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Relationships between Different Dimensions of a Measure

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Abstract. We establish various relationships of the Hausdorff dimension, entropy dimension and L^p-dimension of a measure without assuming that the local dimension of μ exists μ-a.e. These extend a well known result of Young.

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1. Introduction

In the study of fractals and dynamical systems, there are two most frequently used dimensions for a probability measure μ on $\mathbb{R}^d$, namely the *Hausdorff dimension* $\dim_H\mu$ and the *entropy dimension* $\dim_e\mu$. The former is defined by

$$\dim_H\mu = \inf\{\dim F : \mu(F^c) = 0\}$$

where $\dim F$ denotes the Hausdorff dimension of F. To define the latter, let $\mathscr{P}_n$ be the partition of $\mathbb{R}^d$ into grid boxes $\prod_{i=1}^d [2^{-n}j_i, 2^{-n}(j_i+1))$ with $j_i \in \mathbb{Z}$. Let

$$H_n(\mu) = -\sum_{Q\in\mathscr{P}_n} \mu(Q)\log\mu(Q)$$

and define

$$\dim_e\mu = \lim_{n\to\infty} \frac{H_n(\mu)}{\log 2^n}.$$

A well known theorem of Young [20] states that

Theorem 1.1. *If the local dimension*

$$D(\mu,x) = \lim_{r\to 0^+} \frac{\log\mu(B_r(x))}{\log r} = \alpha \quad \mu\text{-a.e.,} \tag{1.1}$$

then $\dim_e\mu = \dim_H\mu = \alpha$.

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For a probability measure, condition (1.1) does not hold in general, and is not easy to verify except for some special cases. In this note we will give a detailed consideration of the dimensions of measures without assuming (1.1). The key of the proof is the following device. Let $\underline{D}(\mu,x)$ and $\overline{D}(\mu,x)$ be the *lower* and *upper local dimensions* of μ at x as an obvious modification of (1.1). We define

$$\dim_*\mu = \operatorname{ess\,inf}\underline{D}(\mu,x), \quad \dim^*\mu = \operatorname{ess\,sup}\underline{D}(\mu,x),$$
$$\operatorname{Dim}_*\mu = \operatorname{ess\,inf}\overline{D}(\mu,x), \quad \operatorname{Dim}^*\mu = \operatorname{ess\,sup}\overline{D}(\mu,x),$$

where the ess inf and ess sup are taken with respect to μ. It is obvious that

$$\dim_*\mu \leqslant \dim^*\mu \quad \text{and} \quad \operatorname{Dim}_*\mu \leqslant \operatorname{Dim}^*\mu.$$

A systematic study of $\dim_*\mu$ and $\dim^*\mu$ was first carried out by Fan in [7], [8] and the first two statements in the following theorem were proved there. They are related to the Hausdorff dimensions of the supports of the measure, of particular interest is the second statement that $\dim^*\mu$ is the Hausdorff dimension of the measure. A parallel theory concerning the packing dimensions of the supports was then developed by Tamashiro in [18], which contains the last two statements in the following theorem.

Theorem 1.2. *Let μ be a probability measure on $\mathbb{R}^d$, then*

$$\dim_*\mu = \sup\{\alpha \geqslant 0 : \forall E, \dim E < \alpha \Rightarrow \mu(E) = 0\},$$

$$\dim^*\mu = \inf\{\dim F : \mu(F^c) = 0\} \ (= \dim_H \mu),$$

$$\operatorname{Dim}_*\mu = \sup\{\alpha \geqslant 0 : \forall E, \operatorname{Dim} E < \alpha \Rightarrow \mu(E) = 0\},$$

$$\operatorname{Dim}^*\mu = \inf\{\operatorname{Dim} F : \mu(F^c) = 0\},$$

where $\operatorname{Dim} E$ *denotes the packing dimension of E.*

In view of these alternative expressions, we call the first two the *lower* and *upper Hausdorff dimensions* of μ and the last two the *lower* and *upper packing dimensions of μ*. We can also make simple modification of $\dim_e\mu$ to define the *upper* and *lower entropy dimensions* $\overline{\dim}_e\mu$, $\underline{\dim}_e\mu$. Our first theorem is

