j=0}^{N-1} (T + d_j) = T + \mathcal{D}.$$

A more explicit expression of T is given by the radix expansions

$$T = \left\{ \sum_{k=1}^{\infty} A^{-k} d_{j_k} \colon d_{j_k} \in \mathcal{D} \right\} = \sum_{k=1}^{\infty} A^{-k} \mathcal{D}.$$

One of the most interesting cases for the pair $(A, \mathcal{D})$ is when $\#\mathcal{D} = |\det A|$ and $T^\circ \neq \emptyset$. It is known that in this case, T tiles $\mathbb{R}^s$ by translations (e.g., [14, Theorem 1.2]). Because of this we define

Definition 1.1. Let $(A, \mathcal{D})$ be a self-affine pair and $\#\mathcal{D} = |\det A|$. If $T^\circ \neq \emptyset$, then T is called a *self-affine tile* and $\mathcal{D}$ is called a *tile digit set* (with respect to A).

The current theory of tiles was first considered by Dekking [3] and Rauzy [21] via the substitutions on finite alphabets. The foundation of self-affine tiles and self-replicating tilings were laid down by Thurston [24] and Kenyon [10], and the basic properties were proved by Lagarias and Wang [14–17]. Recently there have been extensive investigations in this topic, for example, on the tiling properties, the wavelet properties, the fractal structure of the boundaries, the connectedness and disk-likeness, and the classification of the expanding integral matrices (e.g., [1,2,4,5,7,11–17,19,22,23,25]). However, there are still many problems remaining unsolved. Among them is the following:

Question. For a given expanding matrix $A \in M_s(\mathbb{Z})$, characterize the *tile digit sets* $\mathcal{D}$ with respect to A [15,19].

Regarding this question, the following is well known.

Proposition 1.2 [1]. *Let $(A, \mathcal{D})$ be a self-affine pair and $\#\mathcal{D} = |\det A| = b$. Then $\mathcal{D}$ is a tile digit set if $\mathcal{D}$ is a complete set of coset representatives of $\mathbb{Z}^s / A\mathbb{Z}^s$.*

The condition in the proposition is far from being necessary. In the one-dimensional case, when b is a prime, the characterization is reduced to a modulo condition (see, e.g., [10]):

Proposition 1.3. *On $\mathbb{R}$, let $A = [p]$, where p is a prime and let $\mathcal{D} \subset \mathbb{Z}$ with $\#\mathcal{D} = p$. Then $\mathcal{D}$ is a tile digit set if and only if $\mathcal{D} = \ell \tilde{\mathcal{D}}$, where $\ell \in \mathbb{Z}$ and $\tilde{\mathcal{D}}$ is complete set of coset representatives of $\mathbb{Z} / p\mathbb{Z}$, i.e., $\tilde{\mathcal{D}} \equiv \{0, \ldots, p - 1\} \pmod{p}$.*

The assumption that p is a prime is essential in the proposition. In fact, the more extensive form of tile digit sets for $A = [p^l]$ and $[pq]$, where p, q are primes were characterized by Lagarias and Wang [15] and Lau and Rao [19], respectively. The higher-dimensional case is more difficult even for prime determinant, Proposition 1.3 was partially extended to $\mathbb{R}^s$ [15] under the extra condition $p\mathbb{Z}^s \nsubseteq A^2\mathbb{Z}^s$. It was conjectured that the condition is unnecessary. In this paper we give an answer to this question.

Theorem 1.4. *Let* $(A, \mathcal{D})$ *be a self-affine pair with* $\#\mathcal{D} = |\det A| = p$ *and* $p > 1$ *is a prime. Suppose* $\mathrm{span}(\mathcal{D}) = \mathbb{R}^s$. *Then* $\mathcal{D}$ *is a tile digit set if and only if* $\mathcal{D} = A^n \tilde{\mathcal{D}}$ *for some nonnegative integer n and a complete set of coset representatives* $\tilde{\mathcal{D}}$ *of* $\mathbb{Z}^s / A\mathbb{Z}^s$.

The theorem will be proved in Theorem 4.2. For this we will set up the problem in the more general refinement equation

$$\varphi(x) = \sum_{j=0}^{N-1} w_j \varphi(Ax - d_j), \tag{1.2}$$

where each $w_j \in \mathbb{R}$ and $\sum_{j=0}^{N-1} w_j = |\det A| = b$. Formally, the Fourier transform $\hat{\varphi}(\xi) = \int \varphi(x) e^{2\pi i \langle x, \xi \rangle} \, dx$ satisfies

$$\hat{\varphi}(\xi) = m(B^{-1}\xi)\hat{\varphi}(B^{-1}\xi) = m(B^{-1}\xi) \cdots m(B^{-k}\xi)\hat{\varphi}(B^{-k}\xi) = \hat{\varphi}(0) \prod_{k=1}^{\infty} m(B^{-k}\xi), \tag{1.3}$$

where $B = A^t$ and

$$m(\xi) = b^{-1} \sum_{j=0}^{N-1} w_j e^{2\pi i \langle \xi, d_j \rangle}.$$

We call $m(\xi)$ the *mask* of the refinement equation, and a *positive mask* if the coefficients are positive. The product in (1.3) converges uniformly on compact subsets. It follows that φ exists as a distribution. If φ exists as a compactly supported L^1-solution, we call it a *scaling function*; the existence can be characterized by the Fourier transform $\hat{\varphi}(\xi)$ (Lemma 2.1).

For the self-affine pair $(A, \mathcal{D})$ with $\#\mathcal{D} = |\det A| = b$, if we let $\varphi = \chi_T$, the indicator of $T := T(A, \mathcal{D})$, then φ satisfies

$$\varphi(x) = \sum_{j=0}^{b-1} \varphi(Ax - d_j)$$

and the corresponding mask is $m(\xi) = b^{-1} \sum_{j=0}^{b-1} e^{2\pi i \langle \xi, d_j \rangle}$. It is well known that (Proposition 2.2) T is a tile (equivalently χ_T is a scaling function) if and only if for each $0 \neq v \in \mathbb{Z}^s$, there exists an integer $k > 0$ such that $m(B^{-k}v) = 0$.

Our main idea to prove Theorem 1.4 is that when $|\det A| = p$ is a prime, we can make use of the special properties of the roots of unity for primes to reduce the above criterion to a certain finite set of zeros of the mask $m(\xi)$. Such finite set is called a *tight set* (see Section 3) which is an extension of the "minimal cut set" introduced by Protasov [20]. The tight sets can also be used to study the scaling functions. We prove the following interesting result for the refinement equation with positive rational weights. For the case with positive weights, the reader can refer to [9,18,26] for detail.

Theorem 1.5. *Let* p *be a prime. Suppose* $\{w_j\}_{j=0}^{N-1}$ *are positive rationals and* $\sum_{j=0}^{N-1} w_j = p$. *Then the equation* $\varphi(x) = \sum_{j=0}^{N-1} w_j \varphi(px - d_j)$ *on* $\mathbb{R}$ *has a compactly supported* L^1-*solution if and only if there exists a positive integer k such that* $m(p^{-k}) = 0$, *where* $m(\xi) = p^{-1} \sum_{j=0}^{N-1} w_j e^{2\pi i d_j \xi}$.

For the organization of the paper, in Section 2, we give some preliminaries and study the roots of unity for the prime p in conjunction with the mask $m(\xi)$. We introduce the tight sets in Section 3 and prove some related properties. Theorem 1.4 is proved in Section 4. In Section 5, we make use of the tight sets to study the existence of scaling functions of the refinement equations.

2. Preliminaries

Throughout the paper we assume, unless otherwise specified, that $A \in M_s(\mathbb{Z})$ is expanding, $B = A^t$ and $\mathcal{D} = \{0 = d_0, \ldots, d_{N-1}\} \subset \mathbb{Z}^s$. We need the following lemma of which the necessity is well known, and the sufficiency was proved in [20] on $\mathbb{R}$. The proof here is a higher-dimensional analog and is a simplification of [20, Lemma 2].

Lemma 2.1. *Suppose φ is a scaling function satisfying (1.2), then $\hat{\varphi}(v) = 0$ for any nonzero integral vector $v \in \mathbb{Z}^s$. The converse is also true if the w_j's in (1.2) are nonnegative. In this case φ is a bounded function.*

Proof. By assumption, φ is a compactly supported L^1-solution of (1.2). It follows from (1.3) that

$$\hat{\varphi}(\xi) = m(B^{-1}\xi) \cdots m(B^{-k}\xi)\hat{\varphi}(B^{-k}\xi), \tag{2.1}$$

where $m(\xi) = b^{-1} \sum_{j=0}^{N-1} w_j e^{2\pi i \langle \xi, d_j \rangle}$. For a nonzero $v \in \mathbb{Z}^s$, let $\xi = B^k v$. Then $m(B^{-j}\xi) = m(B^{k-j}v) = 1$, $j \leqslant k$ and the above identity is reduced to $\hat{\varphi}(B^k v) = \hat{\varphi}(v)$. It follows that $\hat{\varphi}(v) = \lim_{k \to \infty} \hat{\varphi}(B^k v) = 0$ by the Riemann–Lebesgue lemma.

Conversely assume that the coefficients of (1.2) are nonnegative, we let $\mathcal{S}$ denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}^s$. For $f \in \mathcal{S}$, let $\tilde{f}(x) = \sum_{\ell \in \mathbb{Z}^s} f(x + \ell)$. Then $\tilde{f}$ is a periodic function and

$$\tilde{f}(x) = \sum_{v \in \mathbb{Z}^s} a_v e^{2\pi i \langle v, x \rangle}, \quad \text{where } a_v = \int_{[0,1]^s} \tilde{f}(x) e^{-2\pi i \langle v, x \rangle} \, \mathrm{d}x = \hat{f}(-v).$$

Since φ (as a distribution) has compact support, we define $\tilde{\varphi}(x) = \sum_{\ell \in \mathbb{Z}^s} \varphi(x + \ell)$. It follows from the assumption $\hat{\varphi}(v) = 0$ and $\hat{\varphi}(0) = 1$ that for any $f \in \mathcal{S}$,

$$\langle \tilde{\varphi}, f \rangle = \langle \varphi, \tilde{f} \rangle = \left\langle \varphi, \sum_{v \in \mathbb{Z}^s} a_v e^{2\pi i \langle v, x \rangle} \right\rangle = \sum_{v \in \mathbb{Z}^s} \bar{a}_v \hat{\varphi}(v) = a_0 = \int_{\mathbb{R}^s} f(x) \, \mathrm{d}x.$$

Hence $\tilde{\varphi}(x) = 1$. φ is nonnegative and $\tilde{\varphi}(x) = \sum_{\ell \in \mathbb{Z}^s} \varphi(x + \ell)$ imply that the compactly supported distribution φ is actually a function on $\mathbb{R}^s$ and is bounded. $\quad \square$

As a direct consequence of Lemma 2.1, we have the following criterion of self-affine tiles [10,15].

Proposition 2.2. *Let $(A, \mathcal{D})$ be a self-affine pair and $\#\mathcal{D} = |\det A| = b$, and let $m(\xi) = b^{-1} \sum_{j=0}^{b-1} e^{2\pi i \langle \xi, d_j \rangle}$, then $T(A, \mathcal{D})$ is a self-affine tile if and only if for any $0 \neq v \in \mathbb{Z}^s$, there exists an integer $k > 0$ such that $m(B^{-k}v) = 0$.*

Proof. Let $\varphi = \chi_T$, the indicator function of $T := T(A, \mathcal{D})$. Then the scaling function φ satisfies $\varphi(x) = \sum_{j=0}^{b-1} \varphi(Ax - d_j)$. The proposition follows directly from Lemma 2.1, (2.1) and $\lim_{k \to \infty} \hat{\varphi}(B^{-k}\xi) = \hat{\varphi}(0) = \mathcal{L}(T) > 0$, where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbb{R}^s$. $\square$

Proposition 2.2 has been used extensively in conjunction with the roots of unity [10,15,19]. For each $N > 1$, let $e^{2\pi i k/N}$ denote the Nth roots of unity and let

$$\Gamma_+^N := \left\{ a = (a_0, \ldots, a_{N-1})^t \in \mathbb{Z}^N \colon a_k \geqslant 0, \ \sum_{k=0}^{N-1} a_k e^{2\pi i k/N} = 0 \right\}$$

be the semigroup of integral vectors generated by the $e^{2\pi i k/N}$, $k = 0, \ldots, N-1$. The *rank* of Γ_+^N is the dimension of the real vector space it spans. It was proved in [15] that: if p is a prime and if $N = p^n$ for some $n > 0$, then Γ_+^N has rank p^{n-1} and is generated by $a^{(r)} \in \Gamma_+^N$, $0 \leqslant r \leqslant p^{n-1} - 1$, where the entries of $a^{(r)}$ are

$$a_k^{(r)} = \begin{cases} 1 & \text{if } k = r + jp^{n-1}, \ j = 0, \ldots, p-1, \\ 0 & \text{otherwise.} \end{cases}$$

More explicitly,

$$a^{(0)} = (\underbrace{1, 0, \ldots, 0}_{p^{n-1}}, \ldots, \underbrace{1, 0, \ldots, 0}_{p^{n-1}}), \qquad a^{(1)} = (\underbrace{0, 1, \ldots, 0}_{p^{n-1}}, \ldots, \underbrace{0, 1, \ldots, 0}_{p^{n-1}}), \qquad \text{etc.,}$$

and the corresponding relation (with $N = p^n$) reduces to

$$\sum_{k=0}^{N-1} a_k^{(r)} e^{2\pi i k/N} \equiv \sum_{j=0}^{p-1} e^{2\pi i (r + jp^{n-1})/p^n} = 0, \quad 0 \leqslant r \leqslant p^{n-1} - 1.$$

It follows easily that

Lemma 2.3. *If p is a prime and for $\{b_j\}_{j=0}^{p-1} \subset \mathbb{Z}$ such that $\sum_{j=0}^{p-1} e^{2\pi i b_j/p^n} = 0$, then subject to a permutation, $b_j \equiv r + jp^{n-1} \pmod{p^n}$ for some r with $0 \leqslant r \leqslant p^{n-1} - 1$.*

In the following and in the next section we will modify the condition $m(B^{-k}\nu) = 0$, $\nu \in \mathbb{Z}^s$ in Proposition 2.2 to another criterion that is more flexible to use.