Theorem 1.3. *For a probability measure μ on $\mathbb{R}^d$, we have*

$$\dim_*\mu \leqslant \underline{\dim}_e\mu \leqslant \overline{\dim}_e\mu \leqslant \operatorname{Dim}^*\mu.$$

We will show by example that $\dim^*\mu$ (and $\operatorname{Dim}_*\mu$) is not comparable with $\underline{\dim}_e\mu$ or $\overline{\dim}_e\mu$ (Section 3). Observe that Young's theorem is a direct consequence of Theorem 1.3 and the second statement of Theorem 1.2.

Now let us consider the L^q-dimensions of a measure. Let $\mathscr{P}_n$ be the dyadic partition of $\mathbb{R}^d$ as above. For $q > 0$, let $s_n(q) = \sum_{Q \in P_n} \mu(Q)^q$ and define the *lower* and *upper L^q-dimensions* ($q \neq 1$) by

$$\underline{\dim}_q\mu = \varliminf_{n\to\infty} \frac{\log s_n(q)}{(q-1)\log 2^{-n}}, \quad \overline{\dim}_q\mu = \varlimsup_{n\to\infty} \frac{\log s_n(q)}{(q-1)\log 2^{-n}}.$$

The concept of L^q-dimension was first introduced by Rényi in order to generalize the entropy dimension [17], chapter 9. Nowadays we use

$$\underline{\tau}(q) = \varliminf_{n\to\infty} \frac{\log s_n(q)}{\log 2^{-n}} \quad \text{and} \quad \overline{\tau}(q) = \varlimsup_{n\to\infty} \frac{\log s_n(q)}{\log 2^{-n}}$$

to investigate the multifractal structure of a measure [6], in particular we use $\underline{\tau}(q)$ (often omitting the underline in literature) because it is a concave function. We can rewrite

$$\underline{\dim}_q\mu = \frac{\underline{\tau}(q)}{q-1}, \quad \overline{\dim}_q\mu = \frac{\overline{\tau}(q)}{q-1} \quad \text{if } q > 1,$$
$$\underline{\dim}_q\mu = \frac{\overline{\tau}(q)}{q-1}, \quad \overline{\dim}_q\mu = \frac{\underline{\tau}(q)}{q-1} \quad \text{if } 0 < q < 1.$$

(Note the reversal of the upper and the lower signs in the second case because $q-1$ is negative). In [14], Ngai showed that

$$\underline{\tau}'_+(1) \leqslant \underline{D}(\mu,x) \leqslant \overline{D}(\mu,x) \leqslant \underline{\tau}'_-(1) \quad \mu\text{-a.e.},$$

where τ'_- and τ'_+ denote the left and right derivatives of τ (later these are obtained independently by Heurteaux [12] and Olsen [15]). Then by the concavity of $\underline{\tau}(q)$ and Theorem 1.2, we have

$$\underline{\dim}_q\mu \leqslant \dim_*\mu \quad \text{if } q > 1 \tag{1.2}$$

and

$$\dim^*\mu \leqslant \overline{\dim}_p\mu \quad \text{if } p < 1. \tag{1.3}$$

We write these inequalities into Theorem 1.3 to get in a more complete comparison.

Theorem 1.4. *For $0 < p < 1 < q$, we have*

$$\underline{\dim}_q\mu \leqslant \dim_*\mu \leqslant \underline{\dim}_e\mu \leqslant \overline{\dim}_e\mu \leqslant \mathrm{Dim}^*\mu \leqslant \overline{\dim}_p\mu. \tag{1.4}$$

We point out that in [14], [12] and [15] the proofs rely on the above mentioned Theorem 1.2 or some proofs of Theorem 1.2 were redone. We will provide a direct proof for the first inequality in Theorem 1.4 without using any established results. So the proof is more accessible and simpler than those in [14], [12] and [15]. The last inequality follows the same argument. Heurteaux recently told us that Theorem 1.4 is also proved in [1].