Let $\mathcal{C} = \{0 = c_0, c_1, \ldots, c_m\} \subset \mathbb{Z}^s$, $m \geqslant 1$ and let $S_j(x) = B^{-1}(x + c_j)$, $j = 0, \ldots, m$. We use the maps $\{S_j\}_{j=0}^m$ to define a tree structure: let $\alpha_0 = 0$ and let

$$\alpha_k = S_{j_k}(\alpha_{k-1}) = B^{-1}(\alpha_{k-1} + c_{j_k}), \quad c_{j_k} \in \mathcal{C},$$

denote the descendants in the kth generation. We call such α_k a $\mathcal{C}$-*state* (or just a *state* if there is no confusion). It is clear that

$$\alpha_k = B^{-k} c_{j_1} + \cdots + B^{-1} c_{j_k}.$$

Let $\mathcal{C}_{B^{-1}}^k$ denote the α_k's and $\mathcal{C}_{B^{-1}}^\infty = \bigcup_{k=0}^\infty \mathcal{C}_{B^{-1}}^k$. The following is the key lemma, it is used to prove Theorem 3.3, and the main result Theorem 4.2 via a special $\mathcal{C}$.

Lemma 2.4. *Let $A \in M_s(\mathbb{Z})$ be an expanding matrix, where $|\det(A)| = p$ is a prime, and let $\mathcal{D} = \{0 = d_0, d_1, \ldots, d_{p-1}\} \subset \mathbb{Z}^s$ such that $\operatorname{span}(\mathcal{D}) = \mathbb{R}^s$. Then, for any $\mathcal{C} \subset \mathbb{Z}^s$ and for $B = A^t$, there are at most finitely many $\alpha_k \in \mathcal{C}_{B^{-1}}^\infty$ that are roots of $m(\xi) = p^{-1} \sum_{j=0}^{p-1} e^{2\pi i \langle \xi, d_j \rangle}$.*

We remark that the lemma is trivial in one dimension, however, in higher dimension, the set of zeros of $m(\xi)$ is a manifold and the assertion is not as obvious.

Proof. Suppose otherwise, we can find a sequence $\{\alpha_{\ell_k}\}_{k=1}^\infty \subset \mathcal{C}_{B^{-1}}^\infty$ with strictly increasing indices such that the α_{ℓ_k}'s are distinct and are roots of $m(\xi)$. Let $B^\dagger$ be the adjoint matrix of B. Then $B^\dagger$ is an integer matrix and $B^{-1} = p^{-1}B^\dagger$. By the definition of states, we can write α_{l_k} as $p^{-l_k}\alpha_{l_k}^*$ with $\alpha_{l_k}^* \in \mathbb{Z}^s$. From $m(\alpha_{l_k}) = 0$, we have

$$0 = \frac{1}{p} \sum_{j=0}^{p-1} e^{2\pi i \langle \alpha_{l_k}, d_j \rangle} = \frac{1}{p} \sum_{j=0}^{p-1} e^{2\pi i \langle \alpha_{l_k}^*, d_j \rangle / p^{l_k}}.$$

For each k, applying Lemma 2.3 with $N = p^{l_k}$, we have $r = 0$ since $d_0 = 0$, and

$$\langle \alpha_{l_k}, d_j \rangle = \frac{\langle \alpha_{l_k}^*, d_j \rangle}{p^{l_k}} = \frac{r_{k,j}}{p} \pmod 1, \quad j = 0, 1, \ldots, p - 1,$$

where the set of $\{r_{k,j}\}_{j=0}^{p-1}$ is a permutation of $\{0, \ldots, p - 1\}$. Since $\{\alpha_{l_k}\}_{k=1}^\infty$ is bounded, we can assume that it converges to some α for simplicity. Using the standard diagonal argument, there exists a subsequence which we still denote by $\{\alpha_{l_k}\}_{k=1}^\infty$, such that for each $j = 0, 1, \ldots, p - 1$,

$$\langle \alpha_{l_k}, d_j \rangle = \frac{r_j}{p} + n_{k,j} \equiv \frac{r_j}{p} \pmod 1,$$

where $\{r_0, r_1, \ldots, r_{p-1}\}$ is a permutation of $\{0, 1, \ldots, p - 1\}$. Assume that $\lim_{k \to \infty} \langle \alpha_{l_k}, d_j \rangle = \langle \alpha, d_j \rangle$ yields a $\bar{k}$ such that for each $0 \leqslant j \leqslant p - 1$, we have

$$n_{k,j} = n_j \quad \text{for } k > \bar{k}.$$

Consequently for $t, t' > \bar{k}$, $t \neq t'$,

$$\langle \alpha_{l_t} - \alpha_{l_{t'}}, d_j \rangle = 0 \quad \forall j = 0, 1, \ldots, p - 1,$$

which implies $\alpha_{l_t} = \alpha_{l_{t'}}$ by the condition $\operatorname{span}(\mathcal{D}) = \mathbb{R}^s$. This contradicts with the assumption that the α_{l_k} are distinct and the proof is complete. $\square$

3. Tight sets

In this section we will give a more detailed consideration on the structure of the $\mathcal{C}$-states in Lemma 2.4 that are roots of the mask $m(\xi)$.

Let $B = A^t$ and let $\mathcal{C} = \{0 = c_0, c_1, \ldots, c_m\} \subset \mathbb{Z}^s$. For a finite sequence $\{c_{j_1}, \ldots, c_{j_n}\}$, let $\alpha_k = B^{-k}c_{j_1} + \cdots + B^{-1}c_{j_k}$, $1 \leqslant k \leqslant n$ and $\alpha_0 = 0$, we call the corresponding finite sequence of states $\gamma = \{\alpha_k\}_{k=0}^n$ a *path* from 0 to α_n. An infinite path $\gamma = \{\alpha_k\}_{k=0}^\infty$ is defined similarly. Likewise we can define a path starting from some state α_0, in this case

$$\alpha_k = B^{-k}\alpha_0 + B^{-k}c_{j_1} + \cdots + B^{-1}c_{j_k}.$$

Note that the path from one state to another may not be unique; if $\alpha_{k_1} = \alpha_{k_2}$, then we can identify all paths starting from α_{k_1} with those from α_{k_2}.

It is easy to see that the states defined by $\{0, c_{j_1}, c_{j_2}, \ldots\}$ and $\{0, \ldots, 0, c_{j_1}, c_{j_2}, \ldots\}$ are exactly the same. We can identify the paths arising from this way and it is an equivalence relation. Therefore, we can treat a path $\{\alpha_k\}_{k=0}^n$ (n is finite or infinite) as an equivalence class and the representative is the one with $c_{j_1} \neq 0$. For convenience we make the following convention.