After proving the theorems, we will use Bernoulli products to construct a class of measures to show that the inequalities in the theorems cannot be improved, and that the four dimensions $\dim^*\mu$, $\underline{\dim}_e\mu$, $\overline{\dim}_e\mu$, $\mathrm{Dim}_*\mu$ are not comparable, except $\underline{\dim}_e\mu \leqslant \overline{\dim}_e\mu$.

2. Proof of the Theorems

Proof of Theorem 1.3. Let $Q_n(x)$ be the box in $\mathscr{P}_n$ containing x and let

$$D_n(x) = \frac{\log\mu(Q_n(x))}{\log 2^{-n}} = -\frac{1}{n}\log_2 \mu(Q_n(x)). \tag{2.1}$$

It is clear that $D_n(x)$ is constant on any box in $\mathscr{P}_n$ and

$$-\frac{H_n(\mu)}{\log 2^{-n}} = \int_{\mathbb{R}^d} D_n(x)d\mu(x).$$

By applying the Fatou lemma we have

$$\underline{\dim}_e\mu = \underline{\lim}_{n\to\infty}\int_{\mathbb{R}^d} D_n(x)d\mu(x) \geqslant \int_{\mathbb{R}^d} \underline{D}(\mu,x)d\mu(x).$$

The definition of $\dim_*\mu$ implies that $\underline{D}(\mu,x) \geqslant \dim_*\mu$ for μ-a.e. x and hence the first inequality follows.

The second inequality is trivial. To prove the third inequality, we assume without loss of generality that μ is supported by the unit cube $[0,1]^d$. Define, for $N \geqslant 1$,

$$A_N = \{x : D_n(x) \leqslant d+1 \text{ for all } n > N\}.$$

Since $\overline{D}(\mu,x) = \overline{\lim}_{n\to\infty}D_n(x) \leqslant d$ for μ-almost every x (see [19]), we have $\mu(\bigcup_{N=1}^{\infty} A_N) = 1$. Notice that A_N is increasing on N so that $\lim_{N\to\infty}\mu(A_N) = 1$. Therefore for any $0 < \varepsilon < 1$, there exists $N \geqslant 1$ such that $\mu(A_N) \geqslant 1-\varepsilon$. Write

$$\int_{\mathbb{R}^d} D_n(x)d\mu(x) = \int_{A_N} D_n(x)d\mu(x) + \int_{A_N^c} D_n(x)d\mu(x).$$

On A_N the functions $D_n(x), n \geqslant N$, are uniformly bounded by $d+1$, hence we can use the Fatou lemma to get

$$\begin{aligned}
\overline{\dim}_e\mu &\leqslant \overline{\lim}_{n\to\infty}\int_{A_N} D_n(x)d\mu(x) + \overline{\lim}_{n\to\infty}\int_{A_N^c} D_n(x)d\mu(x)\\
&\leqslant \mathrm{Dim}^*\mu + \overline{\lim}_{n\to\infty}\int_{A_N^c} D_n(x)d\mu(x).
\end{aligned}$$

For fixed $n > N$, we have

$$\begin{aligned}
\int_{A_N^c} D_n(x)d\mu(x) &= \int_{A_N^c\cap\{D_n(x)\leqslant d+1\}} D_n(x)d\mu(x) + \int_{A_N^c\cap\{D_n(x)>d+1\}} D_n(x)d\mu(x)\\
&\leqslant (d+1)\varepsilon + \sum_{m=d+1}^{\infty}\int_{A_N^c\cap\{m<D_n(x)\leqslant m+1\}} D_n(x)d\mu(x)\\
&= (d+1)\varepsilon + \sum_{m=d+1}^{\infty}\sum_{Q\in\mathscr{P}_n}\int_{Q\cap\{m<D_n(x)\leqslant m+1\}} D_n(x)d\mu(x).
\end{aligned}$$

Notice that $D_n(x) > m$ means $\mu(Q_n(x)) < 2^{-mn}$, hence

$$\sum_{Q\in\mathscr{P}_n}\int_{Q\cap\{m<D_n(x)\leqslant m+1\}} D_n(x)d\mu(x) \leqslant (m+1)2^{-nm}\cdot 2^{nd}.$$

Here we have used the fact that the unit cube $[0,1]^d$ contains 2^{nd} boxes in $\mathscr{P}_n$. However

$$\sum_{m=d+1}^{\infty}(m+1)2^{-mn+dn} = O(2^{-n}).$$

It follows that for any $\varepsilon > 0$, $\overline{\dim}_e\mu \leqslant \mathrm{Dim}^*\mu + (d+1)\varepsilon$. This completes the proof. $\square$

Theorem 1.1 can hence be improved as follows

Corollary 2.1. *If* $D(\mu, x) = \lim_{r\to 0} \frac{\log \mu(B_r(x))}{\log r} = \alpha$ *for* μ*-a.e.* x, *then*

$$\dim_*\mu = \dim^*\mu = \underline{\dim}_e\mu = \overline{\dim}_e\mu = \mathrm{Dim}_*\mu = \mathrm{Dim}^*\mu = \alpha.$$

Proof of Theorem 1.4. We will prove $\underline{\dim}_q\mu \leqslant \dim_*\mu$. For $\alpha > \dim_*\mu$, let

$$B = \left\{x : \varliminf_{n\to\infty} \frac{\log \mu(Q_n(x))}{\log 2^{-n}} < \alpha\right\},$$

then $\mu(B) > 0$. For $n \geqslant 1$, let

$$F_n = \bigcup \{Q \in \mathscr{P}_n : \mu(Q) > 2^{-n\alpha}\}.$$

Note that $B \subseteq \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} F_n$. Then

$$0 < \mu(B) \leqslant \lim_{N\to\infty} \mu\left(\bigcup_{n=N}^{\infty} F_n\right). \tag{2.2}$$

We claim that there exists a subsequence $\{n_j\}$ such that

$$\mu(F_{n_j}) \geqslant \frac{1}{n_j^2} \quad \text{for all } j. \tag{2.3}$$

Otherwise,

$$\lim_{N\to\infty} \mu\left(\bigcup_{n=N}^{\infty} F_n\right) \leqslant \lim_{N\to\infty} \sum_{n=N}^{\infty} \mu(F_n) \leqslant \lim_{N\to\infty} \sum_{n=N}^{\infty} \frac{1}{n^2} = 0$$

which will contradict (2.2). By (2.3) we have

$$\begin{aligned} I_{n_j} &:= -\frac{1}{(q-1)n_j} \log_2 \int_{\mathbb{R}^d} \mu(Q_{n_j}(x))^{q-1} d\mu(x) \\ &\leqslant -\frac{1}{(q-1)n_j} \log_2 \int_{F_{n_j}} \mu(Q_{n_j}(x))^{q-1} d\mu(x) \\ &\leqslant -\frac{1}{(q-1)n_j} \log_2 (2^{-n_j\alpha(q-1)} \mu(F_{n_j})) \\ &= \alpha + \frac{1}{(q-1)} \frac{2\log_2 n_j}{n_j}. \end{aligned}$$

This implies that $\underline{\dim}_q\mu \leqslant \varlimsup_{j\to\infty} I_{n_j} \leqslant \alpha$ so that $\underline{\dim}_q\mu \leqslant \dim_*\mu$.

The inequality $\mathrm{Dim}^*\mu \leqslant \overline{\dim}_p\mu$ with $0 < p < 1$ in (1.2) can be proved similarly by considering

$$B = \left\{x : \varlimsup_{n\to\infty} \frac{\log \mu(Q_n(x))}{\log 2^{-n}} > \beta\right\}$$

where $\beta < \mathrm{Dim}^*\mu$. $\square$

We can establish some further relationship of the L^p-dimension and the entropy dimension, besides Theorem 1.4.