Convention. We assume that for any path $\{\alpha_k\}_{k=0}^n$ with n finite or infinite, $\alpha_1 = B^{-1} c_{j_1} \neq 0$ (except the zero path with all digits $c_{j_i} = 0$).

Using this convention we define the *length* of a path $\{\alpha_k\}_{k=0}^n$ to be n.

Definition 3.1. Let $\mathcal{C} = \{0 = c_0, \ldots, c_m\} \subset \mathbb{Z}^s$. Let $\mathcal{P}$ denote the paths $\{\alpha_k\}_{k=0}^\infty$ that contain infinitely many distinct states. We say that $\mathcal{N} \subset \mathcal{C}_{B^{-1}}^\infty$ is a $\mathcal{C}$-tight set (or just a *tight set*) of $\mathcal{P}$ if $0 \notin \mathcal{N}$ and
 (i) every path $\gamma = \{\alpha_k\}_{k=0}^\infty \in \mathcal{P}$ intersects $\mathcal{N}$ for at least one point, and
(ii) $\mathcal{N}$ is a minimal finite set, i.e., for any $\mathcal{N}' \subsetneq \mathcal{N}$, there exists a path in $\mathcal{P}$ which does not intersect $\mathcal{N}'$.

It is useful to observe that for any $\alpha \in \mathcal{C}_{B^{-1}}^\infty$, there exists a path in $\mathcal{P}$ passing through α. Indeed, this is obvious if $\alpha = 0$. If $\alpha \neq 0$, let $\{0, \alpha_1, \ldots, \alpha_k = \alpha\}$ be a path that reaches α. Since $\alpha \neq B^{-n}\alpha$ for all $n \geq 1$, the path $\{\alpha_1, \ldots, \alpha_k = \alpha, B^{-1}\alpha, B^{-2}\alpha, \ldots\}$ belongs to $\mathcal{P}$.

Proposition 3.2. *Let $B = A^t$. Suppose that $\mathcal{C} = \{0 = c_0, \ldots, c_m\} \subset \mathbb{Z}^s$ with $c_i \notin B\mathbb{Z}^s$ for all $1 \leq i \leq m$. Then all (nonzero) infinite paths are in $\mathcal{P}$, i.e., each infinite path contains infinitely many distinct states.*

Proof. Let $\{\alpha_k\}_{k=0}^\infty$ be a nonzero path. The convention guarantees that $c_{j_1} \neq 0$. If $\alpha_n = \alpha_{n+l}$, then

$$B^{-n} c_{j_1} + \cdots + B^{-1} c_{j_n} = B^{-(n+l)} c_{j_1} + \cdots + B^{-1} c_{j_{n+l}}.$$

By multiplying with $B^{(n+l)}$ and by reshuffling the terms, we see that $c_{j_1} \in B\mathbb{Z}^s$ which contradicts the assumption. Hence all the elements of $\{\alpha_k\}_{k=0}^\infty$ are distinct and the statement follows. $\quad\square$

Theorem 3.3. *Let $A \in M_s(\mathbb{Z})$ be an expanding matrix such that $|\det(A)| = p$ is a prime. Suppose $\mathcal{D} = \{0 = d_0, d_1, \ldots, d_{p-1}\} \subset \mathbb{Z}^s$ is a tile digit set and $\mathrm{span}(\mathcal{D}) = \mathbb{R}^s$. Then for any $\mathcal{C} = \{0 = c_0, c_1, \ldots, c_m\} \subset \mathbb{Z}^s$, $m \geq 1$, there exists a $\mathcal{C}$-tight set $\mathcal{N}$ that consists of the roots of the mask $m(\xi) = p^{-1} \sum_{j=0}^{p-1} e^{2\pi i \langle \xi, d_j \rangle}$.*

Proof. Let $\varphi(x) = \chi_T(x)$ be the characteristic function of the self-affine tile $T(A, \mathcal{D})$. Then $\varphi(x)$ satisfies the refinement equation $\varphi(x) = \sum_{j=0}^{p-1} \varphi(Ax - d_j)$ and $\hat{\varphi}(\xi) = m(B^{-1}\xi) \hat{\varphi}(B^{-1}\xi)$, where $B = A^t$. Iterating this for k-times, we have

$$\hat{\varphi}(\xi) = m(B^{-1}\xi) \cdots m(B^{-k}\xi) \hat{\varphi}(B^{-k}\xi). \tag{3.1}$$

Now for any $\mathcal{C}$-path $\{\alpha_k\}_{k=0}^\infty \in \mathcal{P}$, consider those α_k such that $0 \neq v_k = B^k \alpha_k \in \mathbb{Z}^s$, and thus $v_k = c_{j_1} + \cdots + B^{k-1} c_{j_k}$. The integral periodicity of $m(\xi)$ implies that for $l \leq k$,

$$m(B^{-l} v_k) = m\big((B^{-l} c_{j_1} + \cdots + B^{-1} c_{j_l}) + \cdots + B^{k-l-1} c_{j_k}\big) = m(\alpha_l).$$

Hence from Lemma 2.1, we have for $k \geqslant 1$,

$$0 = \hat{\varphi}(v_k) = m(B^{-1}v_k) \cdots m(B^{-k}v_k)\hat{\varphi}(B^{-k}v_k) = m(\alpha_1) \cdots m(\alpha_k)\hat{\varphi}(\alpha_k).$$

We claim that $m(\alpha_k) = 0$ for some k. If otherwise, $m(\alpha_k) \neq 0$ for all $k \geqslant 1$, and the above identity implies that $\hat{\varphi}(\alpha_k) = 0$ for all $k \geqslant 1$. Using (3.1) again we have

$$0 = \hat{\varphi}(\alpha_k) = m(B^{-1}\alpha_k) \cdots m(B^{-l}\alpha_k)\hat{\varphi}(B^{-l}\alpha_k).$$

Since $T(A, \mathcal{D})$ is a tile by assumption, $\lim_{k \to \infty} \hat{\varphi}(B^{-k}\xi) = \hat{\varphi}(0) = \mathcal{L}(T(A, \mathcal{D})) > 0$. Thus there exists l_0 such that

$$\hat{\varphi}(B^{-l}\alpha) \neq 0 \quad \forall l \geqslant l_0 \text{ and } \alpha \in T(A, \mathcal{C}).$$

By the definition of $\{\alpha_k\}_{k=0}^\infty$ in $\mathcal{P}$, there are infinitely many distinct α_k. It is easy to see that there exist $0 \leqslant l \leqslant l_0$ and a subsequence $\alpha_{k_j'}$ such that $\{B^{-l}\alpha_{k_j'}\}_{j=1}^\infty$ are distinct roots of the mask $m(\xi)$. This contradicts Lemma 2.4 and the claim follows.