Proposition 2.2. *For* $0 < p < 1 < q$, *we have*

$$\overline{\dim}_q\mu \leqslant \overline{\dim}_e\mu \quad \textit{and} \quad \underline{\dim}_e\mu \leqslant \underline{\dim}_p\mu.$$

Proof. For $q > 1$, by the convexity of $-\log x$, we have

$$\begin{aligned}\frac{\log s_n(q)}{(q-1)\log 2^{-n}} &= \frac{1}{(q-1)n}\left(-\log_2 \int \mu(Q_n(x))^{q-1} d\mu(x)\right)\\ &\leqslant \frac{1}{n}\int -\log_2 \mu(Q_n(x)) d\mu(x)\\ &= \int D_n(x) d\mu(x)\end{aligned}$$

where $D_n(x)$ is defined as in (2.1). It follows that $\overline{\dim}_q\mu \leqslant \overline{\dim}_e\mu$. The proof of the second inequality is similar. □

3. Examples and Remarks

For $0 \leqslant x \leqslant 1$, we write

$$h(x) = -x\log_2 x - (1-x)\log_2(1-x),$$

and for a sequence $\{p_n\}$ with $0 < p_n < 1$, we write

$$\underline{h}(\{p_k\}) = \varliminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} h(p_n), \quad \overline{h}(\{p_k\}) = \varlimsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} h(p_n).$$

We define a probability measure μ on [0, 1], called *Bernoulli product*, by

$$\mu(I_n(x)) = p_1^{\varepsilon_1}\cdots p_n^{\varepsilon_n}$$

where $x = \sum_{i=1}^{\infty} \varepsilon_i 2^{-i}$ with $\varepsilon_i = 0$ or 1, $I_n(x) = \sum_{i=1}^{n} \varepsilon_i 2^{-i} + [0, 2^{-n}]$ and $p_n^1 = p_n$, $p_n^0 = 1 - p_n$.

Proposition 3.1. *For the Bernoulli product* μ *defined above*, *we have*

$$\begin{gathered}\dim_*\mu = \dim^*\mu = \underline{\dim}_e\mu = \underline{h}(\{p_n\})\\ \mathrm{Dim}_*\mu = \mathrm{Dim}^*\mu = \overline{\dim}_e\mu = \overline{h}(\{p_n\}).\end{gathered}$$

Proof. First we claim that for μ-almost all x,

$$\underline{D}(\mu, x) = \underline{h}(\{p_n\}), \quad \overline{D}(\mu, x) = \overline{h}(\{p_n\}).$$

Indeed we have

$$D_n(x) := \frac{\log \mu(Q_n(x))}{\log 2^{-n}} = -\frac{1}{n}\sum_{i=1}^{n} \log_2 p_i^{\varepsilon_i}.$$

Since $\{\log_2 p_i^{\varepsilon_i}\}_{i=1}^{\infty}$ are μ-independent and $\mathbb{E}_\mu(-\log_2 p_i^{\varepsilon_i}) = h(p_i)$, we have, by the law of large numbers,

$$\begin{aligned}\varliminf_{n\to\infty} D_n(x) &= -\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(\log_2 p_i^{\varepsilon_i} + h(p_i)) + \varliminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} h(p_i)\\ &= \varliminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} h(p_i). \qquad (3.1)\end{aligned}$$

The claimed expression of $\underline{D}(\mu,x)$ follows. Similarly we get the expression of $\overline{D}(\mu,x)$. Consequently

$$\dim_*\mu = \dim^*\mu = \underline{h}(\{p_n\}), \quad \mathrm{Dim}_*\mu = \mathrm{Dim}^*\mu = \overline{h}(\{p_n\}).$$

For the entropy dimension, it suffices to notice that

$$\frac{H_n(\mu)}{\log 2} = -\sum_{\varepsilon_1,\dots,\varepsilon_n} p_1^{\varepsilon_1}\cdots p_n^{\varepsilon_n}\left(\sum_{j=1}^{n}\log_2 p_j^{\varepsilon_j}\right) = \sum_{j=1}^{n} h(p_j). \qquad (3.2)$$

□

We will see that it is possible to choose $\{p_n\}$ such that $\underline{h}(\{p_n\}) < \overline{h}(\{p_n\})$.

In the following we will consider the sum of two Bernoulli products. Let μ be defined as above and let ν be another Bernoulli product defined by $\{q_n\}$ with $0 < q_n < 1$. Consider the measures

$$\sigma_s = s\mu + (1-s)\nu, \quad (0 < s < 1).$$

By Theorem 1.2, it is easy to see that

$$\dim_*\sigma_s = \min\{\dim_*\mu, \dim_*\nu\}, \qquad (3.3)$$

$$\dim^*\sigma_s = \max\{\dim^*\mu, \dim^*\nu\}, \qquad (3.4)$$

and similarly for $\mathrm{Dim}_*\sigma_s$ and $\mathrm{Dim}^*\sigma_s$. Notice that the right-hand side is independent of s.

Proposition 3.2. *Let σ_s $(0 < s < 1)$ be defined as above, then*

$$\underline{\dim}_e\sigma_s = \varliminf_{n\to\infty}\frac{1}{n}\left(s\sum_{j=1}^{n} h(p_j) + (1-s)\sum_{j=1}^{n} h(q_j)\right),$$

$$\overline{\dim}_e\sigma_s = \varlimsup_{n\to\infty}\frac{1}{n}\left(s\sum_{j=1}^{n} h(p_j) + (1-s)\sum_{j=1}^{n} h(q_j)\right).$$

Proof. First we see that

$$\begin{aligned}H_n(\sigma_s) &= -\int_{\mathbb{R}^d}\log\sigma_s(I_n(x)d\sigma_s(x)\\ &= -s\mathbb{E}_\mu\log\sigma_s(I_n(x)) - (1-s)\mathbb{E}_\nu\log\sigma_s(I_n(x)).\end{aligned}$$

Since

$$\sigma_s(I_n(x)) = sp_1^{\varepsilon_1}\cdots p_n^{\varepsilon_1}\left(1+\frac{1-s}{s}\frac{q_1^{\varepsilon_1}\cdots q_n^{\varepsilon_n}}{p_1^{\varepsilon_1}\cdots p_n^{\varepsilon_n}}\right),$$

then

$$\log\sigma_s(I_n(x)) = \log s + \log(p_1^{\varepsilon_1}\cdots p_n^{\varepsilon_n}) + \log\left(1+\frac{1-s}{s}\frac{q_1^{\varepsilon_1}\cdots q_n^{\varepsilon_n}}{p_1^{\varepsilon_1}\cdots p_n^{\varepsilon_n}}\right),$$

and therefore

$$\mathbb{E}_\mu \log\sigma_s(I_n(x)) = \log s - \sum_{j=1}^{n} h(p_i) + \mathbb{E}_\mu \log\left(1+\frac{1-s}{s}\frac{q_1^{\varepsilon_1}\cdots q_n^{\varepsilon_n}}{p_1^{\varepsilon_1}\cdots p_n^{\varepsilon_n}}\right).$$

Note that the last expression is nonnegative; the concavity of $\log(1+\frac{1-s}{s}x)$ implies that it is bounded by

$$\log\left(1+\frac{1-s}{s}\mathbb{E}_\mu\left(\frac{q_1^{\varepsilon_1}\cdots q_n^{\varepsilon_n}}{p_1^{\varepsilon_1}\cdots p_n^{\varepsilon_n}}\right)\right).$$

Since

$$\mathbb{E}_\mu\left(\frac{q_1^{\varepsilon_1}\cdots q_n^{\varepsilon_n}}{p_1^{\varepsilon_1}\cdots p_n^{\varepsilon_n}}\right) = \sum q_1^{\varepsilon_1}\cdots q_n^{\varepsilon_n} = 1,$$

we see that

$$\mathbb{E}_\mu \log_2(\sigma_s(I_n(x))) = -\sum_{i=1}^{n} h(p_i) + O(1).$$

We can obtain an analogous identity for the expectation with respect to ν. Finally we have

$$\frac{H_n(\sigma_s)}{\log 2^n} = \frac{s}{n}\sum_{i=1}^{n} h(p_i) + \frac{1-s}{n}\sum_{i=1}^{n} h(q_i) + O(1).$$

and the lemma follows. □

For further discussion, we suppose that μ and ν are mutually singular. A theorem of Kakutani [13] states that if $0 < \inf p_n \leqslant \sup p_n < 1$, then

$$\mu \perp \nu \quad \text{iff} \quad \sum_{n=1}^{\infty}(p_n - q_n)^2 = \infty.$$

We will adjust the choice of the $\{p_n\}$ and $\{q_n\}$ to get $\sigma_s = s\mu + (1-s)\nu$ which illustrates various possible relationships of the dimensions of a measure.