The claim implies that any path $\{\alpha_k\}_{k=0}^\infty$ in $\mathcal{P}$ contains at least one root of $m(\xi)$. Let k_0 be the smallest integer such that α_{k_0} is a root of $m(\xi)$ and let $\hat{\mathcal{N}}$ be all such α_{k_0} of the paths $\{\alpha_k\}_{k=0}^\infty \in \mathcal{P}$. Since there are at most finitely many distinct states that are roots of $m(\xi)$ (Lemma 2.4), $\hat{\mathcal{N}}$ must be a finite set. It is clear that we can choose a tight set $\mathcal{N}$ from $\hat{\mathcal{N}}$. $\quad \square$

To conclude this section, we give a general property of the tight sets which will be needed in the proof of the main theorem in the next section.

Proposition 3.4. *Let $B \in M_s(\mathbb{Z})$ be an expanding matrix, and let $\mathcal{C} = \{0 = c_0, \ldots, c_m\} \subset \mathbb{Z}^s$ such that $c_i \notin B\mathbb{Z}^s$ for all $1 \leqslant i \leqslant m$. Then for a $\mathcal{C}$-tight set $\mathcal{N}$, there exists a state α such that all its next-level descendants are in $\mathcal{N}$, i.e., $\{B^{-1}(\alpha + c_j)\}_{j=0}^m \subset \mathcal{N}$. In particular, if $\alpha = 0$, then $\mathcal{N} = \{B^{-1}c_j\}_{j=1}^m$.*

Proof. We claim that $\alpha_k \neq \beta_l$ for any two states α_k and β_l with $k \neq l$. If otherwise $\alpha_k = \beta_l$, we can assume that $k > l$. Recall that in a remark before Definition 3.1, we have identified the paths such that the first term $c_{j_1} \neq 0$, then

$$B^{-k}c_{j_1} + \cdots + B^{-1}c_{j_k} = B^{-l}c_{j_1'} + \cdots + B^{-1}c_{j_l'}.$$

By multiplying B^k and by moving the terms to the right, we see that $c_{j_1} \in B\mathbb{Z}^s$. This contradicts the assumption on the c_j's and the claim follows. We conclude that the index of a state α is equal to the length of the path to reach α (with the convention that $c_{j_1} \neq 0$). (Note that such a state can still be reached by different paths but of the same length.)

Now we consider the tight set $\mathcal{N}$. Since $\mathcal{N}$ is a finite set, we let k be the maximal index so that $\alpha_k \in \mathcal{N}$. Consider the paths from 0 to α_k, by the minimality of $\mathcal{N}$ in Definition 3.1, there exists at least one path starting from 0 and reaching α_k without intersecting $\mathcal{N}$ (except α_k). The path has length k by the conclusion in the above paragraph; let us denote it by $0, \alpha_1, \ldots, \alpha_k$. It follows that

$$\alpha_i \notin \mathcal{N}, \quad i = 0, 1, \ldots, k-1 \quad \text{and} \quad \alpha_k = B^{-1}(\alpha_{k-1} + c_{j_k}).$$

For the case $\alpha_{k-1} \neq 0$, let $\alpha_{k,j} = B^{-1}(\alpha_{k-1} + c_j)$, $j = 0, 1, \ldots, m$. By the remark following Definition 3.1, we see that there are infinite paths in $\mathcal{P}$ passing through $\alpha_{k,j}$ and they must meet $\mathcal{N}$. The maximality of k hence implies that $B^{-1}(\alpha_{k-1} + c_j) \in \mathcal{N}$ for each $0 \leqslant j \leqslant m$. For the case $\alpha_{k-1} = 0$, it reduces to $\mathcal{N} = \{B^{-1}c_j\}_{j=1}^m$ readily (the case $c_1 = 0$ does not appear due to our convention of nonzero paths). $\quad \square$

4. Tile digit sets

In this section, we consider the characterization of the prime tile digit sets.

Lemma 4.1. *Let $A \in M_n(\mathbb{Z})$ be expanding such that $|\det A| = p$ is a prime, let $B = A^t$ and let $B^\dagger = pB^{-1}$ be the adjoint matrix of B. Then p does not divide $\langle B^\dagger v, d \rangle$ for any $d \notin A\mathbb{Z}^s$ and $v \in \mathbb{Z}^s \setminus B\mathbb{Z}^s$.*

Proof. Let $\{0 = v_0, v_1, \ldots, v_{p-1}\}$ be a complete set of coset representatives of $\mathbb{Z}^s / B\mathbb{Z}^s$. By considering $\mathbb{Z}^s / B\mathbb{Z}^s$ as a group, it follows that for $d \notin A\mathbb{Z}^s$, the character $e^{2\pi i \langle B^{-1} \cdot, d \rangle}$ in the dual group satisfies

$$0 = \sum_{j=0}^{p-1} e^{2\pi i \langle B^{-1} v_j, d \rangle} = \sum_{j=0}^{p-1} e^{2\pi i \langle B^\dagger v_j, d \rangle / p}$$

(see, e.g., [8, Lemma 2.1]). Applying Lemma 2.3 for $n = 1$ and using $v_0 = 0$, we obtain, after an rearrangement,

$$\frac{\langle B^\dagger v_j, d \rangle}{p} \equiv \frac{j}{p} \pmod{1}, \quad j = 0, \ldots, p-1.$$

Hence, for $v \in \mathbb{Z}^s \setminus B\mathbb{Z}^s$, there exists $j \neq 0$ such that $v \in v_j + B\mathbb{Z}^s$ and

$$\frac{\langle B^\dagger v, d \rangle}{p} \equiv \frac{j}{p} \pmod{1}.$$

The lemma follows from this. $\square$

Theorem 4.2. *Let $A \in M_s(\mathbb{Z})$ be an expanding matrix such that $|\det(A)| = p$ is a prime. Let $\mathcal{D} = \{0 = d_0, d_1, \ldots, d_{p-1}\} \subseteq \mathbb{Z}^s$ and $\mathrm{span}(\mathcal{D}) = \mathbb{R}^s$. Then $\mathcal{D}$ is a tile digit set if and only if $\mathcal{D} = A^n \tilde{\mathcal{D}}$ for some $n \geqslant 0$ and $\tilde{\mathcal{D}}$ is a complete set of coset representatives of $\mathbb{Z}^s / A\mathbb{Z}^s$.*

Proof. The sufficiency follows from Proposition 1.2 and $T(A, \mathcal{D}) = A^n T(A, \tilde{\mathcal{D}})$. We only need to prove the necessity. Let $n \geqslant 0$ be the largest integer such that $\mathcal{D} = A^n \tilde{\mathcal{D}}$ with $\tilde{\mathcal{D}} \subset \mathbb{Z}^d$ and hence $\tilde{\mathcal{D}} \not\equiv 0$ (mod $A\mathbb{Z}$). In view of $T(A, \mathcal{D}) = A^n T(A, \tilde{\mathcal{D}})$, we can assume, without loss of generality, that $\mathcal{D} = \tilde{\mathcal{D}}$, hence $d_l \notin A\mathbb{Z}^s$ for some $1 \leqslant l \leqslant p - 1$. Let $m(\xi) = p^{-1} \sum_{j=0}^{p-1} e^{2\pi i \langle \xi, d_j \rangle}$ be the mask of T. For

$$v \in \mathbb{Z}^s \setminus B\mathbb{Z}^s \quad \text{and} \quad \mathcal{C} = \{0, v, \ldots, (p-1)v\},$$

according to Theorem 3.3 there exists a tight set $\mathcal{N}$ of $\mathcal{P}$ which consists of the roots of $m(\xi)$. By Proposition 3.4, there exists a state α such that $\{B^{-1}(\alpha + tv)\}_{t=0}^{p-1}$ (for $\alpha \neq 0$) or $\{tB^{-1}v\}_{t=1}^{p-1}$ (for $\alpha = 0$) are the roots of the mask $m(\xi)$.

We first consider the case $\alpha = 0$: for $t = 1$, $B^{-1}v$ is a root of $m(\xi)$. Hence

$$0 = m(B^{-1}v) = p^{-1} \sum_{j=0}^{p-1} e^{2\pi i \langle B^{-1}v, d_j \rangle} = p^{-1} \sum_{j=0}^{p-1} e^{2\pi i \langle B^\dagger v, d_j \rangle / p}.$$

By applying Lemma 2.3 (with $n = 1$) and noting that there is one term in the above summand equal to 1, we conclude that

$$\{\langle B^\dagger v, d_j \rangle \colon 0 \leqslant j \leqslant p - 1\} \equiv \{0, 1, \ldots, p - 1\} \pmod{p}.$$

It follows that $d_i - d_j \notin A\mathbb{Z}^s$ for any $i \neq j$ (otherwise, if $d_i - d_j = Aw \in A\mathbb{Z}^s$ for some i, j, then

$$\langle B^\dagger v, d_i \rangle - \langle B^\dagger v, d_j \rangle = \langle B^\dagger v, Aw \rangle = p \langle v, w \rangle \equiv 0 \pmod{p},$$

which is impossible). This implies that $\mathcal{D}$ is a complete set of coset representatives of $\mathbb{Z}^s / A\mathbb{Z}^s$.

Next we show that the case $\alpha \neq 0$ cannot happen and the theorem will follow. Since α is a $\mathcal{C}$-state,

$$\alpha = j_1 B^{-1} v + \cdots + j_k B^{-k} v = \frac{1}{p^k} \left(j_1 p^{k-1} B^\dagger v + \cdots + j_k (B^\dagger)^k v \right) := \frac{\alpha'}{p^k} \neq 0,$$

where $0 \leqslant j_i \leqslant p - 1$. By substituting the roots $B^{-1}(\alpha + tv)$ into the mask $m(\xi)$, we have

$$\sum_{j=0}^{p-1} e^{2\pi i \langle B^\dagger \alpha' + t p^k B^\dagger v, d_j \rangle / p^{k+1}} = 0 \quad \forall t = 0, 1, \ldots, p - 1. \tag{4.1}$$

Hence for $t = 0$, (4.1) reduces to

$$\sum_{j=0}^{p-1} e^{2\pi i \langle B^\dagger \alpha', d_j \rangle / p^{k+1}} = 0.$$

Lemma 2.3 (use $n = k + 1$ and $r = 0$ since $0 \in \mathcal{D}$) shows that with a rearrangement

$$\langle B^\dagger \alpha', d_j \rangle = j p^k + p^{k+1} w_j \quad \forall 0 \leqslant j \leqslant p - 1,$$

where $w_j \in \mathbb{Z}$. Substituting these into (4.1) we have

$$\sum_{j=0}^{p-1} e^{2\pi i (j/p + w_j + (t/p) \langle B^\dagger v, d_j \rangle)} = 0 \quad \forall t = 0, 1, \ldots, p - 1.$$

Again we apply Lemma 2.3 to each $0 \leqslant t \leqslant p - 1$,

$$\left\{ j + t \langle B^\dagger v, d_j \rangle \colon 0 \leqslant j \leqslant p - 1 \right\} \equiv \{0, 1, \ldots, p - 1\} \pmod{p}.$$

Since $d_0 = 0$, we see that for each $0 \leqslant t \leqslant p - 1$,

$$j + t \langle B^\dagger v, d_j \rangle \equiv 0 \pmod{p} \quad \text{if and only if} \quad j = 0. \tag{4.4}$$

On the other hand, by Lemma 4.1, $p \nmid \langle B^\dagger v, d_l \rangle$. This implies that

$$\left\{ t \langle B^\dagger v, d_l \rangle \colon 0 \leqslant t \leqslant p - 1 \right\} \equiv \{0, 1, \ldots, p - 1\} \pmod{p}.$$

Since $l \neq 0$, there exists $1 \leqslant \tilde{t} \leqslant p - 1$ such that $\tilde{t} \langle B^\dagger v, d_l \rangle \equiv -l \pmod{p}$, i.e., $l + \tilde{t} \langle B^\dagger v, d_l \rangle \equiv 0 \pmod{p}$. This contradicts (4.4) and completes the proof of the claim, and hence the theorem. $\quad \square$

We remark that Theorem 4.2 was proved in [15] under the additional assumption that $p\mathbb{Z}^s \nsubseteq A^2\mathbb{Z}^s$. The assumption was used to show that the above d_l has the specific form $A^\dagger d_l \equiv j d_l \pmod{p}$ for some $j \neq 0$. It is known that there are matrices that do not satisfy the assumption (see [12, Proposition 6.2] for examples with $|\det A| = 3$ in $\mathbb{R}^2$). Our proof here bypasses this by using the tightness property.

The condition $\text{span}(\mathcal{D}) = \mathbb{R}^s$ in the theorem is only used in Lemma 2.4; we conjecture that it can be removed. Indeed this is true for $\mathbb{R}^2$:

Corollary 4.3. *Let $A \in M_2(\mathbb{Z})$ be an expanding matrix such that $|\det(A)| = p$ is a prime. Let $\mathcal{D} = \{0 = d_0, d_1, \ldots, d_{p-1}\} \subseteq \mathbb{Z}^2$. Then $\mathcal{D}$ is a tile digit set if and only if $\mathcal{D} = A^n \tilde{\mathcal{D}}$ for $n \geqslant 0$ and $\tilde{\mathcal{D}}$ is a complete set of coset representatives of $\mathbb{Z}^2 / A\mathbb{Z}^2$.*

Proof. If $\mathrm{span}(\mathcal{D}) = \mathbb{R}^2$, the assertion follows from the above theorem. Otherwise, $\mathcal{D}$ is collinear (i.e., $\mathcal{D} = \{0, i_1 v, \ldots, i_{p-1} v\}$ for some $v \in \mathbb{Z}^2$ and for all $i_j \in \mathbb{Z}$). The assertion was proved in [11, Theorem 3.1]. □

If $|\det A|$ is not a prime, then it is harder to characterize the tile digit sets. For example, if we take $A = [4]$ and $\mathcal{D} = \{0, 1, 8, 9\}$ in $\mathbb{R}$, it is easy to show that $T = [0, 1] \cup [2, 3]$; note that $\mathcal{D} \equiv \{0, 1\} \pmod{4}$ and the criterion in Theorem 4.2 does not hold. Besides the case for A to have prime determinant, we only know two other cases on $\mathbb{R}$ that the tile digit sets can be completely characterized, namely, the cases $A = [p^l]$ [15] and $A = [pq]$ [19], where p, q are distinct primes. In the second case, the characterization is quite simple: let $b = pq$,

$$\mathcal{D} \equiv \mathcal{E}_1 + b^{k-1}\mathcal{E}_2 \pmod{b^k}$$

for some integer $k \geqslant 1$, where $\mathcal{E}_1 = \{0, \ldots, p-1\}$, $\mathcal{E}_2 = \{0, p, \ldots, p(q-1)\}$ (or interchange the role of p and q). We do not know if this can be extended to more general $A = [b]$ or to higher dimensions.