Let $a, b, c, d, \in (0, \frac{1}{2})$ be fixed and distinct, define

$$p_n = \begin{cases} a & \text{if} \quad 2\cdot 2^k \leqslant n < 2\cdot 2^k + 2^k, \\ b & \text{if} \quad 2\cdot 2^k + 2^k \leqslant n < 2\cdot 2^{k+1}, \end{cases}$$

$$q_n = \begin{cases} c & \text{if} \quad 2\cdot 2^k \leqslant n < 2\cdot 2^k + 2^k, \\ d & \text{if} \quad 2\cdot 2^k + 2^k \leqslant n < 2\cdot 2^{k+1}. \end{cases}$$

Then the criterion of Kakutani implies that μ and ν are mutually singular. The local dimension $\underline{D}(\sigma_s, x)$ (as well as $\overline{D}(\sigma_s, x)$) takes two values, one on a support of μ and the other on a support of ν. More precisely, there are two disjoint Borel sets A and B with $\mu(A) = 1$ and $\nu(B) = 1$ such that for $0 < s < 1$

$$\underline{D}(\sigma_s, x) = \underline{h}(\{p_n\})1_A(x) + \underline{h}(\{q_n\})1_B(x) \quad \sigma_s\text{-a.e.},$$
$$\overline{D}(\sigma_s, x) = \overline{h}(\{p_n\})1_A(x) + \overline{h}(\{q_n\})1_B(x) \quad \sigma_s\text{-a.e.}$$

where

$$\underline{h}(\{p_n\}) = \min\left\{\frac{2h(a)+h(b)}{3}, \frac{h(a)+h(b)}{2}\right\},$$
$$\overline{h}(\{p_n\}) = \max\left\{\frac{2h(a)+h(b)}{3}, \frac{h(a)+h(b)}{2}\right\}.$$

and similar expressions hold for $\underline{h}(\{q_n\})$ and $\overline{h}(\{q_n\})$. We have the liberty to choose $a, b, c, d \in (0, 1/2)$ to exhibit different situations. For example,

1. $\dim^*\sigma_1 < \mathrm{Dim}_*\sigma_1$ if $a < b$.
2. $\dim^*\sigma_s > \mathrm{Dim}_*\sigma_s$ if $0 < s < 1, a < b, c < d$ and $\frac{2h(a)+h(b)}{3} > \frac{h(c)+h(d)}{2}$.

We can construct more examples by the following procedure. Assume $0 < a < b < \frac{1}{2}$ and $0 < d < c < \frac{1}{2}$ (it is not $c < d$!). Then

$$\underline{A} = \varliminf_{n\to\infty} \sum_{i=1}^{n} h(p_i) = \frac{2h(a)+h(b)}{3}, \quad \overline{A} = \varlimsup_{n\to\infty} \sum_{i=1}^{n} h(p_i) = \frac{h(a)+h(b)}{2},$$

$$\underline{B} = \varliminf_{n\to\infty} \sum_{i=1}^{n} h(q_i) = \frac{h(c)+h(d)}{2}, \quad \overline{B} = \varlimsup_{n\to\infty} \sum_{i=1}^{n} h(q_i) = \frac{2h(c)+h(d)}{3}.$$

Hence for $\sigma_s = s\mu + (1-s)\nu$, we have

$$\dim^*\sigma_s = \max\{\underline{A}, \underline{B}\},$$
$$\mathrm{Dim}_*\sigma_s = \min\{\overline{A}, \overline{B}\},$$
$$\underline{\dim}_e\sigma_s = \min\{s\overline{A} + (1-s)\underline{B}, s\underline{A} + (1-s)\overline{B}\},$$
$$\overline{\dim}_e\sigma_s = \max\{s\overline{A} + (1-s)\underline{B}, s\underline{A} + (1-s)\overline{B}\}.$$

For these four dimensions, we can choose a, b, c, d, so that

3. We can choose $a < b, d < c$ so that $\underline{A} < \overline{A} < \underline{B} < \overline{B}$. Then

$$\mathrm{Dim}_*\sigma_s = \overline{A} < \underline{B} = \underline{\dim}_e\sigma_s.$$

We can adjust the s to fit $\underline{\dim}_e\sigma_s$ and $\overline{\dim}_e\sigma_s$ in between $\overline{A}$ and $\underline{B}$:

$$\underline{\dim}_e\sigma_s < \mathrm{Dim}_*\sigma_s < \overline{\dim}_e\sigma_s < \dim^*\sigma_s, \quad \text{for } s \text{ near } 1,$$
$$\mathrm{Dim}_*\sigma_s < \underline{\dim}_e\sigma_s < \dim^*\sigma_s < \overline{\dim}_e\sigma_s, \quad \text{for } s \text{ near } 0,$$
$$\mathrm{Dim}_*\sigma_s < \underline{\dim}_e\sigma_s < \overline{\dim}_e\sigma_s < \dim^*\sigma_s, \quad \text{for } s \text{ suitable}.$$

4. For $\underline{A} < \underline{B} < \overline{A} < \overline{B}$, then we have similarly

$$\begin{aligned}
\underline{\dim}_e \sigma_s < \dim^* \sigma_s < \overline{\dim}_e \sigma_s < \dim_* \sigma_s, &\quad \text{for } s \text{ near } 1,\\
\dim^* \sigma_s < \underline{\dim}_e \sigma_s < \mathrm{Dim}_* \sigma_s < \overline{\dim}_e \sigma_s, &\quad \text{for } s \text{ near } 0,\\
\dim^* \sigma_s < \underline{\dim}_e \sigma_s < \overline{\dim}_e \sigma_s < \mathrm{Dim}_* \sigma_s, &\quad \text{for } s \text{ suitable.}
\end{aligned}$$

We conclude from these examples that the four dimensions $\dim^* \mu$, $\underline{\dim}_e \mu$, $\overline{\dim}_e \mu$, $\mathrm{Dim}_* \mu$ are not comparable, except $\underline{\dim}_e \mu \leqslant \overline{\dim}_e \mu$.

The idea of using Bernoulli measure comes from [7] where it was called dyadic Riesz product. It is then natural to use barycenters σ_s of Bernoulli measures to construct more exotic measures. Such constructions also appear in [12], [1], [2]. There are some results similar to ours in [12], [1].

Finally we remark that for the invariant measure μ generated by any contractive IFS [10], or for measures which are ergodic with respect to a map preserving Hausdorff dimension [9],

$$\underline{D}(\mu, x) = \underline{d} \quad \text{and} \quad \overline{D}(\mu, x) = \overline{d} \quad \mu\text{-a.e.}$$

Thus $\dim_* \mu = \dim^* \mu = \underline{d}$, $\mathrm{Dim}_* \mu = \mathrm{Dim}^* \mu = \overline{d}$. A natural question is to know whether $\underline{D}(\mu, x) = \overline{D}(\mu, x) = \text{constant}\, \mu$-a.e. for self-similar measures of self-conformal measures. The statement is true if the conformal IFS satisfies the open set condition (see [11]). It is conjectured that the open set condition is superfluous. Eckmann and Ruelle [5] suggested that $\underline{D}(\mu, x) = \overline{D}(\mu, x) = \text{constant}\, \mu$-a.e. for any ergodic measure. Later, Cutler [4] showed by a counterexample that it is not true in general. However, an affirmative answer was obtained by Barreira, Pesin and Schmeling for hyperbolic measures [3].

For L^q-dimension and entropy dimension, Peres and Solomyak [16] have recently proved that $\underline{\dim}_q \mu = \overline{\dim}_q \mu$ and $\underline{\dim}_e \mu = \overline{\dim}_e \mu$ for any self-conformal measure without the open set condition.

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