5. Scaling functions

In this section we will consider the $\mathcal{C}$-tight set in regard to the existence of the scaling function

$$\varphi(x) = \sum_{j=0}^{N-1} w_j \varphi(Ax - d_j) \tag{5.1}$$

($N \geqslant |\det A|$) as in Lemma 2.1. In one dimension, let $A = [b]$ with positive integer $b > 1$. It is well known that a necessity for (5.1) to have compactly supported L^1-solution φ is $\sum_{j=0}^{N-1} w_j = b^{m+1}$ for some $m \geqslant 0$; in this case, there exists g such that $(\mathrm{d}^m g(x))/(\mathrm{d}x^m) = \varphi(x)$ and $g(x) = \sum_{j=0}^{N-1} b^{-m} w_j g(bx - d_j)$. Hence we can assume that $\sum_{j=0}^{N-1} w_j = |\det A| = b$ as usual.

For any set $\mathcal{C} \subset \mathbb{Z}^s$, let $\mathcal{C}_{B,1} = \mathcal{C}$, $\mathcal{C}_{B,k} = \mathcal{C} + B\mathcal{C}_{B,k-1}$, and $\mathcal{C}_B = \bigcup_{k=1}^{\infty} \mathcal{C}_{B,k}$.

Theorem 5.1. *Let $B = A^t$, and let $\mathcal{C} = \{0 = c_0, c_1, \ldots, c_m\} \subset \mathbb{Z}^s$ with $\mathcal{C}_B = \mathbb{Z}^s$. Suppose the weights w_j's in (5.1) are positive and suppose there exists a $\mathcal{C}$-tight set $\mathcal{N}$ consisting of the roots of the mask $m(\xi) = b^{-1} \sum_{j=0}^{N-1} w_j e^{2\pi i \langle \xi, d_j \rangle}$. Then the scaling function φ in (5.1) exists.*

Proof. Let $0 \neq v \in \mathbb{Z}^s = \mathcal{C}_B$, then $v \in \mathcal{C}_k$ for some k and

$$v = c_{j_1} + Bc_{j_2} + \cdots + B^{k-1}c_{j_k}.$$

Without loss of generality, we assume that $c_{j_1} \neq 0$. Let $\alpha_i = B^{-i}c_{j_1} + B^{-i+1}c_{j_2} + \cdots + B^{-1}c_{j_i}$, $i = 1, 2, \ldots, k$, be the $\mathcal{C}$-states, then $v = B^k \alpha_k$. If we let α_{k+l} be the states defined by $c_{j_1}, \ldots, c_{j_k}, 0, 0, \ldots$, then $\{\alpha_k\}_{k=1}^{\infty}$ contains infinitely many distinct states and $v = B^{k+l}\alpha_{k+l}$ also. By (5.1), we have

$$\hat{\varphi}(v) = m(B^{-1}v) \cdots m(B^{-n}v)\hat{\varphi}(B^{-n}v), \quad n \geqslant 1.$$

The integral periodicity of $m(\xi)$ implies that

$$\hat{\varphi}(\nu) = m(\alpha_1) \cdots m(\alpha_{k+l}) \hat{\varphi}(\alpha_{k+l})$$

for all nonnegative integers l. By the tightness property of $\mathcal{N}$, one of the states in the path $\{\alpha_k\}_{k=0}^{\infty}$ is a root of $m(\xi)$. Hence $\hat{\varphi}(\nu) = 0$. Since $0 \neq \nu \in \mathbb{Z}^s$ is arbitrary, the second part of Lemma 2.1 implies the theorem. $\square$

For the hypothesis $\mathcal{C}_B = \mathbb{Z}^s$ in the above theorem, a sufficient condition is that $\mathcal{C} = (\mathcal{E} - \mathcal{E})$, where $\mathcal{E}$ is a tile digit set with respect to B and $\mathcal{L}(T(B, \mathcal{E})) = 1$ (this implies $(\mathcal{E} - \mathcal{E})_B = \mathbb{Z}^s$ [15]). It was also known that for a given expanding matrix $B \in M_s(\mathbb{Z})$, such $\mathcal{E}$ exists for $s \leqslant 3$ [17]; but for $s > 3$, Potiopa gave an example that this is not true (see the addendum of [17]). Here since we have no restriction on the number of elements on $\mathcal{C}$, we can guarantee the existence of such $\mathcal{C}$ by the following proposition.

Proposition 5.2. *For an expanding matrix $B \in M_s(\mathbb{Z})$, there exists a digit set $\mathcal{C} \subset \mathbb{Z}^s$ with $0 \in \mathcal{C}$ and $\mathcal{C}_B = \mathbb{Z}^s$.*

Proof. Let $\mathcal{E}$ be a complete set of coset representatives of $\mathbb{Z}^s / B\mathbb{Z}^s$ and assume that $0 \in \mathcal{E}$. Then there exists $\mathcal{J} \subset (\mathcal{E} - \mathcal{E})_B$ with $0 \in \mathcal{J}$ such that $T(A, \mathcal{E})$ tiles $\mathbb{R}^s$ using $\mathcal{J}$ as a tiling set [14].

Let $\mathcal{F} = T(A, \mathcal{E}) \cap \mathbb{Z}^s$, then $0 \in \mathcal{F}$. For any $\nu \in \mathbb{Z}^s$ there exists $t \in \mathcal{J}$ such that $\nu - t \in \mathcal{D}$, this implies that $\mathbb{Z}^s \subseteq \mathcal{J} + \mathcal{F}$. Now let $\mathcal{C} = \mathcal{E} - \mathcal{E} + \mathcal{F}$. It is clear that

$$\mathbb{Z}^s \supseteq \mathcal{C}_B \supseteq (\mathcal{E} - \mathcal{E})_B + \mathcal{F} \supseteq \mathcal{J} + \mathcal{F} \supseteq \mathbb{Z}^s.$$

Consequently $\mathcal{C}_B = \mathbb{Z}^s$. $\square$

We do not know whether the necessity of Theorem 5.1 is true; a special case is Theorem 3.3 for $\varphi = \chi_T$, where T is determined by the prime tile digit sets. Nevertheless the necessity is true for the refinement equation with one variable.

Proposition 5.3. *Suppose the refinement equation (5.1) defined on $\mathbb{R}$ has a compactly supported L^1-solution. Then for any $\mathcal{C} = \{0, c_i, \ldots, c_m\} \subset \mathbb{Z}$ with $m \geqslant 1$, there exists a $\mathcal{C}$-tight set consisting of roots of $m(\xi) = b^{-1} \sum_{j=0}^{N-1} w_j e^{2\pi i d_j \xi}$.*

Proof. The proof is essentially the same as that of Theorem 3.3; we need only replace the application of Lemma 2.4 at the end of the proof by the fact that for the one variable ξ, the trigonometric polynomial $m(\xi)$ can have at most finitely many zeros in a bounded region. $\square$

As an application of the tight sets, we have an interesting one-dimensional result on the scaling functions with prime dilation.

Recall that a cyclotomic polynomial $F_n(x)$ is the minimal polynomial of the algebraic integer $e^{2\pi i/n}$ over the rational field. If $n = p$ is a prime, then $F_p(x) = 1 + x + \cdots + x^{p-1}$ and $F_{p^k}(x) = F_p(x^{p^{k-1}})$.

Theorem 5.4. *Let p be a prime. Suppose $\{w_j\}_{j=0}^{N-1}$ are positive rationals and $\sum_{j=0}^{N-1} w_j = p$. Then the equation $\varphi(x) = \sum_{j=0}^{N-1} w_j \varphi(px - d_j)$ on $\mathbb{R}$ has a compactly supported L^1-solution if and only if there exists a positive integer k such that $m(p^{-k}) = 0$, where $m(\xi) = p^{-1} \sum_{j=0}^{N-1} w_j e^{2\pi i d_j \xi}$.*

Proof. We first prove the sufficiency. Let $M(z) = p^{-1} \sum_{j=0}^{N-1} w_j z^{d_j}$, then $M(e^{2\pi i \xi}) = m(\xi)$. For $\mathcal{C} = \{0, 1, \ldots, p-1\}$, let

$$\alpha_{i_1,\ldots,i_n} := \frac{i_1 + i_2 p + \cdots + i_n p^{n-1}}{p^n},$$

where $1 \leqslant i_1 \leqslant p-1$, $0 \leqslant i_t \leqslant p-1$, $t = 2, 3, \ldots, n$. It is clear that the set of all $\alpha_{i_1,\ldots,i_n}$ is a tight set. Suppose that $m(p^{-k}) = 0$ for some $k > 0$, then the minimal property of $F_{p^k}(x)$ over the rational field implies that $F_{p^k}(x)$ divides $M(x)$. Note that

$$F_{p^k}(x) = 1 + x^{p^{k-1}} + x^{2p^{k-1}} + \cdots + x^{(p-1)p^{k-1}}. \tag{5.2}$$

Then $F_{p^k}(e^{2\pi i \alpha_{i_1,\ldots,i_k}}) = 0$. It follows that $\alpha_{i_1,\ldots,i_k}$ is a root of the mask $m(\xi)$. The set $\mathcal{N}$ of $\alpha_{i_1,\ldots,i_k}$ is therefore a tight set of zeros of the mask $m(\xi)$, and Theorem 5.1 implies that the scaling function φ exists.

Conversely Proposition 5.3 implies that there is a $\mathcal{C}$-tight set $\mathcal{N}$ that are roots of the mask $m(\xi)$. Since $\alpha \in \mathcal{N}$ can be expressed as $(i_1 + i_2 p + \cdots + i_k p^{k-1})/p^k$, $i_j \in \mathcal{C}$, $i_1 \neq 0$, it follows that $e^{2\pi i \alpha}$ is a primitive p^kth root of unity. Hence $F_{p^k}(x)$ divides $M(x)$, so that $m(p^{-k}) = M(e^{2\pi i/p^k}) = 0$. $\quad\square$

We remark that the above theorem can be modified for $A = [p^\lambda]$: the scaling function of (5.1) exists if and only if there exist integers k_l, $l = 1, \ldots, \lambda$, such that $m(p^{l+k_l \lambda}) = 0$.

The proof is essentially the same; all we need to do is to adjust states into a slightly more complicated form: the 1st level is

$$\frac{i_1 + \cdots + i_j p^{j-1} + \cdots + i_\lambda p^{\lambda-1}}{p^\lambda},$$

where $0 \leqslant i_j \leqslant p-1$ and not all $i_j = 0$, $1 \leqslant j \leqslant \lambda$. Equivalently the states are of the form

$$\frac{i_1 + \cdots + i_j p^{j-1} + \cdots + i_l p^{l-1}}{p^l}, \quad 0 \leqslant i_j \leqslant p-1, \ i_1, i_l \neq 0, \ 1 \leqslant l \leqslant \lambda.$$

Then $(n+1)$th states have the form

$$r_{l,n}/p^{l+n\lambda}, \quad 1 \leqslant l \leqslant \lambda,$$

where $r_{l,n}$ and p are co-prime. Now we can use the same argument as in the above theorem to verify the assertion.

For $\#\mathcal{D} \geqslant |\det A|$, we call $T := T(A, \mathcal{D})$ a *self-affine region* if $T^\circ \neq \emptyset$. Note that this gives a necessary condition for the existence of the scaling functions defined by the pair (A, D). We have studied the sufficient conditions for such pair $(A, \mathcal{D})$ to be a self-affine region [6,7]; but we do not have a characterization even for the prime case in one dimension. An interesting example is when $A = [3]$, $\mathcal{D} = \{0, 1, 3, 4\}$, then $T = [0, 2]$ is a self-affine region, but $\mathcal{D}$ is not a complete residue set modulo 3 [6].

Normally, we should expect that on a self-affine region, we can assign weights with certain flexibility to the refinement equation to yield scaling functions. It is interesting and unexpected to see that in the above example, the refinement equation

$$\varphi(x) = w_0 \varphi(3x) + w_1 \varphi(3x - 1) + w_2 \varphi(3x - 3) + w_3 \varphi(3x - 4),$$

where w_i's are nonnegative rationals and $w_0 + w_1 + w_2 + w_3 = 3$ has no nontrivial L^1-solution. Indeed the mask is

$$m(\xi) = w_0 + w_1 e^{2\pi i \xi} + w_2 e^{6\pi i \xi} + w_3 e^{8\pi i \xi}.$$

Using Theorem 5.4, we see that an L^1-solution $\varphi(x)$ exists if and only if there exists $k > 0$ such that $m(3^{-k}) = 0$. But the condition cannot be satisfied because for $k = 1$,

$$m\left(\frac{1}{3}\right) = w_0 + w_2 + (w_1 + w_3)e^{(2/3)\pi i} \neq 0$$

for $k > 1$, if $m(3^{-k}) = w_0 + w_1 e^{2\pi i/3^k} + w_2 e^{6\pi i/3^k} + w_3 e^{8\pi i/3^k} = 0$, we can assume that all the w_j's are integers without loss of generality. Using Lemma 2.3 for $N = 3^k$, the vector $(w_0, w_1, 0, w_2, w_3, 0, 0, \ldots, 0) \in \Gamma_+^{3^k}$ can be expressed as integral combination of the vectors $\{a^{(l)} \colon 0 \leqslant l \leqslant 3^{k-1} - 1\}$, which is impossible. Hence $m(3^{-k}) \neq 0$ for $k > 1$. We, therefore, conclude that (5.1) has no L^1-solution.

Acknowledgments

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